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Research Article

New Exact Solitary Solutions for Stochastic Graphene Sheets Equation Using Two Different Methods

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Abstract: The Stochastic Graphene Sheets Equation (SGSE) is taken into consideration in this study. By applying two different methods such as the Sardar subequation method and Jacobi Elliptic Function (JEF) method, we obtain new periodic solitons, dark and bright solitons, anti-Kink and Kink solitons solutions for the SGSE. Because graphene sheets are important in many fields, such as electronics, photonics, and energy storage, the solutions of the stochastic graphene sheets model are beneficial in understanding several fascinating scientific phenomena. Using MATLAB, we exhibit several 2D and 3D graphs that illustrate the impact of multiplicative noise on the exact solutions of SGSE.

Keywords: stochastic PDE, stability by noise, exact solutions, Sardar subequation method, Jacobi elliptic function method

MSC: 35A20, 60H15, 60H10, 35Q51

1. Introduction

Partial Differential Equations (PDEs) serve as fundamental tools in modeling a variety of physical phenomena, making their solutions crucial for real-life applications. Solutions to PDEs enable us to model complex phenomena including fluid dynamics, heat transfer, quantum mechanics, financial mathematics, and more. The importance of these solutions lie in their ability to provide insight into the behavior of systems governed by these equations, allowing for predictions and optimizations are crucial for both theoretical understanding and practical applications. Recently, many practical and effective methods, such as the sine-Gordon expansion technique [1], the F-expansion method [2, 3], the Riccati equation method [4], Jacobi elliptic function expansion [5], modified extended tanh function method [6, 7], (G'/G)-expansion [8, 9], exp-function method [10], $exp(-\phi(\varsigma))$-expansion method [11], sine-cosine method [12, 13], modified extended direct algebraic method [14, 15], Kudryashov-expansion method [16], extended tanh method [17], Hirota bilinear method [18], (G'/G^2)-expansion method [19], modified auxiliary equation method [20], generalized exponential rational functional method [21], etc, have recently been devised to get exact solutions for PDEs.

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Graphene, a two-dimensional allotrope of carbon, has garnered significant attention in material science due to its unique properties, including extraordinary electrical conductivity, exceptional mechanical strength, and remarkable thermal conductivity [22–24]. Stochastic graphene sheets, which incorporate a probabilistic approach to modeling the arrangement and behavior of graphene atoms, have emerged as a fascinating area of research. By incorporating randomness and variability into the design and manufacturing processes, researchers are able to explore innovative applications and enhance the functionalities of graphene-based materials.

The concept of stochasticity in graphene sheets allows scientists to consider the inherent irregularities and fluctuations that occur during the synthesis of these materials. Traditional approaches often assume a perfectly ordered lattice structure, neglecting the influence of defects and irregularities. However, real-world graphene often contains vacancies, structural defects, and varying atom placements that can significantly alter its properties. By employing stochastic models, researchers can predict how these variations affect the electronic, mechanical, and thermal characteristics of graphene sheets, providing a more accurate representation of their behavior in practical applications.

Stochastic graphene sheets hold great promise for various applications in electronics, energy storage, and composite materials. For instance, in the field of electronics, incorporating stochastic features can lead to the development of novel transistors with enhanced performance, as the random arrangement of dopants may optimize charge carrier mobility. In energy storage, stochastic graphene structures have the potential to improve the efficiency of supercapacitors and batteries by allowing for more efficient ion transport through irregular pathways in the material. Similarly, in composite materials, the incorporation of stochastic graphene can enhance the mechanical properties, providing better strength and flexibility by enabling more effective load distribution.

In this study, we consider the $(2+1)$-dimensional Stochastic Graphene Sheets Equation (SGSE) in the following form:

$$\mathscr{G}_{tx}+(\mathscr{G}_{xxx}+\mathscr{G}\mathscr{G}_{x}+\alpha_1\mathscr{G}_{x})_x+\alpha_2\mathscr{G}_{yy}=\mu\mathscr{G}_x\mathscr{W}_t, \tag{1}$$

where $\mathscr{G}(x,\ y,\ t)$ denotes the amplitude of the real wave function in the graphene sheet; $\mathscr{W}(t)$ is the Brownian motion; α_1 and α_2 are real constants; μ is the intensity of noise. Recently, Khater et al. [25] used many approaches, including the generalized rational, Khater II, and Khater III methods, to achieve the exact solutions of Eq. (1) with $\mu=0$. While, Mohammed et al. [26] obtained the stochastic exact solutions for SGSE (1) by using the extended tanh function method.

The aim of this paper is to determine the exact stochastic solutions of SGSE (1) by utilizing the JEF method and Sardar subequation method. The SGSE (1) has vital applications in domains such as electronics, photonics, and energy storage; therefore, the derived solutions may be utilized to examine many important scientific phenomena. We also give several 2D and 3D graphical representations using the MATLAB application to study how noise impacts the analytical solutions of the SGSE (1).

The paper is arranged as follows: In section 2, the wave equation for the SGSE (1) is obtained. In section 3, the stochastic solution of the SGSE (1) is acquired. In section 4, we discuss the impact of Brownian motion on the obtained solutions of the SGSE (1). Finally, the conclusions of this study are given.

2. Traveling wave Eq. for SGSE

The next wave transformation

$$\mathscr{G}(x,\ y,\ t)=\mathscr{Z}(\xi)e^{[\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t]}, \quad \xi=\xi_1 x+\xi_2 y+\xi_3 t, \tag{2}$$

where $\mathscr{Z}$ is a deterministic function, $\xi_1,\ \xi_2$ and ξ_3 are unknown constants, is utilized to create the wave equation for SGSE (1). We see that

$$\mathscr{G}_t = [\xi_3 \mathscr{Z}' + \mu \mathscr{Z} \mathscr{W}_t + \frac{1}{2}\mu^2 \mathscr{Z} - \frac{1}{2}\mu^2 \mathscr{Z}] e^{[\mu \mathscr{W}(t) - \frac{1}{2}\mu^2 t]}$$

$$= [\xi_3 \mathscr{Z}' + \mu \mathscr{Z} \mathscr{W}_t] e^{[\mu \mathscr{W}(t) - \frac{1}{2}\mu^2 t]}, \tag{3}$$

and

$$\mathscr{G}_{xt} = [\xi_1 \xi_3 \mathscr{Z}'' + \mu \xi_1 \mathscr{Z}' \mathscr{W}_t] e^{[\mu \mathscr{W}(t) - \frac{1}{2}\mu^2 t]}, \ \mathscr{G}_x = \xi_1 \mathscr{Z}' e^{[\mu \mathscr{W}(t) - \frac{1}{2}\mu^2 t]}, \tag{4}$$

$$\mathscr{G}_{yy} = \xi_2^2 \mathscr{Z}'' e^{[\mu \mathscr{W}(t) - \frac{1}{2}\mu^2 t]}, \ \mathscr{G}_{xxxx} = \xi_1^4 \mathscr{Z}'''' e^{[\mu \mathscr{W}(t) - \frac{1}{2}\mu^2 t]}. \tag{5}$$

Plugging Eq. (2) into Eq. (1) and using (3-5), we attain

$$\xi_1^4 \mathscr{Z}'''' + (\xi_1 \xi_3 + \alpha_1 \xi_1^2 + \alpha_2 \xi_2^2) \mathscr{Z}'' + \xi_1^2 (\mathscr{Z} \mathscr{Z}')' e^{[\mu \mathscr{W}(t) - \frac{1}{2}\mu^2 t]} = 0. \tag{6}$$

Taking the expectations, we have

$$\xi_1^4 \mathscr{Z}'''' + (\xi_1 \xi_3 + \alpha_1 \xi_1^2 + \alpha_2 \xi_2^2) \mathscr{Z}'' + \xi_1^2 (\mathscr{Z} \mathscr{Z}')' e^{-\frac{1}{2}\mu^2 t} \mathbb{E} e^{[\mu \mathscr{W}(t)]} = 0. \tag{7}$$

Since $\mathscr{W}(t)$ has a Normal distribution, then $\mathbb{E}(e^{\mu \mathscr{W}(t)}) = e^{\frac{1}{2}\mu^2 t}$. Hence, Eq. (7) tends to

$$\xi_1^4 \mathscr{Z}'''' + (\xi_1 \xi_3 + \alpha_1 \xi_1^2 + \alpha_2 \xi_2^2) \mathscr{Z}'' + \xi_1^2 (\mathscr{Z} \mathscr{Z}')' = 0. \tag{8}$$

After twice integrating Eq. (8) and neglecting the integration constant, we obtain

$$\mathscr{Z}'' + \mathscr{A}_1 \mathscr{Z} + \mathscr{A}_2 \mathscr{Z}^2 = 0, \tag{9}$$

where

$$\mathscr{A}_1 = \frac{\xi_1 \xi_3 + \alpha_1 \xi_1^2 + \alpha_2 \xi_2^2}{\xi_1^4} \ \text{and} \ \mathscr{A}_2 = \frac{1}{2\xi_1^2}.$$

3. Exact solutions of SGSE

We solve the wave equation (9) by using the JEF method and the Sardar subequation method. The SGSE solutions (1) may then be derived by applying the transformation (2).

3.1 *JEF-method*

Here, we apply the JEF-method (see [27]). Assuming the solutions to Eq. (9) have the form:

$$\mathscr{Z}(\xi) = \sum_{j=0}^{m} a_j [\chi(\xi)]^j, \tag{10}$$

where $\chi(\xi) = sn(\xi,\, \tilde{n})$, for $0 < \tilde{n} < 1$, is Jacobi elliptic sine function; $a_0,\ a_1,\ ....,\ a_M$ are unknown constants and $a_M \neq 0$. To determine m, we balance $\mathscr{Z}^2$ with $\mathscr{Z}''$ in Eq. (9) to have

$$2m = m + 2,$$

hence

$$m = 2. \tag{11}$$

By utilizing Eq. (11), we rewrite Eq. (10) as

$$\mathscr{Z}(\xi) = a_0 + a_1\chi(\xi) + a_2\chi^2(\xi), \tag{12}$$

Differentiating Eq. (12) twice

$$\mathscr{Z}''(\xi) = 2a_2 - a_1(\tilde{n}^2+1)\chi - 4a_2(\tilde{n}^2+1)\chi^2 + 2a_1\tilde{n}^2\chi^3 + 6a_2\tilde{n}^2\chi^4. \tag{13}$$

Substituting Eqs (12) and (13) into Eq. (9), we attain

$$(6\tilde{n}^2a_2 + \mathscr{A}_2a_2^2)\chi^4 + (2\tilde{n}^2a_1 + 2\mathscr{A}_2a_1a_2)\chi^3 + (2a_0\mathscr{A}_2a_2 - 4a_2(\tilde{n}^2+1) + \mathscr{A}_1a_2 + \mathscr{A}_2a_1^2)\chi^2$$

$$-[(\tilde{n}^2+1)a_1 - \mathscr{A}_1a_1 - 2\mathscr{A}_2a_0a_1]\chi + (2a_2 + \mathscr{A}_1a_0 + \mathscr{A}_2a_0^2) = 0.$$

Putting the coefficient of χ^n equal to zero, we get for $n = 4,\ 3,\ 2,\ 1,\ 0$:

$$6\tilde{n}^2a_2 + \mathscr{A}_2a_2^2 = 0,$$

$$2\tilde{n}^2a_1 + 2\mathscr{A}_2a_1a_2 = 0,$$

$$2a_0\mathscr{A}_2a_2 - 4a_2(\tilde{n}^2+1) + \mathscr{A}_1a_2 + \mathscr{A}_2a_1^2 = 0,$$

$$(\tilde{n}^2+1)a_1-\mathscr{A}_1a_1-2\mathscr{A}_2a_0a_1=0,$$

an

$$2a_2+\mathscr{A}_1a_0+\mathscr{A}_2a_0^2=0.$$

The following two sets are obtained by solving these equations:
First set:

$$\begin{cases} a_0=\dfrac{2(\tilde{n}^2+1)-2\sqrt{\tilde{n}^4-\tilde{n}^2+1}}{\mathscr{A}_2}, \\ a_1=0, \\ a_2=\dfrac{-6\tilde{n}^2}{\mathscr{A}_2}, \\ \xi_2=\dfrac{-\xi_1}{1+4\xi_1^2\sqrt{\tilde{n}^4-\tilde{n}^2+1}}. \end{cases}$$

Second set:

$$\begin{cases} a_0=\dfrac{2(\tilde{n}^2+1)+2\sqrt{\tilde{n}^4-\tilde{n}^2+1}}{\mathscr{A}_2}, \\ a_1=0, \\ a_2=\dfrac{-6\tilde{n}^2}{\mathscr{A}_2}, \\ \xi_2=\dfrac{-\xi_1}{1-4\xi_1^2\sqrt{\tilde{n}^4-\tilde{n}^2+1}}. \end{cases}$$

For the first set, the solution of SGSE (1) in the elliptic function form, utilizing (12), is

$$\mathscr{G}(x,\ y,\ t)=\Big(\frac{2(\tilde{n}^2+1)-2\sqrt{\tilde{n}^4-\tilde{n}^2+1}}{\mathscr{A}_2}-\frac{6\tilde{n}^2}{\mathscr{A}_2}sn^2(\xi,\ \tilde{n})\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2t)}, \tag{14}$$

where $\xi=\xi_1x-\dfrac{\xi_1}{1+4\xi_1^2\sqrt{\tilde{n}^4-\tilde{n}^2+1}}t$. If $\tilde{n}\to 1$, then Eq. (14) becomes

$$\mathscr{G}(x,\ y,\ t)=\left[\frac{2}{\mathscr{A}_2}-\frac{6}{\mathscr{A}_2}\tanh^2(\xi)\right]e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2t)}, \tag{15}$$

where $\xi=\xi_1x-\dfrac{\xi_1}{1+4\xi_1^2}t$.

For the second set, the solutions of SGSE (1), utilizing (12), is

$$\mathscr{G}(x,\ y,\ t)=\Big(\frac{2(\tilde{n}^2+1)+2\sqrt{\tilde{n}^4-\tilde{n}^2+1}}{\mathscr{A}_2}-\frac{6\tilde{n}^2}{\mathscr{A}_2}sn^2(\xi,\ \tilde{n})\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2t)}\ , \tag{16}$$

where $\xi = \xi_1 x - \dfrac{\xi_1}{1-4\xi_1^2\sqrt{\tilde{n}^4-\tilde{n}^2+1}}t$.

If $\tilde{n} \to 1$, then Eq. (16) becomes

$$\mathscr{G}(x,\ y,\ t) = \Big(\frac{6}{\mathscr{A}_2} - \frac{6}{\mathscr{A}_2}\tanh^2(\xi)\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)} = \frac{6}{\mathscr{A}_2}\operatorname{sech}^2(\xi)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}, \tag{17}$$

where $\xi = \xi_1 x - \dfrac{\xi_1}{1-4\xi_1^2}t$.

Similarly, we can change $sn(\xi,\ \tilde{n})$ in (12) by $cn(\xi,\ \tilde{n})$ to get the following elliptic solution of the SGSE (1):

$$\mathscr{G}(x,\ y,\ t) = \Big(\frac{(2-4\tilde{n}^2)-2\sqrt{\tilde{n}^4-\tilde{n}^2+1}}{\mathscr{A}_2} + \frac{6\tilde{n}^2}{\mathscr{A}_2}cn^2(\xi,\ \tilde{n})\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}, \tag{18}$$

or

$$\mathscr{G}(x,\ y,\ t) = \Big(\frac{(2-4\tilde{n}^2)+2\sqrt{\tilde{n}^4-\tilde{n}^2+1}}{\mathscr{A}_2} + \frac{6\tilde{n}^2}{\mathscr{A}_2}cn^2(\xi,\ \tilde{n})\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}. \tag{19}$$

If $\tilde{n} \to 1$, then the solutions (18) tends to

$$\mathscr{G}(x,\ y,\ t) = \Big(\frac{-4}{\mathscr{A}_2} + \frac{6}{\mathscr{A}_2}\operatorname{sech}^2(\xi)\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}, \tag{20}$$

or

$$\mathscr{G}(x,\ y,\ t) = \frac{6}{\mathscr{A}_2}\operatorname{sech}^2(\xi)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}, \tag{21}$$

3.2 *Sardar subequation method*

Supposing that the solution of Eq. (9) takes the next, with $m = 2$, form:

$$\mathscr{X}(\xi) = \ell_0 + \ell_1\mathscr{P} + \ell_2\mathscr{P}^2, \tag{22}$$

where $\mathscr{P}$ solves

$$\mathscr{P}' = \sqrt{\mathscr{P}^4 + \hbar_1\mathscr{P}^2 + \hbar_2}, \tag{23}$$

where $\hbar 1$, and $\hbar_2$ are real constants. For the solutions of Eq. (23), there are several cases that rely on $\hbar_1$, and $\hbar_2$ as follows:

Case 1: If $\hbar_1 > 0$, and $\hbar_2 = 0$, then

$$\mathscr{P}_1(\xi) = \pm\sqrt{pq\hbar_1}\text{sech}_{pq}(\sqrt{\hbar_1}\xi), \tag{24}$$

and

$$\mathscr{P}_2(\xi) = \pm\sqrt{pq\hbar_1}\text{csch}_{pq}(\sqrt{\hbar_1}\xi), \tag{25}$$

where

$$\text{sech}_{pq}(\theta) = \frac{2}{pe^{\theta}+qe^{-\theta}} \text{ and } \text{csch}_{pq}(\theta) = \frac{2}{pe^{\theta}-qe^{-\theta}}.$$

Case 2: If $\hbar_1 < 0$, and $\hbar_2 = 0$, then

$$\mathscr{P}_3(\xi) = \pm\sqrt{-pq\hbar_1}\sec_{pq}(\sqrt{-\hbar_1}\xi), \tag{26}$$

and

$$\mathscr{P}_4(\xi) = \pm\sqrt{-pq\hbar_1}\csc_{pq}(\sqrt{-\hbar_1}\xi), \tag{27}$$

where

$$\sec_{pq}(\theta) = \frac{2}{pe^{\theta i}+qe^{-\theta i}} \text{ and } \csc_{pq}(\theta) = \frac{2}{pe^{\theta i}-qe^{-\theta i}}.$$

Case 3: If $\hbar_1 < 0$, and $\hbar_2 = \frac{\hbar_1^2}{4}$, then

$$\mathscr{P}_5(\xi) = \pm\sqrt{\frac{-\hbar_1}{2}}\tanh_{pq}(\sqrt{\frac{-\hbar_1}{2}}\xi), \tag{28}$$

$$\mathscr{P}_6(\xi) = \pm\sqrt{\frac{-\hbar_1}{2}}\coth_{pq}(\sqrt{\frac{-\hbar_1}{2}}\xi), \tag{29}$$

$$\mathscr{P}_7(\xi) = \pm\sqrt{\frac{-\hbar_1}{2}}\Big(\coth_{pq}(\sqrt{-2\hbar_1}\xi)\pm\sqrt{pq}\text{csch}_{pq}(\sqrt{-2\hbar_1}\xi)\Big), \tag{30}$$

and

$$\mathscr{P}_8(\xi) = \pm\sqrt{\frac{-\hbar_1}{8}}\Big(\tanh_{pq}(\sqrt{\frac{-\hbar_1}{8}}\xi) + \coth_{pq}(\sqrt{\frac{-\hbar_1}{8}}\xi)\Big), \tag{31}$$

where

$$\tanh_{pq}(\theta) = \frac{pe^{\theta} - qe^{-\theta}}{pe^{\theta} + qe^{-\theta}} \text{ and } \coth_{pq}(\theta) = \frac{pe^{\theta} + qe^{-\theta}}{pe^{\theta} - qe^{-\theta}}.$$

Case 4: If $\hbar_1 > 0$, and $\hbar_2 = \frac{\hbar_1^2}{4}$, then

$$\mathscr{P}_9(\xi) = \pm\sqrt{\frac{\hbar_1}{2}}\tan_{pq}(\sqrt{\frac{\hbar_1}{2}}\xi), \tag{32}$$

$$\mathscr{P}_{10}(\xi) = \pm\sqrt{\frac{\hbar_1}{2}}\cot_{pq}(\sqrt{\frac{\hbar_1}{2}}\xi), \tag{33}$$

$$\mathscr{P}_{11}(\xi) = \pm\sqrt{\frac{\hbar_1}{2}}\Big(\tan_{pq}(\sqrt{2\hbar_1}\xi) \pm \sqrt{pq}\sec_{pq}(\sqrt{2\hbar_1}\xi)\Big), \tag{34}$$

$$\mathscr{P}_{12}(\xi) = \pm\sqrt{\frac{\hbar_1}{2}}\Big(\cot_{pq}(\sqrt{2\hbar_1}\xi) \pm \sqrt{pq}\csc_{pq}(\sqrt{2\hbar_1}\xi)\Big), \tag{35}$$

and

$$\mathscr{P}_{13}(\xi) = \pm\sqrt{\frac{\hbar_1}{8}}\Big(\tan_{pq}(\sqrt{\frac{\hbar_1}{8}}\xi) + \cot_{pq}(\sqrt{\frac{\hbar_1}{8}}\xi)\Big), \tag{36}$$

where

$$\tan_{pq}(\theta) = -i\frac{pe^{\theta i} - qe^{-\theta i}}{pe^{\theta i} + qe^{-\theta i}} \text{ and } \cot_{pq}(\theta) = i\frac{pe^{\theta i} + qe^{-\theta i}}{pe^{\theta i} - qe^{-\theta i}}.$$

Now, we try to solve the wave equation (9). By differentiating Eq. (22) twice and utilizing (23), we get

$$\mathscr{Z}'' = \ell_1(\hbar_1\mathscr{P} + 2\mathscr{P}^3) + 2\ell_2(\hbar_2 + 2\hbar_1\mathscr{P}^2 + 3\mathscr{P}^4). \tag{37}$$

Setting Eq. (22) and Eq. (37) into Eq. (9) we have

$$(6\ell_2 + \mathscr{A}_2\ell_2^2)\mathscr{P}^4 + (2\ell_1 + 2\ell_1\ell_2\mathscr{A}_2)\mathscr{P}^3 + (4\ell_2\hbar_1 + 2\mathscr{A}_2\ell_0\ell_2 + \ell_1^2$$

$$+\ell_2\mathscr{A}_1)\mathscr{P}^2+(\ell_1\hbar_1+2\mathscr{A}_2\ell_0\ell_1+\mathscr{A}_1\ell_1)\mathscr{P}+(2\hbar_2\ell_2+\mathscr{A}_1\ell_0+\mathscr{A}_2\ell_0^2)=0.$$

If we set all of the coefficients of $\mathscr{P}^k$ to zero, we have an algebraic system of equations. When we solve this system for $\hbar_1^2-3\hbar_2>0$, we obtain the following two sets:

Set I:

$$\ell_0=\Big(\frac{-2\hbar_1-2\sqrt{(\hbar_1^2-3\hbar_2)}}{\mathscr{A}_2}\Big),\ \ \ell_1=0,\ \ell_2=\frac{-6}{\mathscr{A}_2},\ \ \mathscr{A}_1=4\sqrt{(\hbar_1^2-3\hbar_2)}. \tag{38}$$

Set II:

$$\ell_0=\Big(\frac{-2\hbar_1+2\sqrt{(\hbar_1^2-3\hbar_2)}}{\mathscr{A}_2}\Big),\ \ \ell_1=0,\ \ell_2=\frac{-6}{\mathscr{A}_2},\ \ \mathscr{A}_1=-4\sqrt{(\hbar_1^2-3\hbar_2)}. \tag{39}$$

Set I: The solution of Eq. (9) by using (38) is

$$\mathscr{Z}(\xi)=\frac{-2\hbar_1-2\sqrt{(\hbar_1^2-3\hbar_2)}}{\mathscr{A}_2}-\frac{6}{\mathscr{A}_2}\mathscr{P}^2(\xi). \tag{40}$$

As a result, the SGSE solution (1) is

$$\mathscr{G}(x,\ y,\ t)=\Big(\frac{-2\hbar_1-2\sqrt{(\hbar_1^2-3\hbar_2)}}{\mathscr{A}_2}-\frac{6}{\mathscr{A}_2}\mathscr{P}^2(\xi)\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2t)}. \tag{41}$$

Now, we obtain by using Eqs (24)-(36):

Case I-1: If $\hbar_1>0$, and $\hbar_2=0$, then

$$\mathscr{G}(x,\ y,\ t)=\Big(\frac{-4\hbar_1}{\mathscr{A}_2}-\frac{6pq\hbar_1}{\mathscr{A}_2}\mathrm{sech}_{pq}^2(\sqrt{\hbar_1}\xi)\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2t)}, \tag{42}$$

and

$$\mathscr{G}(x,\ y,\ t)=\Big(\frac{-4\hbar_1}{\mathscr{A}_2}-\frac{6pq\hbar_1}{\mathscr{A}_2}\mathrm{csch}_{pq}^2(\sqrt{\hbar_1}\xi)\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2t)}, \tag{43}$$

Case I-2: If $\hbar_1<0$, and $\hbar_2=0$, then

$$\mathscr{G}(x,\ y,\ t)=\frac{6pq\hbar_1}{\mathscr{A}_2}\sec_{pq}^2(\sqrt{-\hbar_1}\xi)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2t)}, \tag{44}$$

and

$$\mathscr{G}(x,\ y,\ t)=\frac{6pq\hbar_1}{\mathscr{A}_2}\operatorname{csc}_{pq}(\sqrt{-\hbar_1}\xi)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}. \tag{45}$$

Case I-3: If $\hbar_1 < 0$, and $\hbar_2 = \dfrac{\hbar_1^2}{4}$, then

$$\mathscr{G}(x,\ y,\ t)=\Big(\frac{-\hbar_1}{\mathscr{A}_2}+\frac{3\hbar_1}{\mathscr{A}_2}\tanh_{pq}^2(\sqrt{\frac{-\hbar_1}{2}}\xi)\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}, \tag{46}$$

$$\mathscr{G}(x,\ y,\ t)=\Big(\frac{-\hbar_1}{\mathscr{A}_2}+\frac{3\hbar_1}{\mathscr{A}_2}\coth_{pq}^2(\sqrt{\frac{-\hbar_1}{2}}\xi)\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}, \tag{47}$$

$$\mathscr{G}(x,\ y,\ t)=\Big(\frac{-\hbar_1}{\mathscr{A}_2}+\frac{3\hbar_1}{\mathscr{A}_2}(\coth_{pq}(\sqrt{-2\hbar_1}\xi)\pm\sqrt{pq}\operatorname{csch}_{pq}(\sqrt{-2\hbar_1}\xi))^2\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}, \tag{48}$$

and

$$\mathscr{G}(x,\ y,\ t)=\Big(\frac{-\hbar_1}{\mathscr{A}_2}+\frac{3\hbar_1}{4\mathscr{A}_2}(\tanh_{pq}(\sqrt{\frac{-\hbar_1}{8}}\xi)+\coth_{pq}(\sqrt{\frac{-\hbar_1}{8}}\xi))^2\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}. \tag{49}$$

Case I-4: If $\hbar_1 > 0$, and $\hbar_2 = \dfrac{\hbar_1^2}{4}$, then

$$\begin{aligned}\mathscr{G}(x,\ y,\ t)&=\left(\frac{-3\hbar_1}{\mathscr{A}_2}-\frac{3\hbar_1}{\mathscr{A}_2}\tan_{pq}^2\left(\sqrt{\frac{\hbar_1}{2}}\xi\right)\right)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}\\&=\frac{-3\hbar_1}{\mathscr{A}_2}\sec_{pq}^2\left(\sqrt{\frac{\hbar_1}{2}}\xi\right)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)},\end{aligned} \tag{50}$$

$$\begin{aligned}\mathscr{G}(x,\ y,\ t)&=\left(\frac{-3\hbar_1}{\mathscr{A}_2}-\frac{3\hbar_1}{\mathscr{A}_2}\cot_{pq}^2\left(\sqrt{\frac{\hbar_1}{2}}\xi\right)\right)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}\\&=\frac{-3\hbar_1}{\mathscr{A}_2}\csc_{pq}^2\left(\sqrt{\frac{\hbar_1}{2}}\xi\right)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)},\end{aligned} \tag{51}$$

$$\mathscr{G}(x,\ y,\ t)=\left(\frac{-3\hbar_1}{\mathscr{A}_2}-\frac{3\hbar_1}{\mathscr{A}_2}\left(\tan_{pq}(\sqrt{2\hbar_1}\xi)\pm\sqrt{pq}\sec_{pq}\left(\sqrt{2\hbar_1}\xi\right)\right)^2\right)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}, \tag{52}$$

$$\mathscr{G}(x,\ y,\ t)=\left(\frac{-3\hbar_1}{\mathscr{A}_2}-\frac{3\hbar_1}{\mathscr{A}_2}\left(\cot_{pq}(\sqrt{2\hbar_1}\xi)\pm\sqrt{pq}\csc_{pq}\left(\sqrt{2\hbar_1}\xi\right)\right)^2\right)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}, \tag{53}$$

and

$$\mathscr{G}(x,\ y,\ t)=\Big(\frac{-3\hbar_1}{\mathscr{A}_2}-\frac{3\hbar_1}{4\mathscr{A}_2}\left(\tan_{pq}\left(\sqrt{\frac{\hbar_1}{8}}\xi\right)+\cot_{pq}(\sqrt{\frac{\hbar_1}{8}}\xi)\right)^2\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2t)}. \tag{54}$$

Set II: The solution of Eq. (9) by using (39) is

$$\mathscr{Z}(\xi)=\frac{-2\hbar_1+2\sqrt{(\hbar_1^2-3\hbar_2)}}{\mathscr{A}_2}-\frac{6}{\mathscr{A}_2}\mathscr{P}^2(\xi). \tag{55}$$

As a result, the SGSE solution (1) is

$$\mathscr{G}(x,\ y,\ t)=\left(\frac{-2\hbar_1+2\sqrt{(\hbar_1^2-3\hbar_2)}}{\mathscr{A}_2}-\frac{6}{\mathscr{A}_2}\mathscr{P}^2(\xi)\right)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2t)}. \tag{56}$$

Now, we obtain by using Eqs (24)-(36):
Case II-1: If $\hbar_1>0$, and $\hbar_2=0$, then

$$\mathscr{G}(x,\ y,\ t)=-\frac{6pq\hbar_1}{\mathscr{A}_2}\operatorname{sech}^2_{pq}(\sqrt{\hbar_1}\xi)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2t)}, \tag{57}$$

and

$$\mathscr{G}(x,\ y,\ t)=-\frac{6pq\hbar_1}{\mathscr{A}_2}\operatorname{csch}^2_{pq}(\sqrt{\hbar_1}\xi)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2t)}, \tag{58}$$

Case II-2: If $\hbar_1<0$, and $\hbar_2=0$, then

$$\mathscr{G}(x,\ y,\ t)=\Big(\frac{-4\hbar_1}{\mathscr{A}_2}+\frac{6pq\hbar_1}{\mathscr{A}_2}\sec^2_{pq}(\sqrt{-\hbar_1}\xi)\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2t)}, \tag{59}$$

and

$$\mathscr{G}(x,\ y,\ t)=\Big(\frac{-4\hbar_1}{\mathscr{A}_2}+\frac{6pq\hbar_1}{\mathscr{A}_2}\csc_{pq}(\sqrt{-\hbar_1}\xi)\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2t)}. \tag{60}$$

Case II-3: If $\hbar_1<0$, and $\hbar_2=\dfrac{\hbar_1^2}{4}$, then we get the following hyperbolic solutions

$$\mathscr{G}(x,\ y,\ t)=\Big(\frac{-\hbar_1}{\mathscr{A}_2}+\frac{3\hbar_1}{\mathscr{A}_2}\tanh^2_{pq}(\sqrt{\frac{-\hbar_1}{2}}\xi)\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2t)}, \tag{61}$$

$$\mathscr{G}(x,\ y,\ t) = \Big(\frac{-\hbar_1}{\mathscr{A}_2} + \frac{3\hbar_1}{\mathscr{A}_2}\coth_{pq}^2(\sqrt{\frac{-\hbar_1}{2}}\xi)\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}, \tag{62}$$

$$\mathscr{G}(x,\ y,\ t) = \Big(\frac{-\hbar_1}{\mathscr{A}_2} + \frac{3\hbar_1}{\mathscr{A}_2}(\coth_{pq}(\sqrt{-2\hbar_1}\xi) \pm \sqrt{pq}\operatorname{csch}_{pq}(\sqrt{-2\hbar_1}\xi))^2\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}, \tag{63}$$

and

$$\mathscr{G}(x,\ y,\ t) = \Big(\frac{-\hbar_1}{\mathscr{A}_2} + \frac{3\hbar_1}{4\mathscr{A}_2}(\tanh_{pq}(\sqrt{\frac{-\hbar_1}{8}}\xi) + \coth_{pq}(\sqrt{\frac{-\hbar_1}{8}}\xi))^2\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}. \tag{64}$$

Case II-4: If $\hbar_1 > 0$, and $\hbar_2 = \frac{\hbar_1^2}{4}$, then we get the following trigonometric solutions

$$\mathscr{G}(x,\ y,\ t) = \Big(\frac{-\hbar_1}{\mathscr{A}_2} - \frac{3\hbar_1}{\mathscr{A}_2}\tan_{pq}^2(\sqrt{\frac{\hbar_1}{2}}\xi)\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}, \tag{65}$$

$$\mathscr{G}(x,\ y,\ t) = \Big(\frac{-\hbar_1}{\mathscr{A}_2} - \frac{3\hbar_1}{\mathscr{A}_2}\cot_{pq}^2(\sqrt{\frac{\hbar_1}{2}}\xi)\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}, \tag{66}$$

$$\mathscr{G}(x,\ y,\ t) = \Big(\frac{-\hbar_1}{\mathscr{A}_2} - \frac{3\hbar_1}{\mathscr{A}_2}(\tan_{pq}(\sqrt{2\hbar_1}\xi) \pm \sqrt{pq}\sec_{pq}(\sqrt{2\hbar_1}\xi))^2\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}, \tag{67}$$

$$\mathscr{G}(x,\ y,\ t) = \Big(\frac{-\hbar_1}{\mathscr{A}_2} - \frac{3\hbar_1}{\mathscr{A}_2}(\cot_{pq}(\sqrt{2\hbar_1}\xi) \pm \sqrt{pq}\csc_{pq}(\sqrt{2\hbar_1}\xi))^2\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}, \tag{68}$$

and

$$\mathscr{G}(x,\ y,\ t) = \Big(\frac{-\hbar_1}{\mathscr{A}_2} - \frac{3\hbar_1}{4\mathscr{A}_2}(\tan_{pq}(\sqrt{\frac{\hbar_1}{8}}\xi) + \cot_{pq}(\sqrt{\frac{\hbar_1}{8}}\xi))^2\Big)e^{(\mu\mathscr{W}(t)-\frac{1}{2}\mu^2 t)}. \tag{69}$$

4. Discussion and impacts of noise

Discussion: In this work, we acquired the exact solutions of the SGSE (1) by using two methods, including the JEF method and the Sardar subequation method. The JEF method has provided elliptic solutions such as Eq. (14) and Eq. (18). While the Sardar subequation method has provided solitary hyperbolic solutions such as Eqs (42) and (43), and solitary trigonometric solutions such as Eqs (44) and (45). The study of solitary exact solutions for graphene sheet equations has far-reaching implications in both theoretical and applied physics. Understanding these nonlinear wave phenomena can pave the way for the development of graphene-based devices that exploit solitons for data transmission, energy harvesting, and sensing applications. Furthermore, insights gained from these mathematical models contribute to the broader understanding of nonlinear systems in condensed matter physics, fostering innovations that harness the unique properties of two-dimensional materials.

Impacts of noise: We utilize the Matlab program to present some figures for some selected solutions of the SGSE (1), including Eqs (14), (42), and (46). Also, we examine how multiplicative noise affects the solitary wave solutions of the SGSE (1) by providing some 2D and 3D graphs for appropriate parametric selections and for various values of the noise amplitude σ as follows:

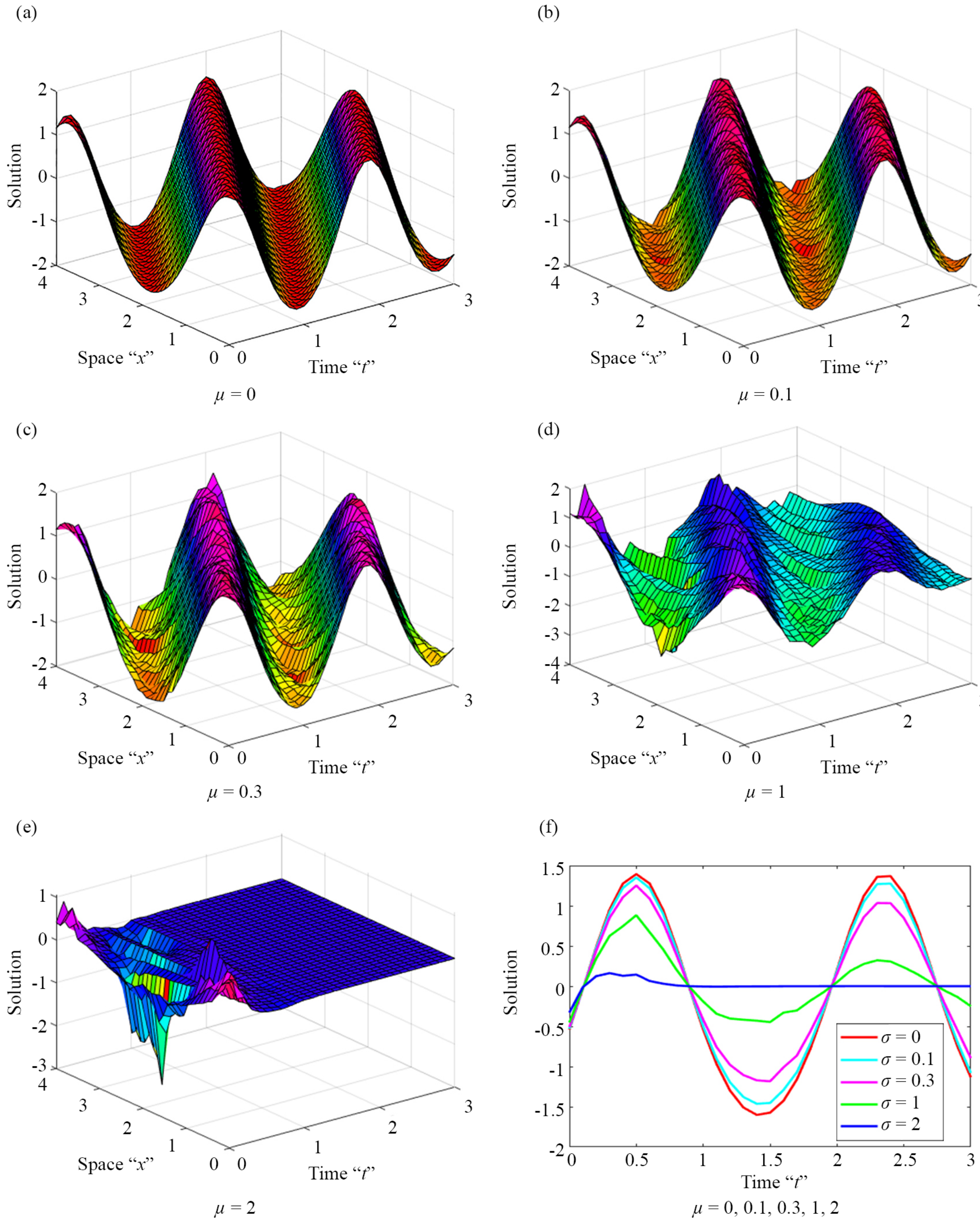

Figure 1. (a-e) display 3D-shape for the periodic solution $\mathscr{G}(x, y, t)$ stated in Eq. (14) with $y = 0$, $\check{n} = 0.5$, $\xi_1 = \xi_2 = 1$, $\xi_3 = -2$, $t \in [0, 3]$, $x \in [0, 4]$ and with distinct σ (f) displays 2D-shape of Eq. (14) with different μ

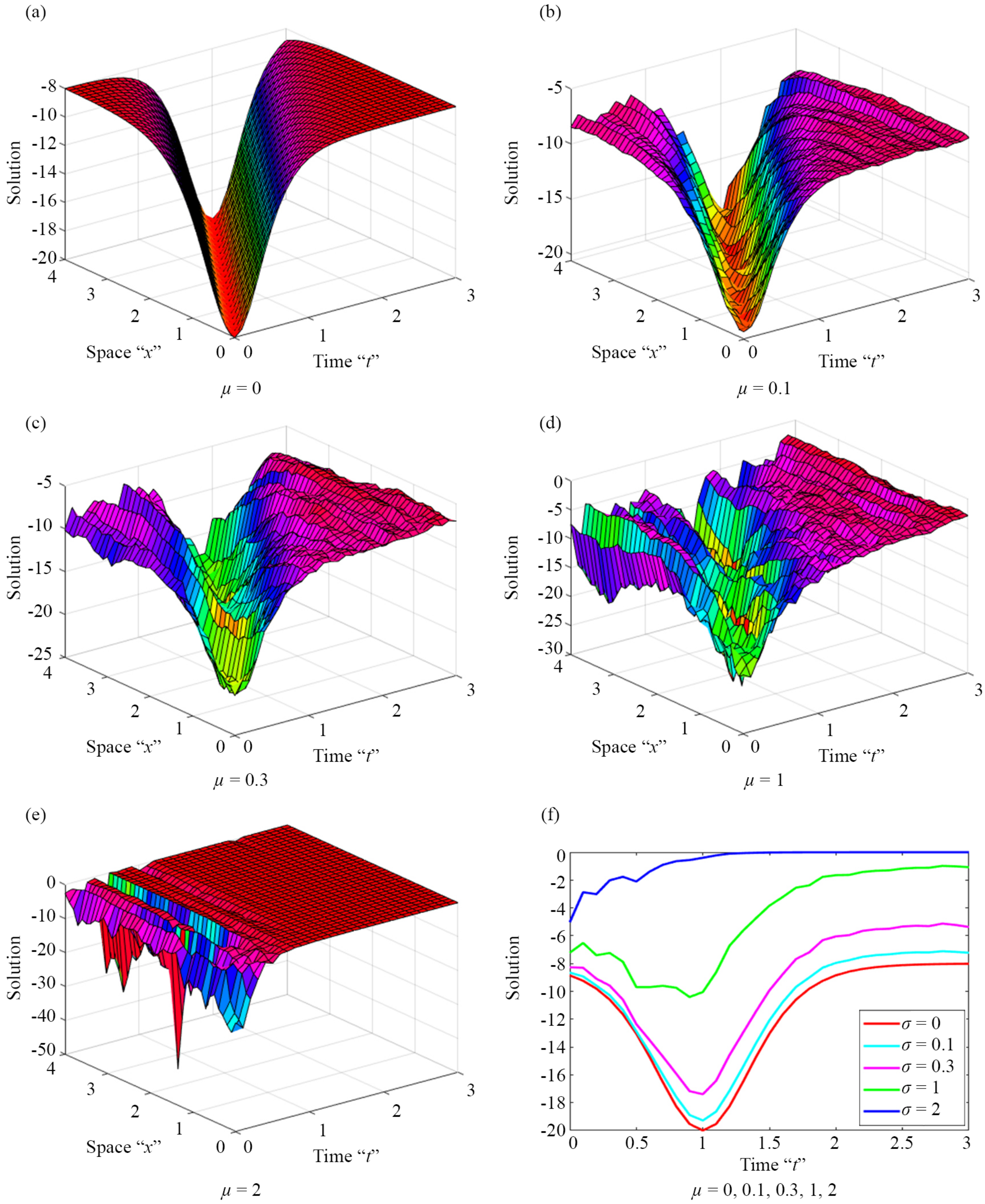

Figure 2. (a-e) display 3D-shape for the dark solution $\mathscr{G}(x, y, t)$ stated in Eq. (42) with $y = 0$, $\xi_1 = \xi_2 = 1$, $\xi_3 = -2$, $t \in [0, 3]$, $x \in [0, 4]$ and with distinct μ (f) displays 2D-shape of Eq. (42) with different μ

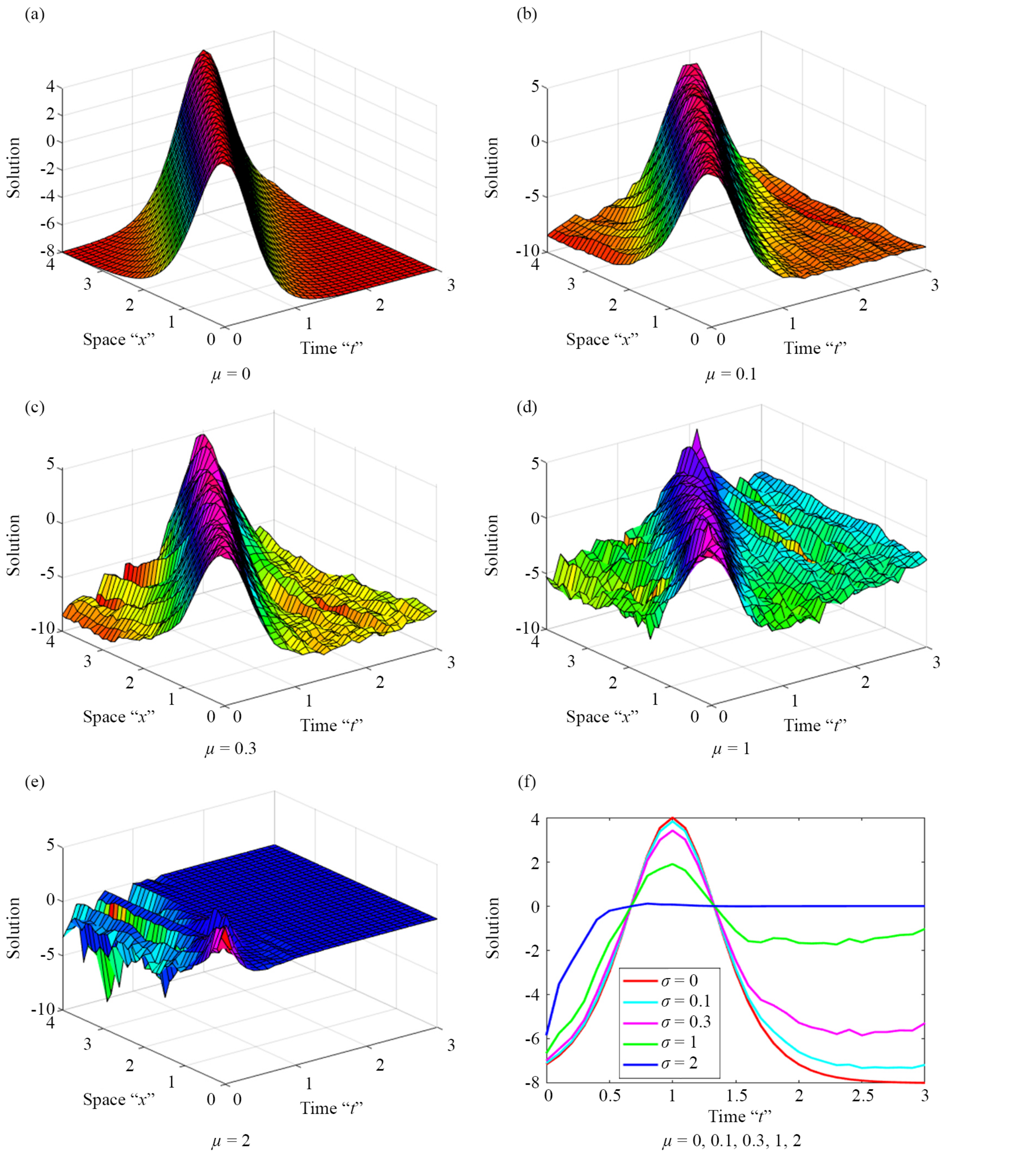

Figure 3. (a-e) display 3D-shape for the bright solution $\mathscr{G}(x,\ y,\ t)$ stated in Eq (46) with $y=0,\ \ \xi_1=\xi_2=1,\ \xi_3=-2,\ t\in[0,\ 3],\ x\in[0,\ 4]$ and with distinct μ (f) displays 2D-shape of Eq. (46) with different μ

Figures 1-3 indicate that when noise is eliminated (i.e. when $\mu=\ 0$), there are several solutions, such as periodic, bright, dark, soliton, and others. Introducing noise with $\mu=\ 0.1,\ 0.3,\ 1,\ 2$ causes the surface to become significantly flatter, following minor transit patterns, as seen by the 2D graph. This indicates that when noise is present, the solutions of SGSE (1) tend to converge toward zero.

5. Conclusions

In this study, we acquired the solutions of the Stochastic Graphene Sheet Model (SGSE) induced in the Itô sense by multiplicative noise. We obtained new periodic soliton, dark and bright solitons, anti-kink and kink solitons solutions for SGSE by applying two different techniques, namely the Sardar subequation method and Jacobi elliptic function method. Because graphene sheets are crucial in a variety of domains, including energy storage, photonics, and electronics, the solutions to the stochastic graphene sheets model help explain a number of intriguing scientific occurrences. Several 2D and 3D graphs that we display using MATLAB illustrate the effect of multiplicative noise on the analytical solutions of the SGSE (1). We determined that the white noise stabilized the solutions at zero. In future studies, we can acquire the exact solutions for the stochastic graphene sheet model with additive or multiplicative color noise.

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Data availability statement

The data used to support the findings of this study are included within the article.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Baskonus HM, Bulut H, Sulaiman TA. New complex hyperbolic structures to the Lonngren-Wave equation by using sine-Gordon expansion method. *Applied Mathematics and Nonlinear Science*. 2019; 4(1): 129-138. Available from: https://doi.org/10.2478/AMNS.2019.1.00013.

[2] Lu B. The first integral method for some time fractional differential equations. *Applied Mathematics and Nonlinear Science*. 2012; 395(2): 684-693. Available from: https://doi.org/10.1016/j.jmaa.2012.05.066.

[3] Khalifa AS, Badra NM, Ahmed HM, Rabie WB. Retrieval of optical solitons in fiber Bragg gratings for high-order coupled system with arbitrary refractive index. *Optik*. 2023; 287: 171116. Available from: https://doi.org/10.1016/j.ijleo.2023.171116.

[4] Al-Askar FM, Mohammed WW, Alshammari M. Impact of Brownian motion on the analytical solutions of the space-fractional stochastic approximate long water wave equation. *Symmetry*. 2022; 14(4): 740. Available from: https://doi.org/10.3390/sym14040740.

[5] Yan Z. Abunbant families of Jacobi elliptic function solutions of the (2 + 1)-dimensional integrable Davey-Stewartson-type equation via a new method. *Chaos Solitons Fractals*. 2003; 18(2): 299-309. Available from: https://doi.org/10.1016/S0960-0779(02)00653-7.

[6] Hussein HH, Ahmed HM, Alexan W. Analytical soliton solutions for cubic-quartic perturbations of the Lakshmanan-Porsezian-Daniel equation using the modified extended tanh function method. *Ain Shams Engineering Journal*. 2024; 15(3): 102513. Available from: https://doi.org/10.1016/j.asej.2023.102513.

[7] Rabie WB, Ahmed HM, Darwish A, Hussein HH. Construction of new solitons and other wave solutions for a concatenation model using modified extended tanh-function method. *Alexandria Engineering Journal*. 2023; 74: 445-451. Available from: https://doi.org/10.1016/j.aej.2023.05.046.

[8] Wang ML, Li XZ, Zhang JL. The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Physics Letters A*. 2008; 372(4): 417-423. Available from: https://doi.org/10.1016/j.physleta.2007.07.051.

[9] Mohammed WW, Ahmad H, Hamza AE, Aly ES, El-Morshedy M, Elabbasy EM. The exact solutions of the stochastic Ginzburg-Landau equation. *Results in Physics*. 2021; 23: 103988. Available from: https://doi.org/10.1016/j.rinp.2021.103988.

[10] He JH, Wu XH. Exp-function method for nonlinear wave equations. *Chaos Solitons Fractals*. 2006; 30(3): 700-708. Available from: https://doi.org/10.1016/j.chaos.2006.03.020.

[11] Khan K, Akbar MA. The $exp(-\phi(\varsigma))$-expansion method for finding travelling wave solutions of Vakhnenko-Parkes equation. *International Journal of Dynamical Systems and Differential Equations*. 2014; 5(1): 72-83. Available from: https://doi.org/10.1504/IJDSDE.2014.067119.

[12] Wazwaz AM. The sine-cosine method for obtaining solutions with compact and noncompact structures. *Applied Mathematics a Computation*. 2004; 159(2): 559-576. Available from: https://doi.org/10.1016/j.amc.2003.08.136.

[13] Jaradat I, Alquran M, Ali M, Al-deiakeh R. Modeling synchronized propagation of two symmetric waves in a new two-mode extension of the (1 + 1)-dimensional chaffee-infante model. *International Journal of Theoretical Physics*. 2025; 64: 45. Available from: https://doi.org/10.1007/s10773-025-05914-w.

[14] Ahmed KK, Badra NM, Ahmed HM, Rabie WB. Unveiling optical solitons and other solutions for fourth-order (2 + 1)-dimensional nonlinear Schrödinger equation by modified extended direct algebraic method. *Journal of Optics*. 2024: 1-13. Available from: https://doi.org/10.1007/s12596-024-01690-8.

[15] Ahmed KK, Ahmed HM, Rabie WB, Shehab MF. Effect of noise on wave solitons for (3 + 1)-dimensional nonlinear Schrödinger equation in optical fiber. *Indian Journal of Physics*. 2024; 98(14): 4863-4882. Available from: https://doi.org/10.1007/s12648-024-03222-3.

[16] Alquran M. Physical properties for bidirectional wave solutions to a generalized fifth-order equation with third-order time-dispersion term. *Results in Physics*. 2021; 28: 104577. Available from: https://doi.org/10.1016/j.rinp.2021.104577.

[17] Alquran M. Variation of the influence of Atangana-conformable time-derivative on various physical structures in the fractional KP-BBM model. *International Journal of Theoretical Physics*. 2024; 63(9): 225. Available from: https://doi.org/10.1007/s10773-024-05767-9.

[18] Alquran M. Investigating fluctuation varieties in the propagation of the perturbed KdV equation with time-dependent perturbation coefficient. *Partial Differential Equations in Applied Mathematics*. 2025; 14: 101206. Available from: https://doi.org/10.1016/j.padiff.2025.101206.

[19] Baber MZ, Rezazadeh H, Iqbal MS, Ahmed N, Yasin MW, Hosseinzadeh MA. Investigation of soliton solutions for the NWHS model with temperature distribution in an infinitely long and thin rod. *Modern Physics Letters B*. 2025; 39(2): 2450392. Available from: https://doi.org/10.1142/S0217984924503925.

[20] Baber MZ, Ahmed N, Xu C, Iqbal MS, Sulaiman TA. A computational scheme and its comparison with optical soliton solutions for the stochastic Chen-Lee-Liu equation with sensitivity analysis. *Modern Physics Letters B*. 2025; 39(1): 2450376. Available from: https://doi.org/10.1142/S0217984924503767.

[21] Baber MZ, Yasin MW, Ahmed N, Ali SM, Ali M. Dynamical analysis and optical soliton wave profiles to GRIN multimode optical fiber under the effect of noise. *Nonlinear Dynamics*. 2024; 112: 20183-20198. Available from: https://doi.org/10.1007/s11071-024-10075-2.

[22] Taya SA, Daher MG, Almawgani AHM, Hindi AT, Colak I, Patel SK, et al. Absorption properties of a one-dimensional photonic crystal with a defect layer composed of a left-handed metamaterial and two monolayer graphene. *Physica Status Solidi Applied Research*. 2023; 220(14): 2300164. Available from: https://doi.org/10.1002/pssa.202300164.

[23] Bradley AN, Thorp SG, Mayonado G, Coporan A, Elliott E, Graham MW. Photoreduced graphene oxide recovers graphene hot electron cooling dynamics. *Physical Review B*. 2023; 107: 224309. Available from: https://doi.org/10.1103/PhysRevB.107.224309.

[24] Model JCM, Veit HM. Development of a more sustainable hybrid process for lithium and cobalt recovery from lithiumion batteries. *Minerals*. 2023; 13(6): 798. Available from: https://doi.org/10.3390/min13060798.

[25] Khater MMA, Alfalqi SH, Vokhmintsev A. High-Precision computational solutions for nonlinear evolution models in graphene sheets. *Scientific Reports*. 2025; 15: 4013. Available from: https://doi.org/10.1038/s41598-025-85263-0.

[26] Mohammed WW, Hassan TS, Sidaoui R, Alshammary H, Algolam MS. Abundant exact traveling-wave solutions for stochastic graphene sheets model. *Axioms*. 2025; 14(6): 477. Available from: https://doi.org/10.3390/axioms14060477.

[27] Peng YZ. Exact solutions for some nonlinear partial differential equations. *Physics Letters A*. 2013; 314: 401-408. Available from: https://doi.org/10.1016/s0375-9601(03)00909-5.