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Research Article

Exploring Collective Extension of Aydi-Lolo-Piri-Rasham Type Contractions on a Closed Ball with Applications

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Abstract: The main purpose of this manuscript is to prove some new fixed point theorems for a pair of multi dominated operators involving simulation function that satisfy a novel extension of Aydi-Lolo-Piri-Rasham type locally contractions in the setting of complete partial b-metric spaces. Also new results for multi-ordered dominated mappings and multi-graph dominated mappings are established on a closed ball in such spaces. To back up our findings, we give non-trivial illustrative examples for multi fixed points in partial b-metric spaces. Applications to the system of nonlinear Volterra type integral equations and fractional differential equations are given to show the novelty of our new results. Our hypothesis will be critical to the theory of fixed points. Our findings can be expanded and improved in various ways by employing multiple sorts of mappings in partial b-metric spaces with applications.

***Keywords*:** multi fixed points, simulation function, multivalued nonlinear dominated operators, multi graph-dominated operators, nonlinear integral-equations, fractional-differential equations

MSC: 47H10, 47H04, 45P05

1. Introduction

Fixed Point (FP) theory is an essential branch of functional analysis and it has a large number of applications in mathematical sciences. A wide range of pure and applied fields of mathematics, including mathematical modeling, dynamic systems theory, fractals, coding theory, approximation theory, game theory, nonlinear optimization problems as well as variational inequalities problems are dependent on FP theory. Banach FP theorem [1] is a fundamental base of metric FP theory and has an inspiring role due to its continuity in numerous disciplines of mathematical sciences. Furthermore, by proving FP theorem for multivalued contraction, Nadler [2] extended Banach's contraction principle. Later, several authors worked on Banach contraction principle and proposed its various extensions. The idea of $\flat$-metric space ($\flat\mathscr{M}S$) was first discussed by Bakhtin [3] and formally described by Czerwik [4] to the extension of Banach contraction principle. Lateral, Matthews [5] presented a significant result in partial metric space ($p\mathscr{M}S$).

The idea of FP theorems in partially ordered metric space ($\mathscr{M}S$) was firstly given by Ran and Reurings [6]. However, a new notion known as metric like space ($\mathscr{M}\mathscr{L}S$) was introduced by Amini-Harandi [7] as a generalization of partial

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metric space ($p\mathscr{MS}$) and dislocated metric space ($D\mathscr{MS}$). The notion of $ b$-metric like space ($ b\mathscr{MLS}$) was given by Alghamdi et al. [8] which is a generalization of $p\mathscr{MS}$. They also illustrated several related FP results and deduced the notion of $\mathscr{MS}$. Lateral, Wardowski [9] proved a new FP theorem related to $\mathscr{F}$-contractions which generalized the Banach's FP theorem. In order to solve several functional and integral equations Sgroi et al. [10] introduced FP theorems for multivalued $\mathscr{F}$-contraction and they presented an application for solving integral equations. The topological relationships between $p\mathscr{MS}$ and $\mathscr{MS}$ were discussed by Shukla [11] and explored the notion of partial b-metric space ($p b\mathscr{MS}$). By simplifying the partial's metric triangle property, he introduced the concept of a partial $p b\mathscr{MS}$ and discussed the FPs of Kannan and Banach contraction in these spaces.

Khojasteh et al. [12] established the novel class of function known as simulation function in the field of metric FP theory. Furthermore, they proposed the new innovation of Z-contraction in terms of simulation function that generalizes the Banach's result and unifies various well-known contractions including the combinations of $d(Sl, Sm)$ and $d(l, m)$. The FP theorems satisfying a generalized locally $\mathscr{F}Z$-contractive multivalued mappings in a complete $ b\mathscr{MLS}$ involving a closed ball were established by Rasham et al. [13]. After this, by combining the concept of the contractive inequality and the abstract structure of Banach's result Karapinar et al. [14] introduced two new types of contraction mappings via simulation functions including rational terms in the framework of $p b\mathscr{MS}$. Their results not only extend, but also generalized and unified the existing results discussed in [13] and [15].

A couple of multivalued dominated locally contractive mappings in $ b$-multiplicative metric space ($ b\mathscr{MMS}$) were explored by Rasham et al. [16]. They established some new FP results on locally contractions instead of globally contractive mappings in $ b\mathscr{MMS}$. Also, some latest FP theorems with multi-graph dominated mappings were discussed in these spaces. Recently, the study of $\mathscr{F}Z$-contractions was initiated by Moussaoui et al. [17] by involving a novel structure of simulation function to obtain a unique FP in the framework of fuzzy $\mathscr{MS}$. They also established some new fixed and best proximity point results in the framework of a M-complete fuzzy $\mathscr{MS}$. By using Banach contraction Lolo et al. [18] presented an important result for a pair of nonlinear contractions via simulation function in partially ordered $\mathscr{MS}$. In recent years, many authors have been proposed numerous innovative and diverse generalizations of simulation function presented in [19–23].

This article explores novel FP theorems involving simulation function satisfying a generalized nonlinear hybrid type rational contraction in the setting of $p b\mathscr{MS}$. Our strategy integrates two distinct mapping techniques: one involve for multi dominated nonlinear operators and second is the weaker class of simulation function. To validate the new findings some illustrative examples are provided. Moreover, to show the originality of our findings, applications on system of nonlinear Volterra type integral and fractional differential equations are presented.

To facilitate understanding, this article is structured as follows: Sec. 2 provides comprehensive explanations of key definitions, supported by relevant examples. Sec. 3 covers innovative FP theorems involving simulation function satisfying a generalized nonlinear hybrid type rational contraction along with nontrivial examples. In Sec. 4 we propose FP theorems for multi-graph dominated operators equipped with graphical structures. In Sec. 5 we apply our primary findings to the system of nonlinear integral equations. In Sec. 6 we demonstrate the applicability of our main results to fractional differential equations. Finally, we summarize our findings and outline future research avenues in Sec. 7.

2. Preliminaries

Definition 1 [24] Let $S \neq \varnothing$ and a mapping $d_b : S \times S \to R^+$ is said partial $ b$-metric with coefficient $\beta > 1$, if the following assumptions hold;

i. $\mathfrak{f} = \mathfrak{g} \Leftrightarrow d_b(\mathfrak{f}, \mathfrak{f}) = d_b(\mathfrak{f}, \mathfrak{g}) = d_b(\mathfrak{g}, \mathfrak{g})$,

ii. $d_b(\mathfrak{f}, \mathfrak{f}) \leq d_b(\mathfrak{f}, \mathfrak{g})$,

iii. $d_b(\mathfrak{f}, \mathfrak{g}) = d_b(\mathfrak{g}, \mathfrak{f})$,

iv. $d_b(\mathfrak{f}, \mathfrak{g}) \leq \beta\,[d_b(\mathfrak{f}, \mathfrak{h}) + d_b(\mathfrak{h}, \mathfrak{g})] - d_b(\mathfrak{h}, \mathfrak{h})$, for all $\mathfrak{f}, \mathfrak{g}, \mathfrak{h} \,\varepsilon\, S$.

Then the pair (S, d_b) is classified as partial $ b$-metric space abbreviated as $p b\mathscr{MS}$. For any $e \in S$ there is $\varepsilon > 0$, $\overline{B_{d_b}(e, \varepsilon)} = \{r \in S : d_b(e, r) \leq \varepsilon\}$ be a closed ball in $p b\mathscr{MS}$.

Definition 2 [25] Let (S, d_b) be a $pb\mathscr{M}S$. Then

i. A sequence $\{z_n\}$ in S is defined as to be convergent to a point $z \in S$ such that $\lim_{n\to+\infty} d_b(z_n, z) \to 0$.

ii. The sequence $\{z_n\}$ in (S, d_b) is nominated as Cauchy sequence when there is any $\varepsilon > 0$ such that $n(\varepsilon) \in M$ for each $n, m \in n(\varepsilon)$ we must have the distance $d_b(z_n, z_m) < \varepsilon$.

iii. Completeness of (S, d_b) is given as every Cauchy sequence $\{z_n\}$ in S converges to any limit point $z \in S$.

Definition 3 [26] Let (S, d) be a dislocated b-metric space. Let $Y \neq \varnothing$ where $Y \subseteq S$, then any point $f_\circ \in Y$ is said best approximation in Y when $d(n, Y) = (n, f_\circ)$, with $d(n, Y) = \inf_{f_\circ \in Y} d(n, f_\circ)$ and $n \in S$.

Here $P(S)$ defines the set consists all closed bounded subsets of S.

Let Ψ_b represents the class of all non-increasing functions $\Psi_b : [0, +\infty) \to [0, +\infty)$ for which $\sum_{k=1}^{+\infty} {\Psi_b}^k(g) < +\infty$ and $\Psi_b(g) < g$, where ${\Psi_b}^k$ denotes the k^{th} iterative term of Ψ_b.

Definition 4 [27] Let (S, d_b) be a $p\mathscr{M}S$. The mapping $H_{d_b} : P(S) \times P(S) \to \mathbb{R}^+$ defined as $H_{d_b}(\dot{A}, H) = \max\{\sup_{\alpha \varepsilon A} d_b(\alpha, H), \sup_{y \in Y} d_b(\dot{A}, y)\}$ for each $\dot{A}, H \in P(S)$, is called partial Hausdorff b-metric on $P(S)$.

Definition 5 [16] Let S be a nonempty set and $U, \pounds : S \to P(S)$ be a distinct couple of set-valued mappings. Consider $\alpha : S \times S \to R^+$ as a real-valued function. Then the mappings U and $\pounds$ are classified as α_*-admissible if for each $n, g \in S$ the following property hold;

$$\alpha(n, g) \geq 1 \;\Rightarrow\; \alpha_*(Un,\; \pounds g) \geq 1 \;\text{ and }\; \alpha_*(\pounds g, Un\,) \geq 1, \tag{1}$$

whenever $\alpha_*(Un, \pounds g) = \inf\{\alpha(n, g) : n \in Un, g \in \pounds g\} > 1$.

Definition 6 [26] Let S be a nonempty set and $Y \subseteq S$. Let $U : S \to P(S)$ is a set-valued mapping and $\alpha : S \times S \to R^+$ be a positive real valued function. Then is classified as an α_*-dominated mapping on Y for each $\rho \in Y$, satisfying the following;

$$\alpha_*(\rho, U(\rho)) = \inf\{\alpha(\rho, \mu) : \mu \in U(\rho)\} \geq 1. \tag{2}$$

If $U : S \to S$ is a self-mapping then U becomes α-dominated mapping on Y.

Example 1 [16] Suppose S be a nonempty set. Define the function $\alpha : S \times S \to R^+$ as

$$\alpha(m, u) = \begin{cases} 1 & \text{if } m > u \\ \\ \frac{1}{2} & \text{otherwise.} \end{cases} \tag{3}$$

The mappings $U, \pounds S \to P(S)$ are specified as $Uj = [j-2, j-1]$ and $\pounds z = [z-5, z-4]$.

Hence, U and $\pounds$ fulfilled α_*-dominated criteria but fails α_*-admissibility.

Definition 7 [28] Let (S, d) be a $\mathscr{M}S$. A mapping $\mathfrak{B} : S \to S$ is designated as $\mathscr{F}$-contraction if there are $\mu > 0$, $d(\mathfrak{B}_f, \mathfrak{B}_g) > 0$ and a function $\mathscr{F}$ of strictly increasing satisfying the given property.

$$\mu + \mathscr{F}\left(\left(\mathfrak{B}_f, \mathfrak{B}_g\right)\right) \leq \mathscr{F}\left(d(f, g)\right) \text{ for all } f, g \in R. \tag{4}$$

Definition 8 [28] The mapping $\mathscr{F} : S^+ \to S$ meets the given assumptions;

$(\mathscr{F}_1)$ $\mathscr{F}$ is a strictly increasing, for each $\gamma, \delta \in R^+$ so that $\gamma < \delta \Rightarrow \mathscr{F}(\gamma) < \mathscr{F}(\delta)$;

$(\mathscr{F}_2)$ For every sequence $\beta_n \in R^+$, then $\lim_{n\to\infty} \beta_n = 0 \Leftrightarrow \lim_{n\to\infty} \mathscr{F}(\beta_n) = -\infty$;

($\mathscr{F}_3$) There is $\gamma \in (0, 1)$ so that $\lim_{\beta \to 0^+} \beta^\gamma \mathscr{F}(\beta) = 0$.

Definition 9 [18] Let (S, d) be a $\mathscr{M}S$. A function $\eta_b : [0, \infty) \times [0, \infty) \to S$ is said a simulation function if the following conditions hold:

(η_{b1}) $\eta_b(\mathfrak{f}, \mathfrak{g}) < \mathfrak{g} - \mathfrak{f}$ for all $\mathfrak{f}, \mathfrak{g} > 0$,

(η_{b2}) If $\mathfrak{f}_n$ and $\mathfrak{g}_n$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} \mathfrak{f}_n = \lim_{n\to\infty} \mathfrak{g}_n = p_1 > 0$, then $\lim_{n\to\infty} \sup\ \eta_b(\mathfrak{f}_n, \mathfrak{g}_n) < 0$.

Example 2 [18] Consider a mapping $\eta_b : [0, \infty) \times [0, \infty) \to S$ defined by $\eta_b(q, b) = \mu(b) - \rho(q)$ for all $q, b \in R^+$ such that $\mu(\mathfrak{f}) = \rho(\mathfrak{f}) \Leftrightarrow \mathfrak{f} = 0$ and $\mu(\mathfrak{f}) < \mathfrak{f} \le \rho(\mathfrak{f})$ for all $\mathfrak{f} > 0$.

Here $\mu, \rho : [0, \infty) \to [0, \infty)$ are continuous functions.

Lemma 1 Let (S, d_b) be a $p b\mathscr{M}S$. Let $(P(S), H_{d_b})$ be a partial Hausdroff $b\mathscr{M}S$ on $P(S)$. For each $\lambda \in \Upsilon$, there is $\eta_\lambda \in Q$ so that $H_{d_b}(\Upsilon, Q) \ge d_b(\lambda, \eta_\lambda)$.

Proof.

For all $\lambda \in \Upsilon$, there is $\eta_\lambda \in Q$ such that $H_{d_b}(\Upsilon, Q) = d_b(\lambda, \eta_\lambda)$. We have

$$H_{d_b}(\Upsilon, Q) \ge \sup_{\lambda \in \Upsilon} d_b(\lambda, Q) = d_b(\lambda, \eta_\lambda). \tag{5}$$

Hence the result is verified. □

3. Main results

Let (S, d_b) is a $p b\mathscr{M}S$, $\nu_\circ \in \mathrm{S}$ and $P, Q : \mathrm{S} \to P(\mathrm{S})$ be a pair of multivalued mappings. Let $\nu_1 \in P(\nu_\circ)$ be an element such that $d_b(\nu_\circ, P(\nu_\circ)) = d_b(\nu_\circ, \nu_1)$. Let $\nu_2 \in Q(\nu_1)$ be an element so that $d_b(\nu_1, Q(\nu_1)) = d_b(\nu_1, \nu)$. Let $\nu_3 \in P(\nu_2)$ be such that $d_b(\nu_2, P(\nu_2)) = d_b(\nu_2, \nu_3)$. Proceeding in a similar way, we generate a sequence ν_n iteratively in S such that $\nu_{2n+1} \in P(\nu_{2n})$ and $\nu_{2n+2} \in Q(\nu_{2n+1})$, for $n = 0, 1, 2, 3, 4, \ldots$. Furthermore, $d_b(\nu_{2n}, P(\nu_{2n})) = d_b(\nu_{2n}, \nu_{2n+1})$ and $d_b(\nu_{2n+1}, Q(\nu_{2n+1})) = d_b(\nu_{2n+1}, \nu_{2n+2})$. We denote this sequence as $\{QP(\nu_n)\}$. For $\nu, \omega \in \mathrm{S}$ and $\alpha > 0$, we define $D(\nu, \omega)$ as

$$D(\nu, \omega) = \max\left\{ d_b(\nu, \omega), \frac{d_b(\nu, P(\nu)) \cdot d_b(\omega, Q(\omega))}{\alpha + d_b(\nu, \omega)}, d_b(\nu, P(\nu)), d_b(\omega, Q(\omega)) \right\}. \tag{6}$$

Theorem 1 Let (S, d_b) be a $p\mathscr{M}S$. Let $\alpha : \mathrm{S} \times \mathrm{S} \to \mathbb{R}^+$ be a function and $P, Q : \mathrm{S} \to P(\mathrm{S})$ be a couple of α_*-dominated multivalued operators on $\overline{B_{d_b}(\nu_\circ, r)}$ where $r > 0$ and $\nu_\circ \in \overline{B_{d_b}(\nu_\circ, r)}$. Suppose the mapping $\mathscr{F}$ is strictly increasing with a constant $\tau > 0$ then for all $\nu, \omega \in \overline{B_{d_b}(\nu_\circ, r)} \cap \{QP(\nu_n)\}$ and $\alpha(\nu, \omega) \ge 1$ there is $\Delta_b \in \Psi_b$, such that the given conditions hold:

i.

$$\tau + \mathscr{F}(H_{d_b}(P(\nu), Q(\omega)) \le \mathscr{F}(\Delta_b(D(\nu, \omega))), \tag{7}$$

where $\eta_b(d_b(P(\nu), Q(\omega)), D(\nu, \omega)) \ge 0$ and $H_{d_b}(P(\nu), Q(\omega)) > 0$. For all $n \in N \cup \{0\}$ and $b \ge 1$, we have:

ii.

$$\sum_{i=0}^{n} b^{i+1}\left\{\Delta_b^i(d_b(\nu_\circ, P(\nu_\circ))\right\} \leq r, \tag{8}$$

Then the sequence $\{QP(\nu_n)\}$ in $\overline{B_{d_b}(\nu_\circ, r)}$, $\alpha(\nu_n, \nu_{n+1}) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$ and $\{QP(\nu_n)\} \to \nu^* \in \overline{B_{d_b}(\nu_\circ, r)}$.

iii. If (7) is true for ν^* and either $\alpha(\nu_n, \nu^*) \geq 1$ or $\alpha(\nu^*, \nu_n) \geq 1$ for all naturals. Then P and Q have a common multi-FP ν^* in $\overline{B_{d_b}(\nu_\circ, r)}$.

Proof. Take a sequence $\{QP(\nu_n)\}$. From (8) we have, $d_b(\nu_\circ, \nu_1) \leq \sum_{i=0}^{n} b^{i+1}\left\{\Delta_b^i(d_b(\nu_\circ, P(\nu_\circ))\right\} \leq r$.

It follows that $\nu_1 \in \overline{B_{d_b}(\nu_\circ, r)}$. Let $\nu_1, \nu_2, \nu_3, \ldots, \nu_j \in \overline{B_{d_b}(\nu_\circ, r)}$ for any $j \in \mathbb{N}$. Assuming $j = 2i+1$, for $i = 1, 2, 3, \ldots, \frac{j-1}{2}$. Since $P, Q : \mathtt{S} \to P(\mathtt{S})$ be two α_*-dominated multivalued mappings on $\overline{B_{d_b}(\nu_\circ, r)}$. So $\alpha_*(\nu_{2i}, P(\nu_{2i})) \geq 1$ and $\alpha_*(\nu_{2i+1}, Q(\nu_{2i+1})) \geq 1$.

Since $\alpha_*(\nu_{2i}, P(\nu_{2i})) \geq 1$, this shows that $\inf\{\alpha(\nu_{2i}, h) \geq 1; h \in P(\nu_{2i}) \geq 1$. Also $\nu_{2i+1} \in P(\nu_{2i})$. So, $(\nu_{2i}, \nu_{2i+1}) \geq 1$. Furthermore, from Lemma 1 and inequality (7), we obtain

$$\begin{aligned}
\tau + \mathscr{F}(d_b(\nu_{2i+1}, \nu_{2i+2})) &\leq \tau + \mathscr{F}(H_{d_b}(P(\nu_{2i}), Q(\nu_{2i+1}))), \\
&\leq \mathscr{F}(\Delta_b((\nu_{2i}, \nu_{2i+1}))), \\
&\leq \mathscr{F}\left(\Delta_b\left(\max\left\{\begin{array}{c} d_b(\nu_{2i}, \nu_{2i+1}), \ \frac{d_b(\nu_{2i}, P(\nu_{2i}))\cdot d_b(\nu_{2i+1}, Q(\nu_{2i+1}))}{\alpha + d_b(\nu_{2i}, \nu_{2i+1})}, \\ d_b(\nu_{2i}, P(\nu_{2i})), \\ d_b(\nu_{2i+1}, Q(\nu_{2i+1})) \end{array}\right\}\right)\right), \\
&\leq \mathscr{F}\left(\Delta_b\left(\max\left\{\begin{array}{c} d_b(\nu_{2i}, \nu_{2i+1}), \ \frac{d_b(\nu_{2i}, \nu_{2i+1})\cdot d_b(\nu_{2i+1}, \nu_{2i+2})}{\alpha + d_b(\nu_{2i}, \nu_{2i+1})}, \\ d_b(\nu_{2i}, \nu_{2i+1}), \\ d_b(\nu_{2i+1}, \nu_{2i+2}) \end{array}\right\}\right)\right), \\
&\leq \mathscr{F}(\Delta_b(\max(d_b(\nu_{2i}, \nu_{2i+1}), d_b(\nu_{2i+1}, \nu_{2i+2})))).
\end{aligned} \tag{9}$$

Moreover,

$$\eta_b(d_b(P_{\nu_{2i}}, Q_{\nu_{2i+1}}), (\nu_{2i}, \nu_{2i+1})) \geq 0. \tag{10}$$

Now, by considering (η_{b_1}), the above inequality changes to $0 < \Delta_b(((\nu_{2i}, \nu_{2i+1})) - \Delta_b(d_b((P_{\nu 2i}, Q_{\nu_{2i+1}}))$, or $\Delta_b(d_b((P_{\nu 2i}, Q_{\nu 2i+1})) < \Delta_b(((\nu_{2i}, \nu_{2i+1})) = \Delta_b(\{\max\{d_b(\nu_{2i}, \nu_{2i+1}), d_b(\nu_{2i+1}, \nu_{2i+2})\}\})$.

Consequently, due to monotone property of Δ_b we have $d_b(\nu_{2i}, \nu_{2i+1}) < \{\max\{d_b(\nu_{2i}, x_{2i+1}), d_b(\nu_{2i+1}, \nu_{2i+2})\}$.

If, $\max\{d_b(\nu_{2i}, \nu_{2i+1}), d_b(\nu_{2i+1}, \nu_{2i+2})\} = d_b(\nu_{2i+1}, \nu_{2i+2})$, then

$$\tau + \mathscr{F}\left(d_b\left(\nu_{2i+1},\, \nu_{2i+2}\right)\right) \leq \mathscr{F}\left(\Delta_b\left(d_b\left(\nu_{2i+1},\, \nu_{2i+2}\right)\right)\right). \tag{11}$$

Since $\mathscr{F}$ is a strictly increasing function, we obtain

$$d_b\left(\nu_{2i+1},\, \nu_{2i+2}\right) < \Delta_b\left(d_b\left(\nu_{2i+1},\, \nu_{2i+2}\right)\right).$$

This inequality leads to contradiction, as $\Delta_b(w) < w$ for $w > 0$. So, we obtain $\max\{d_b\left(\nu_{2i},\, \nu_{2i+1}\right), d_b\left(\nu_{2i+1},\, \nu_{2i+2}\right)\}$ = $d_b\left(\nu_{2i},\, \nu_{2i+1}\right)$. Now, we have

$$\begin{aligned} &\mathscr{F}\left(d_b\left(\nu_{2i+1},\, \nu_{2i+2}\right)\right) < \mathscr{F}\left(\delta_b\left(d_b\left(\nu_{2i},\, \nu_{2i+1}\right)\right)\right), \\ &d_b\left(\nu_{2i+1},\, x_{2i+2}\right) < \delta_b\left(d_b\left(\nu_{2i+1},\, \nu_{2i+2}\right)\right), \end{aligned} \tag{12}$$

Since, $\alpha_*\left(\nu_{2i-1}, Q\left(\nu_{2i-1}\right)\right) \geq 1$ and $\nu_{2i} \in Q\left(x_{2i-1}\right)$, so $\alpha_*(\nu_{2i-1},\, \nu_{2i}) \geq 1$. Furthermore, by using Lemma 1 and inequality (7), we obtain

$$\begin{aligned} \tau + \mathscr{F}\left(d_b\left(\nu_{2i},\, \nu_{2i+1}\right)\right) &\leq \tau + \mathscr{F}\left(H_{d_b}\left(Q\left(\nu_{2i-1}\right), P\left(\nu_{2i}\right)\right)\right), \\ &\leq \mathscr{F}\left(\Delta_b\left(D\left(\nu_{2i},\, \nu_{2i-1}\right)\right)\right), \\ &\leq \mathscr{F}\left(\Delta_b\left(\max\left\{\begin{array}{c} d_b\left(\nu_{2i},\, \nu_{2i-1}\right),\ \frac{d_b\left(\nu_{2i},\, Q\left(\nu_{2i}\right)\right)\cdot d_b\left(\nu_{2i-1},\, Q\left(\nu_{2i-1}\right)\right)}{\alpha + d_b\left(\nu_{2i},\, \nu_{2i-1}\right)}, \\ d_b\left(\nu_{2i},\, Q\left(\nu_{2i}\right)\right), \\ d_b\left(\nu_{2i-1},\, P(\nu_{2i-1})\right) \end{array}\right\}\right)\right), \\ &\leq \mathscr{F}\left(\Delta_b\left(\max\left\{\begin{array}{c} d_b\left(\nu_{2i},\, \nu_{2i-1}\right),\ \frac{d_b\left(\nu_{2i},\, \nu_{2i+1}\right)\cdot d_b\left(\nu_{2i-1},\, \nu_{2i}\right)}{\alpha + d_b\left(\nu_{2i},\, \nu_{2i-1}\right)}, \\ d_b\left(\nu_{2i},\, \nu_{2i+1}\right), \\ d_b\left(\nu_{2i-1},\, \nu_{2i}\right) \end{array}\right\}\right)\right), \\ &\leq \mathscr{F}\left(\Delta_b\left(\max\left\{d_b\left(\nu_{2i},\, \nu_{2i+1}\right), d_b\left(\nu_{2i-1},\, \nu_{2i}\right)\right\}\right)\right). \end{aligned} \tag{13}$$

Since $\mathscr{F}$ is a strictly increasing, we have

$$d_b\left(\nu_{2i},\, \nu_{2i+1}\right) < \Delta_b\left(\max\left\{d_b\left(\nu_{2i-1},\, \nu_{2i}\right), d_b\left(\nu_{2i},\, \nu_{2i+1}\right)\right\}\right). \tag{14}$$

Moreover,

$$\eta_b(\mathcal{d}_b(P_{\nu_{2i}}, Q_{\nu_{2i}-1}), D(\nu_{2i}, \nu_{2i-1}) \geq 0. \tag{15}$$

Now, by considering (η_{b1}) the above inequality changes to

$$0 < \Delta_b\left((\nu_{2i}, \nu_{2i-1})\right) - \Delta_b\left(\mathcal{d}_b\left(\left(P_{\nu 2i},\ Q_{\nu_{2i-1}}\right)\right)\right),$$

or $\Delta_b(\mathcal{d}_b\left(\left(P_{\nu 2i},\ Q_{\nu_{2i-1}}\right)\right) < \Delta_b\left(\left(\left(\nu_{2i}, \nu_{2i-1}\right)\right) = \Delta_b\ \left(\{\max\left\{\mathcal{d}_b\left(\nu_{2i-1}, \nu_{2i}\right), \mathcal{d}_b\left(\nu_{2i}, \nu_{2i+1}\right)\right)\right.$.

Hence, due to monotone property of Δ_b we get

$$\mathcal{d}_b\left(\nu_{2i}, \nu_{2i+1}\right) < \left\{\max\left\{\mathcal{d}_b\left(\nu_{2i-1}, \nu_{2i}\right), \mathcal{d}_b\left(\nu_{2i}, \nu_{2i+1}\right)\right\}\right.. \tag{16}$$

If, $\max\left\{\mathcal{d}_b\left(\nu_{2i-1}, \nu_{2i}\right), \mathcal{d}_b\left(\nu_{2i}, \nu_{2i+1}\right)\right\} = \mathcal{d}_b\left(\nu_{2i}, \nu_{2i+1}\right)$, then $\mathcal{d}_b\left(\nu_{2i}, \nu_{2i+1}\right) < \delta_b\left(\mathcal{d}_b\left(\nu_{2i}, \nu_{2i+1}\right)\right)$.

This inequality leads to contradiction, as $\Delta_b(w) < w$ for $w > 0$. Hence, we obtain

$$\mathcal{d}_b\left(\nu_{2i}, \nu_{2i+1}\right) < \Delta_b\left(\mathcal{d}_b\left(\nu_{2i-1}, \nu_{2i}\right)\right). \tag{17}$$

As Δ_b is non-decreasing,

$$\Delta_b\left(\mathcal{d}_b\left(\nu_{2i}, \nu_{2i+1}\right)\right) < \Delta_b\left(\mathcal{d}_b\left(\nu_{2i-1}, \nu_{2i}\right)\right). \tag{18}$$

From above inequality (12), we conclude that $\Delta_b\left(\mathcal{d}_b\left(\nu_{2i}, \nu_{2i+1}\right)\right) < {\Delta_b}^2\left(\mathcal{d}_b\left(\nu_{2i-1}, \nu_{2i}\right)\right)$.

Continuing in this way, we get

$$\mathcal{d}_b\left(\nu_{2i+1}, \nu_{2i+2}\right) < {\delta_b}^{2i+1}\left(\mathcal{d}_b\left(\nu_{\circ},\ \nu_1\right)\right). \tag{19}$$

Instead, if $j = 2i$ for $i = 1,\ 2,\ 3,\ \ldots,\ \frac{j}{2}$, by repeating the same steps and utilizing (18), we obtain

$$\mathcal{d}_b\left(\nu_{2i}, \nu_{2i+1}\right) < {\Delta_b}^{2i}\left(\mathcal{d}_b\left(\nu_{\circ},\ \nu_1\right)\right). \tag{20}$$

Now, (19) and (20) can be jointly written as

$$\mathcal{d}_b\left(\nu_j, \nu_{j+1}\right) < {\Delta_b}^j\left(\mathcal{d}_b\left(\nu_{\circ},\ \nu_1\right)\right), \quad \text{for all} \quad j \in \mathcal{N}. \tag{21}$$

Furthermore, by inserting triangular inequality and using (21), we get,

$$
\begin{aligned}
d_b\left(\nu_j, \nu_{j+1}\right) \leq & \, b d_b\left(\nu_\circ, \nu_1\right)+b^2 d_b\left(\nu_1, \nu_2\right)+b^3 d_b\left(\nu_2, \nu_3\right)+\cdots+b^{j+1} d_b\left(\nu_j, \nu_{j+1}\right), \\
< & \, b d_b\left(\nu_\circ, \nu_1\right)+b^2 \Delta_b\left(d_b\left(\nu_\circ, \nu_1\right)\right)+b^3 \Delta_b{}^2\left(d_b\left(\nu_\circ, \nu_1\right)\right)+ \\
& b^4 \Delta_b{}^3\left(d_b\left(\nu_\circ, \nu_1\right)\right)+\cdots+b^{j+1} \Delta_b{}^j\left(d_b\left(\nu_\circ, \nu_1\right)\right), \\
< & \sum_{i=0}^{j} b^{i+1}\left(\Delta_b{}^i\left(d_b\left(\nu_\circ, \nu_1\right)\right)\right)<r .
\end{aligned} \tag{22}
$$

Thus, $\nu_{j+1} \in \overline{B_{d_b}(\nu_\circ, r)}$. Hence, $\nu_n \in \overline{B_{d_b}(\nu_\circ, r)}$ for all $n \in \mathbb{N}$. Consequently, the sequence $\{QP(\nu_n)\} \to \mathfrak{r} \in \overline{B_{d_b}(\nu_\circ, r)}$. Since P and Q are α_*-dominated set-valued operators on $\overline{B_{d_b}(\nu_\circ, r)}$, thus $\alpha_*(\nu_{2n}, P(\nu_{2n})) \geq 1$ and $\alpha_*(\nu_{2n+1}, Q(\nu_{2n+1})) \geq 1$. This implies that $\alpha(\nu_n, \nu_{n+1}) \geq 1$. Now inequality (21) can be written as,

$$
d_b\left(\nu_n, \nu_{n+1}\right)<\Delta_b{}^n\left(d_b\left(\nu_\circ, \nu_1\right)\right), \text{ for all } n \in \mathbb{N}. \tag{23}
$$

As $\sum_{k=1}^{\infty} b^k \Delta_b{}^k(Q)<+\infty$, then for any $q \in \mathbb{N}$, $\sum_{k=1}^{\infty} b^k \Delta_b{}^k\left(\psi_b^{q-1}\left(d_b\left(\nu_\circ, \nu_1\right)\right)\right)$ converges. As $\Delta_b(w)<w$. So, for all natural q, we have

$$
b^{n+1} \delta_b{}^{n+1}\left(\Delta_b{}^{q-1}\left(\left(d_b\left(\nu_\circ, \nu_1\right)\right)\right)\right)<b^n \Delta_b{}^n\left(\Delta_b{}^{q-1}\left(\left(d_b\left(\nu_\circ, \nu_1\right)\right)\right)\right). \tag{24}
$$

Fix $\varepsilon>0$ there is $q(\varepsilon) \in \mathbb{N}$, so that

$$
b \Delta_b\left(\delta_b{}^{q(\varepsilon)-1}\left(\left(d_b\left(\nu_\circ, \nu_1\right)\right)\right)\right)+b \Delta_b{}^2\left(\Delta_b{}^{q(\varepsilon)-1}\left(\left(d_b\left(\nu_\circ, \nu_1\right)\right)\right)\right)+\cdots<\varepsilon. \tag{25}
$$

For any natural number y, x where $x>y>q(\varepsilon)$, we have

$$
\begin{aligned}
& d_b\left(\nu_y, \nu_x\right) \leq d_b\left(\nu_y, \nu_{y+1}\right)+b^2 d_b\left(\nu_{y+1}, \nu_{y+2}\right)+\cdots+b^{x-y} d_b\left(\nu_{x-1}, \nu_x\right), \\
& <\Delta_b{}^y\left(d_b\left(\nu_\circ, \nu_1\right)\right)+b^2 \Delta_b{}^{y+1}\left(d_b\left(\nu_\circ, \nu_1\right)\right)+\cdots+b^{x-y} \Delta_b{}^{x-1}\left(d_b\left(\nu_\circ, \nu_1\right)\right), \\
& <\Delta_b\left(\Delta_b{}^{q(\varepsilon)-1}\left(d_b\left(\nu_\circ, \nu_1\right)\right)\right)+b^2\left(\Delta_b{}^2\left(\Delta_b{}^{q(\varepsilon)-1}\left(d_b\left(\nu_\circ, \nu_1\right)\right)\right)\right)+\cdots<\varepsilon.
\end{aligned} \tag{26}
$$

Hence, $\{QP(\nu_n)\}$ is the Cauchy sequence in $\overline{B_{d_b}(\nu_\circ, r)}$. Since $(\overline{B_{d_b}(\nu_\circ, r)}, d_b)$ be a complete $pb\mathcal{M}S$, so there exists $\mathfrak{r} \in \overline{B_{d_b}(\nu_\circ, r)}$ so that $\{QP(\nu_n)\} \to \mathfrak{r}$ as $n \to +\infty$, then

$$
\lim_{n \to +\infty} d_b\left(\nu_n, \mathfrak{r}\right)=0. \tag{27}
$$

Now, by using triangular inequality of $p\flat\mathscr{M}S$, we have

$$\begin{aligned} \partial_\flat(\mathrm{r}, P(\mathrm{r})) &\le \partial_\flat(\mathrm{r}, \nu_{2n+2}) + \partial_\flat(\nu_{2n+2}, P(\mathrm{r})) \\ &\le \partial_\flat(\mathrm{r}, \nu_{2n+2}) + H_{\partial_\flat}(Q(\nu_{2n+1}), P(\mathrm{r})). \end{aligned}$$

By supposition $\alpha(\nu_n, \mathrm{r}) \ge 1$. Suppose that $\partial_\flat(\mathrm{r}, P(\mathrm{r})) > 0,$ then for positive natural h such that $\partial_\flat(\nu_n, P(\mathrm{r})) > 0$ for all $n \ge h$. For $n \ge h$, we have

$$\partial_\flat(\mathrm{r}, P(\mathrm{r})) < \partial_\flat(\mathrm{r}, \nu_{2n+2}) + \flat\left(\Delta_\flat\left(\max\left\{\begin{array}{c} \partial_\flat(\mathrm{r}, \nu_{2n+1}), \frac{\partial_\flat(\nu_{2n+1}, \nu_{2n+2})\cdot\partial_\flat(\mathrm{r}, P(\mathrm{r}))}{\alpha+\partial_\flat(\mathrm{r}, P(\mathrm{r}))}, \\ \partial_\flat(\mathrm{r}, P(\mathrm{r})), \\ \partial_\flat(\nu_{2n+1}, \nu_{2n+2}) \end{array}\right\}\right)\right). \tag{28}$$

Taking $n \to +\infty$ and using the inequality (27), we obtain

$$\partial_\flat(\mathrm{r}, P(\mathrm{r})) < \Delta_\flat(\partial_\flat(\mathrm{r}, P(\mathrm{r}))) < \partial_\flat(\mathrm{r}, P(\mathrm{r})). \tag{29}$$

Moreover,

$$\eta_\flat(\partial_\flat(P_\mathrm{r}, Q(\nu_{2n+1})), (\mathrm{r}, \nu_{2n+1})) \ge 0. \tag{30}$$

Now, by considering $(\eta_{\flat 1})$ the above inequality changes to

$$0 < \Delta_\flat((D(\mathrm{r}, \nu_{2n+1}))) - \Delta_\flat(\partial_\flat(P_\mathrm{r}, Q_{\nu_{2n+1}})). \tag{31}$$

or

$$\begin{aligned} \Delta_\flat(\partial_\flat(P_\mathrm{r}, Q_{\nu_{2n+1}})) &< \Delta_\flat((\mathrm{r}, \nu_{2n+1})) \\ &= \Delta_\flat\left(\max\left\{\begin{array}{c} \partial_\flat(\mathrm{r}, \nu_{2n+1}), \frac{\partial_\flat(\nu_{2n+1}, \nu_{2n+2})\cdot\partial_\flat(\mathrm{r}, P(\mathrm{r}))}{\alpha+\partial_\flat(\mathrm{r}, P(\mathrm{r}))}, \\ \partial_\flat(\mathrm{r}, P(\mathrm{r})), \\ \partial_\flat(\nu_{2n+1}, \nu_{2n+2}) \end{array}\right\}\right). \end{aligned} \tag{32}$$

Taking $n \to +\infty$ and using the inequality (27), we obtain

$$d_b(r, P(r) < d_b(r, P(r)). \tag{33}$$

This is not true. So our supposition doesn't hold. Consequently, $d_b(r, P(r)) = 0$ or $r \in P(r)$. Likewise, by applying Lemma 1 and (27) we can also show that $r \in Q(r)$.

Hence, r is a common multi-FP of both P and Q in $\overline{B_{d_b}(\nu_\circ, r)}$. □

Definition 10 [29] Let $\preccurlyeq$ be the partial order on $\mathtt{S}$. Then the mapping $F : S \rightarrow P(S)$ is known as ordered multi dominated on H where $H \subseteq \mathtt{S}$ and $S \neq \varnothing$ if $p \preccurlyeq \mathtt{S}(p)$ for each $p \in H \subseteq \mathtt{S}$. We suppose that $p \preccurlyeq H$ where for each $x \in H$, we have $p \preccurlyeq x$.

Theorem 2 Let $(\mathtt{S}, \preccurlyeq, d_b)$ be an ordered complete $p\mathscr{M}S$. Let $P, Q : \mathtt{S} \rightarrow P(\mathtt{S})$ be a coupled $\preccurlyeq -$dominated multivalued mappings on $\overline{B_{d_b}(\nu_\circ, r)}$ where > 0 , $\nu_\circ \in \overline{B_{d_b}(\nu_\circ, r)}$. Suppose the mapping $\mathscr{F}$ is strictly increasing with a constant $\tau > 0$ then for all $\nu, \omega \in \overline{B_{d_b}(\nu_\circ, r)} \cap \{QP(\nu_n)\}$ there is $\Delta_b \in \Psi_b$, so that the following restrictions hold:

i.

$$\tau + \mathscr{F}(H_{d_b}(P(\nu), Q(\omega))) \leq \mathscr{F}(\Delta_b((\nu, \omega))), \tag{34}$$

where $\eta_b(d_b(P(\nu), Q(\omega)), (\nu, \omega)) \geq 0$, $\nu \preccurlyeq \omega$ and $H_{d_b}(P(\nu), Q(\omega)) > 0$.

Also, for all $n \in N \cup \{0\}$ and $b \geq 1$, we have:

ii.

$$\sum_{i=0}^{n} b^{i+1}\left\{\Delta_b^i d_b(\nu_\circ, P(\nu_\circ))\right\} \leq r, \tag{35}$$

Then the sequence $\{QP(\nu_n)\}$ in $\overline{B_{d_b}(\nu_\circ, r)}$ for each $n \in N \cup \{0\}$ and $\{QP(\nu_n)\} \rightarrow \nu^* \in \overline{B_{d_b}(\nu_\circ, r)}$.

iii. If (34) holds for ν^* and either $\nu_n \preccurlyeq \nu^*$ or $\nu^* \preccurlyeq \nu_n$ for all $n \in N \cup \{0\}$. Consequently, P and Q both share a common multi-FP ν^* in a $\mathbb{C}$-ball $\overline{B_{d_b}(\nu_\circ, r)}$ and $d_b(\nu^*, \nu^*) = 0$.

Proof. Let $\alpha : \mathtt{S} \times \mathtt{S} \rightarrow R^+$ is a mapping so that $\alpha(\nu, \omega) = 1$ for all $\nu \in \overline{B_{d_b}(\nu_\circ, r)}$, $\nu \preccurlyeq \omega$ and $\alpha(\nu, \omega) = 0$ for every $\nu, \omega \in \mathtt{S}$. Since $P, Q : \mathtt{S} \rightarrow P(\mathtt{S})$ are coupled multi $\preccurlyeq$-dominated operators on $\overline{B_{d_b}(\nu_\circ, r)}$, so $\nu \preccurlyeq P(\nu)$ and $\nu \preccurlyeq Q(\nu)$ for all $\nu \in \overline{B_{d_b}(\nu_\circ, r)}$. This implies $\nu \preccurlyeq b$ for all $b \in P(\nu)$ and $\nu \preccurlyeq \mathfrak{p}$ for all $\mathfrak{p} \in Q(\nu)$. So, $(\nu, b) = 1$ for all $\mathfrak{b} \in P(\nu)$ and $\alpha(\nu, \mathfrak{p}) = 1$ for each $\mathfrak{p} \in Q(\nu)$. This implies that $\inf\{\alpha(\nu, \omega) : \omega \in P(\nu)\} = 1$ and $\inf\{\alpha(\nu, \omega) : \omega \in Q(\nu)\} = 1$. So, $\alpha_*(\nu, P(\nu)) = 1$ and $\alpha_*(\nu, Q(\nu)) = 1$ for each $\nu \in \overline{B_{d_b}(\nu_\circ, r)}$. Since $P, Q : \mathtt{S} \rightarrow P(\mathtt{S})$ be a couple of α_*-dominated set-valued maps on $\overline{B_{d_b}(\nu_\circ, r)}$. So, inequality (34) can be written as

$$\tau + \mathscr{F}(H_{d_b}(P(\nu), Q(\omega)) \leq \mathscr{F}(\Delta_b((\nu, \omega))), \tag{36}$$

$\forall \nu, \omega \in \overline{B_{d_b}(\nu_\circ, r)} \cap \{QP(\nu_n)\}$, $\alpha(\nu, \omega) \geq 1$. Also inequality (35) holds.

So from Theorem 2, we have a sequence $\{QP(\nu_n)\}$ in $\overline{B_{d_b}(\nu_\circ, r)}$ and $\{QP(\nu_n)\} \rightarrow \nu^* \in \overline{B_{d_b}(\nu_\circ, r)}$. Now, $\nu_n, \nu^* \in \overline{B_{d_b}(\nu_\circ, r)}$, $\forall n \in N$ and either $\nu_n \preccurlyeq \nu^*$ or $\nu^* \preccurlyeq \nu_n$ implies that $\alpha(\nu_n, \nu^*) \geq 1$ or $\alpha(\nu^*, \nu_n) \geq 1$. Consequently all the restrictions of Theorem 2 are satisfied. So by Theorem 2, P and Q both admit a common multi-FP ν^* in $\overline{B_{d_b}(\nu_\circ, r)}$ and $d_b(\nu^*, \nu^*) = 0$. □

Corollary 1 Let $(\mathtt{S}, \preccurlyeq, d_b)$ be an ordered complete $p\mathscr{M}S$. Let $P : \mathtt{S} \rightarrow P(\mathtt{S})$ be a $\preccurlyeq$-dominated maps on $\mathtt{S}$. Consider that $\mathscr{F}$ is strictly increasing function such that for some $\Delta_b \in \Psi_b$, there is $\tau > 0$, so that the following restrictions hold:

$$\tau+\mathscr{F}\left(H_{d_b}\left(P(\nu), P(\nu)\right)\right) \leq \mathscr{F}\left(\Delta_b\left(D(\nu, \omega)\right)\right), \tag{37}$$

for all $\nu, \ \omega \in \{\mathtt{S}\, P(\nu_n)\}$, $H_{d_b}\left(P(\nu), P(\nu)\right) > 0$ and $\nu \preccurlyeq \omega$. Then $\{\mathtt{S}\, P(\nu_n)\} \to \nu^* \in \mathtt{S}$, for each $n \in N \cup \{0\}$ and either $(\nu_n \preccurlyeq \nu^*)$ or $(\nu^* \preccurlyeq \nu_n)$. If (37) holds for ν^* then P has a common multi- FP ν^* in $\mathtt{S}$ and $d_b(\nu^*, \nu^*) = 0$.

Example 3 Let $\mathtt{S} = R^+ \cup \{0\}$ $\mathtt{S} = R^+ \cup \{0\}$ and the function d_b: $\mathtt{S} \times \mathtt{S} \to \mathtt{S}$ with $b = 2$ defined as: $d_b(\nu, \ \omega) = (\max\{\nu, \ \omega\})^2$ for all $\nu, \ \omega \in \mathtt{S}$.

Let $P, \ Q : \mathtt{S} \to P(\mathtt{S})$ be the multi-valued mappings defined by $[P(\nu)] = \left[\frac{\nu}{3}, \ \frac{2\nu}{5}\right]$ and $[Q(\nu)] = \left[\frac{\nu}{4}, \ \frac{3\nu}{5}\right]$.

Choosing $\nu_0 = 1$ and $r = 25$. Then $\overline{B_{d_b}(\nu_0, r)} = [1, 4] \cap \mathtt{S}$. Now, we have $d_b(\nu_0, P(\nu_0)) = d_b(1, \frac{1}{3}) = 1$. So, $\nu_1 = \frac{1}{3}$, $d_b(\nu_1, Q(\nu_1)) = d_b\left(\frac{1}{3}, \frac{1}{12}\right)$. As $\nu_2 = \frac{1}{12}$, $d_b(\nu_2, P(\nu_2)) = d_b(\frac{1}{12}, \frac{1}{36})$. So, $\nu_3 = \frac{1}{36}$. Hence, we get the sequence $\{QP(\nu_n)\} = \left\{1, \frac{1}{3}, \ \frac{1}{12}, \ \frac{1}{36}, \ \cdots\right\}$ in $\mathtt{S}$ generated by ν_0. Let $\Delta_b(\mathtt{k}) = \frac{2\mathtt{k}}{9}$, $r = 25$ and $a = 1$. Define $\alpha : \mathtt{S} \times \mathtt{S} \to R^+$ by

$$\alpha(\nu, \omega) = \begin{cases} 1 & \text{if } \nu > \omega \\ \frac{1}{3} & \text{otherwise} \end{cases} \tag{38}$$

Now if $\nu, \omega \in \overline{B_{d_b}(\nu_\circ, r)} \cap \{QP(\nu_n)\}$ with $\alpha(\nu, \omega) \geq 1$. We get,

$$H_{d_b}\left(P(\nu), Q(\omega)\right) = \max\left\{\sup_{c \in P(\nu)} d_b(c, Q(\omega)), \ \sup_{d \in Q(\omega)} d_b(P(\nu), d)\right\} \tag{39}$$

$$H_d\left(P(\nu), Q(\omega)\right) = \max\left\{d\left(\frac{2\nu}{5}, \left[\frac{\omega}{4}, \frac{3\omega}{5}\right]\right), d\left(\left[\frac{\nu}{3}, \frac{2\nu}{5}\right], \frac{3\omega}{5}\right)\right\} \tag{40}$$

$$\therefore d(y, A) = \inf_{g_\circ \in A}(y, g_\circ)$$

$$= \max\left\{d\left(\frac{2\nu}{5}, \frac{\omega}{4}\right), d\left(\frac{\nu}{3}, \frac{3\omega}{5}\right)\right\},$$

$$= \max\left\{(\max\{\frac{2\nu}{5}, \frac{\omega}{4}\})^2, (\max\{\frac{\nu}{3}, \frac{3\omega}{5}\})^2\right\},$$

$$\leq \Delta_b\left(\max\left\{\begin{array}{c} d(\nu, \omega), \frac{d(\nu, P(\nu)) \cdot d(\omega, Q(\omega))}{\alpha + d(\nu, \omega)}, \\ d(\nu, P(\nu)), \\ d(\omega, Q(\omega)) \end{array}\right\}\right),$$

$$\leq \Delta_b \left(\max \left\{ \begin{array}{c} d(\nu, \omega), \frac{d\left(\nu, \left[\frac{\nu}{3}, \frac{2\nu}{5}\right]\right) \cdot d\left(\omega, \left[\frac{\omega}{4}, \frac{3\omega}{5}\right]\right)}{\alpha + d(\nu, \omega)}, \\ d\left(\nu, \left[\frac{\nu}{3}, \frac{2\nu}{5}\right]\right), \\ d\left(\omega, \left[\frac{\omega}{4}, \frac{3\omega}{5}\right]\right) \end{array} \right\} \right),$$

$$\leq \Delta_b \left(\max \left\{ \begin{array}{c} \left(\max\{\nu, \omega\}\right)^2, \frac{\left(\max\left\{\nu, \frac{\nu}{3}\right\}\right)^2 \cdot \left(\max\left\{\omega, \frac{\omega}{4}\right\}\right)^2}{1 + \left(\max\{\nu, \omega\}\right)^2}, \\ \left(\max\left\{\nu, \frac{\nu}{3}\right\}\right)^2, \\ \left(\max\left\{\omega, \frac{\omega}{4}\right\}\right)^2 \end{array} \right\} \right). \tag{41}$$

Taking $\nu > \omega$, let $\nu = 3, \ \omega = 2,$ we have;

$$\begin{aligned} &\leq \Delta_b \left(\max \left\{ \begin{array}{c} \nu^2, \frac{\nu^2 \cdot \omega^2}{1+\nu^2}, \\ \nu^2, \omega^2 \end{array} \right\} \right), \\ &\leq \Delta_b \left(\max \left\{ \begin{array}{c} 3^2, \frac{3^2 \cdot 2^2}{1+3^2}, \\ 3^2, 2^2 \end{array} \right\} \right), \\ &\leq \Delta_b \left(\max \{9, 3.6, 9, 4\} \right), \\ &\leq \Delta_b(9), \\ &\leq \Delta_b(d(3, 2)). \end{aligned} \tag{42}$$

Now take $2, 3 \in \overline{B_{d_b}(\nu_0, r)} \cap \mathtt{S},$ then $\alpha(3, 2) > 1$. But we have,

$$H_{d_b}\left(P(\nu), Q(\omega)\right) < \ \Delta_b\left(D(\nu, \omega)\right). \tag{43}$$

This shows if $\tau \in \left(0, \frac{11}{91}\right]$ and $\mathscr{F}$ is strictly increasing mapping defined as $\mathscr{F}(c) = \ln c$, we obtain

$$H_{d_b}\left(P(\nu), Q(\omega)\right) \cdot e^{\tau} \leq \Delta_b(d_b(\nu, \omega)),$$

$$\ln\left(H_{d_b}\left(P(\nu), Q(\omega)\right) \cdot e^{\tau}\right) \leq \ln \Delta_b(d_b(\nu, \omega)),$$

$$\ln\left(H_{\mathcal{d}_b}\left(P(\nu),\, Q(\omega)\right)\cdot e^{\tau}\right) \leq \ln\Delta_b\left(\mathcal{d}_b(\nu,\, \omega)\right),$$

$$\ln\left(H_{\mathcal{d}_b}\left(P(\nu),\, Q(\omega)\right)\right) + \tau \leq \ln\Delta_b\left(\mathcal{d}_b(\nu,\, \omega)\right),$$

$$\tau + \mathscr{F}\left(H_{\mathcal{d}_b}\left(P(\nu),\, Q(\omega)\right)\right) \leq \mathscr{F}\left(\Delta_b\left((\nu,\, \omega)\right)\right),$$

$$\tau + \mathscr{F}\left(H_{\mathcal{d}_b}\left(P(3),\, Q(2)\right)\right) \leq \mathscr{F}\left(\, \Delta_b\left((3,\, 2)\right)\right),$$

$$\tau + \mathscr{F}\left(\max\left\{\mathcal{d}_b\left(\frac{2\nu}{5},\, \frac{\omega}{4}\right),\, \mathcal{d}_b\left(\frac{\nu}{3},\, \frac{3\omega}{5}\right)\right\}\right) \leq \mathscr{F}\left(\Delta_b((3,\, 2))\right),$$

$$\tau + \mathscr{F}\left(\max\left\{(\max\{\frac{2\nu}{5},\, \frac{\omega}{4}\})^2,\, (\max\{\frac{\nu}{3},\, \frac{3\omega}{5}\})^2\right\}\right) \leq \mathscr{F}\left(\Delta_b\left((3,\, 2)\right)\right),$$

$$\tau + \mathscr{F}\left(\max\left\{(\max\{\frac{6}{5},\, \frac{2}{4}\})^2,\, (\max\{\frac{3}{3},\, \frac{6}{5}\})^2\right\}\right) \leq \mathscr{F}\left(\Delta_b((3,\, 2))\right),$$

$$\tau + \mathscr{F}\left(\max\left\{\left(\frac{6}{5}\right)^2,\, \left(\frac{6}{5}\right)^2\right\}\right) \leq \mathscr{F}\left(\Delta_b\left((3,\, 2)\right)\right),$$

$$\tau + \mathscr{F}(1.44) \leq \mathscr{F}\left(\Delta_b(9)\right).$$

$$\therefore F(c) = \ln c.$$

So $\tau + \ln(1.44) \leq \ln(\Delta_b(9))$,

$$0.1 + \ln(1.44) + 1.44 \leq \ln\left(\frac{2\times 9}{9}\right),$$

$$0.1 + 0.3646 \leq \ln(2),$$

$$0.4646 < 0.6931.$$

So, condition (7) hold on $\overline{B_{\mathcal{d}_b}(\nu_0,\, r)} \cap \mathtt{S}$. Now we take $5,\, 6 \in \mathtt{S}$ then $\alpha(5,\, 6) \geq 1$. We have,

$$\tau + \mathscr{F}\left(H_{d_b}\left(P(5), Q(6)\right)\right) > \mathscr{F}\left(\Delta_b\left((5, 6)\right)\right),$$

$$\tau + \mathscr{F}\left(\max\left\{d_b\left(\frac{2\nu}{5}, \frac{\omega}{4}\right), d_b\left(\frac{\nu}{3}, \frac{3\omega}{5}\right)\right\}\right) > \mathscr{F}\left(\Delta_b((\nu, \omega))\right),$$

$$\tau + \mathscr{F}\left(\max\left\{\left(\max\left\{\frac{2\nu}{5}, \frac{\omega}{4}\right\}\right)^2, \left(\max\left\{\frac{\nu}{3}, \frac{3\omega}{5}\right\}\right)^2\right\}\right) > \mathscr{F}\left(\Delta_b(\nu, \omega)\right),$$

$$\tau + \mathscr{F}\left(\max\left\{\left(\max\left\{\frac{10}{5}, \frac{6}{4}\right\}\right)^2, \left(\max\left\{\frac{5}{3}, \frac{18}{5}\right\}\right)^2\right\}\right) > \mathscr{F}\left(\Delta_b(6, 5)\right),$$

$$\tau + \mathscr{F}\left(\max\left\{\left(\frac{10}{5}\right)^2, \left(\frac{18}{5}\right)^2\right\}\right) > \mathscr{F}\left(\Delta_b(6, 5)\right),$$

$$\tau + \mathscr{F}\left(\max\{4, 12.96\}\right) > \mathscr{F}\left(\Delta_b(36)\right),$$

$$\tau + \mathscr{F}(12.96) > \mathscr{F}\left(\Delta_b(36)\right),$$

$$\therefore \mathscr{F}(c) = \ln c.$$

So $\tau + \ln(12.96) > \ln(\Delta_b(36))$,

$$0.1 + 2.5619 > \ln\left(\frac{2}{9} \times 36\right),$$

$$2.6619 > 2.0794.$$

Thus, (7) fails to hold on S. Furthermore, for each $n \in \mathcal{N} \cup \{0\}$,

$$\sum_{i=0}^{n} b^{i+1}\left(\Delta_b{}^i\left(d_b\left(\nu_0, \ \nu_1\right)\right)\right) = 2\sum_{i=0}^{n}\left(\frac{4}{9}\right)^i < 25 = r. \tag{44}$$

Hence, P and Q satisfies all the requirements of Theorem 2 only for $\nu, \omega \in \overline{B_{d_b}(\nu_0, r)} \cap \{QP(\nu_n)\}$ with $\alpha(\nu, \omega) \geq 1$. Therefore P and Q have a common multi-FP in $\overline{B_{d_b}(\nu_0, r)}$.

4. Existence results for graph theory

In this portion, we illustrate an application of Theorem 2 in graph theory. Bojor [30] introduced a graph structure to prove some FP results. Moreover, we can find the similar results in [15, 24, 31–33] which further explored FP results in metric spaces endowed with a graph.

Definition 11 [34] Suppose $G = (\mathcal{E}(G), \mathcal{W}(G))$ is a graph so that for $\mathtt{S} \neq \varnothing$, we have $\mathcal{E}(G) = \mathtt{S}$ and $\mathtt{U} \subset \mathtt{S}$. Then a multi-graph dominated mapping on $\mathtt{U}$ is defined by a mapping $P : \mathtt{S} \longrightarrow P(\mathtt{S})$ if $(\nu, \omega) \in \mathcal{W}(G)$ for every $\omega \in P(\nu)$ and $\nu \in \mathtt{U}$.

Theorem 3 Let $(\mathtt{S}, d_b)$ be a $p_b\mathcal{MS}$ equipped with a graph $\mathcal{G}$. Let $P, Q : \mathtt{S} \longrightarrow P(\mathtt{S})$ denotes the multi maps on $\overline{B_{d_b}(\nu_0, r)}$ and $r > 0$, $\nu_0 \in \overline{B_{d_b}(\nu_0, r)}$. Assume that $\delta_b \in \Psi_b$ the given conditions hold:

i. P and Q are multi-graph dominated on $\overline{B_{d_b}(\nu_0, r)} \cap \{QP(\nu_n)\}$.

ii. A strictly increasing function $\mathscr{F}$ exists with $\tau > 0$, satisfying the following restrictions;

$$\tau + \mathscr{F}(H_{d_b}(P(\nu), Q(\omega))) \leq \mathscr{F}(\Delta_b(D(\nu, \omega))), \tag{45}$$

where $\nu, \omega \in \overline{B_{d_b}(\nu_0, r)} \cap \{QP(\nu_n)\}$. $\nu, \omega \in \mathcal{W}(G)$ and $H_{d_b}(P(\nu), Q(\omega)) > 0$.

iii. $\sum_{i=0}^{n} b^{i+1}\left(\Delta_b{}^i(d_b(\nu_\circ, \nu_1))\right) < r$.

Then for any sequence $\{QP(\nu_n)\}$ in $\overline{B_{d_b}(\nu_0, r)}$, $(\nu_n, \nu_{n+1}) \in \mathcal{W}(G)$ and $\{QP(\nu_n)\} \longrightarrow e^*$. Also if (45) satisfies for e^* and $(\nu_n, e^*) \in \mathcal{W}(G)$ or $(e^*, \nu_n) \in \mathcal{W}(G)$ for every $n \in \mathcal{N}$ then P and Q have a common multi-FP e^* in $\overline{B_{d_b}(\nu_0, r)}$.

Proof. Let $\alpha : \mathtt{S} \times \mathtt{S} \to R^+$ is defined by

$$\alpha(\nu, \omega) = \begin{cases} 1 \ \ if \ \nu \in \overline{B_{d_b}(\nu_0, r)}, & \nu, \omega \in \mathcal{W}(G) \\ 0 & \text{otherwise.} \end{cases} \tag{46}$$

Let P and Q are multi graph-dominated on $\overline{B_{d_b}(\nu_0, r)}$, then for $\nu \in \overline{B_{d_b}(\nu_0, r)}$, $(\nu, \omega) \in \mathcal{W}(G)$ for each $\omega \in P(\nu)$ and $\omega \in Q(\nu)$. Then, for each $\omega \in P(\nu)$ and $\omega \in Q(\nu)$ we get $\alpha(\nu, \omega) = 1$. It implies that $\inf\{\alpha(\nu, \omega) : \omega \in P(\nu)\} = 1$ and $\inf\{\alpha(\nu, \omega) : \omega \in Q(\nu)\} = 1$. Thus, $\alpha_*(\nu, P(\nu)) = 1$, $\alpha_*(\nu, Q(\nu)) = 1$ for each $\nu \in \overline{B_{d_b}(\nu_0, r)}$. Consequently, P, $Q : \mathtt{S} \longrightarrow P(\mathtt{S})$ are the couple of semi α_*−dominated multi-maps on $\overline{B_{d_b}(\nu_0, r)}$. Also, inequality (45) can be expressed as,

$$\tau + \mathscr{F}(H_{d_b}(P(\nu), Q(\omega))) \leq \mathscr{F}(\Delta_b(D(\nu, \omega))), \tag{47}$$

where $\nu, \omega \in \overline{B_{d_b}(\nu_0, r)} \cap \{QP(\nu_n)\}$, $\alpha(\nu, \omega) \geq 1$ and $H_{d_b}(P(\nu), Q(\omega)) > 0$.

Also, condition (iii) holds. Then from Theorem 2, we find a sequence $\{QP(\nu_n)\}$ in $\overline{B_{d_b}(\nu_0, r)}$ and $\{QP(\nu_n)\} \longrightarrow e^*$ in $\overline{B_{d_b}(\nu_0, r)}$. Now $(\nu_n, e^*) \in \overline{B_{d_b}(\nu_0, r)}$ and either $(\nu_n, e^*) \in \mathcal{W}(G)$ or $(e^*, \nu_n) \in \mathcal{W}(G)$ represents that either $\alpha(\nu_n, e^*) \geq 1$ or $\alpha(e^*, \nu_n) \geq 1$. Hence, all assumptions of Theorem 2 are fully satisfied. Then, by Theorem 2 both P and Q admits a common multi-FP, e^* in $\overline{B_{d_b}(\nu_0, r)}$ and $d_b(e^*, e^*) = 0$. □

5. Application on Volterra integral equations

Here, we will show an application to explore nonlinear Volterra integral equations. We will also prove the existence and uniqueness of solution for integral equations. Some relevant results involving applications of these integral equations can be seen in [2, 35–37].

Theorem 4 Let $(\mathtt{S}, \mathscr{d}_b)$ be a $p\flat\mathscr{M}S$. Suppose for any $\nu \in \mathtt{S}$ and $P,\ Q : \mathtt{S} \to \mathtt{S}$ be the mappings there are $\tau > 0$, a strictly increasing function $\mathscr{F}$ such that for some $\Delta_b \in \psi_b$, the following condition holds;

$$\tau + \mathscr{F}(H_{\mathscr{d}_b}(P(\nu), Q(\omega)) \leq \mathscr{F}(\Delta_b(D(\nu, \omega))), \tag{48}$$

where $\nu,\ \omega \in \{QP(\nu_n)\}$ and $H_{\mathscr{d}_b}(P(\nu), Q(\omega)) > 0$. Then $\{QP(\nu_n)\} \to \omega \in \mathtt{S}$. Also if (48) holds for $\nu, \omega \in \{u\}$. Then u is the common FP of P and Q in $\mathtt{S}$.

Proof. The proof of Theorem 4 is identical to that of Theorem 2. In this case, we will prove the uniqueness only. Suppose γ is another FP of mapping P and Q. Assume that $\mathscr{d}_b(P(\nu), Q(\omega)) > 0$. Then we have,

$$\tau + \mathscr{F}(H_{\mathscr{d}_b}(P(\nu), Q(\omega)) \leq \mathscr{F}(\Delta_b(D(\nu, \omega))),$$

$$\Rightarrow \mathscr{d}_b(\nu, \omega) < \Delta_b \mathscr{d}_b(\nu, \omega) < \Delta_b \mathscr{d}_b(\nu, \omega) < \mathscr{d}_b(\nu, \omega),$$

which is contradiction. So,

$$\mathscr{d}_b(P(\nu), Q(\omega)) = 0 \tag{49}$$

Hence, $\omega = \gamma$. Now, in order to get the unique common solution of Volterra Is, we are going to prove the application of Theorem 2 which is presented as:

$$\nu(\theta) = \int_0^\theta \rho_1(\theta, \mu, \nu(\mu))\, d\mu, \tag{50}$$

$$\omega(\theta) = \int_0^\theta \rho_2(\theta, \mu, \omega(\mu))\, d\mu, \tag{51}$$

$\forall\ \theta \in [0, 1]$. Now we solve (50) and (51). Let $\mathtt{S} = \mathcal{C}([0, 1],\ R^+)$ consists of all continuous functions on $[0, 1]$ with the complete $p\flat\mathscr{M}S$. For any $\nu \in \mathcal{C}([0, 1], R^+)$, identify norm as follows: $\|\nu_\tau\| = \left[\sup_{\theta\in[0,1]}\left\{|\nu(\theta)|e^{-\tau\theta}\right\}\right]$, where $\tau > 0$ is chosen arbitrary. Then, $\mathscr{d}_\tau(\nu, \omega) = \left[\sup_{\theta\in[0,1]}\left\{\max\left\{|\nu(\theta), \omega(\theta)|e^{-\tau\theta}\right\}\right\}\right]^2 = \|\nu\|_\tau^2$, $\forall\ \nu, \omega \in \mathcal{C}([0, 1],\ R^+)$, under these arrangements $(\mathcal{C}([0, 1], R^+), \mathscr{d}_\tau)$ forms a complete $p\flat\mathscr{M}S$.

The existence of solution to the I$\mathcal{E}$ will be presented through the following theorem. □

Theorem 5 Consider the following conditions hold;

(i) $\mathscr{M}_1 \times \mathscr{M}_2 : [0, 1] \times [0, 1] \times \mathcal{C}([0, 1], R^+) \to R$,

(ii) The mappings $P,\ Q : \mathcal{C}([0, 1], R^+) \to \mathcal{C}([0, 1], R^+)$ are defined as

$$(P\nu)(\theta) = \int_0^\theta p_1(\theta, \mu, \nu(\mu))\, d\mu, \tag{52}$$

$$(Q\omega)(\theta) = \int_0^\theta p_2(\theta, \mu, \omega(\mu))\, d\mu\ . \tag{53}$$

Suppose there is $\tau > 0$, so that

$$\max|p_1(\theta, \mu, \nu), p_2(\theta, \mu, \omega)|^2 \le \frac{\Delta_b(D(\nu,\ \omega))}{\tau\, \Delta_b(D(\nu,\ \omega))+1}. \tag{54}$$

$\forall\ \nu,\ \omega \in \mathcal{C}\left([0, 1],\ R^+\right), \theta,\ \in [0, 1]$ and

$$D(\nu,\ \omega) = \max\left\{\ \Delta_b\left(\begin{array}{c} \max\{|\nu,\ \omega|\}^2, \max\{|\nu, P\nu|\}^2,\ max\{|\omega, Q\omega|\}^2, \\ \frac{\max\{|\nu, P\nu|\}^2 . \max\{|\omega, Q\omega|\}^2}{1+\max\{|\nu,\ \omega|\}^2} \end{array}\right)\right\}. \tag{55}$$

Then the IƐs (50) and (51) admit a unique solution in $\mathcal{C}\left([0, 1],\ R^+\right)$.
Proof. From (ii), we have

$$\max\{|P\nu, Q\omega|\}^2 = \int_0^\theta\ \max\{|p_1(\theta, \mu, \nu(\mu)), p_2(\theta, \mu, \omega(\mu))|\}^2\, d\mu,$$

$$\le \int_0^\theta \frac{\Delta_b(D(\nu,\ \omega))}{\tau\, \Delta_b(D(\nu,\ \omega))+1} e^{\tau\mu} d\mu,$$

$$\le \frac{\Delta_b(D(\nu,\ \omega))}{\tau\, \Delta_b(D(\nu,\ \omega))+1} \int_0^\theta e^{\tau\mu} d\mu,$$

$$\le \frac{\delta_b(D(\nu,\ \omega))}{\tau\delta_b(D(\nu,\ \omega))+1} e^{\tau\theta}.$$

This yields,

$$\max|P\nu, Q\omega|^2 e^{-\tau\theta} \le \frac{\Delta_b(D(\nu,\ \omega))}{\tau\, \Delta_b(D(\nu,\ \omega))+1},$$

$$\|P\nu, Q\omega\|_\tau \le \frac{\Delta_b(D(\nu,\ \omega))}{\tau\, \Delta_b(D(\nu,\ \omega))+1},$$

$$\frac{\tau\,\Delta_b(D(\nu,\ \omega))+1}{\Delta_b(D(\nu,\ \omega))} \leq \frac{1}{\|P\nu,\,Q\omega\|_\tau},$$

$$\tau+\frac{1}{\Delta_b(D(\nu,\ \omega))} \leq \frac{1}{\|P\nu,\,Q\omega\|_\tau},$$

which ultimately shows

$$\tau-\frac{1}{\|P\nu,\,Q\omega\|_\tau} \leq \frac{-1}{\Delta_b(D(\nu,\ \omega))}.$$

Thus, all requirements of Theorem 2 are fulfilled for $F(c)=\frac{-1}{c}$; $c>0$ and $d_\tau(\nu,\,\omega)=\|\nu+\omega\|_\tau^2$. Hence, the IEs in (50) and (51) admit a unique common solution. □

6. Application on fractional differential equations

We employ our recent findings to develop the solution of fractional differential equations. Recently, Karapinar et al. [38], Alqahtani et al. [39] and Nashine et al. [40] utilized FP techniques to find the common solutions for fractional differential equations.

Theorem 6 Let $\mathcal{C}\left([0,\,1],\,R^+\right)$ represent all continuous functions. Let the metric $d_b:\mathcal{C}\left([0,\,1],\ R^+\right)\times\mathcal{C}\left([0,\,1],\ R^+\right)\to[0,\ \infty]$ be defined as: $d_b(\nu,\,\omega)=\|\nu-\omega\|_\infty^2=\max_{l\in[0,\,L]}|\nu(l)-\omega(l)|^2$, for all $\nu,\,\omega\in\mathcal{C}\left([0,\,1],\ R^+\right)$. Then $\mathcal{C}\left([0,\,1],\ R^+\right)$ is a complete $p_b\mathcal{M}S$.

Let $H_1,\,H_2:\mathcal{C}\left([0,\,1],\,R^+\right)\to R^+$ be a couple of continuous mappings satisfying $H_1(\mathfrak{r},\,\mathfrak{s})\geq 0$ and $H_2(\mathfrak{t},\,\mathfrak{u})\geq 0$, $\forall\ \mathfrak{r},\,\mathfrak{t}\in[0,\,1]$ and $\mathfrak{s},\,\mathfrak{u}\geq 0$. The investigation of CFDE of order $\mathfrak{z}$ is presented below:

$$D^{\mathfrak{z}}\mathfrak{s}(\mathfrak{r})=H_1[\mathfrak{r},\,\mathfrak{s}(\mathfrak{r})]\ ;\ \mathfrak{s}\in\mathcal{C}\left([0,\,1],\,R^+\right), \tag{56}$$

with boundary conditions $\mathfrak{s}(0)=0$, $I\mathfrak{s}(1)=\mathfrak{s}'(0)$ and

$$D^{\mathfrak{z}}\mathfrak{u}(\mathfrak{t})=H_2[\mathfrak{t},\,\mathfrak{u}(\mathfrak{t})]\ ;\ \mathfrak{u}\in\mathcal{C}\left([0,\,1],\,R^+\right), \tag{57}$$

with boundary conditions $\mathfrak{u}(0)=0$, $I\mathfrak{u}(1)=\mathfrak{u}'(0)$. Hence $D^{\mathfrak{z}}$ represent the CF$\mathcal{D}$ of order $\mathfrak{z}$ expressed as follows

$$D^{\mathfrak{z}}\ \mathfrak{s}(\mathfrak{r})=\frac{1}{\Gamma(\mathfrak{r}-\mathfrak{z})}\int_0^{\mathfrak{r}}(\mathfrak{r}-)^{\mathfrak{y}-\mathfrak{z}-1}\,\mathfrak{s}(\sigma)d\sigma,$$

where $\mathfrak{y}-1<\mathfrak{z}<\mathfrak{y}$ and $\mathfrak{y}=\mathfrak{z}+1$, and $I^{\mathfrak{z}}\mathfrak{s}$ is specified by $I^{\mathfrak{z}}\mathfrak{s}(\mathfrak{r})=\frac{1}{\Gamma(\mathfrak{z})}\int_0^{\mathfrak{r}}(\mathfrak{r}-\sigma)^{\mathfrak{z}-1}\,\mathfrak{s}(\sigma)d\sigma$, with $\mathfrak{z}>0$.

So equation (56) can be represented by

$$\mathfrak{s}(\mathfrak{r}) = \frac{1}{\Gamma(\mathfrak{z})}\int_0^{\mathfrak{r}} (\mathfrak{r}-\mathscr{6})^{\mathfrak{z}-1} H_1[\mathscr{6}, \mathfrak{s}(\mathscr{6})] d\mathscr{6} + \frac{2\mathfrak{r}}{\Gamma(\mathfrak{z})}\int_0^F \int_0^{\mathscr{6}} (\mathscr{6}-p)^{\mathfrak{z}-1} H_1[p, \mathfrak{s}(p)] dp d\mathscr{6}. \tag{58}$$

Similarly equation (57) can be represented by

$$\mathfrak{u}(\mathfrak{t}) = \frac{1}{\Gamma(\mathfrak{z})}\int_0^{\mathfrak{t}} (\mathfrak{t}-\mathscr{6})^{\mathfrak{z}-1} H_2[\mathscr{6}, \mathfrak{u}(\mathscr{6})] d\mathscr{6} + \frac{2\mathfrak{t}}{\Gamma(\mathfrak{z})}\int_0^F \int_0^{\mathscr{6}} (\mathscr{6}-q)^{\mathfrak{z}-1} H_2[q, \mathfrak{u}(q)] dq d\mathscr{6}. \tag{59}$$

Then the mappings H_1, H_2: $\left([0, 1], R^+\right) \to R^+$ meet the following conditions:
i. for all $\mathfrak{s}, \mathfrak{u} \in \mathcal{C}\left([0, 1], R^+\right)$ and $\exists\ \tau > 0$, we obtain $|H_1[\mathfrak{r}, \mathfrak{s}] - H_2[\mathfrak{t}, \mathfrak{u}]| = \frac{e^{-\tau}\Gamma(\mathfrak{z}+1)}{4} |\mathfrak{s} - \mathfrak{u}|$ and $\mathfrak{t}, \mathfrak{s}, \mathfrak{u} > 0$.
ii. $\forall\ \mathfrak{m}, \mathfrak{n} \in \mathcal{C}\left([0, 1], R^+\right) \exists\ \upsilon, \nu \in \mathcal{C}\left([0, 1], R^+\right)$ such that

$$\upsilon(\mathfrak{m}) = \frac{1}{\Gamma(\mathfrak{z})}\int_0^{\mathfrak{m}} (\mathfrak{m}-\mathscr{6})^{\mathfrak{z}-1} H_1[\mathscr{6}, \mathfrak{s}(\mu)] d\mathscr{6} + \frac{2\mathfrak{m}}{\Gamma(\mathfrak{z})}\int_0^F \int_0^{\mathscr{6}} (\mathscr{6}-p)^{\mathfrak{z}-1} H_1[p, \mathfrak{s}(p)] dp d\mathscr{6},$$

and

$$\nu(\mathfrak{n}) = \frac{1}{\Gamma(\mathfrak{z})}\int_0^{n} (\mathfrak{n}-\mathscr{6})^{\mathfrak{z}-1} H_2[\mathscr{6}, \mathfrak{u}(\mathscr{6})] d\mathscr{6} + \frac{2n}{\Gamma(\mathfrak{z})}\int_0^F \int_0^{\mathscr{6}} (\mathscr{6}-q)^{\mathfrak{z}-1} H_2[q, \mathfrak{u}(q)] dq d\mathscr{6}.$$

If the conditions (i) and (ii) are fulfilled then the equations (56) and (57) admits the solution in $\mathcal{C}\left([0, 1], R^+\right)$.
Proof. Let the mappings P, Q: $\mathcal{C}\left([0, 1], R^+\right) \to \mathcal{C}\left([0, 1], R^+\right)$ are denoted as
$P((\mathfrak{m})) = \frac{1}{\Gamma(\mathfrak{z})}\int_0^{\mathfrak{m}} (\mathfrak{m}-\mathscr{6})^{\mathfrak{z}-1} H_1[\mathscr{6}, \mathfrak{s}(\mathscr{6})] d\mathscr{6} + \frac{2\mathfrak{m}}{\Gamma(\mathfrak{z})}\int_0^F \int_0^{\mathscr{6}} (\mathscr{6}-p)^{\mathfrak{z}-1} H_1[q, \mathfrak{s}(q)] dq d\mathscr{6}$, and

$$Q(\nu(\mathfrak{m})) = \frac{1}{\Gamma(\mathfrak{z})}\int_0^{\mathfrak{m}} (\mathfrak{m}-\mathscr{6})^{\mathfrak{z}-1} H_2[\mathscr{6}, \mathfrak{u}(\mathscr{6})] d\mathscr{6} + \frac{2\mathfrak{m}}{\Gamma(\mathfrak{z})}\int_0^F \int_0^{\mathscr{6}} (\mathscr{6}-q)^{\mathfrak{z}-1} H_2[q, \mathfrak{u}(q)] dq d\mathscr{6}.$$

Then by b) $\upsilon, \nu \in \mathcal{C}\left([0, 1], R^+\right)$ so that $P(\upsilon(\mathfrak{m})) = \upsilon(\mathfrak{m})$ and $Q(\nu(\mathfrak{m})) = \nu(\mathfrak{m})$. Then the continuity of H_1, H_2 ensures the continuity of mappings P, Q in $\mathcal{C}\left([0, 1], R^+\right)$. We are going to prove the contractive conditions of 2 Theorem. For this, letting:

$$\sqrt{|P(\upsilon(\mathfrak{m})) - Q(\nu(\mathfrak{m}))|^2}$$

$$= \left| \begin{array}{l} \frac{1}{\Gamma(\mathfrak{z})}\int_0^{\mathfrak{m}} (\mathfrak{m}-\mathscr{6})^{\mathfrak{z}-1} H_1[\mathscr{6}, \mathfrak{s}(\mathscr{6})] d\mathscr{6} + \frac{2\mathfrak{m}}{\Gamma(\mathfrak{z})}\int_0^F \int_0^{\mathscr{6}} (\mathscr{6}-p)^{\mathfrak{z}-1} H_1[q, \mathfrak{s}(q)] dq d\mathscr{6} \\ -\frac{1}{\Gamma(\mathfrak{z})}\int_0^{\mathfrak{m}} (\mathfrak{m}-\mathscr{6})^{\mathfrak{z}-1} H_2[\mathscr{6}, \mathfrak{u}(\mathscr{6})] d\mathscr{6} - \frac{2\mathfrak{m}}{\Gamma(\mathfrak{z})}\int_0^F \int_0^{\mathscr{6}} (\mathscr{6}-q)^{\mathfrak{z}-1} H_2[q, \mathfrak{u}(q)] dq d\mathscr{6} \end{array} \right|$$

$$\leq \left| \frac{1}{\Gamma(\mathfrak{z})}\int_0^{\mathfrak{m}} (\mathfrak{m}-\mathscr{6})^{\mathfrak{z}-1} H_1[\mathscr{6}, \mathfrak{s}(\mathscr{6})] d\mathscr{6} - \frac{1}{\Gamma(\mathfrak{z})}\int_0^{\mathfrak{m}} (\mathfrak{m}-\mathscr{6})^{\mathfrak{z}-1} H_2[\mathscr{6}, \mathfrak{u}(\mathscr{6})] d\mathscr{6} \right|$$

$$+\left|\frac{2\mathfrak{m}}{\Gamma(\mathfrak{z})}\int_0^F\int_0^{\mathfrak{k}}(\mathfrak{k}-p)^{\mathfrak{z}-1}H_1\left[q,\,\mathfrak{s}(q)\,\right]dqd\mathfrak{k}\,-\,\frac{2\mathfrak{m}}{\Gamma(\mathfrak{z})}\int_0^F\int_0^{\mathfrak{k}}(\mathfrak{k}-q)^{\mathfrak{z}-1}H_2\left[q,\,\mathfrak{u}(q)\,\right]dqd\mathfrak{k}\right|$$

$$\leq\left|\int_0^{\mathfrak{m}}\left(\frac{1}{\Gamma(\mathfrak{z})}\,(\mathfrak{m}-\mathfrak{k})^{\mathfrak{z}-1}\,H_1\left[\mathfrak{k},\,\mathfrak{s}(\mathfrak{k})\,\right]\,-\,\frac{1}{\Gamma(\mathfrak{z})}\,(\mathfrak{m}-\mathfrak{k})^{\mathfrak{z}-1}\,H_2\left[\mathfrak{k},\,\mathfrak{u}(\mathfrak{k})\,\right]\right)d\mathfrak{k}\right|$$

$$+\left|\int_0^F\int_0^{\mathfrak{k}}\left(\frac{2\mathfrak{m}}{\Gamma(\mathfrak{z})}(\mathfrak{k}-p)^{\mathfrak{z}-1}H_1\left[q,\,\mathfrak{s}(q)\,\right]\,-\,\frac{2\mathfrak{m}}{\Gamma(\mathfrak{z})}(\mathfrak{k}-q)^{\mathfrak{z}-1}H_2\left[q,\,\mathfrak{u}(q)\,\right]\right)dqd\mathfrak{k}\right|$$

$$\leq\frac{1}{\Gamma(\mathfrak{z})}\frac{e^{-\tau}\Gamma(\mathfrak{z}+1)}{4}\int_0^{\mathfrak{m}}\left(\frac{1}{\Gamma(\mathfrak{z})}\,(\mathfrak{m}-\mathfrak{k})^{\mathfrak{z}-1}\,|\upsilon(\mathfrak{k})\,-\,\nu(\mathfrak{k})|\right)d\mathfrak{k}$$

$$+\frac{2\mathfrak{m}}{\Gamma(\mathfrak{z})}\frac{e^{-\tau}\Gamma(\mathfrak{z}+1)}{4}\int_0^F\int_0^{\mathfrak{k}}\left(\frac{2\mathfrak{m}}{\Gamma(\mathfrak{z})}(\mathfrak{k}-q)^{\mathfrak{z}-1}\,|\upsilon(q)\,-\,\nu(q)|\right)dqd\mathfrak{k},$$

$$\leq\frac{1}{\Gamma(\mathfrak{z})}\frac{e^{-\tau}\Gamma(\mathfrak{z}+1)}{4}|\upsilon\,-\,\nu|\int_0^{\mathfrak{m}}\left(\frac{1}{\Gamma(\mathfrak{z})}\,(\mathfrak{m}-\mathfrak{k})^{\mathfrak{z}-1}\right)d\mathfrak{k}$$

$$+\frac{2\mathfrak{m}}{\Gamma(\mathfrak{z})}\frac{e^{-\tau}\Gamma(\mathfrak{z}+1)}{4}|\upsilon\,-\,\nu|\int_0^F\int_0^{\mathfrak{k}}\left(\frac{2\mathfrak{m}}{\Gamma(\mathfrak{z})}(\mathfrak{k}-q)^{\mathfrak{z}-1}\right)dqd\mathfrak{k},$$

$$\leq\frac{1}{\Gamma(\mathfrak{z})}\frac{e^{-\tau}\Gamma(\mathfrak{z}+1)}{4}|\upsilon\,-\,\nu|\int_0^{\mathfrak{r}}\left(\frac{1}{\Gamma(\mathfrak{z})}\,(\mathfrak{r}-e)^{\mathfrak{z}-1}\right)d\mathfrak{k}$$

$$+\frac{2\mathfrak{m}}{\Gamma(\mathfrak{z})}\frac{e^{-\tau}\Gamma(\mathfrak{z})\,.\Gamma(\mathfrak{z}+1)}{4\Gamma(\mathfrak{z})\,.\Gamma(\mathfrak{z}+1)}|\upsilon\,-\,\nu|\int_0^F\int_0^{\mathfrak{k}}\left(\frac{2\mathfrak{r}}{\Gamma(\mathfrak{z})}(\mathfrak{k}-q)^{\mathfrak{z}-1}\right)dqd\mathfrak{k},$$

$$\leq\left(\frac{e^{-\tau}\Gamma(\mathfrak{z})\,.\Gamma(\mathfrak{z}+1)}{4\Gamma(\mathfrak{z})\,.\Gamma(\mathfrak{z}+1)}\right)|\upsilon\,-\,\nu|+2e^{-\tau}\beta\,(\mathfrak{z}+1,\,1)\,\frac{\Gamma(\mathfrak{z})\,.\Gamma(\mathfrak{z}+1)}{4\Gamma(\mathfrak{z})\,.\Gamma(\mathfrak{z}+1)}|\upsilon\,-\,\nu|,$$

$$\leq\frac{e^{-\tau}}{4}|\upsilon\,-\,\nu|+\frac{e^{-\tau}}{2}|\upsilon\,-\,\nu|\leq e^{-\tau}|\upsilon\,-\,\nu|.$$

Here β is the beta function. So

$$\sqrt{|P\left(\upsilon\left(\mathfrak{m}\right)\right)\,-\,Q\left(\nu\left(\mathfrak{m}\right)\right)|^2}\leq e^{-\tau}|\upsilon\,-\,\nu|.$$

Squaring both sides, we get

$$
\begin{aligned}
&|P(\upsilon(\mathfrak{m})) - Q(\nu(\mathfrak{m}))|^2 \leq e^{-2\tau}|\upsilon - \nu|^2 \leq e^{-\tau}|\upsilon - \nu|^2 \\
&d_b(P(\upsilon(\mathfrak{m})),\ Q(\nu(\mathfrak{m}))) \leq e^{-\tau} d_b(\upsilon,\ \nu).
\end{aligned} \tag{60}
$$

Take $\mathscr{F}(y) = \ln y$, we have $\Delta_b(D(\upsilon, \nu)) = d_b(\upsilon,\ \nu)$. So, we deduce that

$$
d_b(P(\upsilon(\mathfrak{m})), Q(\nu(\mathfrak{m}))) \leq e^{-\tau}(\Delta_b(D(\upsilon,\ \nu))),
$$

$$
e^{\tau}{}_b(P(\upsilon(\mathfrak{m})),\ Q(\nu(\mathfrak{m}))) \leq (\Delta_b(D(\upsilon,\ \nu))),
$$

$$
\ln(e^{\tau}{}_b(P(\upsilon(\mathfrak{m})),\ Q(\nu(\mathfrak{m})))) \leq \ln(\Delta_b(D(\upsilon,\ \nu)),
$$

$$
\tau + \mathscr{F}(d_b(P(\upsilon(\mathfrak{m})),\ Q(\nu(\mathfrak{m}))) \leq \mathscr{F}(\Delta_b(D(\upsilon,\ \nu))).
$$

All conditions of Theorem 2 are fulfilled. Thus, the equations (56) and (57) admit the unique solution. □

7. Conclusion

In this paper, we have established some novel FP theorems satisfying an advanced that satisfy a novel extension of Aydi-Lolo-Piri-Rasham type contractions on a closed ball in the framework of complete $p_b\mathscr{M}S$. New FP results for a couple of multivalued dominated mappings involving simulation function satisfying a generalized locally rational contractions have been proved in a complete $p_b\mathscr{M}S$. Some new findings for graph contraction incorporating with multi-graph dominated mappings and multi-ordered dominated mappings have been introduced. Some illustrative examples are given to exemplify our obtained new findings in the setting of $p_b\mathscr{M}S$. Finally, applications on nonlinear Volterra integral and fractional differential equations are presented to show the novelty of our latest results. The theory of fixed points will be critically dependent on our hypothesis. In future, our results can be extended and refined in many ways by using different types of mappings like fuzzy mappings, L-fuzzy mappings, bipolar fuzzy mappings, intuitionistic fuzzy mappings in $p_b\mathscr{M}S$ with applications.

Author contributions

T. R supervision, visualization, and conceptualization; A. H write the original-draft; S.K.P methodology; A.A funding; N. M and S.K. P review and edit the final draft. All authors approved final paper for submission.

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Confilict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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