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Research Article

Uncertain Multi-Period Efficiency in Data Envelopment Analysis: Performance Evaluation of Healthcare Systems in the Face of COVID-19 Pandemic

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Abstract: Healthcare systems worldwide play a critical role in enhancing public health, ensuring economic equity, and meeting societal expectations. The Coronavirus Disease 2019 (COVID-19) pandemic posed unprecedented challenges, disrupting the recognition and treatment of non-COVID diseases and straining healthcare resources globally. This has underscored the need to evaluate government and healthcare system performance over multiple periods. In this context, Data Envelopment Analysis (DEA) emerges as a robust tool to provide a theoretical and scientific foundation for healthcare policymaking. However, healthcare performance data often involves uncertainties due to variability in socio-cultural conditions, death rates, and expert-driven estimations. This study introduces a novel, uncertain DEA model tailored for multi-period systems to address these challenges. The proposed model simultaneously evaluates overall system and period-specific efficiencies, offering comprehensive insights into the dynamics of healthcare performance over time. The study also provides theoretical proofs to ensure the feasibility, boundedness, and deterministic transformation of the uncertain model. The decomposition of overall efficiency into period-specific components enables the identification of inefficiency sources, providing actionable insights for decision-makers to target process improvements. A sensitivity and stability analysis is conducted to assess the robustness of the proposed method under varying conditions. The model is applied to evaluate the healthcare efficiency of 30 countries during the COVID-19 pandemic across two distinct periods. Results reveal that the uncertain DEA model outperforms traditional deterministic approaches by offering greater discrimination in ranking healthcare units. Furthermore, the analysis highlights an improvement in efficiency for most countries during the second period, potentially attributed to increased preparedness and adaptability in handling the pandemic. This study contributes to the methodological advancements in DEA and provides valuable policy recommendations for enhancing healthcare system resilience in crisis scenarios.

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1. Introduction

Nowadays, the expansion of globalization has resulted in an increase in competition among organizations; simultaneously, the increasing elaboration of the economic environment makes a Decision Maker (DM) consider suitable tools to evaluate the efficiency of an organization. At present, Data Envelopment Analysis (DEA), which was initially introduced by Charnes and Cooper [1], is a mathematical programming procedure for evaluating the relative performance of peer Decision-Making Units (DMUs) with multiple inputs and multiple outputs without any underlying assumption of a functional form for the production function [2].

The Coronavirus Disease 2019 (COVID-19) pandemic was a global epidemic; it affected everyone's life irrespective of religion, language, race, socioeconomic and sociocultural differences, and repressed healthcare systems worldwide. The performance evaluation of a country's healthcare system during the COVID-19 outbreak reveals how much this system has achieved the desired results and can contribute to the health of society. In this context, the performance evaluation using DEA could consolidate the scientific and theoretical basis of health politics handled universally [3–5]. On the other hand, there are many cases in which inputs and outputs are uncertain; for example, the number of infected patients and recovered people estimated by experts during the COVID-19 pandemic cannot be measured precisely. Nevertheless, many surveys have demonstrated that the fuzzy set and the probability theory are not convenient tools for modelling this type of data [6]. Thus, using uncertainty DEA, which was introduced by Liu [7], this study estimates the overall efficiency of 30 countries' healthcare (DMUs) pending the COVID-19 outbreak in a time span covering multiple periods. In traditional DEA, the average data over all periods is used, ignoring individual periods' specific efficiency. To deal with such drawbacks, this paper introduces a new uncertain relational model for multi-period systems, which can measure the overall efficiency system and individual period efficiency by solving one model. Some theorems are presented to prove the proposed model's feasibility and boundedness, and convert the new uncertain model to the deterministic form. Using the decomposition of the overall efficiency system into the period efficiencies, the source of inefficiency in a system in the presence of inaccuracy can be identified, and it can provide a direction for the decision makers to devise improvements to the process that causes the system's inefficiency. The sensitivity and stability analysis of the proposed method within some theorems is provided. In the face of disasters such as COVID-19, the efficiency evaluation of healthcare systems is a significant challenge for health managers, as it can help increase treatment quality. The proposed model is applied to measure the healthcare system efficiency of 30 countries during the COVID-19 pandemic for two periods in 2020. The results illustrated that our proposed method is more discriminative than the crisp ones in ranking the performance of health system units. Inefficient health system units can recognize their deficiency and use the competence of efficient units to preserve and enhance their performance.

The structure of this paper is as follows: Section 2 provides a comprehensive review of key findings in DEA and its extension to Uncertain DEA (UDEA), emphasizing both the strengths and limitations of existing uncertain DEA models. Section 3 introduces fundamental concepts related to multi-period systems and lays the groundwork for understanding uncertainty theory. In Section 4, a novel, uncertain relational model tailored for multi-period systems is proposed to evaluate both overall system efficiency and individual period efficiency, alongside addressing alternative optimal solutions. Section 5 delves into sensitivity and stability analyses of the proposed approach, supported by relevant theorems. Section 6 demonstrates the practical application of the methodology through a simplified illustrative example and a real-world case study conducted during the COVID-19 pandemic. Finally, the concluding section summarizes the findings and highlights potential directions for future research.

2. Literature review

The most widely used DEA model is the CCR model, introduced by Charnes, Cooper, and Rhodes [8]. This foundational model measures efficiency using an oriented radial approach, focusing on either input minimization or output maximization. Subsequently, Banker, Charnes, and Cooper [9] extended the CCR model to incorporate the concept of Variable Returns to Scale (VRS), aligning DEA with the economic notion including banking [10], healthcare [11], supplier selection [12], higher education [13], and educational efficiency evaluation [14], among others.

Panwar et al. [15] recently provided an overview of DEA's evolution over nearly 40 years, summarizing prominent models, their strengths and limitations, and their various applications. Their work highlighted the DEA's versatility in addressing a wide range of efficiency evaluation problems.

However, uncertainty is a significant factor in real-world applications that must be accounted for. Uncertainty theory, a mathematical framework developed to model indeterminacy, has emerged as a critical tool for addressing these challenges [7]. Traditional DEA models, however, assume that all input and output data are precise and deterministic. In practice, many inputs and outputs, such as greenhouse gas emissions, social benefits of organizations, customer satisfaction, and carbon footprints, are inherently uncertain. Applying deterministic DEA models under such circumstances may result in inadequate or misleading conclusions [16–18]. Thus, managing data uncertainty remains one of the fundamental challenges in efficiency measurement.

There is often a misconception that uncertainty is synonymous with probability. Probability theory assumes that the sample size is sufficiently large for estimated probabilities to converge to long-run frequencies, making it a suitable tool under such conditions. However, in cases where the sample size is small, the law of large numbers fails to apply, rendering probability theory ineffective. In these scenarios, uncertainty theory offers a more robust alternative for addressing indeterminate data [7].

Using probability theory, Sengupta [19] attempted to incorporate input and output uncertainty within the DEA framework. However, this approach has two critical limitations: (1) most DEA studies involve relatively small sample sizes, and (2) determining an appropriate probability distribution for the error process is often impractical and challenging [20]. These shortcomings underscore the need for alternative approaches, such as those based on uncertainty theory, to address the complexities of data uncertainty in efficiency evaluation.

Banker [21] provided a foundational statistical framework for constructing asymptotic hypothesis tests in DEA, facilitating robust evaluation of efficiency under statistical scrutiny. Building on this, Land et al. [22] introduced the Concept of Chance-Donstrained DEA (CCDEA), which integrates probabilistic constraints to address uncertainty in inputs and outputs. To apply probability theory's "P-models" in DEA, Cooper et al. [23] proposed models that satisfy DEA constraints under chance settings. These advancements have spurred extensive research and application of probabilistic approaches in DEA [22, 25].

To manage imprecise data, Zadeh [26] introduced fuzzy set theory, which offers a mathematical framework to handle vagueness and linguistic imprecision. Sengupta [27] highlighted that fuzzy sets could effectively model indeterminate inputs and outputs in DEA, incorporating tolerance levels to account for linguistic data. Subsequent studies have extended this concept, yielding significant developments in fuzzy DEA. For instance, Kao and Liu [28] proposed a method to derive fuzzy efficiency scores using membership functions for fuzzy inputs and outputs, while Entani et al. [29] introduced a model that calculates interval efficiency from both pessimistic and optimistic perspectives. Liu and Liu [30] advanced the field by suggesting a fuzzy DEA approach based on credibility measures.

Further contributions include the defuzzification method proposed by Lertworasirikul et al. [31], which converts fuzzy inputs and outputs into numerical values to facilitate DEA analysis. Hatami-Marbini et al. [32] conducted a comprehensive review of fuzzy DEA models, categorising them into four main approaches: tolerance, fuzzy ranking, possibility, and hybrid methodologies. Recent publications [33–36] continue to explore and refine fuzzy DEA models, underscoring their importance in addressing uncertainty.

Despite its widespread application, fuzzy DEA has inherent limitations. For example, consider a Decision-Making Unit (DMU) where carbon emissions, measured as "about 1,000 tons," are represented by a triangular fuzzy variable (800, 1,000, 1,200). According to the possibility measure, the statement "exactly 1,000 tons" is assigned a possibility of

1, while the complementary statement "not exactly 1,000 tons" also holds a possibility of 1. This apparent contradiction highlights the difficulty of using fuzzy concepts to resolve certain types of indeterminacy [37, 38].

When probability distributions cannot be reliably estimated due to a lack of data, expert opinions are often used to assess the likelihood of events. While probability theory or fuzzy set theory can model belief degrees, these approaches can sometimes yield counterintuitive or inconsistent results. To address these challenges, Liu [39] introduced uncertain theory, later refined in [7]. Uncertain theory builds upon three core principles: uncertain measures, uncertain variables, and uncertain distributions. Uncertain theory has emerged as a robust mathematical framework for modeling human uncertainty, offering a logical and consistent approach to handling degrees of belief. It has been successfully applied across various disciplines, including engineering, management, and industrial engineering [18, 40–43]. By addressing the limitations of probabilistic and fuzzy models, uncertain theory provides a comprehensive toolset for managing indeterminacy in DEA and other analytical domains, enabling more accurate and reliable efficiency assessments.

Uncertain DEA methods have emerged as a pivotal approach to addressing problems involving imprecise or uncertain inputs and outputs, drawing upon the principles of uncertainty theory. Wen et al. [44] pioneered this field by proposing the first DEA model capable of operating within an uncertain environment. Building on this foundation, Wen et al. [45] developed additional models to expand the applicability and robustness of uncertain DEA.

Despite these advancements, Lio and Liu [17] identified limitations in the existing uncertain DEA frameworks, noting their inadequacies in fully addressing uncertainty challenges. To overcome these issues, they introduced an enhanced uncertain DEA model designed to address the shortcomings of prior methodologies. Moreover, their study incorporated an analysis of the sensitivity and stability of their model, providing critical insights into its practical reliability and adaptability.

Subsequently, Mohammad Nejad and Ghaffari-Hadigheh [16] highlighted a key limitation in Lio and Liu's approach, emphasizing that the objective function in their model was not explicitly based on the belief function. To address this, they proposed a novel, uncertain DEA model that directly optimizes the highest belief degree for evaluating the efficiency of DMUs. Their approach also facilitated the ranking of DMUs, enhancing decision-makers' ability to differentiate between efficient and inefficient units in uncertain environments.

Jiang et al. [46] further advanced the field by identifying a critical shortcoming in Lio and Liu's model [17]. They noted that it failed to account for scale differences when DMUs operated at varying productive scale sizes, often misclassifying them as inefficient. To rectify this, Jiang et al. introduced an uncertain DEA model capable of separating technical efficiency from scale efficiency for DMUs with imprecise inputs and outputs, offering a more nuanced evaluation framework.

In a more recent contribution, Jiang et al. [47] extended their research by developing two distinct uncertain DEA models to identify scale statuses, specifically Increasing Returns to Scale (IRS) and Decreasing Returns to Scale (DRS). These models provide decision-makers with a powerful tool for diagnosing inefficiencies and determining improvement strategies, even without precise data. Recognising scale effects and incorporating them into the evaluation process marks a significant step forward in applying uncertain DEA methodologies, enabling more informed and effective decision-making.

These advancements underscore the growing sophistication of uncertain DEA models in handling imprecise data and their increasing relevance in various practical settings. From distinguishing scale efficiencies to optimizing belief degrees, the evolution of these models continues to refine the analytical capabilities of DEA in uncertain environments.

To evaluate the efficiency of organizations such as banks, hospitals, and manufacturing companies over time, it is common practice to aggregate data by averaging the inputs and outputs for each period [48]. While this approach simplifies the analysis, it fails to comprehensively understand the efficiency dynamics . Specifically, such an approach may obscure performance variability across different periods. For instance, a DMU may be identified as efficient when evaluated over the entire time horizon despite exhibiting inefficiency during individual periods. This discrepancy highlights a significant limitation in traditional methodologies that rely solely on averaged data for efficiency assessment.

To address this gap, Kao and Liu [49] proposed a model based on the network DEA approach to simultaneously evaluate the overall efficiency of a DMU and its efficiency in individual periods. Their innovative model decomposes the overall efficiency into period-specific efficiencies, offering a more nuanced and accurate depiction of performance over time. A key feature of their model is that the overall efficiency is calculated as a weighted average of the period efficiencies, where the weights are determined to be the most favorable for the DMU under evaluation. This weighting mechanism ensures that the analysis reflects the DMU's best possible performance across periods, accounting for variations in efficiency within the temporal framework.

This approach is important because it distinguishes between overall and period efficiencies, enabling a more detailed identification of inefficiency sources. For example, a DMU might achieve high overall efficiency due to strong performance in certain periods, even if it underperforms in others. By isolating period-specific efficiencies, decision-makers can identify and target specific periods of inefficiency for improvement.

Moreover, this approach has practical implications for strategic decision-making. Organizations can leverage these insights to allocate resources more effectively, optimize operations during periods of inefficiency, and benchmark their performance against competitors or industry standards. Network DEA in this context represents a significant advancement in efficiency measurement, addressing the temporal heterogeneity often overlooked in traditional DEA models. This methodology has broad applicability in sectors where performance fluctuates over time, such as healthcare, manufacturing, and financial services, making it a valuable tool for performance evaluation and improvement.

3. Background (preliminaries)

This study examines two key concepts central to advancing efficiency analysis in uncertain environments: multi-period efficiency measurement and uncertainty theory. Each concept is discussed in a dedicated subsection to provide a comprehensive foundation for our proposed methodology.

Multi-period efficiency measurement

The concept of multi-period efficiency measurement is reviewed to establish a framework for linking performance across individual periods [50, 51]. This approach enables simultaneous calculation of overall and period-specific efficiency for a DMU. By considering the temporal structure of data, this methodology provides a more nuanced understanding of efficiency over time and identifies periods where inefficiencies arise. This dual insight supports decision-makers in targeting specific periods for operational improvements while maintaining a holistic view of performance.

Uncertainty theory

As uncertainty is an inherent characteristic of real-world systems, particularly in efficiency evaluations involving imprecise data, the theory of uncertainty has emerged as a vital branch of mathematics. Developed by Liu [7], uncertainty theory offers a rigorous framework for modeling belief degrees based on subjective or incomplete information. In the second subsection, we delve into the foundational concepts of uncertainty theory, discussing its axioms and theorems. This includes exploring the principles of uncertain measures, uncertain variables, and uncertain distributions, which form the theoretical underpinning for handling indeterminate data in multi-period efficiency analysis. Together, these sections lay the groundwork for integrating multi-period efficiency measurement with uncertainty theory, enabling a robust and innovative approach to evaluate performance under data imprecision and temporal complexity.

3.1 *Multi-period system*

Let the time period for measuring the efficiency of a set of n DMUs, covers q periods, which $x_{ij}^{(p)}$ and $y_{ij}^{(p)}$ denote the inputs and the outputs of DMU$_j$ in the period p with totals of $X_{ij} = \sum_{p=1}^{q} x_{ij}^{(p)}$ and $Y_{rj} = \sum_{p=1}^{q} y_{ij}^{(p)}$, respectively. The relational model for measuring the overall and the period efficiency of the multi-period system is as follows [49]:

$$Max \quad \sum_{r=1}^{s} u_r Y_{rk} \tag{1}$$

$$s.t. \quad \sum_{i=1}^{m} v_i X_{ik} = 1$$

$$\sum_{r=1}^{s} u_r y_{rj}^{(p)} - \sum_{i=1}^{m} v_i x_{ij}^{(p)} \leq 0,\ p = 1,\ ...,\ q,\ j = 1,\ ...,\ n$$

$$u_r \geq 0,\ v_i \geq 0, \quad r = 1,\ ...,\ s,\ i = 1,\ ...,\ m.$$

Since the optimal solutions u_r^*, $(r = 1, \ldots, s)$ and v_i^*, $(i = 1, \ldots, m)$ are obtained using model (11), the overall efficiency, E_k, and the period efficiency $E_k^{(p)}$, $(p = 1, \ldots, q)$ are calculated as:

$$E_k = \frac{\sum_{r=1}^{s} u_r^* Y_{rk}}{\sum_{i=1}^{m} v_i^* X_{ik}} = \sum_{r=1}^{s} u_r^* Y_{rk},\ E_k^{(p)} = \frac{\sum_{r=1}^{s} u_r^* y_{rj}^{(p)}}{\sum_{i=1}^{m} v_i^* x_{ij}^{(p)}},\ (p = 1, \ldots, q).$$

3.2 *Uncertainty theory*

The uncertain measure M is a set function on a σ-algebra L over a non-empty set Γ then the triple $(\Gamma,\ L,\ M)$ is referred to as an uncertainty space, in which the uncertain measure M Satisfies the following three axioms [39]:

Axiom 1 (Normality Axiom) $M\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2 (Duality Axiom) $M\{\Lambda\} + M\{\Lambda^C\} = 1$ for any event Λ.

Axiom 3 (Subadditivity Axiom) For every countable sequence of events $\Lambda_1,\ \Lambda_2,\ \ldots$, we have $M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}$.

Moreover, Liu [7] defined the product uncertain measure as the fourth axiom.

Axiom 4 (Product Axiom) The product uncertain measure M in uncertainty spaces $(\Gamma_k,\ L_k,\ M_k)$is an uncertain measure satisfying $M\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} M_k\{\Lambda_k\}$ where Λ_k are arbitrarily chosen events from L_k for $k = 1,\ 2,\ \ldots$, respectively.

These axioms underpin the uncertainty measure used in constraints of the proposed DEA model in Section 4, ensuring consistent probabilistic interpretations.

Definition 1 Liu [7] let $\{\gamma|\xi(\gamma) \in B\}$ is an event for any Borel set B An uncertain variable is a measurable function $\xi : (\Gamma,\ L,\ M) \to \mathbb{R}$ such that $\{\xi \in B\} = \{\gamma \in \Gamma|\xi(\gamma) \in B\}$. For an uncertain variable ξ, the uncertain distribution Φ is defined by $\Phi(x) = \{\xi \leq x\},\ \forall x \in \mathbb{R}$.

According to these definitions and to describe the uncertain variables, some uncertain distributions have been defined [7]. For example, an uncertain variable is called linear with uncertain distribution for real numbers and with also, for real numbers a and b with an uncertain variable ξ is called zigzag with uncertain distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a \\ (x-a)/(b-a), & \text{if } a < x \le b \\ 1, & \text{if } x > a \end{cases}$$

for real numbers a and b with $a < b$ also, for real numbers $a,\ b$ and c with $a < b < c$, an uncertain variable ξ is called zigzag with uncertain distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a \\ (x-a)/[2(b-a)], & \text{if } a < x \le b \\ (x+c-2b)/[2(c-b)], & \text{if } b < x \le c \\ 1, & \text{if } x > c \end{cases}.$$

An uncertain distribution $\Phi(x)$ is deemed to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$ and $\lim_{x \to -\infty} \Phi(x) = 0$, $\lim_{x \to \infty} \Phi(x) = 1$. Suppose that the uncertain variable ξ is an uncertain variable with regular uncertainty distribution $\Phi(x)$, the inverse function $\Phi^{-1}(\alpha)$ is referred to as the inverse uncertainty distribution of ξ [39]. The inverse uncertain distribution of the linear uncertain distribution $L(a,\ b)$ is as $\Phi^{-1}(\alpha) = (1-\alpha)a + \alpha b$ and the inverse uncertain distribution of the zigzag uncertain distribution $Z(a,\ b,\ c)$ is as follows:

$$\Phi^{-1}(\alpha) = \begin{cases} (1-2\alpha)a + 2\alpha b, & \alpha < 0.5 \\ (2-2\alpha)b + (2\alpha-1)c, & \alpha \ge 0.5. \end{cases}$$

The expected value of an uncertain variable ξ indicates the measurement of uncertain variables and can be computed as $E[\xi] = \int_0^1 \Phi^{-1}(\alpha) d\alpha$. For instance, the expected value of the linear uncertain variable $L(a,\ b)$ and the zigzag uncertain variable $Z(a,\ b,\ c)$ are as $E[L(a,\ b)] = \frac{1}{2}(a+b)$ and $E[Z(a,\ b,\ c)] = \frac{1}{4}(a+2b+c)$, respectively.

Definition 2 (Liu [7]) The uncertain variables $\xi_1,\ \xi_2,\ \ldots,\ \xi_q$ are said to be independent if $M\left\{\bigcap_{i=1}^{q} (\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{q} M(\xi_i \in B_i)$ for any Borel sets $B_1,\ B_2,\ \ldots,\ B_q$ of real numbers.

The inverse uncertainty distribution and the expected value of a strictly monotonous function of independent uncertain variables with regular uncertainty distributions can be computed using the following theorem [7].

Theorem 1 Let $\xi_1,\ \xi_2,\ \ldots,\ \xi_q$ are independent uncertain variables with regular uncertainty distribution $\Phi_1,\ \Phi_2,\ \ldots,\ \Phi_q$ respectively. If $f(\xi_1,\ \xi_2,\ \ldots,\ \xi_q)$ is strictly increasing with respect to $\xi_1,\ \xi_2,\ \ldots,\ \xi_t (t \le q)$ and strictly decreasing with respect to $\xi_{t+1},\ \xi_{t+2},\ \ldots,\ \xi_q$, then

1. $\xi = f(\xi_1,\ \xi_2,\ \ldots,\ \xi_q)$ is an uncertain variable with an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha),\ \ldots,\ \Phi_t^{-1}(\alpha), \Phi_{t+1}^{-1}(1-\alpha),\ \ldots,\ \Phi_q^{-1}(1-\alpha))$$

2. The expected value of $\xi = f(\xi_1,\ \xi_2,\ \ldots,\ \xi_q)$ is as follows:

$$E[\xi]=\int_0^1 f(\Phi_1^{-1}(\alpha),\ \ldots,\ \Phi_t^{-1}(\alpha),\Phi_{t+1}^{-1}(1-\alpha),\ \ldots,\ \Phi_q^{-1}(1-\alpha))d\alpha$$

These axioms underpin the uncertainty measure used in constraints of the proposed DEA model (Section 2, Axioms 1-3), ensuring consistent probabilistic interpretations.

4. Uncertain relational model for multi-period system

Consider a multi-period system composed of q periods where $\tilde{x}_{ij}^{(p)}$ and $\tilde{y}_{rj}^{(p)}$ denote the uncertain inputs and outputs of DMU_j in the period p, respectively. It is important to mention that these data are imprecise and estimated by domain experts based on their knowledge or preference [7].

4.1 *Proposed model*

In all q periods for DMU_j, the total quantities of i^{th} uncertain input and r^{th} uncertain outputs are $\tilde{X}_{ij}=\sum_{p=1}^{q}\tilde{x}_{ij}^{(p)}$ and $\tilde{Y}_{rj}=\sum_{p=1}^{q}\tilde{y}_{rj}^{(p)}$, respectively. Taking this point into account, which distribution function is general, the uncertain relational network model for a multi-period system is as follows:

$$\begin{aligned}
&Max \quad E\left[\sum_{r=1}^{s}u_r\tilde{Y}_{rk}\right] &(2)\\
&s.t. \quad E\left[\sum_{i=1}^{m}v_i\tilde{X}_{ik}\right]=1\\
&\qquad E\left[\sum_{r=1}^{s}u_r\tilde{y}_{rj}^{(p)}-\sum_{i=1}^{m}v_i\tilde{x}_{ij}^{(p)}\right]\leq 0,\ p=1,\ \ldots,\ q,\ j=1,\ \ldots,\ n\\
&\qquad u_r\geq 0,\ v_i\geq 0,\quad r=1,\ \ldots,\ s,\quad i=1,\ \ldots,\ m
\end{aligned}$$

where $u_r\,(r=1,\ \ldots,\ s)$ and $v_i\,(i=1,\ \ldots,\ m)$ are non-negative weight vectors (virtual multipliers). A fundamental remark for the feasibility of model (2) is stated in Theorem 2.

Theorem 2 Model (2) is feasible.

Proof. Considerthen $I=\left\{i\,\middle|\,E[\tilde{X}_{ik}]\neq 0\right\}$, then we define $\bar{v}_i$ as follows:

$$\bar{v}_i=\begin{cases}\dfrac{1}{\alpha E[\tilde{X}_{ik}]}, & E[\tilde{X}_{ik}]\neq 0\\ \beta\quad, & E[\tilde{X}_{ik}]=0\end{cases},$$

where, $card(I)=\alpha$ for $i=1,\ \ldots,\ m$. It implies that

$$E\left[\sum_{i=1}^{m}\bar{v}_i\tilde{X}_{ik}\right]=\sum_{i\in I}\frac{1}{\alpha E[\tilde{X}_{ik}]}\times E[\tilde{X}_{ik}]+\sum_{i\in I^c}\beta\times 0=1$$

As well as, let for all $j,\ p$; $L_{1j}^{(p)}=\left\{r\left|E[\tilde{y}_{rj}^{(p)}]\neq 0\right.\right\}$ and $card(L_{1j}^{(p)})=l_{1j}^{(p)}$.

We assume that $M_1^{(p)}=\max\left\{l_{1j}^{(p)}\mid p=1,\ ...,\ q,\ j=1,\ ...,\ n\right\}$ and by definition of

$$\bar{u}_r=\min\left\{\frac{\sum_{i=1}^{m}\bar{v}_iE[\tilde{x}_{ij}^{(p)}]}{M_1^p\times E[\tilde{y}_{rj}^{(p)}]},\ p=1,\ ...,\ q,\ j=1,\ ...,\ n\right\}.$$

Then, it can be obtained that, $\sum_{r=1}^{s}\bar{u}_rE[\tilde{y}_{rj}^{(p)}]-\sum_{i=1}^{m}\bar{v}_iE[\tilde{x}_{ij}^{(p)}]\leq 0$, so model (2) is always feasible.

The following theorem demonstrates that model (2) has an optimal boundary solution.

Theorem 3 The objective function of the model (2) is bounded.

Proof. Consider the set of the second constraint model (2). By the sum of these constraints, we obtain,

$$\sum_{p=1}^{q}\left(E\left[\sum_{r=1}^{s}u_r\tilde{y}_{rj}^{(p)}\right]-E\left[\sum_{i=1}^{m}v_i\tilde{x}_{ij}^{(p)}\right]\right)\leq 0,\ j=1,\ ...,\ n$$

then, according to the properties of expected value,

$$E\left[\sum_{r=1}^{s}u_r\sum_{p=1}^{q}\tilde{y}_{rj}^{(p)}\right]-E\left[\sum_{i=1}^{m}v_i\sum_{p=1}^{q}\tilde{x}_{ij}^{(p)}\right]\leq 0$$

for all $j=1,\ ...,\ n$, we can get $E\left[\sum_{r=1}^{s}u_r\tilde{Y}_{rj}\right]-E\left[\sum_{i=1}^{m}v_i\tilde{X}_{ij}\right]\leq 0$, for $j=k$ and since $E\left[\sum_{i=1}^{m}v_i\tilde{X}_{ik}\right]=1$. Thus, $E\left[\sum_{r=1}^{s}u_r\tilde{Y}_{rk}\right]\leq 1$ and the theorem is proved.

After the optimal solution u_r^* and v_i^* are obtained by model (2), the overall efficiency, $E_k^{overall}$ and period efficiencies, $E_k^{(p)},\ p=1,\ \ldots,\ q$. For DMU_k are calculated as follows:

$$E_k^{overall}=\frac{E\left[\sum\limits_{r=1}^{s}u_r^*\tilde{Y}_{rk}\right]}{E\left[\sum\limits_{i=1}^{m}v_i^*\tilde{X}_{ik}\right]}=E\left[\sum_{r=1}^{s}u_r^*\tilde{Y}_{rk}\right] \tag{3}$$

$$E_k^{(p)}=\frac{E\left[\sum\limits_{r=1}^{s}u_r^*\tilde{y}_{rk}^{(p)}\right]}{E\left[\sum\limits_{r=1}^{s}v_i^*\tilde{x}_{ik}^{(p)}\right]},\ p=1,\ ...,\ q \tag{4}$$

Definition 3 (Overall Efficient) DMU_k can be considered as overall efficient if and only if the optimal value of the model (2) ($E_k^{overall}$) can achieve 1.

Definition 4 (Efficiency of period p) DMU_k is said to be efficient in period p if and only if $E_k^{(p)}$ can achieve 1. By setting the weight

$$w^{(p)} = \frac{E\left[\sum\limits_{i=1}^{m} v_i^* \tilde{x}_{ik}^{(p)}\right]}{E\left[\sum\limits_{i=1}^{m} v_i^* \tilde{X}_{ik}\right]} \tag{5}$$

as the expected value of the proportion of the aggregate uncertain input consumed in the period p in that of the expected value of all periods, the overall efficiency is equal to the weighted sum of the efficiencies of q period by $w^{(p)}$:

$$\sum_{p=1}^{q} w^{(p)} E_k^{(p)} = \sum_{p=1}^{q} \frac{E\left[\sum\limits_{i=1}^{m} v_i^* \tilde{x}_{ik}^{(p)}\right]}{E\left[\sum\limits_{i=1}^{m} v_i^* \tilde{X}_{ik}\right]} \times \frac{E\left[\sum\limits_{r=1}^{s} u_r^* \tilde{y}_{rk}^{(p)}\right]}{E\left[\sum\limits_{i=1}^{m} v_i^* \tilde{x}_{ik}^{(p)}\right]} \tag{6}$$

$$= \sum_{p=1}^{q} \frac{E\left[\sum\limits_{r=1}^{s} u_r^* \tilde{y}_{rk}^{(p)}\right]}{E\left[\sum\limits_{i=1}^{m} v_i^* \tilde{X}_{ik}\right]} = \frac{E\left[\sum\limits_{r=1}^{s} u_r^* \tilde{Y}_{rk}\right]}{E\left[\sum\limits_{i=1}^{m} v_i^* \tilde{X}_{ik}\right]} = E_k^{overall}$$

Accordingly, based on model (2), we able to not only calculate the overall and period efficiencies of the multi-period system but also we can decompose the overall efficiency system into the q. Period, and identify the sources of inefficiency in a system. This decomposition infers that a multi-period system is efficient if and only if all q period processes are efficient. It also provides a direction for improving processes that cause system inefficiency.

The proposed uncertain DEA model (2) can be applied to specify the overall efficiency and the period efficiency of DMU$_k$ by considering the optimal value of the model. The deterministic form of model (2) is proved as follows:

Theorem 4 For each DMU$_j$, let $\tilde{x}_{ij}^{(1)}, \tilde{x}_{ij}^{(2)}, \ldots, \tilde{x}_{ij}^{(q)}$ $(i = 1, ..., m,\ j = 1, ..., n)$ be independent uncertain variables with corresponding regular uncertainty distributions $\Phi_{1ij}, \Phi_{2ij}, \ldots, \Phi_{qij}$ $(i = 1, ..., m,\ j = 1, ..., n)$. Furthermore, let $\tilde{y}_{rj}^{(1)}, \tilde{y}_{rj}^{(2)}, \ldots, \tilde{y}_{rj}^{(q)}$ $(r = 1, ..., s,\ j = 1, ..., n)$ be independent uncertain variables with corresponding regular uncertainty distributions $\Psi_{1rj}, \Psi_{2rj}, \ldots, \Psi_{qrj}$ $(r = 1, ..., s,\ j = 1, ..., n)$, respectively. Then, the new model (2) is equal to the following form:

$$\max \quad \int_0^1 \sum_{r=1}^{s} u_r \Psi_{rk}^{-1}(\alpha) d(\alpha) \tag{7}$$

$$s.t. \quad \int_0^1 \sum_{i=1}^{m} v_i \Phi_{ik}^{-1}(\alpha) d(\alpha) = 1$$

$$\int_0^1 \left(\sum_{r=1}^{s} u_r \Psi_{prj}^{-1}(\alpha) - \sum_{i=1}^{m} v_i \Phi_{pij}^{-1}(1-\alpha)\right) d(\alpha) \le 0,\ p = 1, ..., q,\ j = 1, ..., n$$

$$u_r \ge 0,\ v_i \ge 0, \quad r = 1, ..., s,\ i = 1, ..., m$$

where, $\Phi_{ij}^{-1}(\alpha)=\sum_{p=1}^{q}\Phi_{pij}^{-1}(\alpha)$, $(i=1, \ldots, m, \ j=1, \ldots, n)$ and $\Psi_{rj}^{-1}(\alpha)=\sum_{p=1}^{q}\Psi_{prj}^{-1}(\alpha)$, $(r=1, \ldots, s, \ j=1, \ldots, n)$.

Proof. for each $i, \ j$ and p since the function $\sum_{p=1}^{q}\tilde{x}_{ij}^{(p)}$ is strictly increasing with respect to $\tilde{x}_{ij}^{(1)}, \tilde{x}_{ij}^{(2)}, \ldots, \tilde{x}_{ij}^{(q)}$ and the function $\sum_{p=1}^{q}\tilde{y}_{rj}^{(p)}$ is strictly increasing concerning $\tilde{y}_{rj}^{(1)}, \tilde{y}_{rj}^{(2)}, \ldots, \tilde{y}_{rj}^{(q)}$ according to Theorem 1, we can infer that the inverse uncertainty distribution of $\tilde{X}_{ij}=\sum_{p=1}^{q}\tilde{x}_{ij}^{(p)}$ and $\tilde{Y}_{rj}=\sum_{p=1}^{q}\tilde{y}_{rj}^{(p)}$ are $\Phi_{ij}^{-1}(\alpha)=\sum_{p=1}^{q}\Phi_{pij}^{-1}(\alpha)$ and $\Psi_{rj}^{-1}(\alpha)=\sum_{p=1}^{q}\Psi_{prj}^{-1}(\alpha)$, respectively.

Similarly, the objective function, $F_k=\sum\limits_{r=1}^{s}u_r\tilde{Y}_{rk}$ is increasing in relation to $\tilde{Y}_{rk}$ and the first constraint $F_k^1=\sum_{i=1}^{m}v_i\tilde{X}_{ik}$ is increasing in relation to $\tilde{X}_{ik}$ it follows from Theorem 1, that the inverse uncertainty distributions of F_k and F_k^1 are $[F_k]^{-1}(\alpha)=\sum_{r=1}^{s}u_r\Psi_{rk}^{-1}(\alpha)$ and $[F_{1k}]^{-1}(\alpha)=\sum_{i=1}^{m}v_i\Phi_{ik}^{-1}(\alpha)$, respectively.

Furthermore, $F_{pj}=\sum_{r=1}^{s}u_r\tilde{y}_{rj}^{(p)}-\sum_{i=1}^{s}v_i\tilde{x}_{ij}^{(p)}$, $\mathrm{p}=1, \ldots, q, \ j=1, \ldots, n$ is increasingly related to $\tilde{y}_{rj}^{(p)}$ and is decreasing related to $\tilde{x}_{ij}^{(p)}$, so, $[F_{pj}]^{-1}(\alpha)=\sum_{r=1}^{s}u_r\Psi_{prj}^{-1}(\alpha)-\sum_{i=1}^{m}v_i\Phi_{pij}^{-1}(1-\alpha)$. According to Theorem 1,

$$E\left[\sum_{r=1}^{s}u_r\tilde{Y}_{rk}\right]=\int_0^1\sum_{r=1}^{s}u_r\Psi_{rk}^{-1}(\alpha)d(\alpha)$$

$$E\left[\sum_{i=1}^{m}v_i\tilde{X}_{ik}\right]=\int_0^1\sum_{i=1}^{m}v_i\Phi_{ik}^{-1}(\alpha)d(\alpha)$$

$$E\left[\sum_{r=1}^{s}u_r\tilde{y}_{rj}^{(p)}-\sum_{i=1}^{m}v_i\tilde{x}_{ij}^{(p)}\right]=\int_0^1(\sum_{r=1}^{s}u_r\Psi_{prj}^{-1}(\alpha)-\sum_{i=1}^{m}v_i\Phi_{pij}^{-1}(1-\alpha))d(\alpha), \ \mathrm{p}=1, \ldots, q, \ j=1, \ldots, n$$

So, the proof is completed.

The uncertain variables in the proposed multiperiod model is formally defined using uncertainty theory. Inputs and outputs $(\tilde{x}_{ij}^{(p)}, \ \tilde{y}_{rj}^{(p)})$ are modeled as independent uncertain variables with regular uncertainty distributions (e.g., linear $L(a, \ b)$ or zigzag $Z(a, \ b, \ c)$ as uncertain variables). Inverse distributions are derived for aggregation across periods (e.g., $\Phi_{ij}^{-1}(\alpha)=\sum\limits_{p=1}^{q}\Phi_{pij}^{-1}(\alpha)$). This ensures measurability and tractability in optimization.

The proposed uncertain DEA model, model (7), is built on several key assumptions and faces some inherent limitations. First, it treats inputs and outputs as independent uncertain variables (such as linear or zigzag distributions), relying on expert estimates or historical data to define these distributions. However, this approach may not fully reflect real-world complexity and variability. Second, the model requires inverse uncertainty distributions to be strictly increasing, meaning irregular distributions would need different analytical methods. Third, it assumes independence between variables across different time periods, which might not hold true in practice, especially for interconnected healthcare metrics like cumulative resource depletion. While the explicit use of uncertain measures helps ensure robustness against data variability, the model still has limitations. For instance, dependence on expert-defined distributions could introduce bias, and the framework currently does not address dynamic correlations over time.

4.2 *Resolution for the alternative optimal solution*

As can be seen, the decomposition of the overall efficiency identifies the sources of inefficiency in a system. Due to multiple solutions of model (2), the decomposition of (6) may not be unique, and the efficiencies of different periods may not be comparable, since for measuring the efficiencies, a common basis is necessary.

One solution to this problem is to find a set of multipliers which produces E_k^t (assuming that period t is the most important period), while maintaining the overall efficiency score at $E_k^{overall}$ calculated from the model (2) [52]. That is:

$$Max \quad E\left[\sum_{r=1}^{s} u_r \tilde{y}_{rk}^{(t)}\right] \tag{8}$$

$$s.t. \quad E\left[\sum_{i=1}^{m} v_i \tilde{x}_{ik}^{(t)}\right] = 1$$

$$E\left[\sum_{r=1}^{s} u_r \tilde{Y}_{rk}\right] = E_k^{overall} \cdot E\left[\sum_{i=1}^{m} v_i \tilde{X}_{ik}\right]$$

$$E\left[\sum_{r=1}^{s} u_r \tilde{y}_{rj}^{(p)} - \sum_{i=1}^{m} v_i \tilde{x}_{ij}^{(p)}\right] \le 0, \; p = 1, \; ..., \; q, \; j = 1, \; ..., \; n$$

$$u_r \ge 0, \; v_i \ge 0, \; r = 1, \; ..., \; s, \; i = 1, \; ..., \; m.$$

Like model (2), model (8) can be converted to a deterministic model.

As can be seen, the feasibility and boundedness are guaranteed by Theorems 2 (non-negative weights ($\bar{v}_i, \; \bar{u}_r$) always exist) and 3, respectively. Therefore, the optimal solution for the model exists. Alternative optima are addressed via model (8). But the efficiency of DMUs under uncertainty, as the optimal value of model (7), is unique.

5. Sensitivity and stability analysis

The basic idea of the sensitivity and stability analysis is to determine the "radius of stability" within which the incidence of data variations will not change the classification of DMUs from efficient to inefficient status and vice versa. In practical cases, based on a new strategy, the manager wants to know how to modify an inefficient unit to be efficient or maintain the efficient unit at the same level. We consider the following two subsections to analyze the sensitivity and stability of the uncertain proposed model (2).

5.1 *Stability radius of inefficient DMUs*

Assume that $(\tilde{X}_k, \; \tilde{Y}_k) = (\tilde{X}_{1k}, \; \ldots, \; \tilde{X}_{mk}, \; \tilde{Y}_{1k1}, \; \ldots, \; \tilde{Y}_{sks})$ is inefficient by model (7), where for all $i, \; k$ and r, $\tilde{X}_{ik} = \sum_{p=1}^{q} \tilde{x}_{ik}^{(p)}$ and $\tilde{Y}_{rk} = \sum_{p=1}^{q} \tilde{y}_{rk}^{(p)}$. We want to find the minimum values of variation in data (δ = radius of stability) namely, increasing the outputs and decreasing the inputs, such that the unit $(\tilde{X}_k - e_m\delta, \; \tilde{Y}_k + e_s\delta) = (\tilde{X}_{1k} - \delta, \; \ldots, \; \tilde{X}_{mk} - \delta, \; \tilde{Y}_{1k1} + \delta, \; \ldots, \; \tilde{Y}_{sks} + \delta)$ be efficient, where e_m and e_s are m and s dimensional vectors, respectively, with all components equal to 1. Therefore, we need to solve the following model:

$$\delta^* = \min \; \delta \tag{9}$$

$$E\left[\sum_{r=1}^{s} u_r(\tilde{Y}_{rk} + \delta)\right] = 1$$

$$s.t.\ \ E\left[\sum_{i=1}^{m} v_i(\tilde{X}_{ik}-\delta)\right] = 1$$

$$E\left[\sum_{r=1}^{s} u_r\tilde{y}_{rj}^{(p)} - \sum_{i=1}^{m} v_i\tilde{x}_{ij}^{(p)}\right] \leq 0,\ p=1,\ ...,\ q,\ j=1,\ ...,\ n$$

$$\tilde{X}_{ik} \geq \delta,\ \ i=1,\ ...,\ m$$

$$u_r \geq 0,\ v_i \geq 0, \quad r=1,\ ...,\ s,\ i=1,\ ...,\ m.$$

The constraints of $\tilde{X}_{ik} \geq \delta$ $(i=1,\ ...,\ m)$ ensures that the inputs of the evaluated units in all periods are not allowed to be negative. Using these assumptions, a deterministic form of model (9) is presented with the following theorem.

Theorem 5 For each DMU$_j$, let $\tilde{x}_{ij}^{(1)},\ \tilde{x}_{ij}^{(2)},\ \ldots,\ \tilde{x}_{ij}^{(q)}$ $(i=1,\ ...,\ m\ ,\ j=1,\ ...,\ n)$ be independent uncertain variables with corresponding regular uncertainty distributions $\Phi_{1ij},\ \Phi_{2ij},\ \ldots,\ \Phi_{qij}$ $(i=1,\ ...,\ m\ ,\ j=1,\ ...,\ n)$. Furthermore, let $\tilde{y}_{rj}^{(1)},\ \tilde{y}_{rj}^{(2)},\ \ldots,\ \tilde{y}_{rj}^{(q)}$ $(r=1,\ ...,\ s\ ,\ j=1,\ ...,\ n)$ be independent uncertain variables with corresponding regular uncertainty distributions $\Psi_{1rj},\ \Psi_{2rj},\ \ldots,\ \Psi_{qrj}$ $(r=1,\ ...,\ s\ ,\ j=1,\ ...,\ n)$, respectively. Then, the uncertain model (9) is equivalent to the following deterministic model.

$$\delta^* = \min\ \delta \tag{10}$$

$$\sum_{r=1}^{s} u_r \int_0^1 (\Psi_{rk}^{-1}(\alpha)+\delta)d\alpha = 1$$

$$s.t.\ \ \sum_{i=1}^{m} v_i \int_0^1 (\Phi_{ik}^{-1}(\alpha)-\delta)d\alpha\ = 1$$

$$\sum_{r=1}^{s} u_r \int_0^1 \Psi_{prj}^{-1}(\alpha)d\alpha - \sum_{i=1}^{m} v_i \int_0^1 \Phi_{pij}^{-1}(1-\alpha)d\alpha\ \leq 0,\ p=1,\ ...,\ q,\ j=1,\ ...,\ n$$

$$\Phi_{ik}^{-1}(1-\alpha) \geq \delta,\ i=1,\ ...,\ m$$

$$u_r \geq 0,\ v_i \geq 0, \quad r=1,\ ...,\ s,\ i=1,\ ...,\ m$$

where, $\Phi_{ij}^{-1}(\alpha) = \sum_{p=1}^{q} \Phi_{pij}^{-1}(\alpha)$, $i=1,\ ...,\ m,\ j=1,\ ...,\ n$ and $\Psi_{rj}^{-1}(\alpha) = \sum_{p=1}^{q} \Psi_{prj}^{-1}(\alpha)$, $r=1,\ ...,\ s,\ j=1,\ ...,\ n.$

Proof. Since is a real variable for each $i(i=1,\ \ldots,\ m)$, $r(r=1,\ \ldots,\ s)$ and $j(j=1,\ \ldots,\ n)$ the inverse uncertainty distribution for the uncertain variables $\tilde{X}_{ik}-\delta$ and $\tilde{Y}_{rk}-\delta$ are $\Phi_{ik}^{-1}(\alpha)-\delta$ and $\Psi_{rk}^{-1}(\alpha)+\delta$, respectively.

Let $\Upsilon_k = \sum_{r=1}^{s} u_r(\tilde{Y}_{rk}+\delta)$ and $\eta_k = \sum_{i=1}^{m} v_i(\tilde{X}_{ik}-\delta)$. As can be seen, Υ_k and η_k are strictly increasing concerning $\tilde{Y}_{rk}$ and $\tilde{X}_{ik}$. Therefore, the inverse uncertainty distribution for Υ_k and η_k are $\Upsilon_k^{-1} = \sum_{r=1}^{s} u_r(\Psi_{rk}^{-1}(\alpha)+\delta)$ and $\eta_k^{-1} =$

$\sum_{i=1}^{m} v_i(\Phi_{ik}^{-1}(\alpha)-\delta)$, respectively. Therefore, $E[\Upsilon_k]=\sum_{r=1}^{s} u_r\int_0^1(\Psi_{rk}^{-1}(\alpha)+\delta)d\alpha$, $E[\eta_k]=\sum_{i=1}^{m} v_i\int_0^1(\Phi_{ik}^{-1}(\alpha)-\delta)d\alpha$ and models (9) and (10) are equivalent.

The validity of model (10) is demonstrated using Theorem 6.

Theorem 6 Assume that the unit $(\tilde{X}_k,\ \tilde{Y}_k)$ is inefficient by using model (7) and δ^* is the optimal value of the model (10). Then,

1. for each $\delta,\ \delta<\delta^*$ the unit $(\tilde{X}_k-e_m\delta,\ \tilde{Y}_k+e_s\delta)$ is inefficient and
2. for each $\delta,\ \delta\geq\delta^*$ the unit $(\tilde{X}_k-e_m\delta,\ \tilde{Y}_k+e_s\delta)$ is efficient.

Proof 1 To prove part 1, assume that the unit $(\tilde{X}_k-e_m\delta,\ \tilde{Y}_k+e_s\delta)\geq 0$ is efficient by model (7). Therefore, there is $(u,\ v,\ \delta)$ such that:

$$\sum_{r=1}^{s} u_r\int_0^1(\Psi_{rk}^{-1}(\alpha)+\delta)d\alpha=1$$

$$\sum_{i=1}^{m} v_i\int_0^1(\Phi_{ik}^{-1}(\alpha)-\delta)d\alpha\ =1$$

$$\sum_{r=1}^{s} u_r\int_0^1\Psi_{prj}^{-1}(\alpha)d\alpha-\sum_{i=1}^{m} v_i\int_0^1\Phi_{pij}^{-1}(1-\alpha)d\alpha\ \leq 0,\ p=1,\ ...,\ q,\ j=1,\ ...,\ n$$

$$\Phi_{ik}^{-1}(1-\alpha)\geq\delta,\ i=1,\ ...,\ m$$

$$u_r\geq 0,\ v_i\geq 0,\quad r=1,\ ...,\ s,\ i=1,\ ...,\ m.$$

This means that $(u,\ v,\ \delta)$ is a feasible solution for the model (10). On the other hand, δ^* is the optimal value of the model (10). Therefore, $\delta^*\leq\delta$ and for any $\delta\leq\delta^*$ the unit $(\tilde{X}_k-e_m\delta,\ \tilde{Y}_k+e_s\delta)$ is inefficient.

Proof 2 To prove part 2, suppose that is $(v^*,\ u^*,\ \delta^*)$ an optimal solution of model (9). For evaluation $(\tilde{X}_k-e_m\delta,\ \tilde{Y}_k+e_s\delta)$ by model (7) with $\delta=\delta^*$ the vector $(v=v^*,\ u=u^*)$ is a feasible solution of model (7) with $\sum_{r=1}^{s} u_r\int_0^1(\Psi_{rk}^{-1}(\alpha)+\delta)d\alpha=1$ and $\sum_{i=1}^{m} v_i\int_0^1(\Phi_{ik}^{-1}(\alpha)-\delta)d\alpha\ =1$. This means that $(\tilde{X}_k-e_m\delta,\ \tilde{Y}_k+e_s\delta)$ is efficient by model (7).

Now, suppose that $\delta>\delta^*$. The function $f(\delta)=\sum_{r=1}^{s} u_r^*\int_0^1(\Psi_{rk}^{-1}(\alpha)+\delta)d\alpha$ is strictly increasing by and the function $g(\delta)=\sum_{i=1}^{m} v_i^*\int_0^1(\Phi_{ik}^{-1}(\alpha)-\delta)d\alpha$ is strictly decreasing by δ, where $\delta\leq\min\left\{\Phi_{ik}^{-1}(1-\alpha): i=1,\ \ldots,\ m\right\}$. Therefore, by $\delta,\ \delta>\delta^*$ we have

$$f(\delta)=\sum_{r=1}^{s} u_r^*\int_0^1(\Psi_{rk}^{-1}(\alpha)+\delta)d\alpha\geq\sum_{r=1}^{s} u_r^*\int_0^1(\Psi_{rk}^{-1}(\alpha)+\delta^*)d\alpha=1$$

$$g(\delta)=\sum_{i=1}^{m} v_i^*\int_0^1(\Phi_{ik}^{-1}(\alpha)-\delta)d\alpha\leq\sum_{i=1}^{m} v_i^*\int_0^1(\Phi_{ik}^{-1}(\alpha)-\delta^*)d\alpha=1.$$

We define

$$\bar{u}_r = u_r^* \Bigg/ \left(\sum_{r=1}^{s} u_r^* \int_0^1 (\Psi_{rk}^{-1}(\alpha) + \delta) d\alpha \right),\ r = 1,\ \ldots,\ s$$

$$\bar{v}_i = v_i^* \Bigg/ \left(\sum_{i=1}^{m} v_i^* \int_0^1 (\Phi_{io}^{-1}(\alpha) - \delta) d\alpha \right),\ i = 1,\ \ldots,\ m.$$

As can be seen, for $(\bar{v},\ \bar{u},\ \delta) = (\bar{v}_1,\ \ldots,\ \bar{v}_m,\ \bar{u}_1,\ \ldots,\ \bar{u}_s,\ \delta) \geq 0$ we have

$$\sum_{r=1}^{s} \bar{u}_r \int_0^1 (\Psi_{ro}^{-1}(\alpha) + \delta) d\alpha = 1, \sum_{i=1}^{m} \bar{v}_i \int_0^1 (\Phi_{io}^{-1}(\alpha) - \delta) d\alpha = 1.$$

On the other hand, for each $p,\ j (p = 1,\ \ldots,\ q,\ j = 1,\ \ldots,\ n)$ we have $\sum_{r=1}^{s} \bar{u}_r \int_0^1 \Psi_{prj}^{-1}(\alpha) d\alpha - \sum_{i=1}^{m} \bar{v}_i \int_0^1 \Phi_{pij}^{-1}(1-\alpha) d\alpha \leq 0$. Therefore, $(\bar{v},\ \bar{u},\ \delta)$ is a feasible solution of model (7), and according to this, the unit $(\tilde{X}_k - e_m\delta,\ \tilde{Y}_k + e_s\delta)$ is efficient by model (7) for any $\delta \geq \delta^*$. The theorem is verified.

5.2 *Stability radius of efficient DMUs*

Assume that $(\tilde{X}_k,\ \tilde{Y}_k) = (\tilde{X}_{1k},\ \ldots,\ \tilde{X}_{mk},\ \tilde{Y}_{1k1},\ \ldots,\ \tilde{Y}_{sks})$ is efficient by model (7), where for all $i,\ k$ and r, $\tilde{X}_{ik} = \sum_{p=1}^{q} \tilde{x}_{ik}^{(p)}$ and $\tilde{Y}_{rk} = \sum_{p=1}^{q} \tilde{y}_{rk}^{(p)}$. We want to find the minimum values of variation in data (δ =radius of stability) namely, increasing the inputs and decreasing the outputs, such that the unit $(\tilde{X}_k + e_m\delta,\ \tilde{Y}_k - e_s\delta)$ be efficient, where e_m and e_s are m and s dimensional vectors, respectively, with all components equal to 1. Therefore, we need to solve the following model:

$$\delta^* = \max\ \delta \tag{11}$$

$$E\left[\sum_{r=1}^{s} u_r(\tilde{Y}_{rk} - \delta)\right] = 1$$

$$s.t.\quad E\left[\sum_{i=1}^{m} v_i(\tilde{X}_{ik} + \delta)\right] = 1$$

$$E\left[\sum_{r=1}^{s} u_r\tilde{y}_{rj}^{(p)} - \sum_{i=1}^{m} v_i\tilde{x}_{ij}^{(p)}\right] \leq 0,\ p = 1,\ldots,q,\ j = 1,\ldots,n,\ j \neq k$$

$$\tilde{Y}_{rk} \geq \delta,\ r = 1,\ \ldots,\ s$$

$$u_r \geq 0,\ v_i \geq 0,\quad r = 1,\ \ldots,\ s,\ i = 1,\ \ldots,\ m$$

where $j \neq k$ means that DMU_k in all of periods should be excluded when $(\tilde{X}_k + e_m\delta,\ \tilde{Y}_k - e_s\delta)$ is evaluated. A deterministic counterpart to the model (11) is provided using the following theorem.

Theorem 7 For each DMU_j, let $\tilde{x}_{ij}^{(1)},\ \tilde{x}_{ij}^{(2)},\ \ldots,\ \tilde{x}_{ij}^{(q)}$ $(i = 1,\ \ldots,\ m,\ j = 1,\ \ldots,\ n)$ be independent uncertain variables with corresponding regular uncertainty distributions $\Phi_{1ij},\ \Phi_{2ij},\ \ldots,\ \Phi_{qij}$ $(i = 1,\ \ldots,\ m,\ j = 1,\ \ldots,\ n)$. Furthermore, let $\tilde{y}_{rj}^{(1)},\ \tilde{y}_{rj}^{(2)},\ \ldots,\ \tilde{y}_{rj}^{(q)}$ $(r = 1,\ \ldots,\ s,\ j = 1,\ \ldots,\ n)$ be independent uncertain variables with corresponding regular uncertainty distributions $\Psi_{1rj},\ \Psi_{2rj},\ \ldots,\ \Psi_{qrj}$ $(r = 1,\ \ldots,\ s,\ j = 1,\ \ldots,\ n)$, respectively. Then, the uncertain model (11) is equivalent to the following deterministic model.

$$\delta^* = \max\ \delta \tag{12}$$

$$\sum_{r=1}^{s} u_r \int_0^1 (\Psi_{rk}^{-1}(\alpha) - \delta) d\alpha = 1$$

$$s.t.\quad \sum_{i=1}^{m} v_i \int_0^1 (\Phi_{ik}^{-1}(\alpha) + \delta) d\alpha \ = 1$$

$$\sum_{r=1}^{s} u_r \int_0^1 \Psi_{prj}^{-1}(\alpha) d\alpha - \sum_{i=1}^{m} v_i \int_0^1 \Phi_{pij}^{-1}(1-\alpha) d\alpha \ \leq 0,\ p = 1,\ \ldots,\ q,\ j = 1,\ \ldots,\ n,\ j \neq k$$

$$\Psi_{rk}^{-1}(1-\alpha) \geq \delta,\ r = 1,\ \ldots,\ s$$

$$u_r \geq 0,\ v_i \geq 0, \quad r = 1,\ \ldots, s,\ i = 1,\ \ldots,\ m$$

where, $\Phi_{ij}^{-1}(\alpha) = \sum_{p=1}^{q} \Phi_{pij}^{-1}(\alpha)$, $i = 1,\ \ldots,\ m,\ j = 1,\ \ldots,\ n$ and $\Psi_{rj}^{-1}(\alpha) = \sum_{p=1}^{q} \Psi_{prj}^{-1}(\alpha)$, $r = 1,\ \ldots,\ s,\ j = 1,\ \ldots,\ n$.

Proof. For each $i (i = 1,\ \ldots,\ m)$ and $r (r = 1,\ \ldots,\ s)$ the inverse uncertainty distribution for the uncertain variables $\tilde{X}_{ik} + \delta$ and $\tilde{Y}_{rk} + \delta$ are $\Phi_{ik}^{-1}(\alpha) + \delta$ and $\Psi_{rk}^{-1}(\alpha) - \delta$, respectively. The uncertain variables $\Upsilon_k = \sum_{r=1}^{s} u_r(\tilde{Y}_{rk} - \delta)$ and $\eta_k = \sum_{i=1}^{m} v_i(\tilde{X}_{ik} + \delta)$ are strictly increasing with respect to $\tilde{Y}_{rk}$ and $\tilde{X}_{ik}$, respectively. Therefore, the inverse uncertainty distribution for Υ_k and η_k are $\Upsilon_k^{-1} = \sum_{r=1}^{s} u_r(\Psi_{rk}^{-1}(\alpha) - \delta)$ and $\eta_k^{-1} = \sum_{i=1}^{m} v_i(\Phi_{ik}^{-1}(\alpha) + \delta)$, respectively. Therefore, $E[\Upsilon_k] = \sum_{r=1}^{s} u_r \int_0^1 (\Psi_{rk}^{-1}(\alpha) - \delta) d\alpha$, $E[\eta_k] = \sum_{i=1}^{m} v_i \int_0^1 (\Phi_{ik}^{-1}(\alpha) + \delta) d\alpha$ and models (10) and (11) are equivalent.

The validity of model (12) is proved using Theorem 8.

Theorem 8 Assume that the unit $(\tilde{X}_k,\ \tilde{Y}_k)$ is efficient using model (7) and δ^* is the optimal value of the model (12). Then,

1. for each $\delta,\ \delta \leq \delta^*$ the unit $(\tilde{X}_k + e_m\delta,\ \tilde{Y}_k - e_s\delta)$ is efficient and
2. for each $\delta,\ \delta > \delta^*$ the unit $(\tilde{X}_k + e_m\delta,\ \tilde{Y}_k - e_s\delta)$ is inefficient.

Proof 1 To prove 1, suppose that $(v^*,\ u^*, \delta^*)$ is an optimal solution of model (12). Consider model (7) for evaluation $(\tilde{X}_k + e_m\delta,\ \tilde{Y}_k - e_s\delta)$. As can be seen, for $\delta = \delta^*$ the vector $(v = v^*,\ u = u^*)$ is a feasible solution of model (7) with $\sum_{r=1}^{s} u_r \int_0^1 (\Psi_{rk}^{-1}(\alpha) - \delta) d\alpha = 1$. Therefore, $(\tilde{X}_k + e_m\delta^*,\ \tilde{Y}_k - e_s\delta^*)$ is efficient by model (7). Now, suppose $\delta < \delta^*$. The function $f(\delta) = \sum_{r=1}^{s} u_r^* \int_0^1 (\Psi_{rk}^{-1}(\alpha) - \delta) d\alpha$ is strictly decreasing by δ and the function $g(\delta) = \sum_{i=1}^{m} v_i^* \int_0^1 (\Phi_{ik}^{-1}(\alpha) + \delta) d\alpha$ is strictly increasing by δ. Therefore, for $\delta < \delta^*$ we have

$$f(\delta) = \sum_{r=1}^{s} u_r^* \int_0^1 (\Psi_{rk}^{-1}(\alpha) - \delta) d\alpha \geq \sum_{r=1}^{s} u_r^* \int_0^1 (\Psi_{rk}^{-1}(\alpha) - \delta^*) d\alpha = 1$$

$$g(\delta) = \sum_{i=1}^{m} v_i^* \int_0^1 (\Phi_{ik}^{-1}(\alpha) + \delta) d\alpha \leq \sum_{i=1}^{m} v_i^* \int_0^1 (\Phi_{ik}^{-1}(\alpha) + \delta^*) d\alpha = 1.$$

In this case, we define

$$\bar{u}_r = u_r^* \bigg/ \left(\sum_{r=1}^{s} u_r^* \int_0^1 (\Psi_{rk}^{-1}(\alpha) - \delta) d\alpha \right), \; r = 1, \; ..., \; s$$

and

$$\bar{v}_i = v_i^* \bigg/ \left(\sum_{i=1}^{m} v_i^* \int_0^1 (\Phi_{io}^{-1}(\alpha) + \delta) d\alpha \right), \; i = 1, \; ..., \; m.$$

As can be seen, $(\bar{v}, \; \bar{u}, \; \delta) \geq 0$ and this vector is a feasible solution of model (7) with $\sum_{r=1}^{s} \bar{u}_r \int_0^1 (\Psi_{ro}^{-1}(\alpha) + \delta) d\alpha = 1$. Therefore, the unit $(\tilde{X}_k + e_m\delta, \; \tilde{Y}_k - e_s\delta)$ is efficient by model (2).

Proof 2 To prove part 2, assume that the unit $(\tilde{X}_k + e_m\delta, \; \tilde{Y}_k - e_s\delta)$ is efficient by model (7). Therefore, there is $(u, \; v, \; \delta)$ such that:

$$\sum_{r=1}^{s} u_r \int_0^1 (\Psi_{rk}^{-1}(\alpha) - \delta) d\alpha = 1$$

$$\sum_{i=1}^{m} v_i \int_0^1 (\Phi_{ik}^{-1}(\alpha) + \delta) d\alpha \; = 1$$

$$\sum_{r=1}^{s} u_r \int_0^1 \Psi_{prj}^{-1}(\alpha) d\alpha - \sum_{i=1}^{m} v_i \int_0^1 \Phi_{pij}^{-1}(1-\alpha) d\alpha \; \leq 0, \; p = 1, \; ..., \; q, \; j = 1, \; ..., \; n, \; j \neq k$$

$$\Psi_{rk}^{-1}(1-\alpha) \geq \delta, \; r = 1, \; ..., \; s$$

$$u_r \geq 0, \; v_i \geq 0, \quad r = 1, \; ..., \; s, \; i = 1, \; ..., \; m.$$

Hence, $(u, \; v, \; \delta)$ is a feasible solution of model (12). On the other hand, δ^* is the optimal value of model (12), so $\delta \leq \delta^*$ and for any $\delta > \delta^*$ the unit $(\tilde{X}_k + e_m\delta, \; \tilde{Y}_k - e_s\delta)$ is inefficient and the proof is completed.

The model's computational complexity and scalability characteristics are important considerations. The deterministic transformation approach (Theorem 4) simplifies the problem by converting it into linear integrals that can be solved using standard linear programming methods like the simplex algorithm. In terms of scalability, the solution time grows proportionally with the number of DMUs (n), time periods (q), and inputs/outputs (m/s), while the memory requirements increase quadratically with these parameters. However, when dealing with large periods of time (large q) or high-dimensional datasets, the computational increases significantly. In such cases, approximation methods such as Monte Carlo integration may be necessary, particularly for handling non-linear distributions efficiently.

6. Numerical example

This section presents two illustrative examples to demonstrate the proposed method's effectiveness and practical relevance. These examples highlight the method's reliability and versatility in addressing real-world problems involving uncertain and multi-period efficiency analysis.

Example 1 Evaluating stability and performance.

The first example assesses the proposed method's performance by evaluating the efficiency of five DMUs. Additionally, the example explores the method's capability to determine the radius of stability for each DMU. The radius of stability represents the extent to which input and output data can vary without altering the efficiency status of the DMU. This analysis provides critical insights into the robustness and reliability of efficiency classifications, ensuring that the method performs consistently under conditions of uncertainty or data perturbations.

Example 2 Healthcare system efficiency during COVID-19.

The second example applies the proposed method to a practical and timely context: measuring the efficiency of healthcare systems in 30 countries during the COVID-19 pandemic. This example involves analyzing two distinct periods, capturing the dynamic and evolving nature of healthcare responses to the pandemic. By integrating uncertain and multi-period data, the method identifies overall efficiency and period-specific efficiencies for each country. The results reveal significant insights into the preparedness and adaptability of healthcare systems, highlighting variations in efficiency across different countries and timeframes.

These examples demonstrate the proposed method's practicality in addressing complex efficiency evaluation problems. The first example emphasizes methodological robustness, while the second showcases its applicability to critical real-world challenges, such as assessing healthcare system performance during a global crisis.

6.1 *Example 1 (illustrative example)*

In this section, a simple example is presented to give an illustration of the periods and overall efficiency, and the sensitivity and stability analysis for the proposed model (7). Table 1 provides information on 5 DMUs. There are three uncertain inputs with linear uncertain variables denoted by $L(a,\ b)$ and three uncertain outputs with zigzag uncertain variables denoted by $Z(a,\ b,\ c)$ in two periods of time. Table 2 depicts the periods and overall efficiency of the proposed model (7) and the stability radius for inputs and outputs with the uncertain proposed models (10) and (12).

As this table shows, the overall efficiency of DMUs (DMU2, DMU5) is also high in each period.

The last column of Table 2 displays the radius of stability for each DMU, and we can observe that the radius of stability of DMU1 is 16.56, which indicates that DMU1 can become overall efficient for at least 16.54 units increment to the mean values of all outputs and decrement to the mean values of all inputs. We can get the same interpretation for inefficient DMU3 and DMU4. In this way, consider DMU5, it remains efficient overall if the increases of all the mean input values and reductions of all the mean output values are less than or equal to 57.2568. We can get the same interpretation for efficient DMU2.

Columns 2, 3 and 4 of Table 3 show the expected values of the inputs by $E[\tilde{X}_{ik}] = \int_0^1 \Phi_{ik}^{-1}(\alpha)d\alpha$ for any i $(i = 1,\ 2,\ 3)$ and columns 5, 6 and 7 of Table 3 present the expected values of the outputs by $E[\tilde{Y}_{rk}] = \int_0^1 \Psi_{rk}^{-1}(\alpha)d\alpha$ for any r $(r = 1,\ 2,\ 3)$.

The last 6 columns of Table 3 illustrate the new values of inputs and outputs of DMUs, which have been obtained by the stability radius. For instance, the number 13.44 in column 8 is the first new input of inefficient DMU1 and is determined by X_{newj11}= $E[\tilde{X}_{ik}] - \delta^* = 30 - 16.56 = 13.44$, while the number 53.432 in column 8 is the new input 1 of efficient DMU2 and is calculated using $X_{newj21} = E[\tilde{X}_{ik}] + \delta^* = 41 + 12.432 = 53.432$. Also, the number 66.56 in column 11 is the first new output of DMU1 and is calculated by Y_{newj11}= $E[\tilde{Y}_{rk}] + \delta^* = 50 + 16.56 = 66.56$.

Table 1. Five DMUs with three uncertain linear inputs and three uncertain zigzag outputs as uncertain variables

DMUs	The inputs and outputs of the first period					
	$\tilde{x}_{11}^{(1)}$	$\tilde{x}_{21}^{(1)}$	$\tilde{x}_{31}^{(1)}$	$\tilde{y}_{11}^{(1)}$	$\tilde{y}_{21}^{(1)}$	$\tilde{y}_{31}^{(1)}$
DMU1	$L(12, 16)$	$L(13, 20)$	$L(26, 27)$	$Z(24, 25, 30)$	$Z(25, 27, 29)$	$Z(26, 27, 31)$
DMU2	$L(18, 23)$	$L(28, 35)$	$L(9, 12)$	$Z(42, 45, 46)$	$Z(80, 91, 94)$	$Z(95, 100, 103)$
DMU3	$L(28, 35)$	$L(42, 45)$	$L(32, 34)$	$Z(20, 21, 26)$	$Z(16, 22, 24)$	$Z(25, 30, 41)$
DMU4	$L(19, 24)$	$L(8, 13)$	$L(10, 12)$	$Z(41, 42, 44)$	$Z(45, 46, 49)$	$Z(45, 50, 74)$
DMU5	$L(9, 13)$	$L(9, 14)$	$L(9, 12)$	$Z(90, 105, 140)$	$Z(50, 52, 54)$	$Z(82, 90, 94)$
DMUs	The inputs and outputs of the second period					
	$\tilde{x}_{11}^{(2)}$	$\tilde{x}_{21}^{(2)}$	$\tilde{x}_{31}^{(2)}$	$\tilde{y}_{11}^{(2)}$	$\tilde{y}_{21}^{(2)}$	$\tilde{y}_{31}^{(2)}$
DMU1	$L(12, 20)$	$L(14, 19)$	$L(26, 31)$	$Z(22, 23, 28)$	$Z(21, 23, 24)$	$Z(25, 27, 34)$
DMU2	$L(15, 26)$	$L(27, 35)$	$L(8, 13)$	$Z(42, 44, 48)$	$Z(82, 85, 104)$	$Z(95, 100, 103)$
DMU3	$L(35, 36)$	$L(41, 46)$	$L(35, 39)$	$Z(16, 21, 23)$	$Z(18, 20, 26)$	$Z(23, 28, 33)$
DMU4	$L(16, 27)$	$L(12, 15)$	$L(12, 16)$	$Z(37, 40, 43)$	$Z(44, 45, 48)$	$Z(49, 52, 55)$
DMU5	$L(8, 14)$	$L(9, 13)$	$L(8, 13)$	$Z(95, 100, 145)$	$Z(48, 52, 56)$	$Z(75, 92, 97)$

Table 2. The results of calculating the periods and overall efficiency, and stability radius for inputs and outputs

DMU	The efficiency of the first period	The efficiency of the second period	Overall efficiency	Radius of stability
1	0.3894	0.2903	0.3365 (Inefficient)	16.56
2	1	1	1 (Efficient)	12.432
3	0.1346	0.1194	0.1266 (Inefficient)	48.688
4	0.9368	0.7286	0.8197 (Inefficient)	3.5714
5	1	1	1 (Efficient)	57.2568

Table 3. The expected value of the inputs and outputs and their modified values by radius stability

DMU	$E\left[\tilde{X}_{1k}\right]$	$E\left[\tilde{X}_{2k}\right]$	$E\left[\tilde{X}_{3k}\right]$	$E\left[\tilde{Y}_{1k}\right]$	$E\left[\tilde{Y}_{2k}\right]$	$E\left[\tilde{Y}_{3k}\right]$	X_{newj1}	X_{newj2}	X_{newj3}	Y_{newj1}	Y_{newj2}	Y_{newj3}
1	30	33	55	50	50	56.5	13.44	16.44	38.44	66.56	66.56	73.06
2	41	63	21	89	178	199	53.432	75.432	33.432	76.568	165.568	186.568
3	67	87	70	42	42	59.5	18.312	38.312	21.312	90.688	90.688	108.188
4	43	24	25	82.75	93	106.75	39.4286	20.4286	21.4286	86.3214	96.5714	110.3214
5	21	22	21	220	104	178	78.2568	79.2568	78.2568	162.7432	46.7432	120.7432

6.2 *Example 2 (Case study)*

The uncertain proposed model can have different significances for healthcare system managers and decision-makers. Healthcare efficiency in response to pandemics such as COVID-19 using our proposed method can enable discernment, trace inefficiency resources, and remedial deficiency [53–56].

Input-output factors

This subsection demonstrates the practical application of the uncertain relational multi-period model by assessing the effectiveness of healthcare systems during the COVID-19 pandemic. The evaluation uses a dataset comprising 30 countries classified as high-income and upper-middle-income economies, with data from the World Bank Data Box [57].

Study timeframes

The analysis spans two distinct timeframes, corresponding to critical phases of the pandemic: 1. First Period: March 1 to July 31, 2020, reflecting the initial outbreak and early response strategies. 2. Second Period: August 1 to December 31, 2020, capturing adaptive responses and potential improvements in system efficiency.

Indicators for efficiency measurement

The choice of input and output variables is pivotal for deriving accurate insights into healthcare efficiency. Based on a comprehensive literature review, this study considers both internal and external factors that influence healthcare performance:

Input variables

1. GDP (Nominal) per Capita (2020): Data reported by Worldometer [57].

2. Population Density (2020): Measured as population per unit of land area, sourced from Population Pyramid [58].

3. Healthcare Resources: Sourced from the World Bank Database [57], including:

Hospital Beds (per 1,000 people)

Physicians (per 1,000 people)

Nurses (per 1,000 people)

These input variables capture the economic and demographic context, as well as the healthcare infrastructure capacity, to address the pandemic.

Output Variables

Common metrics frequently used to evaluate healthcare performance during the COVID-19 pandemic include:

• Confirmed Cases

• Death Cases

• Recovered Cases

Such metrics reflect the system's effectiveness in mitigating the pandemic's impact, as highlighted in prior studies [60].

Data representation

For this study, the input data is treated as crisp (precise) and summarized in Columns 3, 4, 5, 6, and 7 of Table 4. These columns present key variables, enabling a comprehensive assessment of the healthcare systems across the two periods.

Significance of the study

This empirical evaluation provides critical insights into the temporal and cross-national variations in healthcare system efficiency during a global health crisis. By incorporating internal healthcare indicators and broader socioeconomic factors, the study showcases the robustness and versatility of the proposed uncertain relational multi-period model. This approach enables researchers and policymakers to identify periods of inefficiency and develop targeted strategies for system improvement.

Data consistency remains a challenge given the discrepancies in information products published by entities such as the WHO, national public health authorities, and other sources-stemming from variations in inclusion criteria, data cut-off times, and reporting strategies. These variations influence the counts of confirmed cases, deaths, and recovered cases. To address this, we incorporate expert opinions about these indices and employ uncertain theory to evaluate the performance of healthcare systems during the COVID-19 pandemic.

Handling undesirable outputs and data assumptions

In this study, death cases, typically regarded as an undesirable output, are considered an input to reflect the healthcare system's burden. Three key uncertain indices are incorporated:

1. Confirmed Cases (Input): Reflecting the strain on healthcare systems.

2. Death Cases (Output): Highlighting the adverse outcomes of the pandemic.

3. Recovered Cases (Output): Demonstrating healthcare system effectiveness. These indices are sourced from reliable data repositories such as Johns Hopkins GitHub data [61] and the World Health Organization (WHO) [62].

Data structure

The last two columns of Table 4 report the confirmed cases for the two periods under study. Table 5, detailed below, presents the uncertain outputs for the two periods, including death and recovered cases.

Table 4. The crisp and uncertain inputs for 30 countries about the COVID-19 pandemic

#	Country	GDP	Publication density	Beds	Physicians	Nurses	Confirmed cases of Period 1	Confirmed cases of Period 2
1	Azerbaijan	4,151	118.76	4.8 (2014)	3.1 (2019)	6.3 (2014)	L(174.7285, 237.8205)	L(984.2657, 1,443.9434)
2	Bahrain	23,433	1,920	1.7 (2017)	0.8 (2016)	2.3 (2016)	L(231.9681, 300.2411)	L(971.8462, 1,435.8612)
3	Belgium	45,426	378.6	5.6 (2019)	3.2 (2020)	20.1 (2020)	L(366.5575, 522.3968)	L(3,011.5223, 4,511.2359)
4	Brazil	6,923	25.04	2.1 (2017)	2.3 (2019)	7.4 (2019)	L(14,273.0475, 19,938.5211)	L(30,152.0868, 34,633.6910)
5	Denmark	60,975	135.73	2.6 (2019)	4.3 (2019)	10.5 (2019)	L(76.4804, 102.8921)	L(806.4653, 1,120.8680)
6	Finland	49,170	16.34	3.6 (2018)	4.3 (2020)	22.3 (2020)	L(40.6431, 56.7556)	L(163.2055, 211.9579)
7	France	40,927	117.43	5.9 (2018)	3.3 (2020)	12.2 (2020)	L(639.1501, 2,155.9741)	L(12,815.1652, 18,321.9459)
8	Germany	46,678	233.17	8.0 (2017)	4.5 (2020)	12.4 (2020)	L(1,104.9020, 1,610.9412)	L(8,162.0082, 11,164.6192)
9	Iran	2,746	50.02	1.6 (2017)	1.5 (2018)	2.0 (2018)	L(1,855.2780, 2,078.5390)	L(5,351.8948, 6,602.9941)
10	Iraq	4,251	97.82	1.3 (2017)	0.9 (2020)	2.3 (2020)	L(635.7199, 949.2474)	L(2,888.8110, 3,252.7968)
11	Ireland	86,098	70.43	3.0 (2018)	3.5 (2020)	18.4 (2020)	L(129.9690, 210.2401)	L(364.1131, 473.6908)
12	Italy	31,885	197.45	3.1 (2018)	4.0 (2020)	6.6 (2020)	L(1,328.9793, 1,887.0992)	L(10,083.5587, 13,918.4283)
13	Japan	40,311	331.37	13.0 (2018)	2.6 (2020)	12.4 (2020)	L(184.5812, 265.4188)	L(1,126.7295, 1,421.7803)
14	Malaysia	10,161	100.36	1.9 (2017)	2.2 (2020)	3.4 (2019)	L(48.6438, 68.1536)	L(563.1462, 763.7688)
15	Netherlands	52,183	419.71	3.2 (2018)	3.8 (2020)	11.3 (2020)	L(297.6926, 411.9937)	L(4,278.4475, 5,424.1539)
16	New Zealand	42,000	18.91	2.6 (2018)	3.4 (2020)	11.2 (2020)	L(6.7333, 13.6458)	L(3.2100, 4.5155)
17	Norway	68,335	13.97	3.5 (2018)	5.1 (2020)	18.6 (2020)	L(47.4525, 72.7175)	L(222.7598, 297.1095)
18	Oman	16,707	14.68	1.5 (2017)	2.0 (2020)	4.4 (2020)	L(429.5889, 605.0909)	L(258.3241, 389.5191)
19	Poland	15,598	122.9	6.5 (2018)	3.7 (2020)	6.8 (2020)	L(271.4908, 317.1497)	L(6,748.6069, 9,404.6741)
20	Portugal	22,240	111.66	3.5 (2018)	5.6 (2020)	7.6 (2020)	L(295.4390, 369.5022)	L(1,985.1307, 2,655.1177)
21	Qatar	52,315	237.76	1.3 (2017)	2.5 (2018)	7.2 (2018)	L(625.4227, 818.4858)	L(207.5504, 222.8548)
22	Romania	12,930	81.56	6.9 (2017)	3.0 (2020)	7.4 (2017)	L(279.7888, 368.4203)	L(3,321.7455, 4,221.4571)
23	Russi	10,253	8.52	7.1 (2018)	3.8 (2020)	6.2 (2020)	L(4,870.4254, 6,018.3459)	L(13,344.1663, 16,218.6703)
24	Saudi Arabia	20,398	16.75	2.2 (2017)	2.6 (2020)	5.5 (2020)	L(1,559.2777, 2,003.9903)	L(511.0683, 642.7749)
25	South Korea	31,716	517	12.4 (2018)	2.5 (2020)	8.5 (2020)	L(59.8842, 103.3184)	L(242.1798, 338.9444)
26	Spain	26,961	93.62	3.00 (2018)	4.6 (2020)	6.3 (2020)	L(1,458.1670, 2,272.3559)	L(8,861.8988, 12,336.7417)
27	Switzerland	85,651	209.22	4.6 (2018)	4.4 (2020)	18.7 (2020)	L(176.1506, 281.4180)	L(2,076.2623, 3,318.1560)
28	Turkey	8,561	107.13	2.9 (2018)	2.0 (2020)	3.4 (2020)	L(1,311.6887, 1,693.4224)	L(2,276.7699, 23,388.5765)
29	UAE	37,629	111.09	1.4 (2017)	2.9 (2020)	6.4 (2020)	L(348.4850, 429.7241)	L(871.1354, 997.3875)
30	USA	62,690	34.17	2.9 (2017)	3.6 (2020)	12.5 (2020)	L(26,274.4259, 32,341.2211)	L(89,389.9231, 111,361)

Table 5. The uncertain outputs for 30 countries during the COVID-19 pandemic

DMUs	#	The first period		The second period	
		Deaths	Recovered cases	Deaths	Recovered cases
Azerbaijan	1	L(2.3872, 3.3775)	L(133.4979, 195.4955)	L(11.6517, 16.5966)	L(789.1652, 1,216.1984)
Bahrain	2	L(0.7328, 1.1756)	L(0.3711, 0.8185)	L(1.1042, 1.5756)	L(310.0846, 367.6557)
Belgium	3	L(0.7328, 1.1756)	L(92.2994, 136.6157)	L(51.3714, 74.1188)	L(290.0093, 687.0612)
Brazil	4	L(521.7336, 673.8351)	L(667.4314, 15,911.7450)	L(614.7548, 709.7942)	L(25,406.7406, 36,998.9477)
Denmark	5	L(3.1773, 4.8619)	L(64.2527, 99.4859)	L(3.2338, 5.1453)	L(587.9684, 845.4862)
Finland	6	L(1.4826, 2.5958)	L(20.0302, 70.8064)	L(1.3950, 2.2128)	L(61.7607, 224.6029)
France	7	L(148.7767, 245.1710)	L(374.6882, 561.6647)	L(176.4072, 266.2856)	L(566.1388, 778.4586)
Germany	8	L(47.3988, 71.9999)	L(991.8874, 1,511.8381)	L(112.7187, 187.4905)	L(6,209.8859, 8,776.6076)
Iran	9	L(99.2774, 116.7488)	L(1,556.1694, 1,856.6018)	L(231.3512, 269.6814)	L(4,114.9404, 5,173.3064)
Iraq	10	L(24.2182, 36.8406)	L(427.2807, 690.9676)	L(49.1497, 56.3012)	L(2,759.1140, 3,070.9379)
Ireland	11	L(8.0390, 15.0067)	L(11.5817, 293.8300)	L(2.4324, 3.6199)	L(1,639.5747, 3,825.6743)
Italy	12	L(189.5047, 269.3580)	L(1,122.8866, 1,488.2245)	L(205.2277, 297.5566)	L(6,436.4244, 9,764.6405)
Japan	13	L(5.3487, 7.7363)	L(115.0824, 198.5385)	L(13.4830, 18.2817)	L(1,037.0713, 1,450.0715)
Malaysia	14	L(0.6054, 1.0286)	L(46.8414, 65.4984)	L(1.8212, 2.5971)	L(424.3833, 599.2011)
Netherlands	15	L(31.6352, 48.7178)	L(0.91504, 1.6988)	L(28.3393, 39.3339)	L(25.5275, 39.7582)
NewZealand	16	L(0.0138, 0.2868)	L(7.0168, 12.7740)	L(0.0190, 0.0482)	L(2.8326, 4.4272)
Norway	17	L(1.2073, 2.1261)	L(53.3444, 155.7496)	L(0.7071, 1.6589)	L(60.0256, 140.1735)
Oman	18	L(2.1850, 3.3183)	L(309.3394, 493.5364)	L(5.5383, 8.5271)	L(265.0107, 516.7165)
Poland	19	L(9.8019, 12.5380)	L(189.7434, 250.0344)	L(139.5739, 204.2562)	L(5,227.1059, 7,654.7383)
Portugal	20	L(9.7000, 12.8751)	L(108.0166, 364.4017)	L(28.0681, 38.5332)	L(1,541.2727, 2,283.8312)
Qatar	21	L(0.8857, 1.3496)	L(548.1834, 852.2742)	L(0.3462, 0.5819)	L(213.0531, 227.5054)
Romania	22	L(13.5757, 16.5419)	L(143.6674, 204.1627)	L(78.2771, 94.9647)	L(2,698.0534, 4,146.7388)
Russia	23	L(79.0648, 101.0398)	L(3,482.6912, 4,732.7336)	L(244.8219, 300.9690)	L(10,757.9915, 13,427.0475)
SaudiArabia	24	L(15.6034, 21.2070)	L(1,248.0520, 1,739.7781)	L(20.5507, 23.3971)	L(680.7558, 907.7377)
South Korea	25	L(1.4978, 2.2538)	L(63.6128, 108.3610)	L(2.7690, 4.5381)	L(157.1369, 219.9800)
Spain	26	L(133.6290, 238.1749)	L(750.8410, 1,214.8322)	L(114.4958, 176.2755)	L(2,357.4082, 5,500.619)
Switzerland	27	L(9.0415, 14.8147)	L(147.8286, 258.7073)	L(32.1950, 45.8312)	L(1,003.6074, 2,717.1719)
Turkey	28	L(31.5359, 42.6341)	L(1,185.6208, 1,605.7387)	L(85.7854, 109.6525)	L(12,104.5132, 26,405.1014)
UAE	29	L(1.8517, 2.6843)	L(293.5433, 397.9730)	L(1.8003, 2.2912)	L(744.8032, 900.9500)
USA	30	L(884.9785, 1,112.1065)	L(7,310.2182, 11,175.3766)	L(1,136.5078, 1,391.8190)	L(72,495.9633, 1.52360E+5)

6.3 *Results and discussions*

Table 6 compares the efficiency scores obtained using the proposed uncertain model (7) with those from the classical (crisp) model. The efficiency scores for the first and second periods are listed in the third and fourth columns, respectively. In contrast, using the uncertain model, the fifth column presents the overall efficiency scores calculated across all periods.

Analysis of results

1. Period-specific efficiencies:

• The average efficiency score for Period 1 is 0.2858, while for Period 2, it rises significantly to 0.4879.

• The comparison between the third and fourth columns indicates that the efficiency of most countries improved during the second period.

• This improvement, a notable 70.71% increase in average efficiency, suggests that healthcare systems enhanced their preparedness and response capabilities during the second period of the COVID-19 pandemic.

2. Overall efficiency:

• The fifth column reveals the overall efficiency scores for each healthcare system across all periods.

• Country #28 emerges as the top-performing healthcare system, achieving an impressive overall efficiency score of 0.9519.

• On the other hand, Country #3 ranks lowest, with an overall efficiency score of 0.0945, highlighting significant room for improvement in its healthcare system's performance.

Table 6. Results of the proposed model for 30 countries during the COVID-19 pandemic

DMUs	#	Uncertain efficiency		Crisp overall efficiency	
		The first period	The second period	Deaths	The proposed overall
Azerbaijan	1	0.4033	0.5267	0.5049	0.832
Bahrain	2	0.0013	0.6428	0.3464	1
Belgium	3	0.0645	0.1026	0.0946	0.0393
Brazil	4	0.6945	1	0.8866	1
Denmark	5	0.0611	0.2731	0.2014	0.4635
Finland	6	0.0791	0.192	0.1429	0.274
France	7	0.1521	0.0279	0.042	0.073
Germany	8	0.3962	0.4839	0.469	0.75
Iran	9	0.5714	0.5196	0.5325	0.8791
Iraq	10	0.4367	0.6253	0.5847	0.9367
Ireland	11	0.1293	1	0.7486	1
Italy	12	0.418	0.4402	0.437	0.7108
Japan	13	0.1186	0.4319	0.3333	0.4946
Malaysia	14	0.1363	0.7489	0.5185	1
Netherlands	15	0.0007	0.0039	0.0033	0.0075
NewZealand	16	0.1489	0.1441	0.1475	0.381
Norway	17	0.2036	0.1315	0.1606	0.2089
Oman	18	0.3643	0.4532	0.4033	0.7214
Poland	19	0.2684	0.5207	0.5051	0.8125
Portugal	20	0.2274	0.4795	0.4274	0.6983
Qatar	21	0.6938	0.5257	0.6445	1
Romania	22	0.2179	0.5787	0.5358	0.8699
Russia	23	0.6404	0.7232	0.7002	1
SaudiArabia	24	0.5468	0.6401	0.5759	0.9534
South Korea	25	0.0957	0.156	0.1303	0.2474
Spain	26	0.2835	0.241	0.2484	0.1595
Switzerland	27	0.0833	0.3048	0.2416	0.5012
Turkey	28	0.572	1	0.9519	1
UAE	29	0.2918	0.7205	0.5022	1
USA	30	0.272	1	0.8311	0.6738
Average	-	0.285763	0.487927	0.428383	0.65626

Key insights

• The results demonstrate the advantage of the proposed uncertain model in providing a more nuanced and discriminative efficiency assessment compared to the classical approach.

• The significant increase in efficiency from Period 1 to Period 2 reflects healthcare systems' adaptability and learning curve in responding to the pandemic.

• Identifying high-performing countries (e.g., Country #28) provides benchmarks and best practices for other healthcare systems to emulate, while low-performing countries (e.g., Country #3) highlight areas requiring targeted interventions.

This comprehensive analysis underscores the robustness of the uncertain model in capturing and comparing healthcare system performance, even in scenarios with inherent data uncertainty and variability.

The comparison between the proposed uncertain model (7) and the classical crisp model shows noticeable differences in evaluating healthcare system performance. According to the last column of Table 6, the overall efficiency scores calculated using crisp data show that eight countries are efficient. However, no country achieves overall efficiency when using the proposed uncertain model. This result demonstrates that the uncertain model provides a stricter and more robust framework for evaluation by accounting for the uncertainties inherent in the data.

The average overall efficiency score for the 30 countries using the uncertain model is 0.4284, compared to 0.6564 with the crisp model. The lower average in the uncertain model highlights its ability to provide a more precise and realistic ranking of healthcare systems, avoiding overestimations caused by ignoring data uncertainties. This suggests that the uncertain model is more effective in distinguishing between high-performing and low-performing units.

The proposed uncertain model is significantly more discriminative than the crisp model. This enhanced capability is crucial for identifying subtle inefficiencies within healthcare systems, offering valuable insights for decision-makers to address specific shortcomings.

Figure 1 visually compares the overall efficiency scores of the 30 countries during the COVID-19 pandemic. It illustrates the stricter and more accurate assessment that the uncertain model provides, reinforcing its reliability in evaluating performance. Overall, the uncertain model provides a more realistic and nuanced perspective, ensuring that the assessment considers the complexities and uncertainties of real-world data.

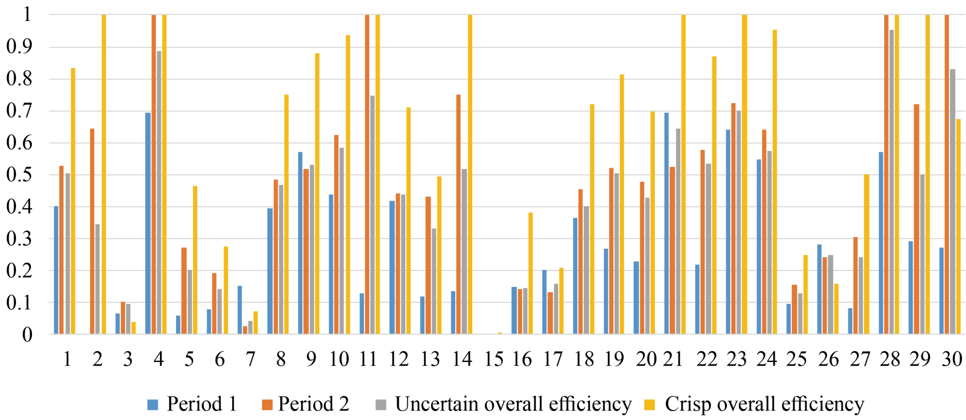

Figure 1. The results of the proposed model for 30 countries during the COVID-19 pandemic

The results generally reveal that most of the studied healthcare systems fail to operate efficiently, primarily due to the suboptimal utilization of available resources. This inefficiency underscores the need for improved resource management and strategic planning in healthcare systems. Additionally, the findings highlight the importance of the World Health Organization (WHO) playing a more proactive role in promoting health awareness and enhancing crisis management capabilities to better address pandemic outbreaks.

For the computational aspect of this study, EXCEL and GAMS software were employed to solve the proposed models effectively. These tools provided robust and efficient platforms for analyzing the data and obtaining accurate results, further validating the applicability and reliability of the proposed methodology.

7. Conclusion and future research

The COVID-19 pandemic, an unprecedented global crisis, has had profound adverse effects worldwide. Healthcare systems have been at the forefront of the critical response in mitigating the pandemic's impact. Efficient evaluation of healthcare systems during disasters like COVID-19 is essential for improving the quality of care during crises. This evaluation is particularly challenging in uncertain environments, yet it is vital for managers to identify inefficiencies and leverage the strengths of effective units to optimize overall performance.

In this study, we proposed a novel, uncertain relational model for multi-period systems that simultaneously measures overall system efficiency and individual period efficiencies. The model incorporates uncertainty by using expected values of uncertain variables to formulate a solvable deterministic equivalent through linear programming methods. A real-world case study was conducted to validate the applicability and reliability of the proposed model.

Studying contributes to the field in several ways. First, it introduced a new relational model for multi-period systems, enabling the evaluation of overall and period-specific efficiencies using a single framework. Second, mathematical theorems were presented to establish the feasibility and boundedness of the model, and the process of converting it into a deterministic equivalent was also demonstrated. Third, the model enables the decomposition of overall efficiency into period-specific efficiencies, helping to identify inefficiency sources and offering guidance to decision-makers for performance improvement. The study also included a sensitivity and stability analysis of the model, adding robustness to the methodology. Finally, a case study conducted during the COVID-19 pandemic illustrated the practical application and reliability of the proposed model in evaluating healthcare systems under uncertainty.

The results demonstrated that the proposed method offers a more discriminative ranking of healthcare systems than deterministic models. Comparing efficiencies between two periods showed a significant increase in efficiency for most countries during the second period, likely reflecting improved readiness and response to the pandemic.

This study opens several avenues for further investigation. First, due to limitations in data collection, this research used eight indices to evaluate healthcare systems in 30 countries in 2020. Future studies could expand the scope by including additional countries or regions facing similar conditions. Second, the findings are based on data from two specific periods in 2020. Analysis over longer or different timeframes could provide deeper insights and allow for more generalizable conclusions. Third, including alternative indices-such as medical staff training, qualifications of personnel, social and cultural factors, surveillance quality, and costs-could yield a more comprehensive evaluation. Many of these inherently uncertain indices align well with the uncertain modeling framework introduced here. Lastly, the methodology could be adapted to evaluate other critical sectors during crises, such as education, logistics, or public safety.

This study highlights the importance of incorporating uncertainty into performance evaluations, especially in the healthcare sector, and demonstrates how such methodologies can improve decision-making during global emergencies. Future work could integrate DEA with epidemic dynamics to explore policy synergies. Several promising avenues emerge for extending this work. First, incorporating dynamic uncertainty through time-series approaches like autoregressive uncertain processes could better capture temporal patterns. Second, machine learning techniques could be employed to train distribution parameters using neural networks, reducing reliance on subjective expert judgments. Finally, applying this model to cross-disciplinary challenges-such as assessing multi-period efficiency of flood barriers amid uncertain rainfall patterns in climate resilience studies-could demonstrate its versatility while addressing pressing real-world problems. Each of these directions offers opportunities to enhance both the theoretical foundations and practical applications of the methodology.

Author contributions

Shabnam Razavyan: Writing-original draft, Validation, Software, Resources, Methodology. Ghasem Tohidi: Writing-review & editing, Visualization, Validation, Farhad Hosseinzadeh Lotfi: Writing-review & editing, Visualization, Validation. Tofigh Allahviranloo: Writing-review & editing, Visualization, Validation, Supervision, Project administration, Investigation, Formal analysis, Conceptualization. Mohammadreza Shahriari: Formal analysis, Data curation, Conceptualization. Sovan Samanta: Methodology, Investigation, Formal analysis. Jeong-Gon Lee: Investigation, Formal analysis. Data curation. Funding Acquisition.

Data availability statement

All data generated or analyzed during this study are included in this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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