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Research Article

Signed Quantum Graphs: A Dynamical System

Sovan Samanta[1,2,3*], Tofigh Allahviranloo[2,4], Alireza Amirteimoori[2], Mohammad Hassan Behzadi[5], Leo Mrsic[3,6]

[1] Department of Mathematics, Tamralipta Mahavidyalaya, Tamluk, W.B, 721636, India
[2] Research Center of Performance and Productivity Analysis, Istinye University, Istanbul, Turkey
[3] Department of Technical Sciences, Algebra Bernays University , Gradiscanska, 24, 10000, Zagreb, Croatia
[4] Quantum Technologies Research Center (QTRC), Science and Research Branch, Islamic Azad University, Tehran, Iran
[5] Department of Statistics, Science and Research Branch, Islamic Azad University, Tehran, Iran
[6] Rudolfovo Science and Technology Centre Novo Mesto, Slovenia
E-mail: ssamantavu@gmail.com

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Abstract: Traditional Graphs, such as weighted graphs, or fuzzy graphs have static edge weights or membership values. These graphs have been widely used to model various complex systems, but they typically do not incorporate dynamic interactions found in evolving systems like brain networks. Temporal graphs are data-based algorithmic time-dependent structures. There is no such structure to represent the dynamic interactions of networks. Quantum graphs are structured graphs based on the dynamic behaviors of vertices and links. This study introduces Signed Quantum Graphs (SQGs) with time-dependent signs that represent dynamical systems. The properties of SQGs have been investigated. Additionally, concepts and properties of triadic closure, balance, domination, and strength of a SQG have been studied. The area of applications in brain networks has been mentioned. We propose a dynamic signed graph framework, where edge signs evolve periodically using functions. Structural properties and convergence results are provided.

***Keywords*:** signed graphs, quantum graphs, dynamical networks, temporal graphs

MSC: 05C90, 81P45

1. Introduction

The dynamic natures of networks cannot be represented by weighted graphs or fuzzy graphs. To include a dynamic nature, temporal graphs are introduced by Kostac [1]. The works have been advanced and implemented in co-authorship networks [2]. But the representation of such graphs is based on time-dependent data. To generalize the dynamic nature of networks, quantum graphs are defined by Samanta and Allahviranloo [3, 4].

A graph $\mathscr{G}$, represented as $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, where $\mathscr{V}$ is the set of nodes and $\mathscr{E}$ is the set of links/edges, is said to be a quantum graph such that:

Each node $v_i \in \mathscr{V}$ has a dynamic weight $\nu_i(t)$ that may change over time.

$$0 \leq \nu_i(t) \leq 1.$$

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The weight is a function, which may be assumed as a sinusoidal function, for example:

$$\nu_i(t) = \frac{1}{2}\left(1 + \cos(\omega_i t + \varphi_i)\right), \tag{1}$$

where ν_i is the frequency and φ_i is the phase shift.

While we illustrate the dynamic weights using sinusoidal functions due to their interpretability and periodic nature, the framework allows any bounded, continuous function over time, such as piecewise, exponential decay, or logistic functions, depending on the modeled system.

Links between vertices v_i and v_j may be entangled or may not. If entangled, the link is defined by a correlation function $\gamma_{ij}(t)$ that represents the relationship between the weights of the connected nodes:

$$\gamma_{ij}(t) = \eta(\nu_i(t), \nu_j(t)), \tag{2}$$

where η is a function defining the nature of the entanglement or correlation.

Edge entanglement refers to the correlation that captures the idea that the state of one vertex can influence the state of another between two nodes. It reflects the entangled nature of quantum systems. The function η describes this relationship and can take various forms depending on the specific nature of the entanglement.

The function η is the correlation between the dynamic weights of two vertices; for example, η could be a simple product of the weights, indicating a direct correlation:

$$\eta(\nu_i(t),\ \nu_j(t)) = \nu_i(t) \cdot \nu_j(t). \tag{3}$$

Alternatively, η could represent a more complex relationship, such as a weighted sum or another function that captures the specific nature of the entanglement.

The study of dynamical systems is important in biological systems [5] and social networking analysis [6]. To measure the stability of a network, the relationship among nodes, sign (positive or negative), is an important factor in dynamical systems. In social networks, stability and influences [7] may be represented by traditional sign graphs, but dynamical systems such as brain networks can not be represented by existing techniques of graph theory. For this purpose, Signed Quantum Graphs (SQGs) have been introduced in this study with several properties such as domination, triadic closures, balanced graphs, and their applications.

Quantum walks [8, 9] are among the research efforts in the area of quantum networks. The concept of edge signs on the behavior of quantum walks has been studied in [10]. Those studies are based on algebraic properties of graphs. Recently, Samanta and Allahviranloo [3] have introduced quantum graphs. That article may be treated as the foundation of quantum graph theory.

A study is available on a weighted adjacency matrix induced by signs to each edge [10]. The dynamic nature in quantum computing covers many real life applications. For instance, the study [11] highlights that quantum decoherence and qubit interconnectivity have advantages in the Noisy Intermediate Scale Quantum (NISQ) era.

Signed graphs are used to represent in various domains, including quantum algorithms [10, 12]. Social network analysis and stability of networks include the topics of triadic closure. Bianconi et al. demonstrated that triadic closure with high clustering coefficients and fat-tailed distributions of node degrees [13]. Huang et al. introduced triadic closure in networks based on the formation of a third tie in a triad. This model plays a significant role in explaining the strength of social ties [14].

The modeling of complex social and biological networks has seen significant advancement with the introduction of signed and temporal graphs. Signed networks were initially explored through the lens of balance theory, and later extended to triadic closure mechanisms and community generation [13]. Huang et al. [14] further analyzed triadic closure in online social networks and its effect on tie reinforcement.

More recently, quantum-inspired frameworks have been proposed for network modeling. Biamonte et al. [6] outlined the transition from classical complex networks to quantum network structures, highlighting the potential for encoding additional structural information using quantum states. Segawa and Yoshie [10] explored quantum search algorithms over signed graphs, proposing that quantum dynamics can enrich structural exploration compared to classical methods.

Several works have integrated fuzzy parameters into graph-based systems. Samanta et al. [7] developed influence measures in social networks using fuzzy dominance and uncertainty modeling. Pal et al. [5] emphasized the role of quantum computing in biological systems, indicating the importance of hybrid paradigms where fuzziness and quantum traits coexist. Gill et al. [11] provided a systematic review of quantum computing architectures, many of which are relevant for information propagation in uncertain environments.

Our work diverges from these prior efforts by combining fuzzy influence modeling with deterministic edge sign oscillations. Unlike traditional temporal signed networks, which use stochastic edge flipping, we introduce periodic functions (e.g., cosine-based transitions) to represent interference-like relational dynamics. This aligns with broader efforts in network neuroscience and quantum sociology, where entanglement or correlated behavior between nodes is central to understanding the evolution of the system [15, 16].

Furthermore, our approach incorporates ideas from quantum walks [8, 9], reinterpreting them in a fuzzy graph-theoretic context. This hybridization enables more expressive models that better reflect both the uncertainty and interconnectedness of real-world relational systems.

Despite the progress made, several challenges are still present in quantum networks. One primary challenge is to develop efficient algorithms. In the quantum graph, the dynamic nature of the weights means that the stability and shortest path calculations must account for changes over time. For instance, the shortest path between two vertices at one moment might not be the shortest path at another moment due to the changing weights, and hence this requires updated algorithms to recalculate paths as the graph parameters are changing [10].

Research gaps

Despite advances in temporal and signed networks, several research gaps remain:

• Existing temporal graphs focus on discrete interactions but do not account for continuous oscillatory dynamics.

• Classical signed graphs capture positive/negative relations but cannot represent dynamics over time.

• Fuzzy graph approaches incorporate uncertainty, yet they lack a principled mechanism to integrate oscillatory or dynamic weights.

• Current models inadequately address stability measures in dynamical systems such as brain networks, where signed interactions evolve continuously.

Major contributions

To bridge these gaps, this article introduces and develops the theory of SQGs. Our contributions are threefold:

• Definition and framework: We formalize the concept of SQGs, where each vertex carries a dynamic weight $\sigma_i(t)$ bounded in $[0, 1]$, represented by periodic or continuous functions, and edges encode entanglement through correlation functions $\gamma_{ij}(t)$.

• Structural properties: We extend notions such as triadic closure, balance, and domination to the Signed Quantum Graph (SQG) framework. These classical social and biological network concepts are redefined under dynamic and signed conditions, and several theoretical properties are proved.

• Applications: We demonstrate how SQGs can model dynamic stability and influence in networks, with a particular focus on brain networks as a case study. This offers an approach for interpreting oscillatory and entangled connectivity patterns in neuroscience.

1.1 *Framework of this article*

The structure of the paper is given below:

Section 1 In the introduction section, we provided an overview of the importance of the quantum theory and quantum graphs/networks, their significance, and a literature review.

Section 2 In this section, SQGs have been defined with examples.

Section 3 Dynamic triadic closure and balanced networks have been discussed in this section. In addition, the concepts of balanced networks and the strength of networks have been discussed. Several properties of the defined concepts have been proved.

Section 4 In Section 4, domination in SQGs have been analysed with its variations. Some important properties have been proved.

Section 5 In this section, the area of application on brain networks have been mentioned.

Section 6 Finally, conclusions with a summary of findings and future research directions have been drawn.

2. Signed Quantum Graphs

Sign (positive or negative) is important in the representations of relations among nodes. Hence, the concept of SQGs is important in representing the signs of dynamical edges. But the measurement of frequent changes of sign in such dynamical graph edges is itself challenging.

A signed quantum graph is a quantum graph where each edge has a positive or negative sign that is dependent and changeable during time intervals. The formal definition of SQGs is given as follows.

Superposition in SQG: Each vertex is not in a definite state but in a superposed probabilistic state described by its dynamic weight function $\sigma_i(t)$.

Definition 1 A quantum graph $G = (V, E)$ where $\mathscr{V}$ is the vertex set and $\mathscr{E}$ is the edge set, is termed a signed quantum graph if it satisfies the following conditions:

1. Each edge $\varepsilon_{ij} \in \mathscr{E}$ connecting vertices v_i and v_j possesses a time-varying sign $\sigma_{ij}(t)$. This sign alternates between $+1$ (positive) and -1 (negative):

$$\sigma_{ij}(t) \in \{+1, -1\}.$$

The sign changes periodically, for instance:

$$\sigma_{ij}(t) = \begin{cases} +1 & \text{if } \cos(\omega_{ij}t + \varphi_{ij}) \geq 0 \\ -1 & \text{if } \cos(\omega_{ij}t + \varphi_{ij}) < 0, \end{cases}$$

where ω_{ij} denotes the frequency and φ_{ij} represents the phase shift of the sign oscillation.

2. The dynamic weight $v_i(t)$ of each vertex $v_i \in \mathscr{V}$ follows a specific pattern. For example:

$$v_i(t) = 12\left(1 + \cos(\omega_i t + \varphi_i)\right).$$

3. The adjacency matrix $\mathscr{A}(t)$ of the signed quantum graph is given by:

$$\mathscr{A}_{ij}(t) = \sigma_{ij}(t) \cdot \gamma_{ij}(t),$$

where $\gamma_{ij}(t)$ is the correlation function.

Remark 1 The choice of binary signs $(+1, -1)$ is motivated by the need to distinguish sharply between cooperative and competitive interactions. While real-valued signed weights are possible, we restrict to binary signs in this foundational framework for interpretability and simplicity. Future extensions may explore continuous sign functions.

Remark 2 The choice of the correlation function $\gamma_{ij}(t)$ is not restricted to a single form. In its simplest version, one may take

$$\gamma_{ij}(t) = v_i(t)v_j(t),$$

which reflects direct multiplicative interaction between the dynamic weights of two vertices. However, more general formulations are possible depending on the system under study. For instance:

• Additive or affine kernels: $\gamma_{ij}(t) = \alpha v_i(t) + \beta v_j(t)$, useful when one node exerts a stronger influence than the other.

• Normalized correlation: $\gamma_{ij}(t) = \dfrac{v_i(t)v_j(t)}{\|v_i(t)\|\|v_j(t)\|}$, analogous to cosine similarity, ensuring values remain in $[-1,\ 1]$.

• Nonlinear kernels: Functions such as $\gamma_{ij}(t) = \exp(-|v_i(t) - v_j(t)|)$ or logistic forms can capture saturation effects or threshold dynamics.

Thus, the framework of SQGs is flexible: the definition accommodates both linear and nonlinear correlation measures, allowing empirical calibration to specific domains such as social influence, neuronal activation, or portfolio performance. This generalization ensures that the signed quantum graph model is not tied to a single assumption about inter-vertex dependencies but instead can be tailored to diverse applications.

Example 1 Let us assume a financial network of three investors as follows. v_1 (Sovan), v_2 (Tofigh), and v_3 (Leo) have the dynamic weights representing their investment portfolios' performance, which may change over time based on market conditions, and positive (collaborative) or negative (competitive) signs are given to their financial interactions.

Here,

$$v_1(t) = \frac{1}{2}(1 + \sin(t)) \quad \text{(Sovan's portfolio performance)},$$

$$v_2(t) = \frac{1}{2}(1 + \sin(2t)) \quad \text{(Tofigh's portfolio performance)},$$

$$v_3(t) = \frac{1}{2}(1 + \sin(3t)) \quad \text{(Leo's portfolio performance)}.$$

The signs of their financial interactions may change over time based on their current investment strategies as follows:

$$\sigma_{12}(t) = \begin{cases} +1 & \text{if } \sin(t) \geq 0 \quad \text{(Sovan and Tofigh are collaborating each other.)} \\ -1 & \text{if } \sin(t) < 0 \quad \text{(Sovan and Tofigh are competing each other.)} \end{cases}$$

$$\sigma_{23}(t) = \begin{cases} +1 & \text{if } \sin(2t) \geq 0 \quad \text{(Tofigh and Leo are collaborating each other.)} \\ -1 & \text{if } \sin(2t) < 0 \quad \text{(Tofigh and Leo are competing each other.)} \end{cases}$$

$$\sigma_{31}(t) = \begin{cases} +1 & \text{if } \sin(3t) \geq 0 \quad \text{(Leo and Sovan are collaborating each other.)} \\ -1 & \text{if } \sin(3t) < 0 \quad \text{(Leo and Sovan are competing each other.)} \end{cases}$$

The first subplot of Figure 1 shows the dynamic weights of three vertices, Sovan, Tofigh, and Leo, by $v_1(t)$, $v_2(t)$, and $v_3(t)$ respectively. The second subplot depicts the time-dependent signs of the edges connecting these vertices, $\sigma_{12}(t)$, $\sigma_{23}(t)$, and $\sigma_{31}(t)$, indicating whether the interactions are collaborative (+1) or competitive (-1) at any given time. The final subplot presents the elements of the adjacency matrix, $\mathscr{A}_{12}(t)$, $\mathscr{A}_{23}(t)$, and $\mathscr{A}_{31}(t)$, which combine the dynamic weights and time-dependent signs to describe the overall state of the financial network.

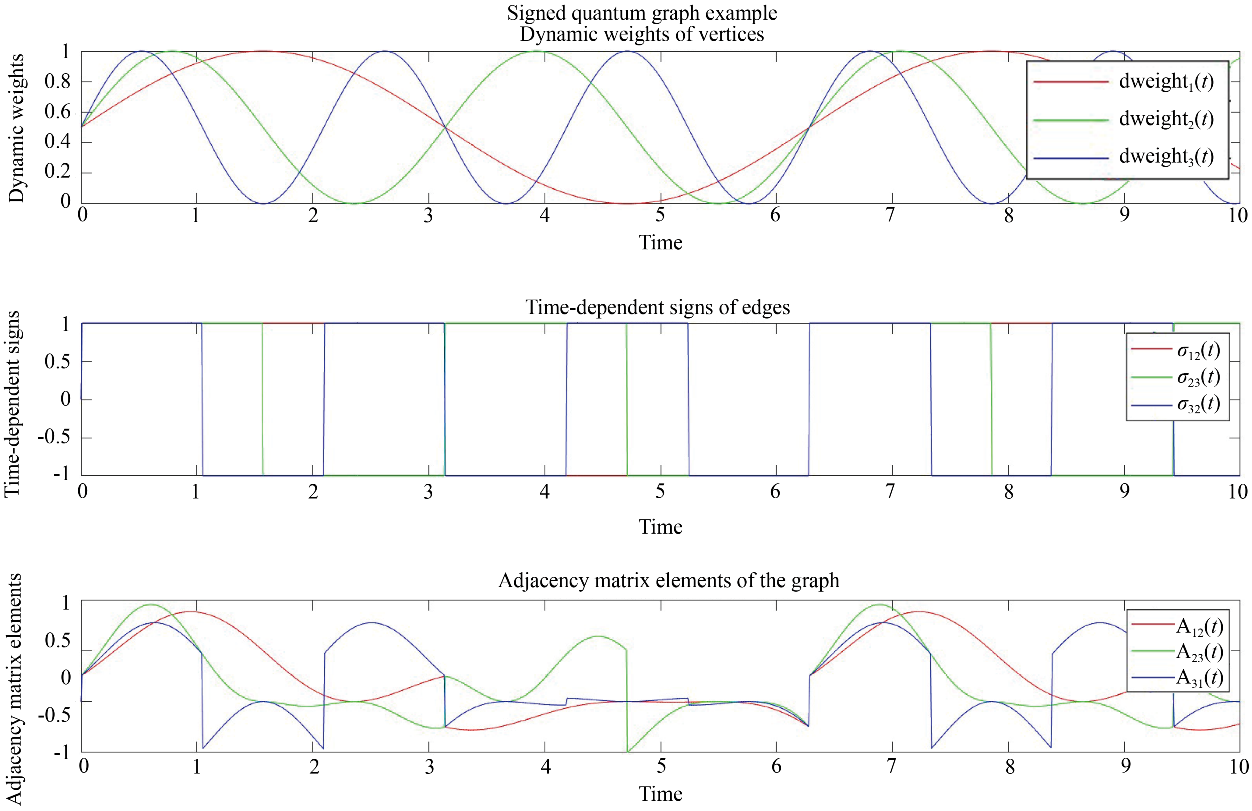

Figure 1. Example of signed quantum graphs

Remark 3 In this framework, the function $\eta(\cdot)$ specifies how vertex weights are correlated beyond simple classical interaction. The simplest case is the multiplicative kernel

$$\eta(v_i, v_j) = v_i \times v_j,$$

which directly couples the dynamic weights of two vertices. However, more general forms may be considered:

• Additive influence: $\eta(v_i, v_j) = \alpha v_i + \beta v_j$, capturing asymmetric correlations.

• Normalized similarity: $\eta(v_i, v_j) = \dfrac{v_i v_j}{\|v_i\| \|v_j\|}$, analogous to cosine similarity, ensuring scale invariance.

• Nonlinear kernels: e.g. $\eta(W_i, W_j) = \exp(-|v_i - v_j|)$, reflecting saturation or threshold effects.

Thus, $\eta(\cdot)$ generalizes the notion of entanglement by allowing flexible correlation structures between vertices.

Dynamic Triadic Closure in SQGs involves the formation or dissolution of triads. The following section describes a dynamic triadic closure.

3. Dynamic triadic closure

Dynamic triadic closure in SQGs refers to the principle that if two vertices v_i and v_j are both connected to a common vertex v_k with entangled edges, there is a tendency for v_i and v_j to also become directly connected for more than $\lambda\%$ (in particular 50%) of the time over a given period. This connection is influenced by the signs of the edges and the domination of vertex weights. The threshold λ (e.g., 50%) is user-defined and can be adjusted based on statistical or empirical calibration. In practice, it may be set based on data-driven analysis, e.g., maximizing temporal stability or minimizing imbalance in labeled datasets. Adaptive thresholding is a potential extension for future work. In any condition, it may be customized for the purpose of that condition.

For vertices v_i, v_j, and v_k, the network demonstrates dynamic triadic closure if:

$$\int_0^T \chi_{ij}(t)\,dt > \frac{T}{2}, \tag{4}$$

where $\chi_{ij}(t)$ is an indicator function defined as:

$$\chi_{ij}(t) = \begin{cases} 1 & \text{if } \sigma_{ik}(t)\cdot\sigma_{jk}(t) > 0 \text{ and } \nu_i(t)+\nu_j(t) > \nu_k(t) \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

This means that the integral of the indicator function over the time interval T is greater than half of T, indicating that the direct connection exists for more than 50% of the time.

Any network tends to form direct connections between nodes with a common vertex as a neighbour, influenced by the signs of the edges and the domination of vertex weights.

Note 1 The choice of 50% as the threshold for defining a triadic closure in SQGs is based on the principle of simple majority. It provides a clear and strong criterion for determining balance. But, there is flexibility to adjust this threshold (λ), for customization based on specific needs in different scenarios.

Example 2 Let us consider a professional networking scenario where three individuals, Ananya (v_i), Bharat (v_j), and Charu (v_k) are part of the same industry. Charu is a well-known industry leader who collaborates with both Ananya and Bharat is on various projects. Over time, due to their mutual connection with Charu, Ananya and Bharat start collaborating directly on more than 50% of their projects. The nature of their collaboration (positive or negative) and the influence of their professional reputations (dynamic weights) determine the strength and frequency of their direct connection.

3.1 *Sensitivity analysis of the triadic threshold λ*

Our dynamic triadic closure and balance conditions use a time-fraction criterion:

$$\frac{1}{T}\int_0^T \chi_{ij}(t)\,dt > \lambda \text{ and } \frac{1}{T}\int_0^T \sigma_{ij}(t)\sigma_{jk}(t)\sigma_{ki}(t)\,dt > \lambda,$$

so that a link/triangle is considered closed/balanced if it is positive for more than a fraction λ of the observation window. Earlier, we instantiated $\lambda = 0.5$ as a simple-majority rule for interpretability and stability, and explicitly noted that λ is user-tunable to context.See the use of "more than $\lambda\%$ (in particular 50%)" and the majority-based integral conditions, together with the note that λ can be adjusted; this was clarified in the revised text. This generalizes the fixed $\frac{T}{2}$ inequalities given in our definitions. Majority interpretation and $\frac{T}{2}$ bounds appear in the paper's balance/closure definitions.

Monotonicity. Let $p_{ij} = \frac{1}{T}\int_0^T \chi_{ij}(t)\,dt \in [0,\ 1]$ be the empirical positive-time fraction for the pair $(i,\ j)$ and let $b_{ijk} = \frac{1}{T}\int_0^T \sigma_{ij}(t)\sigma_{jk}(t)\sigma_{ki}(t)\,dt \in [-1,\ 1]$ for a triad $(i,\ j,\ k)$. Define the triadic-closure decision by $\mathbb{I}\{p_{ij} > \lambda\}$ and the triangle-balance decision by $\mathbb{I}\{b_{ijk} > \lambda\}$. Then, for any fixed data, the number of declared closures/balanced triangles is a non-increasing function of λ. Thus, varying λ traces a threshold curve from permissive (small λ) to stringent (large λ).

Data-based selection of λ. When labeled outcomes exist (e.g., externally verified closures/balances, or application-ground truth), we may select λ by: (i) maximizing predictive validity, (ii) maximizing temporal stability. This complements the simple-majority instance $\lambda = 0.5$ and provides empirical grounding for the chosen threshold.

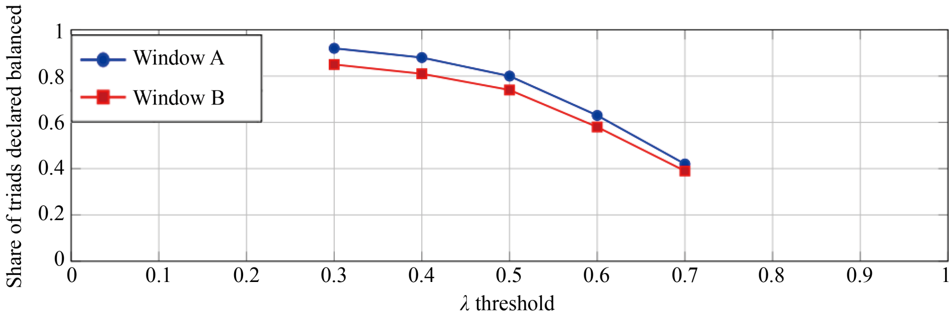

Figure 2. Sensitivity of time-averaged balance rate to the threshold parameter λ

Figure 2 shows that as increases, fewer triads are declared balanced, leading to a monotonic decrease in the balance rate. This demonstrates that the model's outcomes are not fixed by an ad hoc choice of $\lambda = 0.5$ but respond smoothly to parameter variation.

3.2 *Balanced network*

A Balanced network in the context of SQGs is defined as a network where the product of the signs of the edges in any cycle is positive for more than λ% (in particular 50%) of the time over a given period. Mathematically, this can be expressed using integration over a time interval T.

For any three vertices v_i, v_j, and v_k forming a triangle, the network is balanced if:

$$\int_0^T \sigma_{ij}(t)\cdot\sigma_{jk}(t)\cdot\sigma_{ki}(t)\,dt > \frac{T}{2}. \tag{6}$$

This means that the integral of the product of the signs over the time interval T is greater than half of T, indicating that the product is positive for more than 50% of the time. This definition ensures that the network maintains a stable and harmonious structure for the majority of the time, even as the signs of the edges change periodically.

Example 3 Imagine a corporate alliance between three companies: Tata Industries (v_i), Reliance Enterprises (v_j), and Infosys Technologies (v_k). These companies form a strategic partnership where the nature of their relationships (collaborative or competitive) changes over time. For the network to be balanced, the product of the signs of their relationships must be positive for more than 50% of the time. This means that if Tata Industries and Reliance Enterprises have a positive relationship, and Reliance Enterprises and Infosys Technologies also have a positive relationship, then Tata Industries and Infosys Technologies should also have a positive relationship for the majority of the time to maintain a balanced network.

3.3 *Temporal balance in social networks*

Temporal balance in social networks directs the tendency of a signed quantum graph to develop such that the product of the signs of the edges in all cycles is positive over time. For any cycle, involving vertices i, j, and k, this can be expressed as:

$$\sigma_{ij}(t) \cdot \sigma_{jk}(t) \cdot \sigma_{ki}(t) > 0. \tag{7}$$

Next, we discuss two specific scenarios that change the temporal balance: Negative Closure and Mixed Closure. These concepts help in understanding different configurations of edge signs and vertex weights on the stability and structure of the network over time.

Negative closure occurs when all edges in a triad have negative signs (see Figure 3), but the sum of the dynamic weights of two vertices is greater than or equal to the dynamic weight of the common vertex. This implies that the triad might maintain a balanced state for more than 50% of the time.

$$\text{If } \sigma_{ik}(t) < 0,\ \sigma_{jk}(t) < 0 \text{ and } \nu_i(t) + \nu_j(t) \geq \nu_k(t), \text{ then } \int_0^T \chi_{ij}(t)\, dt > \frac{T}{2}. \tag{8}$$

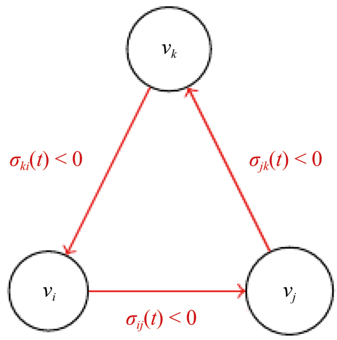

Figure 3. Negative closure (signs of edges indicating negative)

Let $G = (V, E)$ be a signed quantum graph with dynamic weights $\nu_i(t)$ for each vertex $v_i \in V$ and time-dependent signs $\sigma_{ij}(t)$ for each edge $e_{ij} \in E$. Let us consider a triad (v_i, v_j, v_k) in G. If all edges in the triad have negative signs and the dynamic weights of the vertices satisfy certain threshold conditions, then the triad will maintain dynamic triadic closure for more than 50% of the time over a given period T. Formally, if

$$\sigma_{ik}(t) < 0,\ \sigma_{jk}(t) < 0,\ \sigma_{ij}(t) < 0 \text{ and } \nu_i(t) + \nu_j(t) \geq \nu_k(t), \tag{9}$$

then

$$\int_0^T \chi_{ij}(t)\, dt > \frac{T}{2}, \tag{10}$$

where $\chi_{ij}(t)$ is an indicator function defined as:

$$\chi_{ij}(t) = \begin{cases} 1 & \text{if } \sigma_{ik}(t) \cdot \sigma_{jk}(t) > 0 \text{ and } \nu_i(t) + \nu_j(t) \geq \nu_k(t) \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Mixed closure occurs when a triad consists of edges with mixed signs (Figure 4), meaning that the product of the signs of two edges is negative. This indicates a combination of positive and negative interactions within the triad, leading to a dynamic and potentially unstable configuration.

$$\text{If } \sigma_{ik}(t) \cdot \sigma_{jk}(t) < 0, \text{ then the closure condition is influenced by } \nu_i(t),\ \nu_j(t),\ \text{ and } \nu_k(t). \tag{12}$$

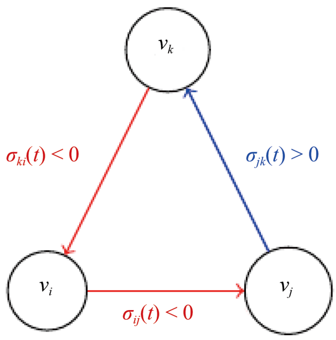

Figure 4. Mixed closure

Let $G = (V,\ E)$ be a signed quantum graph with dynamic weights $\nu_i(t)$ for each vertex $v_i \in V$ and time-dependent signs $\sigma_{ij}(t)$ for each edge $e_{ij} \in E$. Consider a triad $(v_i,\ v_j,\ v_k)$ in G. If the product of the signs of the edges $\sigma_{ik}(t)$ and $\sigma_{jk}(t)$ is negative, then the closure condition is influenced by the dynamic weights of the vertices. Formally, if

$$\sigma_{ik}(t) \cdot \sigma_{jk}(t) < 0, \tag{13}$$

then the stability of the triad depends on the relative magnitudes of $\nu_i(t)$, $\nu_j(t)$, and $\nu_k(t)$.

3.4 *Algorithm for finding a balanced graph*

Algorithm 1 provides a systematic way to test whether a signed quantum graph remains balanced throughout an observation time T. By scanning cycles and accumulating the proportion of time intervals during which the sign product is positive, the algorithm converts dynamic temporal behavior into a clear Boolean outcome. The objective of this algorithm is to identify if a signed quantum graph is balanced, where the product of the signs of the edges in any cycle is positive for more than 50% of the time.

1. Input: Signed quantum graph $\mathscr{G} = (\mathscr{V},\ \mathscr{E})$, time period T.
2. Output: Boolean indicating if the graph is balanced.

Algorithm 1 Cycle Check in Signed Quantum Graph

```
for each cycle C in G do
    positive_product_time ← 0;
    for t ← 0 to T do
```

```
        product ← 1;
        for each edge (i, j) in C do
            product ← product × σ_ij(t);
        end
        if product > 0 then
            positive_product_time ← positive_product_time + 1;
        end
    end
    if positive_product_time ≤ T/2 then
        return False;
    end
end
return True.
```

Complexity and implementation. While the pseudocode presents a stepwise iteration for clarity, the computational cost of updating all vertex states is $O(m)$ per time step for a graph with m edges, yielding a total complexity $O(mT)$ over T time steps. For sparse graphs ($m = O(n)$), this remains efficient, whereas dense graphs ($m = O(n^2)$) demand more care. In practice, adjacency lists or sparse matrices reduce storage, and vectorized linear-algebra operations enable GPU acceleration. For temporal graphs with localized changes, incremental updates offer further savings. These considerations make the proposed framework implementable at scale beyond the illustrative pseudocode.

3.5 *SQG*

The strength of an SQG $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is defined as the average of the absolute values of the weights of the edges over a given time interval T.

The strength $\mathscr{S}(\mathscr{G})$ of the graph over the time period T is given by:

$$\mathscr{S}(\mathscr{G}) = \frac{1}{|\mathscr{E}|} \int_0^T \sum_{(v_i,\ v_j)\in\mathscr{E}} |\mathscr{A}_{ij}(t)|\, dt, \tag{14}$$

where $|\mathscr{E}|$ is the number of edges in the graph.

Example 4 Let us consider an SQG $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ where the node set are Indian companies: Tata Motors (v_1), Infosys (v_2), and Reliance Industries (v_3). The edges represent their financial relationships, which can be either collaborative or competitive over time. The dynamic weights reflect the performance of each company's stock, and the signs indicate the nature of their financial interactions.

The dynamic weights and time-dependent signs are defined as follows:

$$v_1(t) = \frac{1}{2}(1+\cos(t)) \quad \text{(Tata Motors' stock performance)}, \tag{15}$$

$$v_2(t) = \frac{1}{2}(1+\cos(2t)) \quad \text{(Infosys' stock performance)}, \tag{16}$$

$$v_3(t) = \frac{1}{2}(1+\cos(3t)) \quad \text{(Reliance Industries' stock performance)}. \tag{17}$$

The signs of their financial interactions may change over time based on market conditions:

$$\sigma_{12}(t) = \begin{cases} +1 & \text{if } \cos(t) \geq 0 \quad \text{(Tata Motors and Infosys are collaborating with each other.)} \\ -1 & \text{if } \cos(t) < 0 \quad \text{(Tata Motors and Infosys are competing with each other.)} \end{cases} \tag{18}$$

$$\sigma_{23}(t) = \begin{cases} +1 & \text{if } \cos(2t) \geq 0 \quad \text{(Infosys and Reliance Industries are collaborating with each other.)} \\ -1 & \text{if } \cos(2t) < 0 \quad \text{(Infosys and Reliance Industries are competing with each other.)} \end{cases} \tag{19}$$

$$\sigma_{31}(t) = \begin{cases} +1 & \text{if } \cos(3t) \geq 0 \quad \text{(Reliance Industries and Tata Motors are collaborating with each other.)} \\ -1 & \text{if } \cos(3t) < 0 \quad \text{(Reliance Industries and Tata Motors are competing with each other.)} \end{cases} \tag{20}$$

The weights of the edges are given by:

$$\mathscr{A}_{12}(t) = \sigma_{12}(t) \cdot \nu_1(t) \cdot \nu_2(t), \tag{21}$$

$$\mathscr{A}_{23}(t) = \sigma_{23}(t) \cdot \nu_2(t) \cdot \nu_3(t), \tag{22}$$

$$\mathscr{A}_{31}(t) = \sigma_{31}(t) \cdot \nu_3(t) \cdot \nu_1(t). \tag{23}$$

The strength $\mathscr{S}(\mathscr{G})$ of the graph over the time period $T = [0,\ 2\pi]$ is given by:

$$\mathscr{S}(\mathscr{G}) = \frac{1}{|\mathscr{E}|} \int_0^{2\pi} (|\mathscr{A}_{12}(t)| + |\mathscr{A}_{23}(t)| + |\mathscr{A}_{31}(t)|)\, dt, \tag{24}$$

where $|\mathscr{E}|$ is the number of edges in the graph.

This example shows that the companies can be modeled using SQGs,with their dynamic relationships over time.

3.6 *Calculation of the strength of a SQG*

1. Input: SQG $\mathscr{G} = (\mathscr{V},\ \mathscr{E})$, time period T.
2. Output: Strength $S(\mathscr{G})$.

Algorithm 2 Calculate strength of a SQG

Input: Graph $\mathscr{G} = (\mathscr{V},\ \mathscr{E})$, Time period T

Output: Strength $S(\mathscr{G})$

total_weight $\leftarrow 0$;
for $t \leftarrow 0$ **to** T **do**
    for each edge $(i,\ j)$ in $\mathscr{E}$ **do**
        total_weight $\leftarrow$ total_weight $+ |\sigma_{ij}(t) \times \gamma_{ij}(t)|$;
    end
end
$S(\mathscr{G}) \leftarrow \dfrac{\text{total\_weight}}{|\mathscr{E}| \times T}$;
return $S(\mathscr{G})$.

3.7 *Upper and lower bounds of strengths of a SQG*

Let us consider an SQG $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ where $\mathscr{V}$ is the set of nodes and $\mathscr{E}$ is the set of edges. Each vertex $v_i \in \mathscr{V}$ has a dynamic weight $\mathscr{W}_i(t)$ that changes over time, given by:

$$0 \leq v_i(t) \leq 1, \tag{25}$$

with the specific pattern:

$$v_i(t) = \frac{1}{2}\left(1 + \cos(\omega_i t + \varphi_i)\right), \tag{26}$$

where ω_i is the frequency and φ_i is the phase shift.

Each edge $\varepsilon_{ij} \in \mathscr{E}$ connecting vertices v_i and v_j possesses a time-varying sign $\sigma_{ij}(t)$ that alternates between $+1$ and -1:

$$\sigma_{ij}(t) = \begin{cases} +1 & \text{if } \cos(\omega_{ij} t + \varphi_{ij}) \geq 0 \\ -1 & \text{if } \cos(\omega_{ij} t + \varphi_{ij}) < 0. \end{cases} \tag{27}$$

where ω_{ij} denotes the frequency and φ_{ij} represents the phase shift of the sign oscillation.

The strength $\mathscr{S}_i(t)$ of a vertex v_i at time t is defined as the average of the absolute values of the weights of the edges incident to v_i:

$$\mathscr{S}_i(t) = \frac{1}{|\mathscr{N}(i)|} \sum_{j \in \mathscr{N}(i)} |\sigma_{ij}(t) \cdot v_j(t)|, \tag{28}$$

where $\mathscr{N}(i)$ is the set of neighbors of v_i.

3.7.1 *Upper bound*

The upper bound of the strength $\mathscr{S}_i(t)$ of a vertex v_i is obtained when all neighboring vertices v_j have their maximum dynamic weights:

$$\max(v_j(t)) = 1. \tag{29}$$

Thus, the upper bound of $\mathscr{S}_i(t)$ is:

$$\mathscr{S}_i^{\max} = \frac{1}{|\mathscr{N}(i)|} \sum_{j \in \mathscr{N}(i)} |\sigma_{ij}(t) \cdot 1| = \frac{1}{|\mathscr{N}(i)|} \sum_{j \in \mathscr{N}(i)} 1 = 1. \tag{30}$$

3.7.2 *Lower bound*

The lower bound of the strength $\mathscr{S}_i(t)$ of a vertex v_i is obtained when all neighboring vertices v_j have their minimum dynamic weights:

$$\min(v_j(t)) = 0. \tag{31}$$

Thus, the lower bound of $\mathscr{S}_i(t)$ is:

$$\mathscr{S}_i^{\min} = \frac{1}{|\mathscr{N}(i)|} \sum_{j \in \mathscr{N}(i)} |\sigma_{ij}(t) \cdot 0| = \frac{1}{|\mathscr{N}(i)|} \sum_{j \in \mathscr{N}(i)} 0 = 0. \tag{32}$$

Therefore, the strength $\mathscr{S}_i(t)$ of a vertex v_i at any time t is bounded by:

$$0 \leq \mathscr{S}_i(t) \leq 1. \tag{33}$$

Note that the bounds of the strength of a signed graph are similarly calculated.

3.8 *Properties on dynamic triadic closure*

Lemma 1 Dynamic triadic closure with weighted influence. Let $G = (V, E)$ be a signed quantum graph with dynamic weights $v_i(t)$ for each vertex $v_i \in V$ and time-dependent signs $\sigma_{ij}(t)$ for each edge $e_{ij} \in E$. Consider a triad (v_i, v_j, v_k) in G. If all edges in the triad have positive signs and the sum of the dynamic weights of any two vertices, weighted by their influence factors, is greater than the dynamic weight of the remaining vertex, then the triad will exhibit dynamic triadic closure for more than 50% of the time over a given period T. Formally, if

$$\sigma_{ik}(t) > 0, \sigma_{jk}(t) > 0, \sigma_{ij}(t) > 0 \text{ and } \alpha \cdot v_i(t) + \beta \cdot v_j(t) > \gamma \cdot v_k(t), \tag{34}$$

where $\alpha,\ \beta,\ \gamma$ are positive influence factors such that $\alpha + \beta > \gamma$, then

$$\int_0^T \chi_{ij}(t)\, dt > \frac{T}{2}, \tag{35}$$

where $\chi_{ij}(t)$ is an indicator function defined as:

$$\chi_{ij}(t) = \begin{cases} 1 & \text{if } \sigma_{ik}(t) \cdot \sigma_{jk}(t) > 0 \text{ and } \alpha \cdot v_i(t) + \beta \cdot v_j(t) > \gamma \cdot v_k(t) \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

Proof. To prove this theorem, we need to show that the integral of the indicator function $\chi_{ij}(t)$ over the time interval T is greater than $\frac{T}{2}$ under the given conditions.

Since $\sigma_{ik}(t) > 0$, $\sigma_{jk}(t) > 0$, and $\sigma_{ij}(t) > 0$, the product $\sigma_{ik}(t) \cdot \sigma_{jk}(t)$ is positive. Additionally, the product $\sigma_{ik}(t) \cdot \sigma_{jk}(t) \cdot \sigma_{ij}(t)$ is also positive because all three signs are positive.

Given that $\alpha \cdot \nu_i(t) + \beta \cdot \nu_j(t) > \gamma \cdot \nu_k(t)$, the weighted sum of the dynamic weights of vertices v_i and v_j is greater than the dynamic weight of vertex v_k.

The indicator function $\chi_{ij}(t)$ is defined to be 1 if $\sigma_{ik}(t) \cdot \sigma_{jk}(t) > 0$ and $\alpha \cdot \nu_i(t) + \beta \cdot \nu_j(t) > \gamma \cdot \nu_k(t)$. Under the given conditions, $\chi_{ij}(t) = 1$ for more than half of the time interval T.

Since $\chi_{ij}(t) = 1$ for more than half of the time interval T, we have:

$$\int_0^T \chi_{ij}(t)\, dt > \frac{T}{2}.$$

Therefore,

$$\int_0^T \chi_{ij}(t)\, dt > \frac{T}{2}.$$

This completes the proof that the triad will exhibit dynamic triadic closure for more than 50% of the time under the given conditions.

Lemma 2 Let $v_i,\ v_j,\ v_k$ form a triad with $\sigma_{ik}(t) < 0$ and $\sigma_{jk}(t) < 0$ for all t. Assume $\nu_\ell(t) = a_\ell + b_\ell \cos(\omega t + \phi_\ell)$, $0 \le \nu_\ell(t) \le 1$, and define

$$\mathscr{N}_{ij} := \{\, t \in [0,\ T] : \nu_i(t) + \nu_j(t) < \nu_k(t) \,\}, \quad T = \frac{2\pi}{\omega}.$$

If the phases are aligned ($\phi_i = \phi_j = \phi_k$) and $(a_i + a_j - a_k)\,(b_i + b_j - b_k) < 0$, then

$$\frac{\mu(\mathscr{N}_{ij})}{T} \; > \; \frac{1}{2},$$

i.e., the $(i,\ j)$ link fails to close dynamically for strictly more than half of the observation period. Here μ denotes Lebesgue measure on $[0,\ T]$.

Proof. With aligned phases $\phi_i = \phi_j = \phi_k$, we have

$$\nu_i(t) + \nu_j(t) - \nu_k(t) \; = \; C_0 + C_1 \cos(\omega t), \quad C_0 := a_i + a_j - a_k,\ C_1 := b_i + b_j - b_k.$$

The set of non-closure times is

$$\mathscr{N}_{ij} = \{\, t : C_0 + C_1 \cos(\omega t) < 0 \,\}.$$

If $|c| < 1$ with $c = -\dfrac{C_0}{C_1}$, then over one period $\dfrac{2\pi}{\omega}$,

$$\frac{\mu(\mathscr{N}_{ij})}{T} = \frac{1}{2\pi}\, \mu\{\theta \in [0,\ 2\pi] : \cos\theta < c\} = 1 - \frac{1}{\pi} \arccos(c).$$

This ratio exceeds $\frac{1}{2}$ precisely when $c > 0$, i.e. when $C_0C_1 < 0$. Thus, under the stated condition, the non-closure fraction is strictly greater than $\frac{1}{2}$, and the triad fails to close for most of the observation window.

Lemma 3 If the edges in a triad have mixed signs, the dynamic triadic closure depends on the relative magnitudes of the dynamic weights and the signs of the edges.

Proof. Given vertices v_i, v_j, and v_k with mixed signs $\sigma_{ik}(t)$ and $\sigma_{jk}(t)$, we need to analyze the condition:

$$\int_0^T \chi_{ij}(t)\, dt > \frac{T}{2}.$$

If $\sigma_{ik}(t)\cdot\sigma_{jk}(t) < 0$, the indicator function $\chi_{ij}(t) = 0$ regardless of the dynamic weights. However, if $\sigma_{ik}(t)\cdot\sigma_{jk}(t) > 0$ and $v_i(t) + v_j(t) > v_k(t)$, then $\chi_{ij}(t) = 1$ for more than half of the time period T. Therefore, the triadic closure condition depends on the relative magnitudes of the dynamic weights and the signs of the edges.

This triadic closure, i.e., link connection, is important in the context of domination in social networks. The following section describes the domination in SQGs.

4. SQGs

In a SQG $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, a node v_i is said to dominate another node v_j if there exists a time t such that the dynamic weight $\mathscr{W}_i(t)$ of v_i exceeds that of v_j and the sign $\sigma_{ij}(t)$ of the edge ε_{ij} is positive.

Let us assume the dynamic weight $\mathscr{W}_i(t)$ of node v_i is given by:

$$v_i(t) = e^{\alpha_i t}\left(1 + \cos(\theta_i t + \psi_i)\right).$$

Similarly, the dynamic weight $v_j(t)$ of node v_j is:

$$v_j(t) = e^{\alpha_j t}\left(1 + \cos(\theta_j t + \psi_j)\right),$$

where α_i and α_j are constants that affect the exponential growth, θ_i and θ_j are the angular frequencies, and ψ_i and ψ_j are the phase shifts of the cosine functions for nodes v_i and v_j, respectively.

The sign $\sigma_{ij}(t)$ of the edge ε_{ij} is defined as:

$$\sigma_{ij}(t) = \begin{cases} +1 & \text{if } \cos(\kappa_{ij} t + \lambda_{ij}) \geq 0 \\ -1 & \text{if } \cos(\kappa_{ij} t + \lambda_{ij}) < 0, \end{cases}$$

where κ_{ij} is the angular frequency and λ_{ij} is the phase shift in the cosine function for the edge ε_{ij}.

Node v_i dominates node v_j if:

$$v_i(t) > v_j(t) \text{ and } \sigma_{ij}(t) = +1.$$

There exists a time t such that:

$$e^{\alpha_i t}\left(1+\cos(\theta_i t+\psi_i)\right) > e^{\alpha_j t}\left(1+\cos(\theta_j t+\psi_j)\right) \text{ and } \cos(\kappa_{ij} t+\lambda_{ij}) \geq 0.$$

If such a time t exists, then v_i dominates v_j.

4.1 *Total domination in SQGs*

In an SQG $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, a subset of nodes $\mathscr{D} \subseteq \mathscr{V}$ is called a total dominating set if every node $v \in \mathscr{V}$ is adjacent to at least one node in $\mathscr{D}$ with a positive sign at some time t.

A set $\mathscr{D} \subseteq \mathscr{V}$ is a total dominating set if for every node $v \in \mathscr{V}$, there exists a node $\mu \in \mathscr{D}$ such that:

$$\sigma_{\mu v}(t) = +1 \text{ for some time } t.$$

The adjacency matrix $\mathscr{B}(t)$ of the graph is defined as:

$$\mathscr{B}_{ij}(t) = \sigma_{ij}(t) \cdot \eta_{ij}(t),$$

where $\eta_{ij}(t)$ is the correlation function.

For every node $v \in \mathscr{V}$, there exists a node $\mu \in \mathscr{D}$ such that:

$$\cos(\kappa_{\mu v} t+\lambda_{\mu v}) \geq 0.$$

If such a set $\mathscr{D}$ exists, then $\mathscr{D}$ is a total dominating set.

4.2 *Domination number in SQGs*

The domination number $\gamma(\mathscr{G})$ of an SQG $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is the minimum number of nodes in a dominating set.

A set $\mathscr{D} \subseteq \mathscr{V}$ is a dominating set if every node $v \in \mathscr{V} \setminus \mathscr{D}$ is adjacent to at least one node in $\mathscr{D}$ with a positive sign at some time t.

For every node $v \in \mathscr{V} \setminus \mathscr{D}$, there exists a node $\mu \in \mathscr{D}$ such that:

$$\sigma_{\mu v}(t) = +1 \text{ for some time } t.$$

The domination number $\gamma(\mathscr{G})$ is the minimum cardinality of such a set $\mathscr{D}$.

There exists a set $\mathscr{D}$ with minimum cardinality such that:

$$|\mathscr{D}| = \gamma(\mathscr{G}).$$

4.3 *Independent domination in SQGs*

In an SQG $\mathscr{G} = (\mathscr{V},\ \mathscr{E})$, an independent dominating set is a dominating set that is also an independent set, meaning no two nodes in the set are adjacent.

A set $\mathscr{I} \subseteq \mathscr{V}$ is independent if no two nodes in $\mathscr{I}$ are adjacent.

A set $\mathscr{D} \subseteq \mathscr{V}$ is a dominating set if for every node $v \in \mathscr{V} \setminus \mathscr{D}$, there exists a node $\mu \in \mathscr{D}$ such that:

$$\sigma_{\mu v}(t) = +1 \text{ for some time } t.$$

A set $\mathscr{I} \subseteq \mathscr{V}$ is an independent dominating set if it is both independent and dominating.

There exists a set $\mathscr{I}$ such that:

$$\mathscr{I} \text{ is independent and } \mathscr{I} \text{ is a dominating set.}$$

Example 5 Let us consider a signed quantum graph $\mathscr{G} = (\mathscr{V},\ \mathscr{E})$ with the following vertices and edges:
• Vertices: $\mathscr{V} = \{v_1,\ v_2,\ v_3,\ v_4\}$
• Edges with dynamic weights and time-dependent signs:

$$v_1(t) = e^{0.1t}\left(1+\sin\left(0.5t\right)\right),$$

$$v_2(t) = e^{0.2t}\left(1+\sin\left(0.4t+\frac{\pi}{4}\right)\right),$$

$$v_3(t) = e^{0.15t}\left(1+\sin\left(0.3t+\frac{\pi}{2}\right)\right),$$

$$v_4(t) = e^{0.1t}\left(1+\sin\left(0.6t+\frac{\pi}{3}\right)\right),$$

$$\sigma_{ij}(t) = \begin{cases} +1 & \text{if } \cos(0.2t+\varepsilon_{ij}) \geq 0 \\ -1 & \text{if } \cos(0.2t+\varepsilon_{ij}) < 0, \end{cases}$$

where ε_{ij} represents the phase shift for each edge.

Let's consider the domination set $\mathscr{D} = \{v_1,\ v_3\}$. For every node $v \in \mathscr{V} \setminus \mathscr{D}$, there exists a node $\mu \in \mathscr{D}$ such that $\sigma_{\mu v}(t) = +1$ for some time t (see Figure 5).

A set $\mathscr{D} = \{v_1,\ v_2,\ v_3\}$ is a total dominating set if every node $v \in \mathscr{V}$ is adjacent to at least one node in $\mathscr{D}$ with a positive sign at some time t.

An independent dominating set $\mathscr{I} = \{v_1,\ v_4\}$ is both independent and dominating. No two nodes in $\mathscr{I}$ are adjacent, and for every node $v \in \mathscr{V} \setminus \mathscr{I}$, there exists a node $\mu \in \mathscr{I}$ such that $\sigma_{\mu v}(t) = +1$ for some time t.

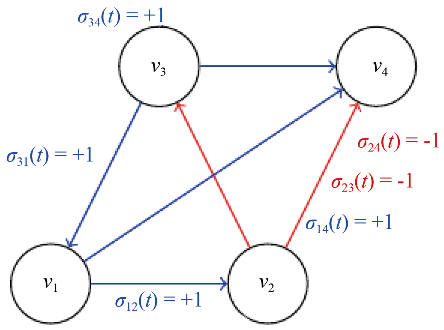

Figure 5. Example of a signed quantum graph with domination sets

Example 6 Figure 6 illustrates the dynamic behavior of a signed quantum graph over time. The first subplot of the first row shows the dynamic weights of three vertices, represented by $v_1(t)$, $v_2(t)$, and $v_3(t)$. These weights oscillate sinusoidally, reflecting changes in the performance of the entities they represent over time. The red, green, and blue lines correspond to the dynamic weights of the three vertices, respectively.

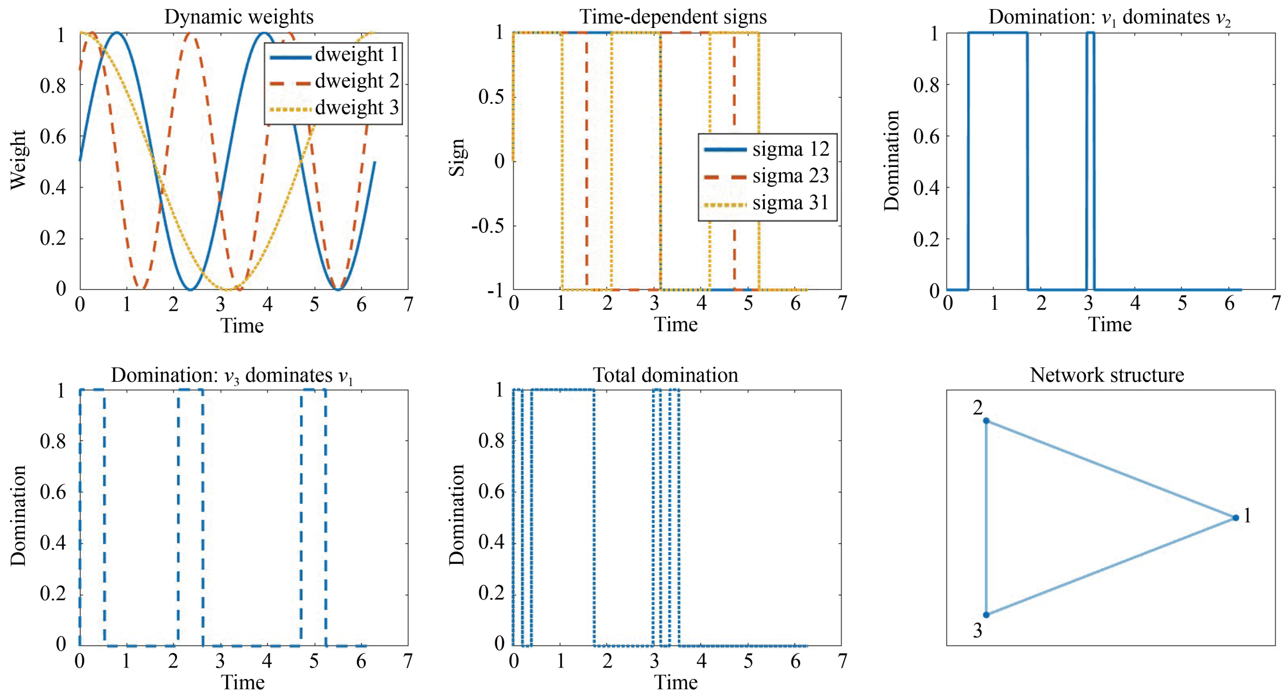

Figure 6. Example of domination in signed quantum graphs

The second subplot depicts the time-dependent signs of the edges connecting the vertices, represented by $\sigma_{12}(t)$, $\sigma_{23}(t)$, and $\sigma_{31}(t)$, indicating (collaborative) or negative (competitive) at any given time with their signs.

The third subplot presents the elements of the adjacency matrix, $\mathscr{A}_{12}(t)$, $\mathscr{A}_{23}(t)$, and $\mathscr{A}_{31}(t)$: red, green, and blue lines correspond to the adjacency matrix elements for the edges between the vertices.

4.4 *Algorithm for searching dominating set*

1. Signed quantum graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$.

2. Dominating set D.

Algorithm 3 Find Dominating Set in a Signed Quantum Graph

Input: Signed quantum graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$

Output: Dominating set D

$D \leftarrow \emptyset$;
$all\_vertices \leftarrow \mathscr{V}$;
while $all\_vertices$ is not empty **do**
    $v \leftarrow$ vertex in $all\_vertices$ with maximum degree;
    $D \leftarrow D \cup \{v\}$;
    $all\_vertices \leftarrow all\_vertices \setminus \{v\}$;
    $all\_vertices \leftarrow all\_vertices \setminus \text{neighbors}(v)$;
end
return D.

4.5 *Upper and lower bounds of DN in SQGs*

The Domination Number (DN) $\gamma(\mathscr{G})$ is the minimum number of nodes in a dominating set. A set $\mathscr{D} \subseteq \mathscr{V}$ is a dominating set if every node $v \in \mathscr{V} \setminus \mathscr{D}$ is adjacent to at least one node in $\mathscr{D}$ with a positive sign at some time t.

If all vertices have maximum dynamic weights and positive correlations, the upper bound is:

$$\gamma(\mathscr{G}) \leq \sum_{i \in \mathscr{V}} \max(\mathscr{W}_i(t)) = |\mathscr{V}|.$$

If at least one vertex has a non-zero dynamic weight and positive correlation, the lower bound is:

$$\gamma(\mathscr{G}) \geq 1.$$

4.5.1 *TDN*

The Total Domination Number (TDN) $\gamma_t(\mathscr{G})$ is the minimum number of nodes in a total dominating set. A set $\mathscr{D} \subseteq \mathscr{V}$ is a total dominating set if every node $v \in \mathscr{V}$ is adjacent to at least one node in $\mathscr{D}$ with a positive sign at some time t.

The upper bound of the total DN, considering maximum dynamic weights and positive correlations, is:

$$\gamma_t(\mathscr{G}) \leq \sum_{i \in \mathscr{V}} \max(v_i(t)) = |\mathscr{V}|.$$

The lower bound of the total DN, considering minimum dynamic weights and positive correlations, is:

$$\gamma_t(\mathscr{G}) \geq 2.$$

4.5.2 *Independent DN*

The independent DN $\gamma_i(\mathscr{G})$ is the minimum number of nodes in an independent dominating set. A set $\mathscr{I} \subseteq \mathscr{V}$ is an independent dominating set if it is both independent (no two nodes in $\mathscr{I}$ are adjacent) and dominating.

The upper bound of the independent DN, considering maximum dynamic weights and positive correlations, is:

$$\gamma_i(\mathscr{G}) \leq \sum_{i \in \mathscr{V}} \max(v_i(t)) = |\mathscr{V}|.$$

The lower bound of the independent DN, considering minimum dynamic weights and positive correlations, is:

$$\gamma_i(\mathscr{G}) \geq 1.$$

Note 2 The strength $\mathscr{S}(\mathscr{G})$ of an SQG $\mathscr{G}$ is bounded below. Let $w_{\min}$ be the minimum absolute value of the edge weights over the period T. The sum of the absolute values of the edge weights is bounded by $|\mathscr{E}| \cdot w_{\min}$. Integrating over the period T, we get:

$$\mathscr{S}(\mathscr{G}) \geq \frac{1}{|\mathscr{E}|} \cdot |\mathscr{E}| \cdot w_{\min} \cdot T = w_{\min} \cdot T.$$

Thus, the lower bound on the strength is $w_{\min} \cdot T$.

Note 3 The DN $\gamma(\mathscr{G})$ of an SQG $\mathscr{G}$ is bounded above by the total number of vertices $|\mathscr{V}|$. A dominating set $\mathscr{D} \subseteq \mathscr{V}$ is a set of vertices such that every vertex not in $\mathscr{D}$ is adjacent to at least one vertex in $\mathscr{D}$. Therefore, the DN $\gamma(\mathscr{G})$ is at most $|\mathscr{V}|$.

Thus, the upper bound on the DN is $|\mathscr{V}|$.

Note 4 The DN $\gamma(\mathscr{G})$ of an SQG $\mathscr{G}$ is bounded below by the minimum degree $\delta(\mathscr{G})$ of the graph. The minimum degree $\delta(\mathscr{G})$ is the smallest number of edges incident to any vertex in the graph. A dominating set must include at least one vertex for every $\delta(\mathscr{G})$ vertices. Therefore, the DN $\gamma(\mathscr{G})$ is at least $\delta(\mathscr{G})$.

Thus, the lower bound on the DN is $\delta(\mathscr{G})$.

Note 5 The strength $\mathscr{S}(\mathscr{G})$ of an SQG $\mathscr{G}$ is bounded by a function of the DN $\gamma(\mathscr{G})$ and the maximum edge weight $w_{\max}$.

Let $\gamma(\mathscr{G})$ be the DN of the graph. Let $w_{\max}$ be the maximum absolute value of the edge weights. The strength is bounded by:

$$\mathscr{S}(\mathscr{G}) \leq \gamma(\mathscr{G}) \cdot w_{\max} \cdot T.$$

Thus, the strength is bounded by $\gamma(\mathscr{G}) \cdot w_{\max} \cdot T$.

5. Area of application of SQGs: modeling neurodegenerative diseases

In this section, we present a hypothetical scenario to illustrate the potential of SQGs in modeling brain networks. However, in future work, we intend to use publicly available brain datasets such as those from the Human Connectome Project or the Alzheimer's Disease Neuroimaging Initiative (ADNI) to validate and calibrate the proposed model.

Neurodegenerative diseases [16] such as Alzheimer's, Parkinson's, and epilepsy are modelled using differential equations [15, 17]. In this paper, we use the representation of SQGs to analyze these disruptions. We investigate how unbalanced networks can be indicative of brain diseases and how the concepts of strength and domination in brain networks can provide deep insights. The brain networks, an unbalanced network may indicate abnormal neural interactions such as Alzheimer's disease, the loss of synaptic connections, and the presence of abnormal protein deposits can lead to unbalanced interactions between brain regions [18]. Using SQGs, we can model the progression of these diseases by observing changes in network balance, strength, and domination over time. SQGs offer a useful way to detect and study issues in brain networks. By using changing weights and signs over time, this method opens up new ways to understand and diagnose neurological disorders. Figure 7 illustrates the dynamic behavior of a signed quantum graph over time, which is used to model neurodegenerative diseases and epilepsy.

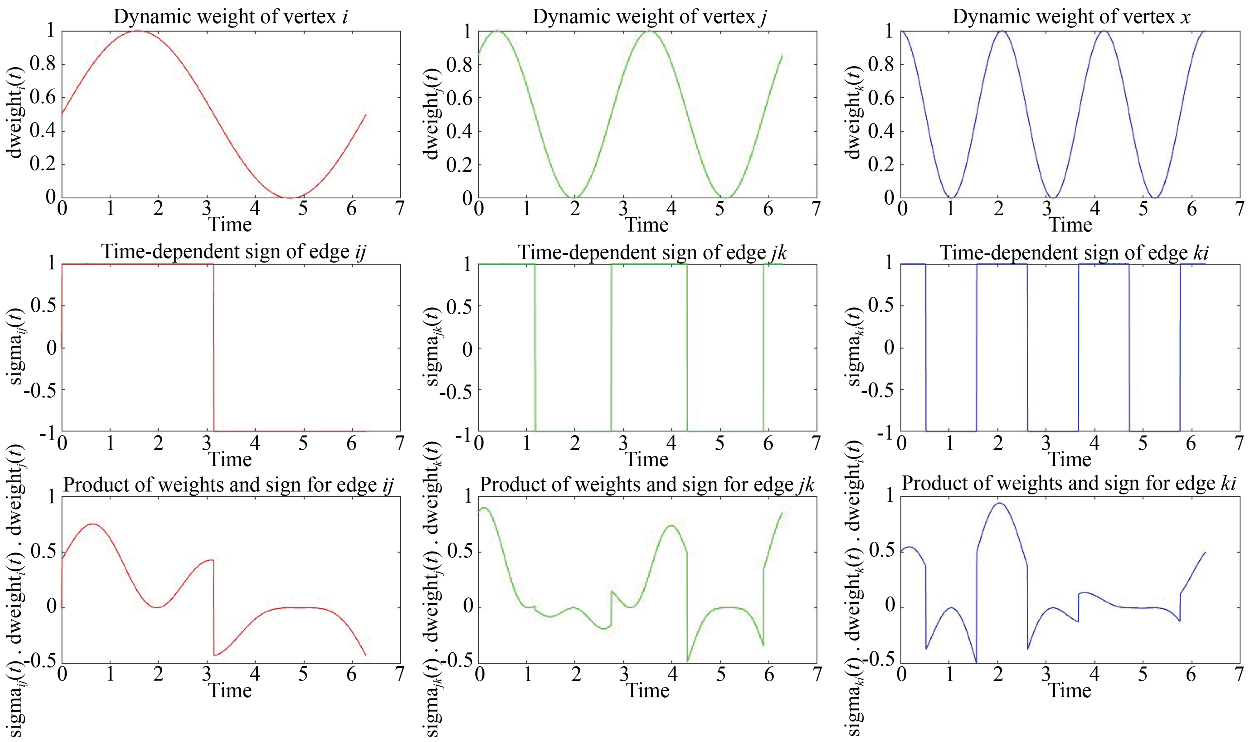

Figure 7. Products of nodes and edges in brain networks (The first three subplots show the time-varying weights of three vertices (i, j, k). These weights follow sinusoidal patterns, reflecting the fluctuating states of neurons over time. The next three subplots display the time-dependent signs of the edges connecting the vertices. These signs change periodically, indicating the dynamic nature of the connections between neurons. The final three subplots depict the product of the dynamic weights and the time-dependent signs for each edge. These products help in understanding the strength and stability of the connections in the brain network)

As the disease progresses, the network may become increasingly unbalanced. This can be quantified by the integral of the product of the signs of the edges in cycles:

$$\int_0^T \sigma_{ij}(t) \cdot \sigma_{jk}(t) \cdot \sigma_{ki}(t)\, dt \leq \frac{T}{2}.$$

Monitoring this integral over time can provide insights into the progression of the disease.

The strength of the brain network is expected to decrease as synaptic connections are lost:

$$S(\mathscr{G}_{\text{diseased}}(t)) = \frac{1}{|\mathscr{E}|} \int_0^t \sum_{(i,\ j)\in\mathscr{E}} |w_{ij}(\tau)|\, d\tau.$$

By comparing the strength at different time points, we can quantify the rate of degeneration.

The domination number may change as key regions of the brain are affected by the disease. For example, in Alzheimer's, the hippocampus is one of the first regions to be affected, which may alter the domination number:

$$\gamma(\mathscr{G}_{\text{diseased}}(t)) \leq \gamma(\mathscr{G}_{\text{healthy}}).$$

Tracking changes in the domination number can help identify critical regions affected by the disease.

Epilepsy is characterized by recurrent seizures, which are often localized to specific regions of the brain. Using signed quantum graphs, we can model the network dynamics during seizures.

During a seizure, the network may become temporarily unbalanced. This can be modeled by observing the product of the signs of the edges in cycles:

$$\int_0^T \sigma_{ij}(t) \cdot \sigma_{jk}(t) \cdot \sigma_{ki}(t)\, dt \leq \frac{T}{2}.$$

Identifying periods of imbalance can help in understanding seizure dynamics.

The strength of the network may fluctuate during a seizure. By calculating the strength over short time intervals, we can capture these fluctuations:

$$S(\mathscr{G}_{\text{seizure}}(t)) = \frac{1}{|\mathscr{E}|} \int_0^t \sum_{(i,\ j) \in \mathscr{E}} |\mathscr{A}_{ij}(\tau)|\, d\tau.$$

Analyzing these fluctuations can provide insights into the mechanisms of seizure propagation.

The domination number may decrease during a seizure as certain regions (seizure foci) dominate the network:

$$\gamma(\mathscr{G}_{\text{seizure}}) \leq \gamma(\mathscr{G}_{\text{healthy}}).$$

Identifying these regions can help in localizing seizure foci for targeted interventions.

6. Conclusion

This study investigates into the dynamic nature of signed quantum graphs, domination of signed quantum graphs, balanced graphs/networks, strengths of signed graphs, bounds of the strengths and domination numbers, highlighting the importance of dynamic weights. By incorporating these dynamic properties, we showed that the model can be useful in dynamical systems such as brain networks. There are some limitations of capturing real data in such dynamical systems. Once such data is available, the theory of signed quantum graphs will be the backbone of the research on brain diseases. Theoretically, there are lots of scopes for advancements in the direction of algorithmic graph theory. The algorithms will be useful to capture the parameters of dynamical systems, such as distance, diameter, influential nodes, etc. The structure of signed quantum graphs provides a time-varying adjacency matrix $\mathscr{A}(t)$, which may serve as a Hamiltonian for defining discrete or continuous-time quantum walks. Future work will include studying how these walks behave over signed quantum topologies and how sign transitions affect hitting times and interference patterns.

Conflict of interest

The authors declare no competing financial interest.

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