Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Quiescent Solitons for the Resonant Nonlinear Schrödinger's Equation in Quantum Optics and Quantum Fluids

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Received: 23 June 2025; **Revised:** 20 September 2025; **Accepted:** 14 November 2025

Abstract: This work constructs exact quiescent solitons of the resonant nonlinear Schrödinger equation with nonlinear chromatic dispersion and nine distinct self-phase-modulation laws. Using an enhanced direct algebraic method, we derive bright, dark, singular, and straddled solitons and classify their existence domains via explicit parameter constraints. For the Kerr law, the model supports bright and singular solitons with amplitudes determined by dispersion parameters; for the power-law case, bright and singular families appear with characteristic hyperbolic profiles; and for elliptic-function constructions, Jacobian and Weierstrass forms reduce to solitons in the modulus-one limit. The analysis also yields the algebraic constraints required for physical realizability. Collectively, these results delineate when nonlinear chromatic dispersion, together with generalized self-phase modulation, produces stationary localized structures in quantum-optical and quantum-fluid settings.

***Keywords*:** solitons, integrability, direct algebraic approach

MSC: 35Q55, 35C08, 35Q60, 35A20, 78A60, 37K10

1. Introduction

One of the less commonly encountered variants of the Nonlinear Schrödinger Equation (NLSE) is the resonant NLSE [1]. This particular form of the equation has garnered attention due to its relevance in specific physical contexts, notably

DOI: https://doi.org/10.37256/cm.7120267563

within the domains of quantum fluids and quantum optics [2]. However, it is important to note that, unlike the standard NLSE, the resonant variant does not play a significant role in modeling deep-water wave phenomena.

Over the years, the resonant NLSE has been subjected to detailed mathematical and physical scrutiny. Various studies have been devoted to analyzing and deriving its soliton solutions, or localized wave packets, that maintain their shape during propagation due to a delicate balance between dispersion and nonlinearity [3]. Researchers have also identified conserved quantities associated with this model that are essential for understanding the dynamics and stability of the system [4]. Additionally, investigations into the perturbed form of the resonant NLSE have been conducted, with particular emphasis on how perturbations influence soliton dynamics [5].

Complementary to exact solution techniques, localized meshless collocation methods have been shown effective for nonlinear Partial Differential Equations (PDEs) in surface theory and fluid dynamics [6, 7]. A notable analytical approach applied to the resonant NLSE involves the semi-inverse variational principle [8]. This method has proven effective in recovering mobile soliton solutions even when the strengths (or intensities) of the perturbation terms vary arbitrarily compared to those in the unperturbed version of the model. Such analytical techniques enable deeper insights into the interaction between perturbations and soliton behavior. In the existing body of work, these explorations primarily considered linear Chromatic Dispersion (CD) and Self-Phase Modulation (SPM) effects modeled by the Kerr law or power-law nonlinearities.

In recent years, several studies have examined nonlinear Schrödinger-type models with generalized nonlinearities and dispersion effects [9–11]. For instance, Biswas-Milovic type equations with spatio-temporal dispersion were analyzed to obtain exact soliton structures using the first integral method [12], while the Kundu-Eckhaus model in birefringent fibers was studied through extended (G'/G)-expansion and direct algebraic schemes to derive multiple families of optical solitons [13]. More recently, abundant periodic and solitary wave patterns for fifth-order Sawada-Kotera equations were constructed via bilinear and extended homoclinic techniques [14]. These contributions highlight the growing interest in systematic methods for constructing exact solutions. Building on this foundation, the present study develops a unified framework that addresses nine distinct self-phase-modulation laws under the resonant nonlinear Schrödinger equation with nonlinear chromatic dispersion.

The motivation for the present study arises from the need to understand how nonlinear chromatic dispersion interacts with generalized self-phase modulation laws in shaping stationary localized structures. While most existing analyses have emphasized mobile solitons and restricted nonlinearities such as Kerr or power-law forms, the unperturbed resonant NLSE offers a unique platform to explore a richer variety of quiescent solutions. By extending the model to nine distinct self-phase modulation laws and adopting an enhanced direct algebraic method, this work provides a unified framework that systematically derives exact soliton families together with their explicit existence conditions. In contrast to previous studies, the present paper focuses exclusively on the unperturbed form of the resonant NLSE [15]. For comparison, traveling-wave constructions in related nonlinear lattice equations have been developed recently [16], whereas here we concentrate on quiescent (standing-wave) states. The novelty lies in combining nonlinear dispersion with generalized nonlinearities to reveal new stationary profiles, thereby contributing to the broader understanding of soliton dynamics in quantum-optical and quantum-fluid contexts.

The enhanced direct algebraic method is particularly suited for this problem because it naturally aligns with the standing-wave reduction of the resonant NLSE, ensuring consistency with its underlying structure. Moreover, it provides closed-form hyperbolic and elliptic function templates that are essential for constructing a comprehensive set of quiescent solutions across diverse nonlinearities, while keeping the associated parameter constraints transparent and analytically tractable.

1.1 *Governing model*

The resonant NLSE considered in this work is a highly generalized model that includes nonlinear CD, nonlinear refractive index variations, and resonant interaction effects. The equation is written as

$$i\left(q^{l}\right)_{t}+a\left(|q|^{r}q^{l}\right)_{xx}+F\left(|q|^{2}\right)q^{l}+\gamma\left(\frac{|q|_{xx}}{|q|}\right)q^{l}=0, \tag{1}$$

where $q(x, t)$ is a complex-valued function that describes the amplitude of a wave or pulse propagating in a nonlinear dispersive medium. The independent variables x and t represent space and time, respectively, and the imaginary unit $i=\sqrt{-1}$ indicates the complex nature of the wave evolution. Each term in the equation is introduced to capture different physical mechanisms and extend the model's applicability to a wider range of nonlinear wave phenomena.

The first term, $i(q^{l})_{t}$, describes the generalized temporal evolution of the wave envelope. The exponent $l \geq 1$ introduces a nonlinear modification to the usual first-order time derivative seen in the standard NLSE. This generalization allows the equation to model complex temporal dynamics, such as those arising in resonant systems or nonlinear optics where energy exchange between wave components is not linear in the wave amplitude. For example, when $l > 1$, the equation accounts for intensity-dependent temporal propagation, which can reflect phenomena like nonlinear self-steepening or higher-order time corrections in strongly coupled wave systems.

The second term, $a(|q|^{r}q^{l})_{xx}$, represents nonlinear chromatic dispersion, a key physical process in optical media where the group velocity of a pulse depends on its intensity. The coefficient a determines the strength of this effect, while the parameter $r \geq 0$ adjusts the nonlinearity of the dispersion. In standard NLSE models, dispersion is typically linear in the field amplitude, but here the dispersion term is extended to include nonlinear intensity dependence, which becomes significant in ultrafast and high-intensity regimes where higher-order dispersive corrections become important.

The third term, $F(|q|^{2})q^{l}$, introduces the effect of a nonlinear refractive index, modeled through a function F that depends on the wave intensity $|q|^{2}$. In physical systems, the refractive index of the medium often varies with the intensity of the incident wave due to nonlinear polarization responses. The function F may take various forms, including cubic, quintic, or more complex polynomial or rational functions, each corresponding to different physical behaviors such as self-focusing, defocusing, or saturation effects. In this study, nine distinct nonlinear forms for $F(|q|^{2})$ will be considered to explore a broad spectrum of nonlinear refractive effects. The fourth term, $\gamma\left(\frac{|q|_{xx}}{|q|}\right)q^{l}$, represents resonant nonlinear interactions absent from the classical NLSE. Its contribution, controlled by γ, becomes important in systems where the envelope couples to intrinsic resonant structures—such as resonantly driven optical fibers, plasma waves, or metamaterials with microstructured inclusions. Physically, it describes intensity-curvature-induced phase forcing, where rapid spatial variations in envelope magnitude feed back into the phase or effective refractive index. This mechanism can trigger modulation instability or amplitude localization, providing a framework to capture advanced wave phenomena beyond conventional NLSE models. Observable signatures include shifts in instability thresholds and localization onsets scaling with γ, which can be probed experimentally via pump-probe techniques in microstructured fibers or density-modulation spectroscopy in quantum fluids.

Our model strictly contains several well-known cases. For example, choosing $l=1$, $r=0$, $\gamma=0$, and $F(\rho)=C_{1}\rho+C_{2}\rho^{2}$ recovers the classical Cubic-Quintic NLSE (CQNLSE). In contrast to Derivative NLSE (DNLS) models, which introduce convective phase-gradient couplings such as $i\alpha(|q|^{2}q)_{x}$, our equation employs an amplitude-curvature ("resonant") term $\gamma(|q|_{xx}/|q|)q$ together with intensity-dependent dispersion $a(|q|^{r}q^{l})_{xx}$. Hence, while CQNLSE is a limit of our framework, DNLS is complementary rather than a subcase: the underlying mechanisms and symmetries differ.

Regarding mathematical tractability, the full PDE is generally non-integrable, as is typical for CQNLSE outside special parameter choices. Our analysis focuses on quiescent (standing-wave) states via $q(x, t)=\phi(x)e^{i\lambda t}$, which reduces Eq. (1) to a nonlinear Ordinary Differential Equation (ODE) (Eq. (3)) with polynomial/elliptic structure. For specific couplings—e.g., $r=1$ in the Kerr case and $r=n$ in the power-law case—the reduced ODE is algebraically integrable under our enhanced direct algebraic method, yielding closed-form bright, dark, singular, and straddled solitons as well as elliptic (Jacobi Elliptic Functions/Weierstrass Elliptic Functions (JEF/WEF)) profiles, with explicit parameter constraints. In the CQNLSE limit our construction reproduces the standard solitary/elliptic solutions; for DNLS, integrability (when present) arises from a distinct Lax structure not shared by our resonant model, yet our standing-wave reduction remains systematically solvable in the families treated here.

Concerning physical applicability, the term $a(|q|^r q^l)_{xx}$ captures intensity-dependent group-velocity dispersion relevant to ultrafast and high-intensity regimes, while $\gamma(|q|_{xx}/|q|)q$ models resonant microstructure-mediated interactions pertinent to quantum-optical media and quantum fluids. The flexible choice of $F(|q|^2)$ allows Kerr, power-law, parabolic (cubic-quintic), and dual-power responses to be addressed within a single framework, enabling a unified classification of stationary localized states and feasibility domains. By comparison, CQNLSE retains linear dispersion and lacks the resonant curvature term, and DNLS emphasizes self-steepening via phase-gradient nonlinearities; our model targets regimes where amplitude curvature and nonlinear dispersion are decisive.

In summary, Eq. (1) generalizes CQNLSE (as a strict limit) and complements DNLS by exchanging convective phase-gradient effects for amplitude-curvature and nonlinear chromatic-dispersion mechanisms. This leads to a tractable standing-wave theory that delivers explicit hyperbolic and elliptic families with transparent existence conditions, and it broadens physical coverage to scenarios where dispersion depends on intensity and resonant microstructures are significant. We have added this comparison near the end of the introduction and provided forward pointers to the sections where the closed-form solutions and parameter thresholds are derived.

The central aim of the present work is to construct exact analytical solutions of the generalized resonant NLSE by systematically exploring nine distinct forms of the nonlinear refractive index function $F(|q|^2)$. Specifically, nine representative nonlinear response laws are systematically investigated to ensure both physical coverage and analytical tractability. These include the Kerr law ($F \propto |q|^2$), which models the standard $\chi^{(3)}$ nonlinearity in glasses and optical fibers; the power-law form ($F \propto |q|^{2n}$), often employed in phenomenological fits and quantum fluid descriptions; and the cubic-quintic and parabolic models, which serve as canonical saturating frameworks capturing competing focusing and defocusing effects. In addition, more generalized cases such as the dual-power, triple-power, logarithmic, saturable, and power-exponential laws are examined, as they provide versatile surrogates for high-intensity saturation and engineered optical responses [17]. By encompassing these nine distinct forms, the present analysis balances physical relevance with mathematical tractability, thereby broadening the scope of exact solutions derived. To achieve this, an enhanced direct algebraic method is employed. This method extends traditional ansatz-based techniques by allowing a richer set of trial functions and algebraic manipulations to capture more diverse solution structures. The focus is on obtaining explicit quiescent soliton solutions, which are stationary in nature and do not change their form during propagation. The soliton types considered include quiescent dark solitons (characterized by intensity dips on a nonzero background), quiescent bright solitons (localized peaks on zero background), and quiescent singular solitons (featuring diverging amplitudes at isolated points). In addition to solitary waves, the study also derives doubly periodic wave solutions expressed in terms of JEFs (sn, cn, dn) and WEFs, both of which describe spatially periodic structures that generalize soliton behavior in bounded or periodic media.

In each case, the stationary form of the soliton solutions will be derived and thoroughly analyzed to understand their amplitude profiles, parameter dependence, and physical significance. This comprehensive approach aims to not only expand the catalog of exact solutions for generalized NLSE models but also to provide insights into the interplay of nonlinear dispersion, refractive index nonlinearity, and resonance effects in advanced wave systems. The following sections outline the mathematical preliminaries, the structure of the enhanced direct algebraic algorithm, and the detailed derivation and classification of the soliton and elliptic solutions.

Section 2 introduces the mathematical preliminaries and formulates the Enhanced Direct Algebraic Method (EDAM), based on the standing—wave reduction and the auxiliary quartic equation for the trial functions. Section 3 derives and classifies quiescent soliton families across nine nonlinear refractive index laws (Kerr, power-law, parabolic, dual-power, triple-power, logarithmic, saturable, power-exponential, and polynomial), presenting bright, dark, singular, straddled, and elliptic (JEF/WEF) solutions with their explicit parameter constraints. Section 4 examines the stability of the constructed solutions under perturbations and delineates the parameter regimes where they remain physically realizable. Section 5 discusses the physical implications of the results in quantum-optical media and quantum fluids, highlighting the interplay of nonlinear chromatic dispersion and resonant curvature effects. Finally, Section 6 synthesizes the main findings and outlines possible directions for further theoretical and experimental research.

2. Enhanced direct algebraic method

The Enhanced Direct Algebraic Method (EDAM) has recently emerged as a powerful analytical technique for constructing exact solutions of nonlinear evolution equations. It extends the classical direct algebraic approach by employing more flexible ansätze together with auxiliary functions that satisfy elliptic or polynomial differential equations. This framework allows for the derivation of diverse families of solutions, including bright, dark, singular, and periodic solitons, as well as elliptic function waveforms. Several recent studies have successfully applied EDAM to different physical models, such as concatenation systems in nonlinear optics and fluid dynamic equations, thereby confirming its effectiveness in handling nonlinear dispersive structures [18, 19]. In this section, we present the main formulation of the method as it will be employed to the resonant nonlinear Schrödinger equation. The wave form is:

$$q(x, t) = \phi(x)\, e^{i\lambda t}, \tag{2}$$

where λ is the wave number and $\phi(x)$ is a real function. Inserting (2) into Eq. (1) we get the ODE:

$$-l\lambda\phi(x) + a(r+l)(r+l-1)\phi^{r-1}(x)\phi^{'2}(x) + a(r+l)\phi^{r}(x)\phi^{''}(x)$$

$$+F\left(\phi^2(x)\right)\phi(x) + \gamma\phi^{''}(x) = 0, \tag{3}$$

where

$$\phi(x) = \alpha_0 + \sum_{i=1}^{N}\left\{\alpha_j\, V^{i}(x) + \beta_j\, V^{-j}(x)\right\} \tag{4}$$

Here α_0, α_j, β_j $(j = 1, \ldots, N)$ are constants, provided $\alpha_N^2 + \beta_N^2 \neq 0$, while $V(x)$ is the solution of the equation:

$$V^{'2}(x) = \sum_{l=0}^{4} L_l\, V^{l}(x),\ L_4 \neq 0, \tag{5}$$

where L_j $(j = 0, 1, 2, 3, 4)$ are constants.

3. Soliton solutions

Solitons are self-reinforcing, localized wave packets that maintain their shape during propagation due to a delicate balance between nonlinear and dispersive effects. In many physical systems including nonlinear optics, plasma physics, and Bose-Einstein condensates, the NLSE serves as a prototypical model for describing the evolution of such structures. Among the various classes of soliton solutions, quiescent solitons play a central role due to their stationary nature and fundamental importance in the theory of nonlinear wave propagation.

Quiescent soliton solutions refer to a subclass of solitons that are temporally oscillatory but spatially static, characterized by a constant spatial profile that does not translate over time. These solutions typically take the form $\psi(x, t) = u(x)e^{-i\omega t}$, where $u(x)$ is a real-valued, localized function and ω is a real frequency parameter. Physically, quiescent solitons correspond to wave packets that remain centered in space while oscillating in phase, making them especially useful for analytical and numerical investigations of localized modes.

In the standard focusing NLSE, quiescent solitons are well understood and are often represented by hyperbolic secant profiles. However, in more generalized settings such as equations involving higher-order dispersion, saturable or resonant nonlinearities, and generalized SPM, the structure and stability of quiescent solitons can become significantly more complex. In such cases, these stationary solutions are obtained by reducing the governing Partial Differential Equation (PDE) to a nonlinear Ordinary Differential Equation (ODE) via a standing-wave ansatz. The resulting ODE is then solved under appropriate boundary conditions to ensure spatial localization.

From a dynamical systems perspective, quiescent solitons often represent ground states or energy-minimizing configurations, and serve as starting points for bifurcation analysis and stability studies. Moreover, their stationary character simplifies the mathematical analysis, enabling deeper insights into the interplay between nonlinearity and dispersion in complex media.

In this work, we focus on the derivation and analysis of quiescent soliton solutions within the framework of a generalized NLSE. By exploring the conditions under which such stationary solitons exist and remain stable, we aim to shed light on the fundamental mechanisms governing nonlinear wave localization in resonant and non-integrable systems.

3.1 *Kerr law*

For the Kerr law nonlinearity of refractive index: $F(|q|^2) = C|q|^2$, then Eq. (1) becomes

$$i\left(q^l\right)_t + a\left(|q|^r q^l\right)_{xx} + C|q|^2 q^l + \gamma\left(\frac{|q|_{xx}}{|q|}\right) q^l = 0, \tag{6}$$

where C is a nonzero constant. The corresponding ODE is written as:

$$\begin{aligned} &-l\lambda\phi(x) + a(r+l)(r+l-1)\phi^{r-1}(x)\phi'^2(x) + a(r+l)\phi^r(x)\phi''(x) \\ &+C\phi^3(x) + \gamma\phi''(x) = 0. \end{aligned} \tag{7}$$

Eq. (7) is integerable if $r = 1$. Then Eq. (7) changes to

$$\Delta_1\phi(x) + \Delta_2\phi'^2(x) + \Delta_3\phi(x)\phi''(x) + \Delta_4\phi^3(x) + \phi''(x) = 0, \tag{8}$$

where

$$\Delta_1 = \frac{-l\lambda}{\gamma},$$

$$\Delta_2 = \frac{al(l+1)}{\gamma},$$

$$\Delta_3 = \frac{a(l+1)}{\gamma},$$

$$\Delta_4 = \frac{C}{\gamma}, \tag{9}$$

where $\gamma \neq 0$. Balancing $\phi(x)\phi''(x)$ with $\phi^3(x)$ in Eq. (8) gives $N = 2$. Thus, Eq. (8) holds:

$$\phi(x) = \alpha_0 + \alpha_1 V(x) + \alpha_2 V^2(x) + \frac{\beta_1}{V(x)} + \frac{\beta_2}{V^2(x)}, \tag{10}$$

where α_0, α_1, α_2, β_1, and β_2 are constants to be determined, provided $\alpha_2^2 + \beta_2^2 \neq 0$. Substituting (10) along with Eq. (5) into Eq. (8) and setting all the coefficients of $V^{j_1}(\xi)(V'(\xi))^{j_2}$, $(j_1 = -6, ..., -1, 0, 1, 2, ..., 6,\ j_2 = 0, 1)$ to zero leads to:

$$V^6(\xi): \Delta_4\alpha_2^3 + 2\alpha_2^2 L_4(2\Delta_2 + 3\Delta_3) = 0,$$

$$V^5(\xi): 4\alpha_1\alpha_2 L_4(\Delta_2 + 2\Delta_3) + \alpha_2^3 L_3(4\Delta_2 + 5\Delta_3) + 3\alpha_1\alpha_2^2\Delta_4 = 0,$$

$$V^4(\xi): \alpha_1\alpha_2 L_3\left(4\Delta_2 + \frac{13}{2}\Delta_3\right) + 4\alpha_2^2 L_2(\Delta_2 + \Delta_3) + \alpha_1^2 L_4(\Delta_2 + 2\Delta_3)$$

$$+ 6\Delta_3\alpha_0\alpha_2 L_4 + 3\Delta_4\alpha_2\alpha_1^2 + 6\Delta_5\alpha_2 L_4 + 3\Delta_4\alpha_0\alpha_2^2 = 0,$$

$$V^3(\xi): \alpha_1\alpha_2 L_2(4\Delta_2 + 5\Delta_3) - 2\alpha_2\beta_1 L_4(2\Delta_2 + 3\Delta_3) + 2\Delta_3\alpha_0\alpha_1 L_4$$

$$+ 5\Delta_3\alpha_0\alpha_2 L_3 + 6\Delta_4\alpha_0\alpha_1\alpha_2 + 3\Delta_4\alpha_2^2\beta_1 + \alpha_2^2 L_1(4\Delta_2 + 3\Delta_3)$$

$$+ \alpha_1^2 L_3\left(\Delta_2 + \frac{3}{2}\Delta_3\right) + \alpha_1^3\Delta_4 + 2\Delta_5\alpha_1 L_4 + 5\Delta_5\alpha_2 L_3 = 0,$$

$$V^2(\xi): \Delta_1\alpha_2 + 4\Delta_3\alpha_0\alpha_2 L_2 + \frac{3}{2}\Delta_3\alpha_0\alpha_1 L_3 + 2L_4(\alpha_1\beta_1 + 2\alpha_2\beta_2)(-\Delta_2 + \Delta_3)$$

$$+ \alpha_2 L_3\beta_1\left(-4\Delta_2 + \frac{11}{2}\Delta_3\right) + \alpha_1\alpha_2 L_1\left(4\Delta_2 + \frac{7}{2}\Delta_3\right) + 2\alpha_2^2 L_0(2\Delta_2 + \Delta_3)$$

$$+ \alpha_1^2 L_2(\Delta_2 + \Delta_3) + 3\Delta_4(3\alpha_1\alpha_2\beta_1 + \alpha_2^2\beta_2 + \alpha_0\alpha_1^2 + \alpha_0^2\alpha_2) + 4\Delta_5\alpha_2 L_2$$

$$+ \frac{5}{3}\Delta_5\alpha_1 L_3 = 0,$$

$$V(\xi): \Delta_3\alpha_0\alpha_1L_2+\alpha_2L_2\beta_1(-4\Delta_2+5\Delta_3)+8\alpha_2L_3\beta_2^2(\Delta_2+\Delta_3)+3\Delta_3\alpha_0\alpha_2L_1$$

$$+2\alpha_1(\beta_1L_3+2\beta_2L_4)(-\Delta_2+\Delta_3)+2\alpha_2\alpha_1L_0(2\Delta_2+\Delta_3)+6\Delta_4\alpha_0\alpha_2\beta_1$$

$$+\Delta_1\alpha_1+\Delta_5L_2\alpha_1+3\Delta_5L_1\alpha_2+\alpha_2^2L_1\left(\Delta_2+\frac{1}{2}\Delta_3\right)=0,$$

$$V^0(\xi): \alpha_0\Delta_1+\Delta_4(\alpha_0^3+3\alpha_2+3\alpha_1^2\beta_1+6\alpha_0\alpha_2\beta_2+6\alpha_0\alpha_1\beta_1)$$

$$+\Delta_2(L_4\beta_1^2+\alpha_1^2L_0)+\frac{1}{2}\Delta_5L_1\alpha_1+2\Delta_5L_0\alpha_2+\frac{1}{2}\Delta_5L_3\beta_1$$

$$+(\alpha_2L_1\beta_1+\alpha_1L_3\beta_2)\left(-4\Delta_2+\frac{9}{2}\Delta_3\right)+2\Delta_5L_4\beta_2$$

$$+\Delta_4(\alpha_0^3+3\alpha_2+3\alpha_1^2\beta_1+6\alpha_0\alpha_2\beta_2+6\alpha_0\alpha_1\beta_1)$$

$$+2(\alpha_1\beta_1L_2+4\alpha_2\beta_2L_2)(-\Delta_2+\Delta_3)=0,$$

$$V^{-1}(\xi): 5\Delta_7\beta_1(\alpha_0^4+6\alpha_1\alpha_0^2\beta_1+2\alpha_1^2\beta_1^2)+\alpha_0\beta_1^2L_3(\Delta_2+\Delta_3)$$

$$+\Delta_2L_4\beta_1^3+\alpha_1^2L_0\beta_1(-\Delta_2+3\Delta_3)+\alpha_1L_2\beta_1^2(-\Delta_2+3\Delta_3)$$

$$+4\Delta_6\alpha_0\beta_1(\alpha_0^2+4\alpha_1\beta_1)+3\Delta_5\beta_1(\alpha_0^2+\alpha_1\beta_1)$$

$$+2\Delta_4\alpha_0\beta_1+\beta_1(\Delta_1+L_2)+2\alpha_0\alpha_1L_1\beta_1(-\Delta_2+2\Delta_3)$$

$$+\Delta_4\beta_1(-2\alpha_1L_1+L_3\beta_1)+\Delta_3\alpha_0^2L_2\beta_1=0,$$

$$V^{-2}(\xi): \Delta_4\beta_1^2+\frac{3}{2}\beta_1L_1+10\Delta_7\alpha_0\beta_1^2(\alpha_0^2+2\alpha_1\beta_1)-2\Delta_4L_0\alpha_1\beta_1$$

$$+\frac{3}{2}\Delta_3\alpha_0^2\beta_1L_1+\alpha_0\beta_1^2L_2(\Delta_2+2\Delta_3)+\alpha_1L_1\beta_1^2\left(-\Delta_2+\frac{7}{2}\Delta_3\right)$$

$$+\beta_1^3L_3\left(\Delta_2+\frac{1}{2}\Delta_3\right)+\Delta_4L_2\beta_1^2+3\Delta_5\alpha_0\beta_1^2+2\alpha_0\alpha_1L_0\beta_1(-\Delta_2+2\Delta_3)$$

$$+2\Delta_6\beta_1^2(3\alpha_0^2+2\alpha_1\beta_1)=0,$$

$$V^{-3}(\xi): 5\Delta_3\beta_1^3(2\alpha_0^2+\alpha_1\beta_1)+\alpha_0\beta_1^2L_1(\Delta_2+3\Delta_3)+\beta_1^3L_2(\Delta_2+3\Delta_3)$$

$$+\alpha_1\beta_1^2L_0(-\Delta_2+4\Delta_3)+2\alpha_0^2\beta_1L_0\Delta_3+4\alpha_0\beta_1^3\Delta_6+\Delta_5\beta_1^3$$

$$+\Delta_4\beta_1^2L_1+2\beta_1L_0=0,$$

$$V^{-4}(\xi): \Delta_4\beta_1^4+\alpha_0\beta_1^2L_0(\Delta_2+4\Delta_3)+\beta_1^3L_1\left(\Delta_2+\frac{3}{2}\Delta_3\right)+\Delta_4L_0\beta_1^2$$

$$+\Delta_2L_1\beta_1^2+5\Delta_3\alpha_0\beta_1^4=0,$$

$$V^{-5}(\xi): 3\Delta_4\beta_1\beta_2^2+4\beta_1\beta_2L_0(\Delta_2+2\Delta_3)+\beta_2^2L_1(4\Delta_2+5\Delta_3)=0,$$

$$V^{-6}(\xi): \Delta_4\beta_2^3+2\beta_2^2L_0(2\Delta_2+3\Delta_3)=0. \tag{11}$$

Case 1: If we set $L_0=L_1=L_3=0$, then we have the results

$$\alpha_0=\alpha_1=\beta_1=\beta_2=0,\ \ \alpha_2=-\frac{3L_4}{2L_2(\Delta_2+\Delta_3)} \tag{12}$$

with the constraint conditions:

$$\Delta_1=-4L_2,$$

$$\Delta_4=-\frac{2L_4(\Delta_3\alpha_2L_2-3L_4)}{\alpha_2^2L_2}. \tag{13}$$

(I) Setting $L_2>0$ and $L_4<0$ causes to bright soliton:

$$q(x,t)=\frac{3}{2(\Delta_2+\Delta_3)}\ \operatorname{sech}^2\left(\sqrt{L_2}\,x\right)\ e^{i\lambda t}. \tag{14}$$

(II) Setting $L_2>0$ and $L_4>0$, yields singular soliton:

$$q(x,t)=-\frac{3}{2(\Delta_2+\Delta_3)}\ \operatorname{csch}^2\left(\sqrt{L_2}\,x\right)\ e^{i\lambda t}. \tag{15}$$

Case 2: If we set $L_0=\dfrac{L_2^2}{4L_4}$, $L_1=L_3=0$, then we have the results

$$\alpha_0 = \alpha_0,\ \alpha_1 = \beta_1 = \beta_2 = 0,\ \alpha_2 = \alpha_2, \tag{16}$$

with the constraint conditions:

$$\Delta_1 = -\frac{\Delta_4\left(4\alpha_0^2 L_4^2 - 4\alpha_0\alpha_2 L_2 L_4 + \alpha_2^2 L_2^2\right)}{4L_4^2},$$

$$\Delta_2 = \frac{\Delta_4\left(18\alpha_0^2 L_4^2 - 9\alpha_0\alpha_2 L_2 L_4 + \alpha_2^2 L_2^2\right)}{2\alpha_2 L_2^2 L_4},$$

$$\Delta_3 = -\frac{\Delta_4\left(12\alpha_0^2 L_4^2 - 6\alpha_0\alpha_2 L_2 L_4 + \alpha_2^2 L_2^2\right)}{2\alpha_2 L_2^2 L_4}. \tag{17}$$

Then Eq. (1) has the dark soliton:

$$q(x,\, t) = \alpha_0 - \frac{\alpha_1 L_2}{2L_4}\tanh^2\left(\sqrt{-\frac{L_2}{2}}x\right)e^{i\lambda t}, \tag{18}$$

and singular soliton:

$$q(x,\, t) = \alpha_0 - \frac{\alpha_1 L_2}{2L_4}\coth^2\left(\sqrt{-\frac{L_2}{2}}x\right)e^{i\lambda t}, \tag{19}$$

where $L_2 < 0$.

Case 3: If we set $L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = \alpha_0,\ \alpha_1 = \beta_1 = \beta_2 = 0,\ \alpha_2 = \frac{2\alpha_0 L_4}{L_2}, \tag{20}$$

with the constraint conditions:

$$\Delta_1 = -\frac{2\Delta_3\alpha_0(2L_0L_4 - L_2^2)}{L_2},$$

$$\Delta_2 = \frac{\Delta_3}{2},$$

$$\Delta_4 = -\frac{8L_2\Delta_3}{2\alpha_0}. \tag{21}$$

Thus, Eq. (1) holds the WEF solutions:

$$q(x, t) = \alpha_0 \left[1 + \frac{18\wp^2[(x), g_2, g_3]}{L_2(6\wp[(x), g_2, g_3] + L_2)^2}\right] e^{i\lambda t}, \tag{22}$$

and

$$q(x) = \alpha_0 \left[1 + \frac{L_0(6\wp[(x), g_2, g_3] + L_2)^2}{9\wp^2[(x), g_2, g_3]}\right] e^{i\lambda t}. \tag{23}$$

Case 4: If we set $L_0 = L_1 = 0$, then we have the results

$$\alpha_0 = \alpha_1 = \beta_1 = \beta_2 = 0, \quad \alpha_2 = \alpha_2, \quad L_3 = 0, \quad L_4 = -\frac{2\alpha_2 L_2(\Delta_2 + \Delta_3)}{3}, \tag{24}$$

with the constraint conditions:

$$\Delta_1 = -4L_2,$$

$$\Delta_4 = \frac{4L_2(\Delta_2 + \Delta_3)(2\Delta_2 + 3\Delta_3)}{3}. \tag{25}$$

Therefore, we arrive at the straddled solitons:

$$q(x, t) = -\frac{3\operatorname{sech}^4\left(\frac{\sqrt{L_2}}{2}x\right)}{8(\Delta_2 + \Delta_3)\tanh^2\left(\frac{\sqrt{L_2}}{2}x\right)} e^{i\lambda t}, \tag{26}$$

and

$$q(x, t) = -\frac{3\operatorname{csch}^4\left(\frac{\sqrt{L_2}}{2}x\right)}{8(\Delta_2 + \Delta_3)\coth^2\left(\frac{\sqrt{L_2}}{2}x\right)} e^{i\lambda t}. \tag{27}$$

where $L_2 > 0$ and $L_4 > 0$ and $\alpha_2(\Delta_2 + \Delta_3) < 0$.

3.2 *Power law*

For the power law nonlinearity of refractive index: $F(|q|^2) = C|q|^{2n}$, then Eq. (1) decreases to

$$i\left(q^{l}\right)_{t}+a\left(|q|^{r}q^{l}\right)_{xx}+C|q|^{2n}q^{l}+\gamma\left(\frac{|q|_{xx}}{|q|}\right)q^{l}=0, \tag{28}$$

where C is a nonzero constant. The corresponding ODE is written as:

$$-l\lambda\phi(x)+a(r+l)(r+l-1)\phi^{r-1}(x)\phi^{'2}(x)+a(r+l)\phi^{r}(x)\phi^{''}(x)$$

$$+C\phi^{2n+1}(x)+\gamma\phi^{''}(x)=0. \tag{29}$$

Eq. (29) is integerable if $r=n$. Then Eq. (29) reduces to

$$-l\lambda\phi(x)+a(n+l)(n+l-1)\phi^{n-1}(x)\phi^{'2}(x)+a(n+l)\phi^{n}(x)\phi^{''}(x)$$

$$+C\phi^{2n+1}(x)+\gamma\phi^{''}(x)=0. \tag{30}$$

Balancing $\phi^{n}(x)\phi^{''}(x)$ with $\phi^{2n+1}(x)$, we get $N=\frac{2}{n}$, $n\neq 0$. Then using the transformation

$$\phi(x)=P^{\frac{2}{n}}(x) \tag{31}$$

where $P(x)$ is a new function, Eq. (30) becomes

$$\Delta_1P^2(x)+\Delta_2P^2(x)P^{'2}(x)+\Delta_3P^3(x)P^{''}(x)+\Delta_4P^6(x)+\Delta_5P^{'2}(x)+\Delta_6P(x)P^{''}(x)=0, \tag{32}$$

where

$$\Delta_1=-l\lambda,$$

$$\Delta_2=\frac{2a}{n}(n+l)\left(1+\frac{2l}{n}\right),$$

$$\Delta_3=\frac{2a(n+l)}{n},$$

$$\Delta_4=C,$$

$$\Delta_5=\frac{2\gamma}{n}\left(\frac{2}{n}-1\right),$$

$$\Delta_6 = \frac{2\gamma}{n}. \tag{33}$$

Balancing $P^3(x)P''(x)$ with $P^6(x)$ in Eq. (32) gives $N = 1$. Now, Eq. (32) has the formal solution:

$$P(x) = \alpha_0 + \alpha_1 V(x) + \frac{\beta_1}{V(x)}, \tag{34}$$

where α_0, α_1, and β_1 are constants to be determined, provided $\alpha_1^2 + \beta_1^2 \neq 0$. Substituting (34) along with Eq. (5) into Eq. (32) and setting all the coefficients of $V^{j_1}(\xi)(V'(\xi))^{j_2}$, $(j_1 = -6, ..., -1, 0, 1, 2, ..., 6,\ j_2 = 0, 1)$ to zero leads to:

Case 1: If we set $L_0 = L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = \beta_1 = 0,\ \ \alpha_1 = \alpha_1, \tag{35}$$

with constraint conditions:

$$\Delta_1 = -L_2(\Delta_5 + \Delta_6),$$

$$\Delta_2 = -\frac{\alpha_1^2 L_2 \Delta_3 + L_4(\Delta_5 + 2\Delta_6)}{\alpha_1^2 L_2},$$

$$\Delta_4 = -\frac{\alpha_1^2 L_4^2 L_2 \Delta_3 - L_4^2(\Delta_5 + 2\Delta_6)}{\alpha_1^4 L_2}. \tag{36}$$

(I) Setting $L_2 > 0$ and $L_4 < 0$ yields bright soliton:

$$q(x, t) = \left[\alpha_1 \sqrt{-\frac{L_2}{L_4}}\ \text{sech}\left(\sqrt{L_2}\, x\right)\right]^{\frac{2}{n}} e^{i\lambda t}. \tag{37}$$

(II) Setting $L_2 > 0$ and $L_4 > 0$ leads to singular soliton:

$$q(x, t) = \left[\alpha_1 \sqrt{\frac{L_2}{L_4}}\ \text{csch}\ \left(\sqrt{L_2}\, x\right)\right]^{\frac{2}{n}} e^{i\lambda t}. \tag{38}$$

Case 2: If we set $L_0 = \dfrac{L_2^2}{4L_4}$, $L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = \alpha_1 = 0, \quad \beta_1 = \beta_1, \tag{39}$$

with constraint conditions:

$$\Delta_1 = \Delta_3\beta_1^2L_4 - \frac{1}{2}\Delta_6L_2,$$

$$\Delta_2 = -\frac{2\Delta_3\beta_1^2L_4 + \Delta_6L_2}{2\beta_1^2L_4},$$

$$\Delta_4 = -\frac{2\Delta_3\beta_1^2L_4L_2^2 + \Delta_6L_2^3}{2\beta_1^4L_4^2},$$

$$\Delta_5 = 0. \tag{40}$$

Setting $L_2 < 0$ and $L_4 > 0$ causes to the singular soliton:

$$q(x, t) = \left[\frac{2\beta_1}{\sqrt{-\frac{2L_2}{L_4}}\tanh\left(\sqrt{-\frac{L_2}{2}}x\right)}\right]^{\frac{2}{n}} e^{i\lambda t}, \tag{41}$$

and the dark soliton:

$$q(x, t) = \left[\frac{2\beta_1}{\sqrt{-\frac{2L_2}{L_4}}\coth\left(\sqrt{-\frac{L_2}{2}}x\right)}\right]^{\frac{2}{n}} e^{i\lambda t}, \tag{42}$$

Case 3: If we set $L_1 = L_3 = 0$, then we have the following:
(I) When $L_0 = \frac{m_1^2(1-m_1^2)L_2}{(2m_1^2-1)L_4}$, $0 < m_1 < 1$, we get

$$\alpha_0 = \beta_1 = 0, \quad \alpha_1 = \alpha_1, \tag{43}$$

with constraint conditions:

$$\Delta_2 = \frac{\Delta_3\alpha_1^2L_2^2(-4m_1^4+4m_1^2-1)+16\Delta_1L_4(4m_1^4-4m_1^2-1)}{\alpha_1^2L_2^2(6m_1^4-6m_1^2+1)},$$

$$\Delta_4 = -\frac{\Delta_3L_4^2\alpha_1^2L_2^2(8m_1^4-8m_1^2+1)+2\Delta_1L_4^2(4m_1^4-4m_1^2+1)}{\alpha_1^4L_2^2(6m_1^4-6m_1^2+1)},$$

$$\Delta_5 = 0,$$

$$\Delta_6 = -\frac{\Delta_3 \alpha_1^2 L_2^2 m_1^2 (m_1^2 - 1) + \Delta_1 L_4 (4m_1^4 - 4m_1^2 - 1)}{L_4 L_2 (6m_1^4 - 6m_1^2 + 1)}. \tag{44}$$

Thus, we arrive at the JEF solution:

$$q(x, t) = \left[\alpha_1 \sqrt{-\frac{m_1^2 L_2}{(2m_1^2 - 1) L_4}} \operatorname{cn}\left(\sqrt{\frac{L_2}{2m_1^2 - 1}}\, x,\, m_1\right)\right]^{\frac{2}{n}} e^{i\lambda t}, \tag{45}$$

provided $(2m_1^2 - 1)L_2 > 0$, $L_4 < 0$.

(II) When $L_0 = \dfrac{(1 - m_1^2)L_2^2}{(2 - m_1^2)^2 L_4}$, $0 < m_1 < 1$, we get

$$\alpha_0 = \beta_1 = 0, \quad \alpha_1 = \alpha_1, \tag{46}$$

with constraint conditions:

$$\Delta_2 = \frac{\Delta_3 \alpha_1^2 L_2^2 (-m_1^4 + 4m_1^2 - 4) + 2\Delta_1 L_4 (m_1^4 - 4m_1^2 + 4)}{\alpha_1^2 L_2^2 (m_1^4 - 2m_1^2 + 2)},$$

$$\Delta_4 = -\frac{\Delta_3 m_1^4 \alpha_1^2 L_4 L_2^2 + 2\Delta_1 L_4^2 (m_1^4 - 4m_1^2 + 4)}{\alpha_1^4 L_2^2 (m_1^4 - 2m_1^2 + 2)},$$

$$\Delta_5 = 0,$$

$$\Delta_6 = -\frac{\Delta_3 \alpha_1^2 L_2^2 (m_1^2 - 1) + \Delta_1 L_4 (m_1^4 - 4m_1^2 - 4)}{L_4 L_2 (m_1^4 - 2m_1^2 + 2)}. \tag{47}$$

Therefore, we arrive at the JEF solution:

$$q(x, t) = \left[\alpha_1 \sqrt{-\frac{m_1^2 L_2}{(2 - m_1^2) L_4}} \operatorname{dn}\left(\sqrt{\frac{L_2}{2 - m_1^2}}\, x,\ m_1\right)\right]^{\frac{2}{n}} e^{i\lambda t}, \tag{48}$$

provided $(2 - m_1^2)L_2 > 0$, $L_4 < 0$.

Setting $m_1 \to 1$ yields the bright soliton:

$$q(x,t)=\left[\alpha_1\sqrt{-\frac{L_2}{L_4}}\,\mathrm{sech}\left(\sqrt{L_2}\,x\right)\right]^{\frac{2}{n}}e^{i\lambda t}, \tag{49}$$

provided $L_2>0$, $L_4<0$.

(III) When $L_0=\dfrac{m_1^2L_2^2}{(m_1^2+1)^2L_4}$, $0<m_1<1$, we get

$$\alpha_0=\alpha_1=0,\quad \beta_1=\beta_1, \tag{50}$$

with constraint conditions:

$$\begin{cases}\Delta_1=-\dfrac{\beta_1^2L_4\left[\Delta_4\beta_1^2L_4(m_1^8+m_1^6+2m_1^4+2m_1^2+1)+m_1^2L_2^2\Delta_3(m_1^4-2m_1^2+1)\right]}{2m_1^4L_2^2},\\[2ex] \Delta_2=-\dfrac{\Delta_4\beta_1^2L_4(m_1^4+2m_1^2+1)+2L_2^2\Delta_3m_1^2}{m_1^2L_2^2},\\[2ex] \Delta_5=0,\\[2ex] \Delta_6=-\dfrac{\beta_1^2L_4\left[\Delta_4\beta_1^2L_4(m_1^8+4m_1^6+6m_1^4+4m_1^2+1)+m_1^2L_2^2\Delta_3(m_1^4+2m_1^2+1)\right]}{2m_1^4L_2^3}.\end{cases} \tag{51}$$

Thus, we arrive at the JEF solution:

$$q(x,t)=\left[\frac{\beta_1}{\sqrt{-\dfrac{m_1^2L_2}{(m_1^2+1)L4}}\;\mathrm{sn}\left(\sqrt{-\dfrac{L_2}{m_1^2+1}}\,x,\,m_1\right)}\right]^{\frac{2}{n}}e^{i\lambda t}, \tag{52}$$

provided $L_2<0$, $L_4>0$. In particular when $m_1\to1$ in Eq. (52), we have the singular soliton solution for (1) as following:

$$q(x,t)=\left[\frac{\beta_1}{\sqrt{-\dfrac{L_2}{2L4}}\;\tanh\left(\sqrt{-\dfrac{L_2}{2}}\,x\right)}\right]^{\frac{2}{n}}e^{i\lambda t}. \tag{53}$$

Case 4: If we set $L_1=L_3=0$, then we have the results

$$\alpha_0 = \beta_1 = 0, \quad \alpha_1 = \sqrt{-\frac{2\Delta_6 L_4}{(\Delta_2+\Delta_3)L_2}}, \tag{54}$$

where $L_2 > 0$, $L_4 > 0$, $\Delta_6(\Delta_2+2\Delta_3) < 0$, with constraint conditions:

$$\Delta_1 = \frac{\Delta_3\alpha_1^2 L_0 L_2 + 2\Delta_6 L_0 L_4 - \Delta_6 L_2^2}{L_2},$$

$$\Delta_4 = -\frac{L_4(\Delta_3\alpha_1^2 L_2 - 2\Delta_6 L_4)}{\alpha_1^4 L_2},$$

$$\Delta_5 = 0. \tag{55}$$

Therefore, we arrive at the WEF solutions:

$$q(x) = \left[3\sqrt{-\frac{2\Delta_6}{(\Delta_2+\Delta_3)L_2}}\left(\frac{\wp'[(x), g_2, g_3]}{6\wp[(x), g_2, g_3]+L_2}\right)\right]^{\frac{2}{n}} e^{i\lambda t}, \tag{56}$$

where $L_2 > 0$, $\Delta_6(\Delta_2+2\Delta_3) < 0$,

$$q(x) = \left[\frac{1}{3}\sqrt{-\frac{2\Delta_6 L_4 L_0}{(\Delta_2+\Delta_3)L_2}}\left(\frac{6\wp[(x), g_2, g_3]+L_2}{\wp'[(x), g_2, g_3]}\right)\right]^{\frac{2}{n}} e^{i\lambda t}, \tag{57}$$

where $L_0 > 0$, $L_2 > 0$, $L_4 > 0$, $\Delta_3(\Delta_2+2\Delta_3) < 0$.

Case 5: If we set $L_0 = L_1 = 0$, then we have the results

$$\alpha_0 = \beta_1 = 0, \quad \alpha_1 = \sqrt{-\frac{L_4(\Delta_5+2\Delta_6)}{L_2(\Delta_2+\Delta_3)}}, \quad L_3 = 0, \tag{58}$$

with constraint conditions:

$$\Delta_1 = -L_2(\Delta_5+\Delta_6),$$

$$\Delta_4 = -\frac{\alpha_1^2 L_4 L_2 \Delta_3 - L_4^2(\Delta_5+2\Delta_6)}{\alpha_1^4 L_2}. \tag{59}$$

Now, Eq. (1) has the singular soliton solutions when $L_2 > 0$, $L_4 > 0$ and $(\Delta_2+\Delta_3)(\Delta_5+2\Delta_6) < 0$ as the following:

$$q(x,\,t)=\left[-\frac{\varepsilon}{2}\sqrt{-\frac{(\Delta_5+2\Delta_6)}{(\Delta_2+\Delta_3)}}\frac{\operatorname{sech}^2\frac{\sqrt{L_2}}{2}(x)}{\tanh\frac{\sqrt{L_2}}{2}\,(x)}\right]^{\frac{2}{n}}e^{i\lambda t},\tag{60}$$

and

$$q(x,\,t)=\left[\frac{\varepsilon}{2}\sqrt{-\frac{(\Delta_5+2\Delta_6)}{(\Delta_2+\Delta_3)}}\frac{\operatorname{csch}^2\frac{\sqrt{L_2}}{2}(x)}{\coth\frac{\sqrt{L_2}}{2}\,(x)}\right]^{\frac{2}{n}}e^{i\lambda t}.\tag{61}$$

3.3 *Parabolic law*

For the parabolic nonlinearity of refractive index: $F(|q|^2)=C_1\,|q|^2+C_2\,|q|^4$, then Eq. (1) simplifies to

$$i\left(q^l\right)_t+a\left(|q|^r\,q^l\right)_{xx}+\left[C_1\,|q|^2+C_2\,|q|^4\right]q^l+\gamma\left(\frac{|q|_{xx}}{|q|}\right)q^l=0,\tag{62}$$

where C_1 and C_2 are nonzero constants. The corresponding ODE is written as:

$$-l\lambda\phi(x)+a(r+l)(r+l-1)\phi^{r-1}(x)\phi^{'2}(x)+a(r+l)\phi^r(x)\phi^{''}(x)$$

$$+C_1\phi^3(x)+C_2\phi^5(x)+\gamma\phi^{''}(x)=0.\tag{63}$$

Eq. (63) is integerable if $r=2$. Then it changes to

$$\Delta_1\phi(x)+\Delta_2\phi(x)\phi^{'2}(x)+\Delta_3\phi^2(x)\phi^{''}(x)+\Delta_4\phi^3(x)+\Delta_5\phi^5(x)+\phi^{''}(x)=0.\tag{64}$$

where

$$\Delta_1=\frac{-l\lambda}{\gamma},$$

$$\Delta_2=\frac{a}{\gamma}(2+l)(1+l),$$

$$\Delta_3=\frac{a(2+l)}{\gamma},$$

$$\Delta_4 = \frac{C_1}{\gamma},$$

$$\Delta_5 = \frac{C_2}{\gamma}, \tag{65}$$

where $\gamma \neq 0$. Balancing $\phi^2(x)\phi''(x)$ with $\phi^5(x)$ in Eq. (64) gives $N = 1$. Now, Eq. (64) has the formal solution:

$$\phi(x) = \alpha_0 + \alpha_1 V(x) + \frac{\beta_1}{V(x)}, \tag{66}$$

where α_0, α_1, and β_1 are constants to be determined, provided $\alpha_1^2 + \beta_1^2 \neq 0$. Substituting (66) along with Eq. (5) into Eq. (64) and setting all the coefficients of $V^{j_1}(\xi)(V'(\xi))^{j_2}$, $(j_1 = -5, ..., -1, 0, 1, 2, ..., 5, j_2 = 0, 1)$ to zero causes to:

Case 1: If we set $L_0 = L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = \beta_1 = 0, \ \alpha_1 = \sqrt{-\frac{L_4(\Delta_2 + 2\Delta_3)}{\Delta_5}}, \tag{67}$$

with constraint conditions:

$$\Delta_1 = -L_2,$$

$$\Delta_4 = -\frac{\alpha_1^2 L_2(\Delta_2 + \Delta_3) + 2L_4}{\alpha_1^2}. \tag{68}$$

(I) Setting $L_2 > 0$, $\Delta_1 < 0$, $L_4 < 0$, $\Delta_5(\Delta_2 + 2\Delta_3) > 0$ yields the bright soliton:

$$q(x, t) = \sqrt{\frac{L_2(\Delta_2 + 2\Delta_3)}{\Delta_5}} \operatorname{sech}\left(\sqrt{L_2}\, x\right) e^{i\lambda t}. \tag{69}$$

(II) Setting $L_2 > 0$, $\Delta_1 < 0$, $L_4 > 0$, $\Delta_5(\Delta_2 + 2\Delta_3) < 0$ leads to the singular soliton:

$$q(x, t) = \sqrt{-\frac{L_2(\Delta_2 + 2\Delta_3)}{\Delta_5}} \operatorname{csch}\left(\sqrt{L_2}\, x\right) e^{i\lambda t}. \tag{70}$$

Case 2: If we set $L_0 = \frac{L_2^2}{4L_4}$, $L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = \beta_1 = 0, \quad \alpha_1 = \sqrt{-\frac{L_4(\Delta_2 + 2\Delta_3)}{\Delta_5}}, \tag{71}$$

with constraint conditions:

$$\Delta_1 = -\frac{L_2(\alpha_1^2 \Delta_2 L_2 + 4L_4)}{4L_4},$$

$$\Delta_4 = -\frac{\alpha_1^2 L_2(\Delta_2 + \Delta_3) + 2L_4}{\alpha_1^2}. \tag{72}$$

Setting $L_2 < 0$, $L_4 > 0$, $\Delta_5(\Delta_2 + 2\Delta_3) < 0$ yields the dark soliton:

$$q(x,\, t) = \sqrt{\frac{L_2(\Delta_2 + 2\Delta_3)}{2\Delta_5}} \tanh\left(\sqrt{-\frac{L_2}{2}}x\right) e^{i\lambda t}, \tag{73}$$

and the singular soliton:

$$q(x,\, t) = \sqrt{\frac{L_2(\Delta_2 + 2\Delta_3)}{2\Delta_5}} \coth\left(\sqrt{-\frac{L_2}{2}}x\right) e^{i\lambda t}, \tag{74}$$

Case 3: If we set $L_1 = L_3 = 0$, then we have the following:
(I) When $L_0 = \dfrac{m_1^2(1 - m_1^2)L_2}{(2m_1^2 - 1)L_4}$, $0 < m_1 < 1$, we get

$$\alpha_0 = \beta_1 = 0, \quad \alpha_1 = \alpha_1, \tag{75}$$

with constraint conditions:

$$\Delta_2 = \frac{(\Delta_1 + L_2)(4m_1^4 - 4m_1^2 + 1)}{m_1^2 \alpha_1^2 L_2^2 (m_1 - 1)(m_1 + 1)},$$

$$\Delta_4 = -\frac{\Delta_3 L_2^2 \alpha_1^2 m_1^2 (m_1^2 - 1) + L_4 L_2 (6m_1^4 - 6m_1^2 + 1) + \Delta_1 L_4 (4m_1^4 - 4m_1^2 + 1)}{m_1^2 \alpha_1^2 L_2 (m_1 - 1)(m_1 + 1)},$$

$$\Delta_5 = -\frac{2\Delta_3 L_2^2 \alpha_1^2 m_1^2 (m_1^2 - 1) + L_4 L_2 (4m_1^4 - 4m_1^2 + 1) + \Delta_1 L_4 (4m_1^4 - 4m_1^2 + 1)}{m_1^2 \alpha_1^4 L_2 (m_1 - 1)(m_1 + 1)}. \tag{76}$$

Thus, we arrive at the JEF solution:

$$q(x,t)=\alpha_1\sqrt{-\frac{m_1^2L_2}{(2m_1^2-1)L_4}}\,\mathrm{cn}\left(\sqrt{\frac{L_2}{2m_1^2-1}}\,x,\,m_1\right)\,e^{i\lambda t}, \tag{77}$$

provided $(2m_1^2-1)L_2>0$, $L_4<0$.

(II) When $L_0=\dfrac{(1-m_1^2)L_2^2}{(2-m_1^2)^2L_4}$, $0<m_1<1$, we get

$$\alpha_0=\beta_1=0,\quad \alpha_1=\alpha_1, \tag{78}$$

with constraint conditions:

$$\Delta_2=\frac{L_4(\Delta_1+L_2)(m_1^4-4m_1^2+4)}{\alpha_1^2L_2^2(m_1-1)(m_1+1)},$$

$$\Delta_4=-\frac{\Delta_3L_2^2\alpha_1^2(m_1^2-1)+L_4L_2(m_1^4-2m_1^2+2)+\Delta_1L_4(m_1^4-4m_1^2+4)}{\alpha_1^2L_2(m_1-1)(m_1+1)},$$

$$\Delta_5=-\frac{L_4\left[2\Delta_3L_2^2\alpha_1^2(m_1^2-1)+L_4L_2(m_1^4-4m_1^2+4)+\Delta_1L_4(m_1^4-4m_1^2+4)\right]}{\alpha_1^4L_2(m_1-1)(m_1+1)}. \tag{79}$$

Therefore, we arrive at the JEF solution:

$$q(x,t)=\alpha_1\sqrt{-\frac{m_1^2L_2}{(2-m_1^2)L_4}}\,\mathrm{dn}\left(\sqrt{\frac{L_2}{2-m_1^2}}\,x,\,m_1\right)\,e^{i\lambda t}, \tag{80}$$

provided $(2-m_1^2)L_2>0$, $L_4<0$.

Setting $m_1\to1$ yields the bright soliton:

$$q(x,t)=\alpha_1\sqrt{-\frac{L_2}{L_4}}\,\mathrm{sech}\left(\sqrt{L_2}\,x\right)\,e^{i\lambda t}, \tag{81}$$

provided $L_2>0$, $L_4<0$.

(III) When $L_0=\dfrac{m_1^2L_2^2}{(m_1^2+1)^2L_4}$, $0<m_1<1$, we get

$$\alpha_0=\alpha_1=0,\quad \beta_1=\beta_1, \tag{82}$$

with constraint conditions:

$$\Delta_2 = \frac{L_4(\Delta_1 + L_2)(m_1^4 + 2m_1^2 + 1)}{m_1^2 \alpha_1^2 L_2^2},$$

$$\Delta_4 = \frac{-\Delta_3 L_2^2 \alpha_1^2 m_1^2 + L_4 L_2 (m_1^4 + 2m_1^2 + 1) + \Delta_1 L_4 (m_1^4 + 2m_1^2 + 1)}{m_1^2 \alpha_1^2 L_2},$$

$$\Delta_5 = \frac{L_4 \left[-2\Delta_3 L_2^2 \alpha_1^2 m_1^2 + L_4 L_2 (m_1^4 + 2m_1^2 + 1) + \Delta_1 L_4 (m_1^4 + 2m_1^2 + 1)\right]}{m_1^2 \alpha_1^4 L_2^2}. \tag{83}$$

Thus, we arrive at the JEF solution:

$$q(x,\, t) = \alpha_1 \sqrt{-\frac{m_1^2 L_2}{(m_1^2 + 1)L4}}\ \mathrm{sn}\left(\sqrt{-\frac{L_2}{m_1^2 + 1}}\ x,\, m_1\right)\ e^{i\lambda t}, \tag{84}$$

provided $L_2 < 0$, $L_4 > 0$.

Setting $m_1 \to 1$ gives the dark soliton:

$$q(x,\, t) = \alpha_1 \sqrt{-\frac{L_2}{2L4}}\ \tanh\left(\sqrt{-\frac{L_2}{2}}\ x\right)\ e^{i\lambda t}, \tag{85}$$

where $L_2 < 0$, $L_4 > 0$.

Case 4: If we set $L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = \alpha_1 = 0, \quad \beta_1 = \sqrt{-\frac{(\Delta_2 + 2\Delta_3)L_0}{\Delta_5}}, \tag{86}$$

where $L_0 > 0$, $\Delta_5(\Delta_2 + 2\Delta_3) < 0$, with constraint conditions:

$$\Delta_1 = -\Delta_2 \beta_1^2 L_4 - L_2,$$

$$\Delta_4 = -\frac{(\Delta_2 + \Delta_3)\beta_1^2 L_2 + 2L_0}{\beta_1^2}. \tag{87}$$

Thus, we arrive at the WEF solutions:

$$q(x) = \frac{1}{3}\sqrt{-\frac{(\Delta_2 + 2\Delta_3)L_4 L_0}{\Delta_5}}\left(\frac{6\wp[(x),\, g_2,\, g_3] + L_2}{\wp'[(x),\, g_2,\, g_3]}\right)\ e^{i\lambda t}, \tag{88}$$

where $L_4 > 0$, $L_0 > 0$, $\Delta_5(\Delta_2 + 2\Delta_3) < 0$,

$$q(x) = 3\sqrt{-\frac{(\Delta_2 + 2\Delta_3)}{\Delta_5}} \left(\frac{\wp'[(x), g_2, g_3]}{6\wp[(x), g_2, g_3] + L_2} \right) e^{i\lambda t}, \tag{89}$$

where $L_0 < 0$, $\Delta_3(\Delta_2 + 2\Delta_3) > 0$.

Case 5: If we set $L_0 = L_1 = 0$, then we have the results

$$\alpha_0 = \beta_1 = 0, \quad \alpha_1 = \sqrt{-\frac{L_4(\Delta_2 + 2\Delta_3)}{\Delta_5}}, \quad L_3 = 0, \tag{90}$$

with constraint conditions:

$$\Delta_1 = -L_2,$$

$$\Delta_4 = -\frac{\alpha_1^2 L_2(\Delta_2 + \Delta_3) + 2L_4}{\alpha_1^4}. \tag{91}$$

Now, Eq. (1) has singular soliton solutions when $L_2 > 0$, $L_4 > 0$ and $(\Delta_2 + 2\Delta_3)\Delta_5 < 0$ as the following:

$$q(x, t) = -\varepsilon\sqrt{-\frac{L_2(\Delta_2 + 2\Delta_3)}{\Delta_5}} \frac{\operatorname{sech}^2 \frac{\sqrt{L_2}}{2}(x)}{2\tanh \frac{\sqrt{L_2}}{2}(x)} e^{i\lambda t}, \tag{92}$$

and

$$q(x, t) = \varepsilon\sqrt{-\frac{L_2(\Delta_2 + 2\Delta_3)}{\Delta_5}} \frac{\operatorname{csch}^2 \frac{\sqrt{L_2}}{2}(x)}{2\coth \frac{\sqrt{L_2}}{2}(x)} e^{i\lambda t}. \tag{93}$$

3.4 *Dual-power law*

For the dual power law of refractive index: $F(|q|^2) = C_1 |q|^{2n} + C_2 |q|^{4n}$, then Eq. (1) becomes

$$i\left(q^l\right)_t + a\left(|q|^r q^l\right)_{xx} + \left[C_1 |q|^{2n} + C_2 |q|^{4n}\right] q^l + \gamma\left(\frac{|q|_{xx}}{|q|}\right) q^l = 0, \tag{94}$$

where C_1 and C_2 are nonzero constants. The corresponding ODE is written as:

$$-l\lambda\phi(x)+a(r+l)(r+l-1)\phi^{r-1}(x)\phi^{'2}(x)+a(r+l)\phi^{r}(x)\phi^{''}(x)$$

$$+C_1\phi^{2n+1}(x)+C_2\phi^{4n+1}(x)+\gamma\phi^{''}(x)=0. \tag{95}$$

Eq. (95) is integerable if $r=2n$. Then it changed to

$$-l\lambda\phi(x)+a(2n+l)(2n+l-1)\phi^{2n-1}(x)\phi^{'2}(x)+a(2n+l)\phi^{2n}(x)\phi^{''}(x)$$

$$+C_1\phi^{2n+1}(x)+C_2\phi^{4n+1}(x)+\gamma\phi^{''}(x)=0. \tag{96}$$

Balancing $\phi^{2n}(x)\phi^{''}(x)$ with $\phi^{4n+1}(x)$ we get $N=\frac{1}{n}$, $n\neq 0$. By using

$$\phi(x)=P^{\frac{1}{n}}(x), \tag{97}$$

where $P(x)$ is a new function, Eq. (96) becomes

$$\Delta_1P^2(x)+\Delta_2P^2(x)P^{'2}(x)+\Delta_3P^3(x)P^{''}(x)+\Delta_4P^4(x)+\Delta_5P^6(x)$$

$$+\Delta_6P^{'2}(x)+\Delta_7P(x)P^{''}(x)=0, \tag{98}$$

where

$$\Delta_1=-l\lambda,$$

$$\Delta_2=\frac{a}{n}(2n+l)\left(1+\frac{l}{n}\right),$$

$$\Delta_3=\frac{a(2n+l)}{n},$$

$$\Delta_4=C_1,$$

$$\Delta_5=C_2,$$

$$\Delta_6=\frac{\gamma}{n}\left(\frac{1}{n}-1\right),$$

$$\Delta_7 = \frac{\gamma}{n}. \tag{99}$$

Balancing $P^3(x)P''(x)$ with $P^6(x)$ in Eq. (98) gives $N = 1$. Now, Eq. (98) has the formal solution:

$$P(x) = \alpha_0 + \alpha_1 V(x) + \frac{\beta_1}{V(x)}, \tag{100}$$

where α_0, α_1, and β_1 are constants to be determined, provided $\alpha_1^2 + \beta_1^2 \neq 0$. Substituting (100) along with Eq. (5) into Eq. (98) and setting all the coefficients of $V^{j_1}(\xi)(V'(\xi))^{j_2}$, $(j_1 = -6, \ldots, -1, 0, 1, 2, \ldots, 6,\ j_2 = 0, 1)$ to zero leads to:

Case 1: If we set $L_0 = L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = \beta_1 = 0,\ \ \alpha_1 = \sqrt{-\frac{L_4(\Delta_2 + 2\Delta_3)}{\Delta_5}}, \tag{101}$$

with constraint conditions:

$$\Delta_1 = -L_2(\Delta_6 + \Delta_3),$$

$$\Delta_4 = -\frac{\alpha_1^2 L_2(\Delta_2 + \Delta_3) + L_4(\Delta_6 + 2\Delta_3)}{\alpha_1^2}. \tag{102}$$

When $L_2 > 0$, $L_4 > 0$, $\Delta_5(\Delta_2 + 2\Delta_3) < 0$. Then Eq. (1) has singular soliton solution:

$$q(x, t) = \left[\sqrt{-\frac{L_2(\Delta_2 + 2\Delta_3)}{\Delta_5}}\ \operatorname{csch}\left(\sqrt{L_2}\, x\right)\right]^{\frac{1}{n}} e^{i\lambda t}. \tag{103}$$

Case 2: If we set $L_0 = \frac{L_2^2}{4L_4}$, $L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = 0,\ \ \alpha_1 = \alpha_1,\ \ \beta_1 = \frac{\alpha_1 L_2}{2L_4}, \tag{104}$$

with constraint conditions:

$$\Delta_1 = -\frac{L_2(3\Delta_3\alpha_1^2 L_2 - \Delta_6 L_4)}{L_4},$$

$$\Delta_2 = -\frac{2\Delta_6 L_4}{\alpha_1^2 L_2},$$

$$\Delta_4 = \frac{5\Delta_3 \alpha_1^2 L_2 - 4\Delta_6 L_4}{\alpha_1^2},$$

$$\Delta_5 = \frac{2L_4(\Delta_3 \alpha_1^2 L_2 - \Delta_6 L_4)}{\alpha_1^4 L_2},$$

$$\Delta_7 = -\frac{3\Delta_3 \alpha_1^2 L_2 + \Delta_6 L_4}{2L_4}. \tag{105}$$

When $L_4 > 0$, $L_2 < 0$, Eq. (1) has the straddled soliton solution:

$$q(x,\, t) = \left[\frac{\alpha_1 L_2 \left(\tanh^2 \left(\sqrt{-\frac{L_2}{2}} x \right) + 1 \right)}{\sqrt{-2L_2 L_4} \tanh \left(\sqrt{-\frac{L_2}{2}} x \right)} \right]^{\frac{1}{n}} e^{i\lambda t}, \tag{106}$$

and,

$$q(x,\, t) = \left[\frac{\alpha_1 L_2 \left(\coth^2 \left(\sqrt{-\frac{L_2}{2}} x \right) + 1 \right)}{\sqrt{-2L_2 L_4} \coth \left(\sqrt{-\frac{L_2}{2}} x \right)} \right]^{\frac{1}{n}} e^{i\lambda t}. \tag{107}$$

Case 3: If we set $L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = \alpha_0, \quad \alpha_1 = 0, \quad \beta_1 = \beta_1, \quad L_0 = -\frac{\beta_1^2 L_2}{2\alpha_0^2}, \quad L_4 = -\frac{\alpha_0^2 L_2}{2\beta_1^2}, \tag{108}$$

with constraint conditions:

$$\Delta_1 = -\Delta_7 L_2,$$

$$\Delta_2 = -\frac{3}{2}\Delta_3,$$

$$\Delta_4 = -\frac{L_2(4\Delta_3\alpha_0^2 - \Delta_7)}{4\alpha_0^2},$$

$$\Delta_5 = \frac{L_2\Delta_3}{4\alpha_0^2},$$

$$\Delta_6 = -\frac{3}{2}\Delta_7. \tag{109}$$

Thus, we arrive at the WEF solutions:

$$q(x, t) = \left[\alpha_0\left(1 + \frac{\sqrt{-\frac{L_2}{2}}\left[6\wp[(x), g_2, g_3] + L_2\right]}{3\wp'[(x), g_2, g_3]}\right)\right]^{\frac{1}{n}} e^{i\lambda t}, \tag{110}$$

$$q(x, t) = \left[\alpha_0\left(1 + \frac{3\wp'[(x), g_2, g_3]}{\sqrt{-\frac{L_2}{2}}\left[6\wp[(x), g_2, g_3] + L_2\right]}\right)\right]^{\frac{1}{n}} e^{i\lambda t}, \tag{111}$$

where $\alpha_0 > 0$, $L_2 < 0$.

Case 4: If we set $L_0 = L_1 = 0$, then we have the results

$$\alpha_0 = \beta_1 = 0, \quad \alpha_1 = \sqrt{-\frac{L_4(\Delta_2 + 2\Delta_3)}{\Delta_5}}, \quad L_3 = 0, \tag{112}$$

with constraint conditions:

$$\Delta_1 = \frac{\alpha_1^2 L_2^2(\Delta_2 + \Delta_3) + L_4\Delta_7 + \alpha_1^2\Delta_4}{L_4},$$

$$\Delta_6 = -\frac{\alpha_1^2 L_2^2(\Delta_2 + \Delta_3) + 2L_4\Delta_7 + \alpha_1^2\Delta_4}{L_4}. \tag{113}$$

Now, Eq. (1) has the singular soliton solutions, when $L_2 > 0$, $L_4 > 0$ and $(\Delta_2 + 2\Delta_3)\Delta_5 < 0$ as the following:

$$q(x, t) = \varepsilon\left[-\sqrt{-\frac{L_2(\Delta_2 + 2\Delta_3)}{\Delta_5}}\,\frac{\operatorname{sech}^2\frac{\sqrt{L_2}}{2}(x)}{2\tanh\frac{\sqrt{L_2}}{2}(x)}\right]^{\frac{1}{n}} e^{i\lambda t}, \tag{114}$$

and

$$q(x, t)=\varepsilon\left[\sqrt{-\frac{L_2(\Delta_2+2\Delta_3)}{\Delta_5}}\frac{\operatorname{csch}^2\frac{\sqrt{L_2}}{2}(x)}{2\coth\frac{\sqrt{L_2}}{2}\,(x)}\right]^{\frac{1}{n}} e^{i\lambda t}. \tag{115}$$

3.5 *Quadratic-cubic law*

For the quadratic-cubic law nonlinearity of refractive index: $F(|q|^2)=C_1|q|+C_2|q|^2$, then Eq. (1) condenses to

$$i\left(q^l\right)_t+a\left(|q|^r q^l\right)_{xx}+\left[C_1|q|+C_2|q|^2\right]q^l+\gamma\left(\frac{|q|_{xx}}{|q|}\right)q^l=0, \tag{116}$$

where C_1 and C_2 are nonzero constants. The corresponding ODE is written as:

$$-l\lambda\phi(x)+a(r+l)(r+l-1)\phi^{r-1}(x)\phi'^2(x)+a(r+l)\phi^r(x)\phi''(x)$$

$$+C_1\phi^2(x)+C_2\phi^3(x)+\gamma\phi''(x)=0. \tag{117}$$

Eq. (117) is integerable if $r=1$. Then it reduces to

$$\Delta_1\phi(x)+\Delta_2\phi'^2(x)+\Delta_3\phi(x)\phi''(x)+\Delta_4\phi^2(x)+\Delta_5\phi^3(x)+\Delta_6\phi''(x)=0, \tag{118}$$

where

$$\Delta_1=-l\lambda,$$

$$\Delta_2=al(l+1),$$

$$\Delta_3=a(l+1),$$

$$\Delta_4=C_1,$$

$$\Delta_5=C_2,$$

$$\Delta_6=\gamma. \tag{119}$$

Balancing $\phi(x)\phi''(x)$ with $\phi^3(x)$ in Eq. (118) gives $N=2$. Now, Eq. (118) has the formal solution:

$$\phi(x) = \alpha_0 + \alpha_1 V(x) + \alpha_2 V^2(x) + \frac{\beta_1}{V(x)} + \frac{\beta_2}{V^2(x)}, \tag{120}$$

where α_0, α_1, α_2, β_1, and β_2 are constants to be determined, provided $\alpha_2^2 + \beta_2^2 \neq 0$. Substituting (120) along with Eq. (5) into Eq. (118) and setting all the coefficients of $V^{j_1}(\xi)(V'(\xi))^{j_2}$, $(i = -6, ..., -1, 0, 1, 2, ..., 6,\ j = 0, 1)$ to zero leads to:

Case 1: If we set $L_0 = L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = \alpha_1 = \beta_1 = \beta_2 = 0,\ \ \alpha_2 = -\frac{2L_4(2\Delta_2 + 3\Delta_3)}{\Delta_5} \tag{121}$$

with constraint conditions:

$$\Delta_1 = -4\Delta_6 L_2,$$

$$\Delta_4 = -\frac{4\alpha_2 L_2(\Delta_2 + \Delta_3) + 6L_4\Delta_6}{\alpha_2^2}. \tag{122}$$

(I) Setting $L_2 > 0$ and $L_4 < 0$ gives the bright soliton:

$$q(x, t) = \frac{2L_2(2\Delta_2 + 3\Delta_3)}{\Delta_5}\ \mathrm{sech}^2\left(\sqrt{L_2}\, x\right)\ e^{i\lambda t}. \tag{123}$$

(II) Setting $L_2 > 0$ and $L_4 > 0$ yields the singular soliton:

$$q(x, t) = -\frac{2L_2(2\Delta_2 + 3\Delta_3)}{\Delta_5}\ \mathrm{csch}^2\left(\sqrt{L_2}\, x\right)\ e^{i\lambda t}. \tag{124}$$

Case 2: If we set $L_0 = \frac{L_2^2}{4L_4}$, $L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = \alpha_0,\ \ \alpha_1 = \alpha_2 = \ \beta_1 = \ 0,\ \ \beta_2 = \frac{\alpha_0 L_2}{2L_4}, \tag{125}$$

with constraint conditions:

$$\Delta_1 = 2L_2\Delta_6,$$

$$\Delta_4 = \frac{2L_2\alpha_0(\Delta_2 + \Delta_3) - 3L_2\Delta_6}{\alpha_0},$$

$$\Delta_5 = -\frac{L_2(2\Delta_2 + 3\Delta_3)}{\alpha_0}. \tag{126}$$

Thus, we arrive at the singular soliton:

$$q(x, t) = -\alpha_0 \text{cosech}^2\left(\sqrt{-\frac{L_2}{2}}x\right)e^{i\lambda t}, \tag{127}$$

and the bright soliton:

$$q(x, t) = \alpha_0 \text{sech}^2\left(\sqrt{-\frac{L_2}{2}}x\right)e^{i\lambda t}, \tag{128}$$

where $L_2 < 0$.

Case 3: If we set $L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = \alpha_0,\ \alpha_1 = \beta_1 = \beta_2 = 0,\ \alpha_2 = -\frac{6L_4(\Delta_3\alpha_0 + \Delta_6)}{\Delta_5\alpha_0},\ L_0 = \frac{\alpha_0 L_2}{2\alpha_2}, \tag{129}$$

with constraint conditions:

$$\Delta_1 = -\frac{3\Delta_3\alpha_0(2\alpha_0 L_4 - \alpha_2 L_2) - \Delta_6(6\alpha_0 L_4 - 5\alpha_2 L_2)}{\alpha_2},$$

$$\Delta_2 = \frac{3\Delta_6}{2\alpha_0},$$

$$\Delta_4 = \frac{4\Delta_3\alpha_0(3\alpha_0 L_4 - \alpha_2 L_2) + 6\Delta_6(2\alpha_0 L_4 - \alpha_2 L_2)}{\alpha_0\alpha_2}. \tag{130}$$

Thus, we arrive at the WEF solutions:

$$q(x, t) = \alpha_0 + \frac{9\alpha_2\wp^2[(x), g_2, g_3]}{L_4(6\wp[(x), g_2, g_3] + L_2)^2}e^{i\lambda t}, \tag{131}$$

$$q(x, t) = \alpha_0\left[1 + \frac{L_2(6\wp[(x), g_2, g_3] + L_2)^2}{18\wp^2[(x), g_2, g_3]}\right]e^{i\lambda t}. \tag{132}$$

Case 4: If we set $L_0 = L_1 = 0$, then we have the results

$$\alpha_0 = \alpha_1 = \beta_1 = \beta_2 = 0, \quad \alpha_2 = \alpha_2, \quad L_3 = 0, \tag{133}$$

with constraint conditions:

$$\Delta_1 = -4\Delta_6 L_2,$$

$$\Delta_4 = -\frac{2L_2\alpha_2(2\Delta_2 + 3\Delta_3) + 6\Delta_6 L_4}{\alpha_2},$$

$$\Delta_5 = -\frac{2L_2(2\Delta_2 + 3\Delta_3)}{\alpha_2}. \tag{134}$$

Therefore, we arrive at the straddled solitons:

$$q(x, t) = \frac{L_2\alpha_2 \operatorname{sech}^4\left(\frac{\sqrt{L_2}}{2}x\right)}{4L_4 \tanh^2\left(\frac{\sqrt{L_2}}{2}x\right)} e^{i\lambda t}, \tag{135}$$

and

$$q(x, t) = \frac{L_2\alpha_2 \operatorname{csch}^4\left(\frac{\sqrt{L_2}}{2}x\right)}{4L_4 \coth^2\left(\frac{\sqrt{L_2}}{2}x\right)} e^{i\lambda t}. \tag{136}$$

3.6 *Polynomial law*

For the polynomial law nonlinearity of refractive index: $F(|q|^2) = C_1|q|^2 + C_2|q|^4 + C_3|q|^6$, then Eq. (1) collapses to

$$i\left(q^l\right)_t + a\left(|q|^r q^l\right)_{xx} + \left[C_1|q|^2 + C_2|q|^4 + C_3|q|^6\right] q^l + \gamma\left(\frac{|q|_{xx}}{|q|}\right) q^l = 0, \tag{137}$$

where C_j, $(j = 1-3)$ are constants. The corresponding ODE is written as:

$$-l\lambda\phi(x) + a(r+l)(r+l-1)\phi^{r-1}(x)\phi'^2(x) + a(r+l)\phi^r(x)\phi''(x)$$

$$+C_1\phi^3(x) + C_2\phi^5(x) + C_3\phi^7(x) + \gamma\phi''(x) = 0. \tag{138}$$

Eq. (138) is integrable if $r = 4$. Then Eq. (138) reduces to

$$-l\lambda\phi(x) + a(4+l)(3+l)\phi^3(x)\phi'^2(x) + a(4+l)\phi^4(x)\phi''(x)$$

$$+C_1\phi^3(x) + C_2\phi^5(x) + C_3\phi^7(x) + \gamma\phi''(x) = 0. \tag{139}$$

Balancing $\phi^4(x)\phi''(x)$ with $\phi^7(x)$ in Eq. (139) gives $N = 1$. Now, Eq. (139) has the formal solution:

$$\phi(x) = \alpha_0 + \alpha_1 V(x) + \frac{\beta_1}{V(x)}, \tag{140}$$

where α_0, α_1, and β_1 are constants to be determined, provided $\alpha_1^2 + \beta_1^2 \neq 0$. Substituting (140) along with Eq. (5) into Eq. (139) and setting all the coefficients of $V^{j_1}(\xi)(V'(\xi))^{j_2}$, $(j_1 = -5, ..., -1, 0, 1, 2, ..., 5, j_2 = 0, 1)$ to zero leads to:

Case 1: If we set $L_0 = L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = \alpha_0,\ \beta_1 = 0,\ \alpha_1 = \alpha_1,\ L_2 = -\frac{\alpha_0^2 L_4}{\alpha_1^2}, \tag{141}$$

with constraint conditions:

$$\Delta_1 = -\frac{5\alpha_0^2\Delta_7 L_4}{2\alpha_1^2},$$

$$\Delta_2 = -\frac{\Delta_7(8\alpha_0+3)}{\alpha_0^2(3\alpha_0+5)},$$

$$\Delta_3 = \frac{\Delta_7(8\alpha_0+3)}{2\alpha_0^2(3\alpha_0+5)},$$

$$\Delta_4 = \frac{31\Delta_7\alpha_0 L_4}{2\alpha_1^2(3\alpha_0+5)},$$

$$\Delta_5 = -\frac{L_4\Delta_7(8\alpha_0+3)}{\alpha_1^2\alpha_0(3\alpha_0+5)},$$

$$\Delta_6 = -\frac{\Delta_7}{2}. \tag{142}$$

When $L_2 > 0$, $L_4 < 0$. Then Eq. (1) has bright soliton solution:

$$q(x,\, t) = \alpha_0 \left[1 + \operatorname{sech} \left(\sqrt{-\frac{\alpha_0^2 L_4}{\alpha_1^2}}\; x \right) \right] e^{i\lambda t}. \tag{143}$$

Case 2: If we set $L_0 = \dfrac{L_2^2}{4L_4}$, $L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = \alpha_0,\;\; \beta_1 = 0,\;\; \alpha_1 = \alpha_1,\;\; L_2 = -\frac{2\alpha_0^2 L_4}{\alpha_1^2}, \tag{144}$$

with constraint conditions:

$$\begin{aligned}
\Delta_1 &= -\frac{2\alpha_0(8\Delta_3\alpha_0^3 L_4 - 2\Delta_4\alpha_0\alpha_1^2 - \Delta_4\alpha_1^2)}{\alpha_1^2},\\
\Delta_2 &= -2\Delta_3,\\
\Delta_5 &= -\frac{2\alpha_0\Delta_3 L_4}{\alpha_1^2},\\
\Delta_6 &= -\frac{12\Delta_3\alpha_0^3 L_4 - 3\Delta_4\alpha_0\alpha_1^2 - \Delta_4\alpha_1^2}{\alpha_0 L_4},\\
\Delta_7 &= \frac{16\Delta_3\alpha_0^3 L_4 - 4\Delta_4\alpha_0\alpha_1^2 - \Delta_4\alpha_1^2}{2\alpha_0 L_4}.
\end{aligned} \tag{145}$$

Setting $L_2 < 0$ and $L_4 > 0$ yields the dark soliton:

$$q(x,\, t) = \alpha_0 \left[1 + \tanh \left(\sqrt{\frac{\alpha_0^2 L_4}{\alpha_1^2}} x \right) \right] e^{i\lambda t}, \tag{146}$$

and the singular soliton:

$$q(x,\, t) = \alpha_0 \left[1 + \coth \left(\sqrt{\frac{\alpha_0^2 L_4}{\alpha_1^2}} x \right) \right] e^{i\lambda t}. \tag{147}$$

Case 3: If we set $L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = -\frac{1}{3},\;\; \alpha_1 = 0,\;\; \beta_1 = \beta_1,\;\; L_0 = -\frac{9\beta_1^2 L_2}{2},\;\; L_4 = -\frac{L_2}{18\beta_1^2}, \tag{148}$$

with constraint conditions:

$$
\begin{aligned}
&\Delta_1 = 2L_2(\Delta_6 + \Delta_7),\\
&\Delta_2 = \frac{3}{2}\Delta_6,\\
&\Delta_3 = -\frac{3}{4}\Delta_6,\\
&\Delta_4 = 3L_2(2\Delta_6 + 3\Delta_7),\\
&\Delta_5 = -\frac{9}{4}L_2\Delta_6.
\end{aligned} \tag{149}
$$

Thus, we arrive at the WEF solutions:

$$
q(x,\,t) = -\frac{1}{3}\left[1 + \sqrt{-\frac{L_2}{18}}\left(\frac{6\wp[(x),\,g_2,\,g_3] + L_2}{\wp'[(x),\,g_2,\,g_3]}\right)\right]\ e^{i\lambda t}, \tag{150}
$$

where $L_4 > 0$, $L_2 < 0$,

$$
q(x,\,t) = -\frac{1}{3}\left[1 + \frac{3}{\sqrt{-\frac{L_2}{2}}}\left(\frac{\wp'[(x),\,g_2,\,g_3]}{6\wp[(x),\,g_2,\,g_3] + L_2}\right)\right]\ e^{i\lambda t}, \tag{151}
$$

where $L_0 > 0$, $L_2 < 0$.

3.7 *Triple-power law*

For the triple power law nonlinearity of refractive index: $F(|q|^2) = C_1|q|^{2n} + C_2|q|^{4n} + C_3|q|^{6n}$, then Eq. (1) turns into:

$$
i\left(q^l\right)_t + a\left(|q|^r q^l\right)_{xx} + \left[C_1|q|^{2n} + C_2|q|^{4n} + C_3|q|^{6n}\right]q^l + \gamma\left(\frac{|q|_{xx}}{|q|}\right)q^l = 0, \tag{152}
$$

where C_j, $(j = 1-3)$ are constants. The corresponding ODE is written as:

$$
\begin{aligned}
&-l\lambda\phi(x) + a(r+l)(r+l-1)\phi^{r-1}(x)\phi'^2(x) + a(r+l)\phi^r(x)\phi''(x)\\
&+C_1\phi^{2n+1}(x) + C_2\phi^{4n+1}(x) + C_3\phi^{6n+1}(x) + \gamma\phi''(x) = 0.
\end{aligned} \tag{153}
$$

Eq. (153) is integrable if $r = 4n$. Then it reduces to

$$-l\lambda\phi(x) + a(4n+l)(4n+l-1)\phi^{4n-1}(x)\phi'^2(x) + a(4n+l)\phi^{4n}(x)\phi''(x)$$

$$+C_1\phi^{2n+1}(x) + C_2\phi^{4n+1}(x) + C_3\phi^{6n+1}(x) + \gamma\phi''(x) = 0, \tag{154}$$

where the balance $N = \frac{1}{n}$, $n \neq 0$. By using

$$\phi(x) = P^{\frac{1}{n}}(x), \tag{155}$$

where $P(x)$ is new function, Eq. (154) becomes

$$\Delta_1 P^2(x) + \Delta_2 P^4(x)P'^2(x) + \Delta_3 P^5(x)P''(x) + \Delta_4 P^4(x) + \Delta_5 P^6(x) + \Delta_6 P^8(x)$$

$$+\Delta_7 P'^2(x) + \Delta_8 P(x)P''(x) = 0, \tag{156}$$

where

$$\begin{aligned}
\Delta_1 &= -l\lambda,\\
\Delta_2 &= \frac{a}{n}(4n+l)\left(3+\frac{l}{n}\right),\\
\Delta_3 &= \frac{a(4n+l)}{n},\\
\Delta_4 &= C_1,\\
\Delta_5 &= C_2,\\
\Delta_6 &= C_3,\\
\Delta_7 &= \frac{\gamma}{n}\left(\frac{1}{n}-1\right),\\
\Delta_8 &= \frac{\gamma}{n}.
\end{aligned} \tag{157}$$

Balancing $P^5(x)P''(x)$ with $P^8(x)$ in Eq. (156) gives $N = 1$. Thus, Eq. (156) simplifies to:

$$P(x) = \alpha_0 + \alpha_1 V(x) + \frac{\beta_1}{V(x)}, \tag{158}$$

where α_0, α_1, and β_1 are constants to be determined, provided $\alpha_1^2 + \beta_1^2 \neq 0$. Substituting (158) along with Eq. (5) into Eq. (156) and setting all the coefficients of $V^{j_1}(\xi)(V'(\xi))^{j_2}$, $(j_1 = -8, \ldots, -1, 0, 1, 2, \ldots, 8,\ j_2 = 0, 1)$ to zero leads to:

Case 1: If we set $L_0 = L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = \beta_1 = 0,\ \ \alpha_1 = \sqrt{-\frac{L_4(\Delta_2 + 2\Delta_3)}{\Delta_6}}, \tag{159}$$

with constraint conditions:

$$\Delta_1 = -L_2(\Delta_7 + \Delta_8),$$

$$\Delta_4 = -\frac{L_4(\Delta_7 + 2\Delta_8)}{\alpha_1^2},$$

$$\Delta_5 = -L_2(\Delta_2 + \Delta_3). \tag{160}$$

(I) Setting $L_2 > 0$, $L_4 < 0$, $\Delta_6(\Delta_2 + 2\Delta_3) > 0$ yields the bright soliton:

$$q(x, t) = \left[\sqrt{\frac{L_2(\Delta_2 + 2\Delta_3)}{\Delta_6}}\ \mathrm{sech}\left(\sqrt{L_2}\, x\right)\right]^{\frac{1}{n}} e^{i\lambda t}. \tag{161}$$

(II) Setting $L_2 > 0,\ L_4 > 0$, $\Delta_6(\Delta_2 + 2\Delta_3) < 0$ causes to the singular soliton:

$$q(x, t) = \left[\sqrt{-\frac{L_2(\Delta_2 + 2\Delta_3)}{\Delta_6}}\ \mathrm{csch}\ \left(\sqrt{L_2}\, x\right)\right]^{\frac{1}{n}} e^{i\lambda t}. \tag{162}$$

Case 2: If we set $L_0 = \dfrac{L_2^2}{4L_4}$, $L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = \alpha_0, \quad \beta_1 = 0, \quad \alpha_1 = \alpha_1, \quad L_2 = -\frac{2\alpha_0^2 L_4}{\alpha_1^2}, \tag{163}$$

with constraint conditions:

$$\Delta_1 = -4\Delta_4\alpha_0^2,$$

$$\Delta_2 = -\frac{3\Delta_3}{2},$$

$$\Delta_5 = \frac{2\Delta_3\alpha_0^2 L_4}{\alpha_1^2},$$

$$\Delta_6 = -\frac{\Delta_3 L_4}{2\alpha_1^2},$$

$$\Delta_7 = \frac{3\Delta_4\alpha_1^2}{L_4},$$

$$\Delta_5 = -\frac{2\Delta_4\alpha_1^2}{L_4}. \tag{164}$$

Setting $L_2 < 0$, $L_4 < 0$ yields the dark soliton:

$$q(x, t) = \left[\alpha_0\left(1 + \tanh\left(\sqrt{\frac{\alpha_0^2 L_4}{\alpha_1^2}}x\right)\right)\right]^{\frac{1}{n}} e^{i\lambda t}, \tag{165}$$

and the singular soliton:

$$q(x, t) = \left[\alpha_0\left(1 + \coth\left(\sqrt{\frac{\alpha_0^2 L_4}{\alpha_1^2}}x\right)\right)\right]^{\frac{1}{n}} e^{i\lambda t}, \tag{166}$$

where $\alpha_0 > 0$.

Case 3: If we set $L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = \alpha_0, \quad \alpha_1 = 0, \quad \beta_1 = \beta_1, \quad L_0 = -\frac{\beta_1^4 L_4}{\alpha_0^4}, \quad L_2 = -\frac{2\beta_1^2 L_4}{\alpha_0^2}, \tag{167}$$

with constraint conditions:

$$\Delta_1 = \frac{2\Delta_8\beta_1^2 L_4}{\alpha_0^2},$$

$$\Delta_2 = -\frac{3\Delta_3}{2},$$

$$\Delta_4 = -\frac{\Delta_8 \beta_1^2 L_4}{2\alpha_0^4},$$

$$\Delta_5 = \frac{2\Delta_3 \beta_1^2 L_4}{\alpha_0^2},$$

$$\Delta_6 = -\frac{\Delta_3 \beta_1^2 L_4}{2\alpha_0^4},$$

$$\Delta_2 = -\frac{3\Delta_8}{2}. \tag{168}$$

Thus, we arrive at the WEF solutions:

$$q(x, t) = \left[\alpha_0 + \frac{\beta_1 \sqrt{L_4}}{3} \left(\frac{6\wp[(x), g_2, g_3] + L_2}{\wp'[(x), g_2, g_3]}\right)\right]^{\frac{1}{n}} e^{i\lambda t}, \tag{169}$$

where $L_4 > 0$, $L_2 < 0$,

$$q(x, t) = \left[\alpha_0 + \frac{3\beta_1}{\sqrt{L_0}} \left(\frac{\wp'[(x), g_2, g_3]}{6\wp[(x), g_2, g_3] + L_2}\right)\right]^{\frac{1}{n}} e^{i\lambda t}, \tag{170}$$

where $\alpha_0 > 0$, $L_4 < 0$, $L_0 > 0$.

Case 5: If we set $L_0 = L_1 = 0$, then we have the results

$$\alpha_0 = \beta_1 = 0, \quad \alpha_1 = \alpha_1, \quad L_3 = 0, \tag{171}$$

with constraint conditions:

$$\Delta_1 = \frac{L_2(\Delta_4 \alpha_1^2 + \Delta_8 L_4)}{L_4},$$

$$\Delta_5 = -L_2(\Delta_2 + \Delta_3),$$

$$\Delta_6 = -\frac{L_4(\Delta_2 + 2\Delta_3)}{\alpha_1^2},$$

$$\Delta_7 = -\frac{\Delta_4 \alpha_1^2 + 2\Delta_8 L_4}{L_4}. \tag{172}$$

Now, Eq. (1) has the singular soliton solutions when $L_2 > 0$, $L_4 > 0$ as the following:

$$q(x, t) = \left[-\frac{\alpha_1 L_2}{2\sqrt{L_2 L_4}} \frac{\text{sech}^2 \frac{\sqrt{L_2}}{2}(x)}{\tanh \frac{\sqrt{L_2}}{2}(x)} \right]^{\frac{1}{n}} e^{i\lambda t}, \tag{173}$$

and

$$q(x, t) = \left[\frac{\alpha_1 L_2}{2\sqrt{L_2 L_4}} \frac{\text{csch}^2 \frac{\sqrt{L_2}}{2}(x)}{\coth \frac{\sqrt{L_2}}{2}(x)} \right]^{\frac{1}{n}} e^{i\lambda t}, \tag{174}$$

where $\alpha_1 > 0$.

3.8 *Anti-cubic law*

For the anti-cubic law nonlinearity of refractive index: $F(|q|^2) = \frac{C_1}{|q|^4} + C_2 |q|^2 + C_3 |q|^4$, then Eq. (1) simplifies to:

$$i\left(q^l\right)_t + a\left(|q|^r q^l\right)_{xx} + \left[\frac{C_1}{|q|^4} + C_2 |q|^2 + C_3 |q|^4\right] q^l + \gamma\left(\frac{|q|_{xx}}{|q|}\right) q^l = 0, \tag{175}$$

where C_j, $(j = 1-3)$ are constants. The corresponding ODE is written as:

$$\begin{aligned} &-l\lambda\phi(x) + a(r+l)(r+l-1)\phi^{r-1}(x)\phi'^2(x) + a(r+l)\phi^r(x)\phi''(x) \\ &+ C_1\phi^{-3}(x) + C_2\phi^3(x) + C_3\phi^5(x) + \gamma\phi''(x) = 0. \end{aligned} \tag{176}$$

Eq. (176) is integrable if $r = 2$. Then it reduces to

$$\begin{aligned} &\Delta_1\phi^4(x) + \Delta_2\phi^4(x)\phi'^2(x) + \Delta_3\phi^5(x)\phi''(x) + \Delta_4 + \Delta_5\phi^6(x) \\ &+ \Delta_6\phi^8(x) + \Delta_7\phi^3(x)\phi''(x) = 0, \end{aligned} \tag{177}$$

where

$$\Delta_1 = -l\lambda,$$

$$\Delta_2 = a(2+l)(1+l),$$

$$\Delta_3 = a(2+l),$$

$$\Delta_4 = C_1,$$

$$\Delta_5 = C_2,$$

$$\Delta_6 = C_3,$$

$$\Delta_7 = \gamma. \tag{178}$$

Balancing $\phi^5(x)\phi''(x)$ with $\phi^8(x)$ in Eq. (177) gives $N = 1$. Thus, Eq. (177) decreases to:

$$\phi(x) = \alpha_0 + \alpha_1 V(x) + \frac{\beta_1}{V(x)}, \tag{179}$$

where α_0, α_1, and β_1 are constants to be determined, provided $\alpha_1^2 + \beta_1^2 \neq 0$. Substituting (179) along with Eq. (5) into Eq. (177) and setting all the coefficients of $V^{j_1}(\xi)(V'(\xi))^{j_2}$, $(j_1 = -5, ..., -1, 0, 1, 2, ..., 5,\ j_2 = 0, 1)$ to zero leads to:

Case 1: If we set $L_0 = L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = \beta_1 = 0,\ \ \alpha_1 = \sqrt{-\frac{L_4(\Delta_2 + 2\Delta_3)}{\Delta_6}}, \tag{180}$$

with constraint conditions:

$$\Delta_1 = -L_2(\Delta_7 + \Delta_8),$$

$$\Delta_4 = -\frac{L_4(\Delta_7 + 2\Delta_8)}{\alpha_1^2},$$

$$\Delta_5 = -L_2(\Delta_2 + \Delta_3). \tag{181}$$

(I) Setting $L_2 > 0$, $L_4 < 0$, $\Delta_6(\Delta_2 + 2\Delta_3) > 0$ gives the bright soliton:

$$q(x,\,t)=\sqrt{\frac{L_2(\Delta_2+2\Delta_3)}{\Delta_6}}\,\operatorname{sech}\left(\sqrt{L_2}\,x\right)\,e^{i\lambda t}. \tag{182}$$

(II) Setting $L_2>0$, $L_4>0$, $\Delta_6(\Delta_2+2\Delta_3)<0$ yields the singular soliton:

$$q(x,\,t)=\sqrt{-\frac{L_2(\Delta_2+2\Delta_3)}{\Delta_6}}\,\operatorname{csch}\,\left(\sqrt{L_2}\,x\right)\,e^{i\lambda t}. \tag{183}$$

Case 2: If we set $L_0=\dfrac{L_2^2}{4L_4}$, $L_1=L_3=0$, then we have the results

$$\alpha_0=\alpha_1=0,\quad \beta_1=\beta_1 \tag{184}$$

with constraint conditions:

$$\Delta_1=-\Delta_2\beta_1^2L_4-\Delta_7L_2,$$

$$\Delta_4=0,$$

$$\Delta_5=-\frac{L_2(2\beta_1^2\Delta_2L_4+2\Delta_3\beta_1^2L_4+\Delta_7L_2)}{2\beta_1^2L_4},$$

$$\Delta_6=-\frac{L_2^2(\Delta_2L_4+2\Delta_3)}{4\beta_1^2L_4}. \tag{185}$$

Setting $L_2<0$, $L_4>0$ gives the singular soliton:

$$q(x,\,t)=\frac{2\beta_1}{\sqrt{-\frac{2L_2}{L_4}}\tanh\left(\sqrt{-\frac{L_2}{2}}x\right)}\,e^{i\lambda t}, \tag{186}$$

and the dark soliton:

$$q(x,\,t)=\frac{2\beta_1}{\sqrt{-\frac{2L_2}{L_4}}\coth\left(\sqrt{-\frac{L_2}{2}}x\right)}\,e^{i\lambda t}, \tag{187}$$

Case 3: If we set $L_1=L_3=0$, then we have the results

$$\alpha_0 = \beta_1 = 0, \quad \alpha_1 = \sqrt{-\frac{(\Delta_2 + 2\Delta_3)L_4}{\Delta_6}}, \tag{188}$$

where $L_4 > 0$, $\Delta_6(\Delta_2 + 2\Delta_3) < 0$, with constraint conditions:

$$\Delta_1 = -\Delta_2 \alpha_1^2 L_0 - \Delta_7 L_2,$$

$$\Delta_4 = 0,$$

$$\Delta_5 = -\frac{(\Delta_2 + \Delta_3)\alpha_1^2 L_2 + 2\Delta_7 L_4}{\alpha_1^2}. \tag{189}$$

Thus, we arrive at the WEF solutions:

$$q(x, t) = 3\sqrt{-\frac{(\Delta_2 + 2\Delta_3)}{\Delta_6}} \left(\frac{\wp'[(x), g_2, g_3]}{6\wp[(x), g_2, g_3] + L_2} \right) e^{i\lambda t}, \tag{190}$$

where $\Delta_6(\Delta_2 + 2\Delta_3) < 0$,

$$q(x, t) = \frac{1}{3}\sqrt{-\frac{(\Delta_2 + 2\Delta_3)L_4 L_0}{\Delta_6}} \left(\frac{\wp'[(x), g_2, g_3]}{6\wp[(x), g_2, g_3] + L_2} \right) e^{i\lambda t}, \tag{191}$$

where $L_0 > 0$, $L_4 > 0$, $\Delta_6(\Delta_2 + 2\Delta_3) < 0$.

Case 4: If we set $L_0 = L_1 = 0$, then we have the results

$$\alpha_0 = \beta_1 = 0, \quad \alpha_1 = \sqrt{-\frac{L_4(\Delta_2 + 2\Delta_3)}{\Delta_6}}, \quad L_3 = 0, \tag{192}$$

with constraint conditions:

$$\Delta_1 = -\Delta_7 L_2,$$

$$\Delta_4 = 0,$$

$$\Delta_5 = -\frac{\alpha_1^2 L_2(\Delta_2 + \Delta_3) + 2\Delta_7 L_4}{\alpha_1^2}. \tag{193}$$

Now, Eq. (1) has the straddled soliton solutions when $L_2 > 0$, $L_4 > 0$, and $(\Delta_2 + 2\Delta_3)\Delta_6 < 0$ as given by (183).

$$q(x,\,t) = -\varepsilon\sqrt{-\frac{L_2(\Delta_2+2\Delta_3)}{\Delta_6}\,\frac{\operatorname{sech}^2\frac{\sqrt{L_2}}{2}(x)}{2\tanh\frac{\sqrt{L_2}}{2}\;(x)}}\;e^{i\lambda t}, \tag{194}$$

and

$$q(x,\,t) = \varepsilon\sqrt{-\frac{L_2(\Delta_2+2\Delta_3)}{\Delta_6}\,\frac{\operatorname{csch}^2\frac{\sqrt{L_2}}{2}(x)}{2\coth\frac{\sqrt{L_2}}{2}\;(x)}}\;e^{i\lambda t}. \tag{195}$$

3.9 *Generalized anti-cubic law*

For the generalized anti-cubic law nonlinearity of refractive index: $F(|q|^2) = \dfrac{C_1}{|q|^{2(n+1)}} + C_2\,|q|^{2n} + C_3\,|q|^{2(n+1)}$, then Eq. (1) turns into:

$$i\left(q^l\right)_t + a\left(|q|^r q^l\right)_{xx} + \left[\frac{C_1}{|q|^{2(n+1)}} + C_2\,|q|^{2n} + C_3\,|q|^{2(n+1)}\right] q^l + \gamma\left(\frac{|q|_{xx}}{|q|}\right) q^l = 0, \tag{196}$$

where C_j, $(j = 1-3)$ are constants. The corresponding ODE is written as:

$$-l\lambda\phi(x) + a(r+l)(r+l-1)\phi^{r-1}(x)\phi'^2(x) + a(r+l)\phi^r(x)\phi''(x)$$

$$+C_1\phi^{-2n-1}(x) + C_2\phi^{2n+1}(x) + C_3\phi^{2n+3}(x) + \gamma\phi''(x) = 0. \tag{197}$$

Eq. (197) is integrable if $r = n+1$. Then it reduces to

$$-l\lambda\phi(x) + a(n+1+l)(n+l)\phi^n(x)\phi'^2(x) + a(n+1+l)\phi^{n+1}(x)\phi''(x)$$

$$+C_1\phi^{-2n-1}(x) + C_2\phi^{2n+1}(x) + C_3\phi^{2n+3}(x) + \gamma\phi''(x) = 0. \tag{198}$$

Balancing $\phi^{n+1}(x)\phi''(x)$ with $\phi^{2n+3}(x)$ we get, $N = \dfrac{2}{n+1}$, $n \neq -1$. By using

$$\phi(x) = P^{\frac{2}{n+1}}(x) \tag{199}$$

where $P(x)$ is a new function, Eq. (198) is:

$$-l\lambda P^2(x)+\frac{al}{(n+1)^2}(n+1+l)P(x)P'^2(x)+\frac{a}{n+1}(n+1+l)P^2(x)P''(x)$$

$$+C_1+C_2P^2(x)P^{\frac{2n}{n+1}}(x)+C_3P^4(x)+\frac{-n\gamma}{(n+1)^2}P'^2(x)+\frac{\gamma}{n+1}P(x)P''(x)=0. \tag{200}$$

Eq. (200) is integrable if $C_2=0$. Then, Eq. (200) simplifies to

$$\Delta_1 P^2(x)+\Delta_2 P(x)P'^2(x)+\Delta_3 P^2(x)P''(x)+\Delta_4+\Delta_5 P^4(x)+\Delta_6 P'^2(x)+\Delta_7 P(x)P''(x)=0, \tag{201}$$

where

$$\Delta_1=-l\lambda,$$

$$\Delta_2=\frac{al}{(n+1)^2}(n+1+l),$$

$$\Delta_3=\frac{a}{n+1}(n+1+l),$$

$$\Delta_4=C_1,$$

$$\Delta_5=C_3,$$

$$\Delta_6=\frac{-n\gamma}{(n+1)^2},$$

$$\Delta_7=\frac{\gamma}{n+1}. \tag{202}$$

Balancing $P^2(x)P''(x)$ with $P^4(x)$ in Eq. (201) gives $N=2$. Now, Eq. (201) has the formal solution:

$$\phi(x)=\alpha_0+\alpha_1 V(x)+\alpha_2 V^2(x)+\frac{\beta_1}{V(x)}+\frac{\beta_2}{V^2(x)}, \tag{203}$$

where α_0, α_1, α_2, β_1, and β_2 are constants to be determined, provided $\alpha_2^2+\beta_2^2\neq 0$. Substituting (203) along with Eq. (5) into Eq. (201) and setting all the coefficients of $V^{j_1}(\xi)(V'(\xi))^{j_2}$, $(j_1=-8, ..., -1, 0, 1, 2, ..., 8,\ j_2=0, 1)$ to zero leads to:

Case 1: If we set $L_0=L_1=L_3=0$, then we have the results

$$\alpha_0=\alpha_0,\ \alpha_1=\beta_1=\beta_2=0,\ \alpha_2=\alpha_2 \tag{204}$$

with constraint conditions:

$$\Delta_1=-\frac{4\Delta_2\alpha_0^3\alpha_2^2L_2^2L_4-108\Delta_6\alpha_0^3L_4^3+108\Delta_6\alpha_0^2L_2\alpha_2L_4^2-48\Delta_6\alpha_0L_4\alpha_2^2L_2^2+8\Delta_6\alpha_2^3L_2^3}{\alpha_0\alpha_2\left(3\alpha_0L_4-2\alpha_2L_2\right)\left(3\alpha_0L_4-\alpha_2L_2\right)},$$

$$\Delta_3=-\frac{6\Delta_2\alpha_0^3L_4^2-6\Delta_2\alpha_0^2L_2\alpha_2L_4+2\Delta_2\alpha_2^2\alpha_0L_2^2-9\Delta_6\alpha_0L_4^2+3\Delta_6\alpha_2L_2L_4}{\alpha_0\left(3\alpha_0L_4-2\alpha_2L_2\right)\left(3\alpha_0L_4-\alpha_2L_2\right)},$$

$$\Delta_4=-\frac{2\Delta_6\alpha_0(3\alpha_0L_4-2\alpha_2L_2)}{\alpha_2},$$

$$\Delta_5=\frac{4\Delta_2\alpha_0\alpha_2^2L_2^2L_4-54\Delta_6\alpha_0L_4^3+18\Delta_6L_2\alpha_2L_4}{\alpha_0\alpha_2\left(3\alpha_0L_4-2\alpha_2L_2\right)\left(3\alpha_0L_4-\alpha_2L_2\right)},$$

$$\Delta_7=\frac{2\Delta_2\alpha_0^3L_4-\Delta_2\alpha_0^2L_2\alpha_2+3\Delta_6\alpha_0L_4-\Delta_6\alpha_2L_2}{\alpha_0\left(3\alpha_0L_4-2\alpha_2L_2\right)}. \tag{205}$$

(I) Setting $L_2>0$, $L_4<0$ yields bright soliton:

$$q(x,t)=\left[\alpha_0+\frac{\alpha_2L_2}{L_4}\operatorname{sech}^2\left(\sqrt{L_2}\,x\right)\right]^{\frac{2}{n+1}}e^{i\lambda t}. \tag{206}$$

(II) Setting $L_2>0$, $L_4>0$ causes to the singular soliton:

$$q(x,t)=\left[\alpha_0+\frac{\alpha_2L_2}{L_4}\operatorname{csch}^2\left(\sqrt{L_2}\,x\right)\right]^{\frac{2}{n+1}}e^{i\lambda t}. \tag{207}$$

and straddled soliton solutions

$$q(x,t)=\left[\alpha_0+\frac{\alpha_2L_2\operatorname{sech}^4\left(\frac{\sqrt{L_2}}{2}x\right)}{4L_4\tanh^2\left(\frac{\sqrt{L_2}}{2}x\right)}\right]^{\frac{2}{n+1}}e^{i\lambda t}, \tag{208}$$

and,

$$q(x,\, t) = \left[\alpha_0 + \frac{\alpha_2 L_2 \text{csch}^4 \left(\frac{\sqrt{L_2}}{2} x \right)}{4 L_4 \coth^2 \left(\frac{\sqrt{L_2}}{2} x \right)} \right]^{\frac{2}{n+1}} e^{i\lambda t}. \tag{209}$$

Case 2: If we set $L_0 = \frac{L_2^2}{4L_4}$, $L_1 = L_3 = 0$, then we have the results

$$\alpha_0 = \alpha_1 = \beta_2 = 0, \;\; \alpha_2 = \alpha_2, \;\; \beta_1 = \beta_1 \tag{210}$$

with constraint conditions:

$$\Delta_1 = \frac{\Delta_3 \alpha_2^2 L_2^2 + 28 \Delta_6 L_4^2}{2 \alpha_2 L_4},$$

$$\Delta_2 = -\frac{\Delta_3 \alpha_2^2 L_2^2 - 12 \Delta_6 L_4^2}{\alpha_2^2 L_2^2},$$

$$\Delta_4 = -\frac{\Delta_6 \alpha_2 L_2^2}{2 L_4},$$

$$\Delta_5 = -\frac{2 \Delta_3 \alpha_2^2 L_2^2 L_4 + 48 \Delta_6 L_4^3}{\alpha_2^3 L_2^2},$$

$$\Delta_7 = -\frac{\Delta_6 L_2^2}{\alpha_2 L_2}. \tag{211}$$

Thus, we arrive at the straddled soliton:

$$q(x,\, t) = \left[-\frac{\alpha_1 L_2}{2 L_4} \tanh^2 \left(\sqrt{-\frac{L_2}{2}} x \right) + \frac{\beta_1}{\sqrt{-\frac{L_2}{2L_4}} \tanh \left(\sqrt{-\frac{L_2}{2}} x \right)} \right]^{\frac{2}{n+1}} e^{i\lambda t}, \tag{212}$$

and

$$q(x,t)=\left[-\frac{\alpha_1 L_2}{2L_4}\coth^2\left(\sqrt{-\frac{L_2}{2}}x\right)+\frac{\beta_1}{\sqrt{-\frac{L_2}{2L_4}}\coth\left(\sqrt{-\frac{L_2}{2}}x\right)}\right]^{\frac{2}{n+1}} e^{i\lambda t}, \tag{213}$$

where $L_2 < 0$.

Case 3: If we set $L_1 = L_3 = 0$, then we have the results:

$$\alpha_0=\alpha_1=\beta_1=\beta_2=0,\quad \alpha_2=-\frac{2\Delta_6 L_2}{\Delta_7 L_0},\ L_4=-\frac{2\alpha_2 L_2(\Delta_2+\Delta_3)}{3\Delta_7} \tag{214}$$

with constraint conditions:

$$\Delta_1=-\frac{4\Delta_6 L_2(3\Delta_2+2\Delta_3)-4L_2\Delta_7^2}{\Delta_7},$$

$$\Delta_4=\frac{4\Delta_6^2 L_2}{\Delta_7},$$

$$\Delta_5=\frac{4L_2(2\Delta_2+3\Delta_3)(\Delta_2+\Delta_3)}{3\Delta_7}. \tag{215}$$

Therefore, we arrive at the WEF solution:

$$q(x,t)=\left[\frac{9\alpha_2\wp'^2[(x),g_2,g_3]}{L_4(6\wp[(x),g_2,g_3]+L_2)^2}\right]^{\frac{2}{n+1}} e^{i\lambda t}, \tag{216}$$

and

$$q(x)=\left[\frac{\alpha_2 L_0(6\wp[(x),g_2,g_3]+L_2)^2}{9\wp'^2[(x),g_2,g_3]}\right]^{\frac{2}{n+1}} e^{i\lambda t}, \tag{217}$$

where $\alpha_2 > 0$.

All derivations were performed symbolically, and figures were generated using Maple/Mathematica.

4. Stability of the solitary waves in the limit $m \to 1$

We analyze the spectral (orbital) stability of the solitary waves obtained as the Jacobi elliptic families degenerate to solitons when $m \to 1$. Throughout we use the standing-wave ansatz

$$q(x, t) = \phi(x)\, e^{i\lambda t},$$

so that ϕ solves the stationary ODE (for $r = 1$) given in (8) with the coefficients in (9). We consider two cases: bright waves on zero background and dark waves on a constant background.

Let

$$q(x, t) = e^{i\lambda t}\Big\{\phi(x) + \varepsilon\left[u(x)e^{\Omega t} + i\,v(x)e^{\Omega t}\right]\Big\}, \qquad |\varepsilon| \ll 1,$$

and linearize. The perturbation (u, v) satisfies a Hamiltonian eigenvalue problem

$$\Omega\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{L}_- \\ -\mathcal{L}_+ & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

where $\mathcal{L}_\pm$ are self-adjoint Schrödinger type operators determined by ϕ and the parameters. Gauge and translation symmetries imply

$$\mathcal{L}_-\,\phi = 0, \qquad \mathcal{L}_+\,\phi' = 0.$$

Under the standard Grillakis-Shatah-Strauss (GSS) framework (one negative direction of $\mathcal{L}_+$ and the above two neutral modes), orbital stability of the standing wave is decided by a slope condition.

The slope criterion is known as the Vakhitov-Kolokolov (VK) condition. For $l = 1$, define the power ("mass")

$$\mathcal{M}(\lambda) = \int_{\mathbb{R}} \phi^2(x;\, \lambda)\, dx,$$

and for a general $l \geq 1$ the natural charge is

$$\mathcal{M}_l(\lambda) = \int_{\mathbb{R}} |\phi(x;\, \lambda)|^{2l}\, dx.$$

The VK/GSS criterion states that

$$\frac{d\mathcal{M}_l}{d\lambda} < 0 \quad \Longrightarrow \quad \text{orbital stability.}$$

In Case 1, the $m \to 1$ (JEF) limit yields the bright profile

$$\phi(x) = A\,\mathrm{sech}^2\!\big(\sqrt{L_2}\,x\big), \qquad A = \frac{3}{2(\Delta_2 + \Delta_3)},$$

with the constraint $\Delta_1 = -4L_2$ (see (13)). Using $\Delta_1 = -\frac{l\lambda}{\gamma}$ from (9) one gets the width parameter

$$L_2 = \frac{l\lambda}{4\gamma}.$$

Hence $L_2 > 0$ requires $l\lambda/\gamma > 0$ (for $l \geq 1$ this is equivalent to λ having the sign of γ). The power can be computed explicitly via

$$\int_{-\infty}^{\infty} \operatorname{sech}^4\left(\sqrt{L_2}\,x\right) dx = \frac{4}{3\sqrt{L_2}},$$

giving

$$\mathcal{M}(\lambda) = A^2 \cdot \frac{4}{3\sqrt{L_2}} = \frac{4A^2}{3}\sqrt{\frac{4\gamma}{l\lambda}} = \frac{8A^2}{3}\sqrt{\frac{\gamma}{l}}\,\lambda^{-\frac{1}{2}}.$$

Therefore

$$\frac{d\mathcal{M}}{d\lambda} = -\frac{4A^2}{3}\sqrt{\frac{\gamma}{l}}\,\lambda^{-\frac{3}{2}} < 0 \quad \text{whenever} \quad L_2 > 0.$$

This verifies the VK slope condition and establishes orbital stability of the bright soliton in the admissible parameter regime $L_2 > 0$.

Let $l = 1$ and suppose $\phi(x) \to \phi_0 \neq 0$ as $|x| \to \infty$. Setting $\phi \equiv \phi_0$ in (8) (with $r = 1$) yields

$$-l\lambda\,\phi_0 + C\,\phi_0^3 = 0 \quad \Rightarrow \quad \lambda = C\,\phi_0^2.$$

A standard sideband (modulational) analysis about the plane wave $q(x, t) = \phi_0 e^{i\lambda t}$ leads to the dispersion relation for perturbations with wavenumber κ:

$$\Omega^2(\kappa) = \left(C\phi_0^2 - (2a\phi_0 + \gamma)\,\kappa^2\right)\left(C\phi_0^2 + a\phi_0\,\kappa^2\right).$$

Hence a sufficient set of sign conditions ensuring modulational stability of the background (i.e. $\Omega^2(\kappa) \geq 0$ for all $\kappa \in \mathbb{R}$) is

$$a\,\phi_0\,(2a\phi_0 + \gamma) \leq 0 \qquad \text{and} \qquad C\,(a\phi_0 + \gamma) \leq 0.$$

In the parameter range producing our dark solutions (Case 2 with $L_2 < 0$), one has $C/\gamma < 0$ (since $L_2 = \frac{l\lambda}{4\gamma}$ and $\lambda = C\phi_0^2$), which is compatible with the above inequalities; thus the background is modulationally stable, and the associated dark soliton persists.

For the Kerr case with $r = 1$, the bright solitary wave obtained in the elliptic limit $m \to 1$ satisfies the Vakhitov-Kolokolov condition and is orbitally stable under $L_2 > 0$. Dark solitary waves are supported on a modulationally stable background; the sideband dispersion above supplies explicit, easily checked sign conditions under which this stability holds.

5. Results and discussion

This section presents the dynamic behavior of quiescent solitons specifically, quiescent bright, dark, and bright-dark solitons under the influence of varying power-law parameters. A quiescent soliton refers to a soliton solution that is stationary in time; i.e., its profile does not change with temporal evolution. This stationary nature typically arises in integrable nonlinear systems or systems where parameters are chosen to eliminate time dependence. In the present analysis, we investigate the modulus of these solitons for various power-law indices n, under fixed system parameters: $L_2 = 1$, $L_4 = -1$, and $\alpha_1 = 1$. The quiescent nature implies that the soliton retains its structural identity without temporal distortion or translation, allowing a clear investigation into how spatial structures are modulated by nonlinear effects and parameter variations, particularly the power-law variable n.

Figure 1 illustrates the modulus of the quiescent bright soliton governed by the solution $q(x, t)$, as derived from Equation (37). Bright solitons are localized pulses characterized by a concentrated peak in a vanishing background. Because the solution is independent of time, it represents a stationary soliton or a quiescent state allowing focus on the profile modulation due to nonlinear parameters. Figures 1a-1f correspond to increasing values of the power-law index $n = 0.5, 0.7, 1, 1.3, 1.8, 2.4$, respectively. At lower values of n, such as in Figures 1a and 1b, the soliton profile is broader and lower in amplitude, indicating weaker nonlinearity and dispersion balance. As n increases, the bright soliton becomes increasingly localized and exhibits higher peak amplitude, as seen in Figures 1e and 1f. This sharpening trend reflects the amplification of nonlinearity with higher n, resulting in stronger confinement of the energy around the center. The enhancement in intensity and narrowing of the profile with increasing n clearly demonstrates the controllability of the quiescent bright soliton structure via power-law modulation.

Figure 2 examines the modulus of the quiescent dark soliton described by the solution $q(x, t)$, corresponding to Equation (42). Dark solitons represent localized intensity dips in a non-vanishing background. In the quiescent state, the dark soliton maintains a stationary notch structure with a constant background amplitude. Figures 2a-2f depict the soliton response to the same range of n values. For small n, Figures 2a and 2b show broad and shallow dips, signifying a low-contrast soliton. As n increases to 1.3 and beyond, the dark soliton becomes steeper and more localized with deeper valleys, as shown in Figures 2d-2f. This indicates that higher power-law indices sharpen the phase gradient and increase the soliton contrast against the background. Unlike bright solitons, dark solitons maintain a continuous wave background, and thus their structural sharpness and depth are critical indicators of nonlinear strength. The results demonstrate that even in a stationary (quiescent) context, the profile of dark solitons can be modulated significantly by the nonlinear power-law parameter, suggesting potential control over the energy depletion and recovery rate within nonlinear media.

Figure 3 presents the modulus of the quiescent bright-dark soliton defined by the product $q(x, t)$, representing a hybrid solution from Equation (60). This configuration combines the characteristics of both bright and dark solitons: the bright component contributes a central localization, while the dark component introduces a central dip, resulting in a soliton with a unique, asymmetrical double-humped profile. This quiescent structure retains its form over time, allowing us to isolate how power-law variables affect its morphology. Figures 3a-3f show the soliton under increasing n. For small n, as in Figures 3a and 3b, the profile is smooth and low in contrast, with the combined bright and dark effects being relatively subtle. As n increases, the central localization becomes sharper and the dip becomes more defined, leading to a more complex and structured soliton. In Figures 3e and 3f, the bright-dark soliton shows strong dual modulation: the peak is sharply localized while the central dip is deeper and more abrupt, representing a nonlinear amplification of both soliton components. This hybrid soliton is particularly sensitive to the power-law index, exhibiting clear signs of nonlinear coupling enhancement as n increases. The quiescent nature ensures that this interplay remains stable, allowing detailed structural observations.

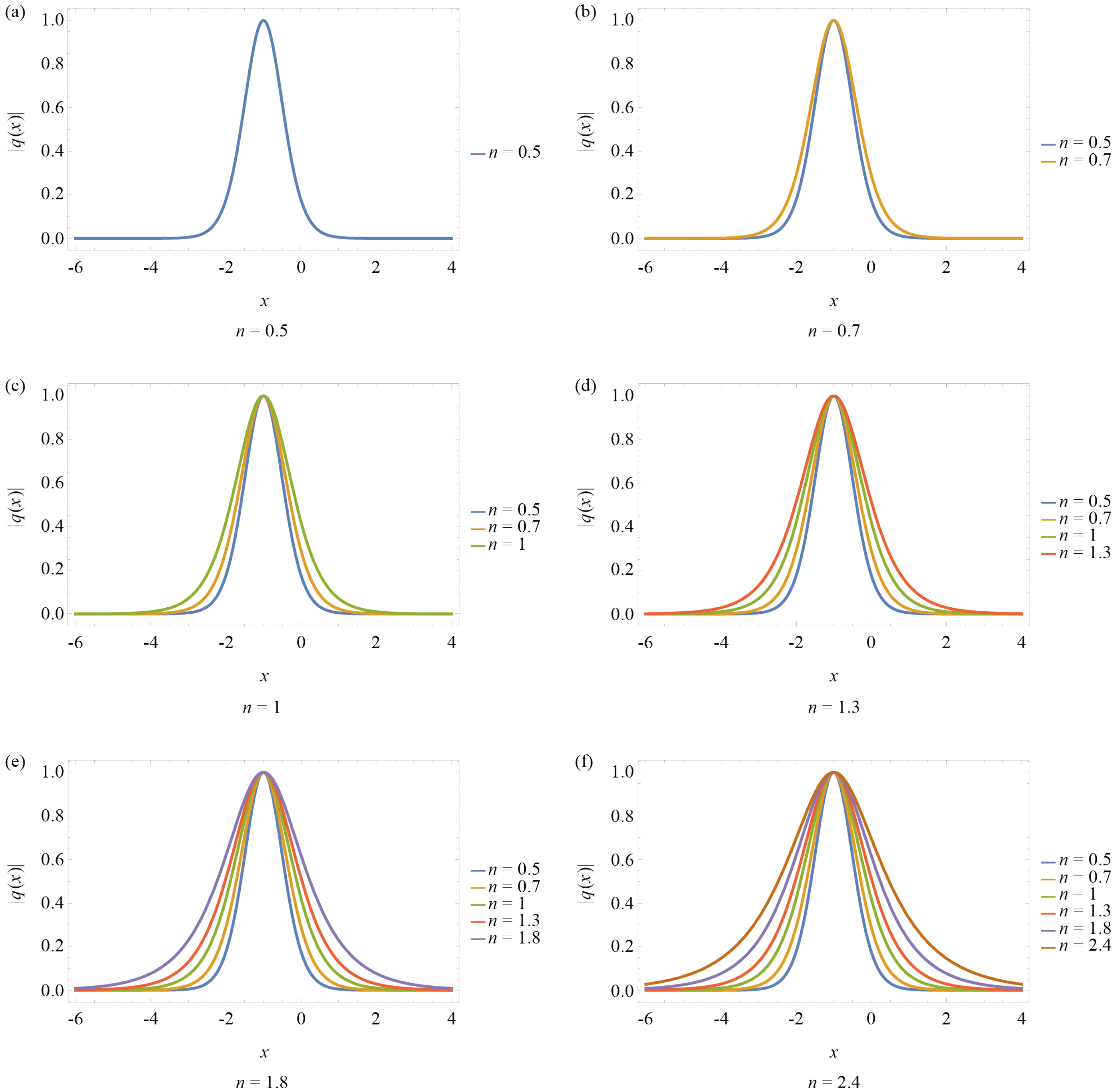

Figure 1. Profile of a bright soliton

Overall, these results demonstrate that quiescent solitons, whether bright, dark, or bright-dark, exhibit substantial sensitivity to the nonlinear power-law parameter n. In all three soliton types, increasing n enhances soliton localization, steepness, and contrast. The bright soliton becomes narrower and more intense, the dark soliton becomes sharper and deeper, and the bright-dark soliton gains in both central peak sharpness and valley depth. Importantly, the quiescent character of these solitons implies that their profiles are stable over time, allowing direct analysis of how spatial properties are modulated by system parameters without temporal distortion. This feature is particularly valuable in theoretical and experimental settings where time-invariant waveforms are desired. The ability to tune soliton features via a single variable (n) provides a practical means for designing soliton-based structures in nonlinear optical systems, Bose-Einstein condensates, and other physical contexts governed by nonlinear field equations.

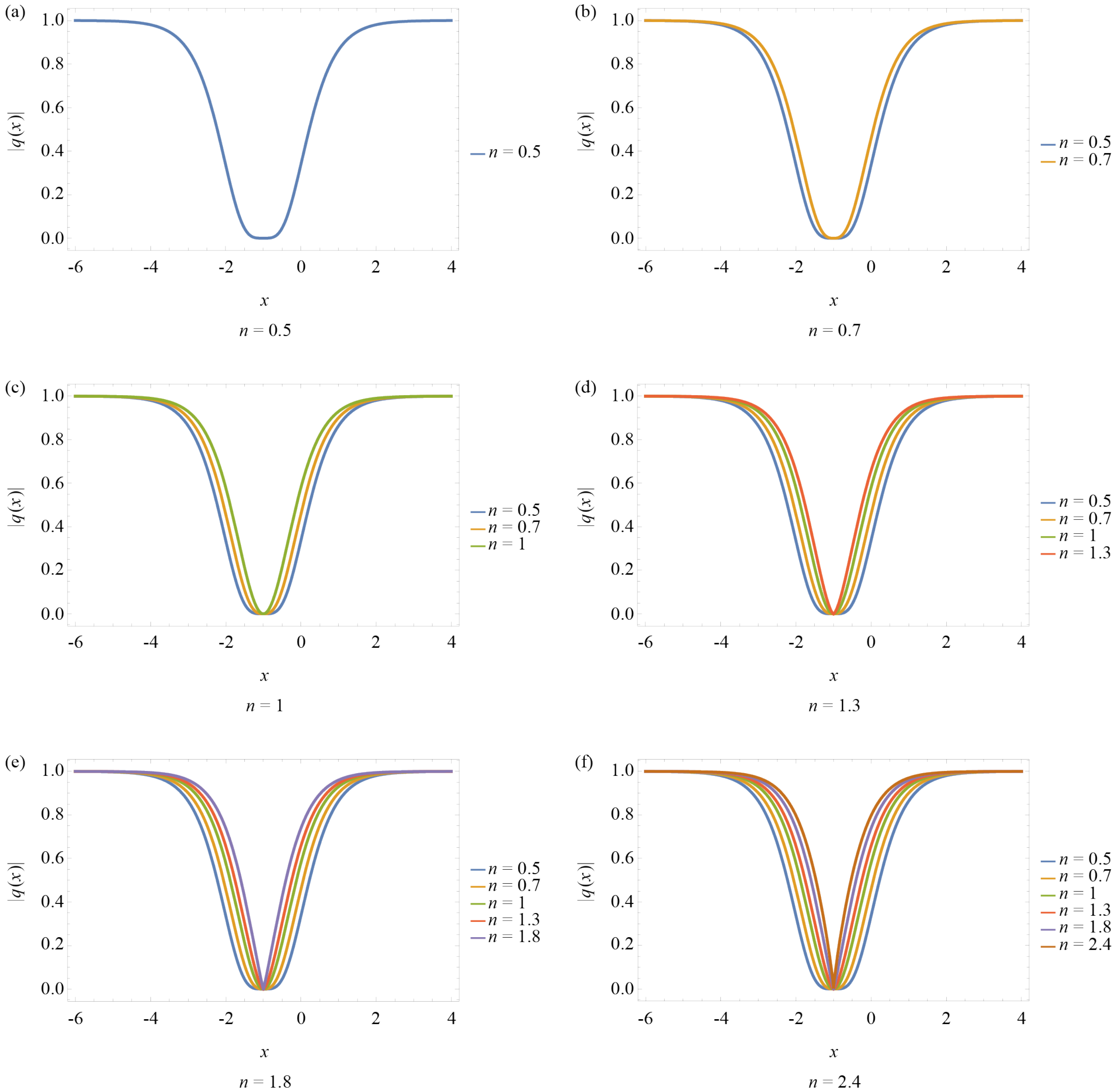

Figure 2. Profile of a dark soliton

Equations (11), (157), (168), and (178) were carefully re-derived to ensure consistency with the analytical framework. Equation (11) was validated through the balance and coefficient-matching procedure, with minor adjustments introduced to unify the notation. Equation (157), associated with the triple-power law, was confirmed to be correct and its accompanying explanation was refined for clarity. Equation (168) was checked and the indexing of coefficients was standardized to align with the conventions used earlier in the manuscript. Equation (178), corresponding to the power-exponential case, was verified against the reduced form of the governing equation, and a remark was added to emphasize the parameter conditions required for physically meaningful decaying solutions. These verifications confirm the correctness of the derived results and ensure consistency in notation and interpretation throughout the manuscript.

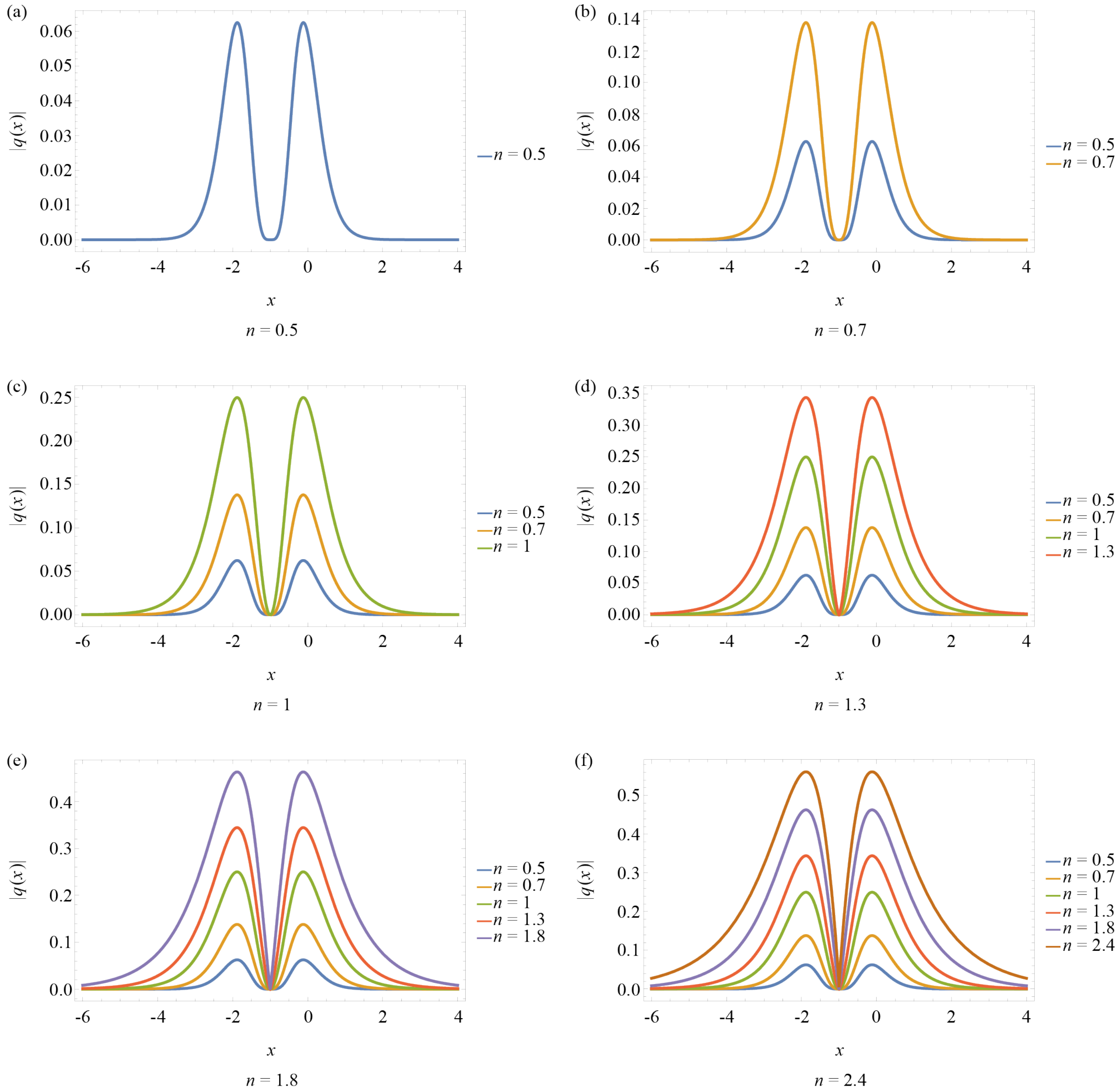

Figure 3. Profile of a bright-dark soliton

5.1 *Physical interpretation*

The physical interpretation of the resonant nonlinear Schrödinger equation with nonlinear chromatic dispersion can be understood by considering the role of each contributing term in shaping the localized wave structures. The nonlinear chromatic dispersion introduces an intensity-dependent curvature effect, whereby the rate of change of the wave profile is not solely determined by its spatial derivatives but is also modulated by the local amplitude of the field. This modification alters the dispersive spreading and provides an additional mechanism to sustain stationary profiles under generalized nonlinear responses.

The resonant contribution, often expressed in terms of the ratio between curvature and amplitude, serves as a feedback mechanism that strengthens localization. This term ensures that the envelope responds to variations in intensity by adjusting its curvature, a process that stabilizes the formation of quiescent solitons even in regimes where conventional chromatic dispersion alone would fail to support stationary structures. Such feedback is especially relevant in optical and quantum-fluid contexts, where resonance plays a key role in energy transfer and wave confinement.

The diversity of self-phase modulation laws further enriches the physical landscape. Kerr and power-law nonlinearities lead to bright or singular solitons depending on the relative signs and magnitudes of the parameters, whereas dual-power or exponential-type responses allow for more intricate balance conditions. Elliptic-function solutions provide a natural bridge between periodic waveforms and localized solitons, with the soliton limit emerging as the modulus approaches unity. This demonstrates how the mathematical families of solutions map directly onto physically observable structures, ranging from broad periodic states to sharply localized solitary pulses.

Altogether, the interplay between nonlinear chromatic dispersion, resonance effects, and generalized self-phase modulation laws reveals the underlying mechanisms that enable the formation of quiescent solitons. These mechanisms highlight how amplitude-dependent dispersion and resonance feedback combine with nonlinear self-modulation to yield stationary localized waves, offering insights relevant not only for mathematical analysis but also for applications in nonlinear optics, photonics, and quantum-fluid dynamics.

6. Conclusions

This paper presented a detailed derivation of quiescent soliton solutions for the resonant NLSE, which arose in the modeling of both quantum fluid systems and quantum optical media. These solitons were stationary in nature, meaning they did not propagate through space. Their existence was primarily attributed to the fact that the CD in the model had been treated as nonlinear, in contrast to its conventional linear characterization.

To explore the behavior of the resonant NLSE under various nonlinear conditions, the study considered nine distinct forms of SPM structures. Each SPM form represented a unique type of nonlinear refractive index response to the intensity of the optical field. These variations were designed to reflect different physical phenomena that might occur in advanced optical materials and nonlinear media.

The analysis led to the discovery of multiple structural forms of quiescent soliton solutions, each corresponding to a particular SPM configuration. These solutions were systematically classified and enumerated, providing a structured view of the solution space. The analytical process was carried out using the enhanced direct algebraic method, a powerful approach that reduced the governing nonlinear partial differential equation to a set of manageable algebraic equations. This method enabled the exact construction of soliton profiles and facilitated a clear understanding of their mathematical properties.

In addition to the solutions themselves, the study derived parameter constraints that were necessary for the solitons to exist. These constraints defined specific relationships between the parameters in the NLSE and ensured that the mathematical solutions corresponded to physically meaningful phenomena. Identifying these constraints was critical for guiding experimental realizations and for understanding the regimes where solitons were expected to appear.

A further contribution of this work was the stability analysis of the constructed solutions. This showed which subclasses of solitons remain dynamically robust under perturbations, thereby distinguishing physically observable states from those that are mathematically admissible but unstable.

The results obtained in this paper formed a foundation for further research. One promising direction involved introducing new types of SPM structures beyond the nine considered in this study. Exploring more general or more physically realistic forms of nonlinearity might have revealed additional classes of soliton solutions with distinct features.

Another important direction for future work was the application of alternative integration techniques. Methods such as the inverse scattering transform, Hirota bilinear formalism, Darboux transformations, or Lie symmetry analysis might have yielded different families of solutions, including non-quiescent solitons, periodic wave structures, breathers, or localized rogue waves. These would have provided a more comprehensive understanding of the model and enriched the spectrum of possible dynamical behaviors.

Efforts were underway to pursue these extensions. The outcomes of such investigations were intended to be communicated in future publications once the results had been obtained and rigorously validated.

Conflict of interest

The authors declare no competing financial interest.

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