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Research Article

Numerical and Graphical Validations of Hermite-Hadamard, Fejér and Pachpatte-Type Integral Inequalities Pertaining to Interval Analysis

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Abstract: Fractional inequalities and interval analysis are interdependent mathematical fields that have received much interest in recent years. Both concepts have a lot of applications in the field of applied sciences such as physics, signal processing, mathematical modeling, control theory, numerical analysis, uncertainty quantification, engineering and optimization problems. In this article, we present a new concept between two intervals related to center-radius ordered relations. First, we introduce the new definition, namely, interval-valued center-radius order generalized preinvex function. Further, we establish the novel perspective of the Hermite-Hadamard type inequality via a newly investigated concept pertaining to the Riemann-Liouville fractional integral operator in the frame of interval analysis. In addition, fractional Fejér type inequality of the first and second kind is also investigated via a newly introduced concept. Finally, we establish the fractional Pachpatte-type inequality in the frame of interval analysis. The graphical, numerical, and literature validity of the presented inequalities are also estimated. Some special instances of our general results are also stated in the form of corollaries and remarks. The present research provides an improved understanding of integral inequalities with fractional integral operators.

***Keywords*:** Hermite-Hadamard inequality, Fejér inequality, Pachpatte-type inequality, interval-valued functions, center-radius order, Riemann-Liouville fractional operator

MSC: 26A33, 26A51, 26D07, 26D10, 26D15

1. Introduction

Interval analysis is a reliable approach to address uncertainty, providing accurate bounds and ensuring precision in complex computations. Ramon Moore for the first time developed the subject interval analysis in the 1960s as a way to deal with errors and uncertainties in numerical calculations. It gives strict bounds on solutions by representing integers as intervals. This method guarantees precision and integrity in intricate systems. Applications for interval analysis are numerous in the applied sciences, such as biology, computer science, physics, engineering, and finance. It is essential for

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risk analysis, uncertainty quantification, and complex system modeling due to its capacity to manage uncertainties and set strict limitations, which spurs innovation in a variety of industries.

The study of integral inequalities and interval analysis together creates a powerful system of analysis that connects the robustness of uncertainty quantification with the accuracy of standard mathematical bounds. Hermite-Hadamard (H-H), Pachpatte and Fejér inequalities provide important techniques for evaluating function behaviors over intervals, while interval analysis offers a methodical approach to managing data uncertainties and computational errors. Together, these two fields provide fresh insights into the stability, convergence, and error analysis of numerical techniques and enable the construction of tighter, more accurate bounds for integrals of uncertain or imprecise functions. In applied sciences and engineering, where accurate results are infrequent and reliable approximations under uncertainty are essential, this synergy is particularly beneficial.

Fractional calculus is an elementary concept in mathematics and applied sciences. Students are encouraged to use fractional calculus to solve a variety of real-world problems. Recently, researchers have been interested in whether it is possible to generalize established integral inequalities via fractional operators. Numerous scientific and engineering phenomena, including viscous fluids, equilibrium, elasticity, heat conduction issues, viscoelastic deformation, and fracture mechanics, are integrated through the use of fractional operators.

Inequality theory and fractional analysis have developed together in the contemporary era. Many researchers have been combining new ideas with fractional analysis over the past decade in order to expand the domain of analysis. In the practical sciences, the idea of inequalities in the frame of fractional calculus has many implementations. Fractional operators have been used by scholars who analyze inequalities to investigate some generalizations and extensions. The analysis of fractional operators has attracted numerous scientists and mathematicians from a vast range of application and scientific communities. Fractional calculus has gained a widespread area of investigation because of its applicability to scientific modeling.

Different fractional forms of H-H, Pachpatte- and Fejér-inequalities have been well researched over time. The sources [1–17] well reveals the close relationship shared between integral inequalities and fractional operators.

Here, we are concerned with applying the ideas of a novel kind of interval-valued mapping involving fractional operator, to introduce our findings. To be more precise, H-H-type inequalities and their improvements, like Fejér and Pachpatte-type inequalities, are addressed involving generalized preinvexity in the mode of interval analysis.

In the recent past, the applications of interval analysis and fractional calculus have grown exponentially in relation to integral inequalities. For advancement in this direction, there is a need to integrate the ideas of interval-valued center-radius order and fractional operator to share new inequalities. There are not sufficient studies conducted with regard to the H-H inequality for interval-valued center-radius order. We strive to demonstrate specific H-H-type inequalities and their estimations using fractional calculus, fuzzy calculus, time-scale calculus, and quantum calculus in order to close this gap in the literature. Specifically in this paper, we will start our research by integrating the interval-valued center-radius order generalized preinvex function via Riemann-Liouville Fractional Integral Operator (RLFIO) to present our results. In later research papers, we will keep in mind applying these ideas in the aforementioned angles.

The novelty of the present research is that we use fractional operators to derive our inequalities with respect to the interval-valued center-radius-ordered. The latest study yields an innovative viewpoint in the field of interval analysis with regards to the employing of integral inequalities such as the H-H, Fejér-, and Pachpatte types associated with this idea. Here, we estimate the intervals utilizing the idea of radius and center stated as $\mathrm{k_r} = \dfrac{\overline{\mathrm{k}} - \underline{\mathrm{k}}}{2}$ and $\mathrm{k_c} = \dfrac{\overline{\mathrm{k}} + \underline{\mathrm{k}}}{2}$, respectively.

The structure of the paper is constructed as follows: Firstly, in Section 2, we rehash a few familiar ideas and terminology that will help us in our investigation in the upcoming parts. In Section 3, we explore a new definition, namely interval-valued center-radius order preinvex function. In Section 4, we investigate the novel sort of H-H-type inequality involving the newly introduced concept via RLFIO in the frame of interval analysis with some interesting corollaries. In Section 5, we explore the novel sort of Fejér inequality of the first and second kind involving interval-valued center-radius order preinvex function via RLFIO. In Section 6, we investigate a novel kind of Pachpatte inequality via RLFIO with some corollaries in the frame of a newly introduced idea. In the final Section 7, we offer a brief outcome and outline certain prospective possibilities for further research.

2. Preliminaries

To facilitate readers' understanding and engagement, this section will focus on a narrow set of definitions, remarks, and theorems that are critical to the discussion. The primary aim of this section is to set the stage by introducing and examining the fundamental concepts and definitions that will be crucial to our analysis in subsequent sections.

The term invexity have a lot of applications in nonlinear optimization, variational inequalities and in the different domains of pure and applied sciences.

Definition 1 [18] Assume that $\Pi : \mathbb{X} \times \mathbb{X} \times (0, 1] \to \mathbb{R}^n$ and $\mathbb{X} \subseteq \mathbb{R}^n$. Then $\mathbb{X}$ is m-invex w.r.t. Π, if

$$m\mathtt{I_a} + \mathfrak{x}\Pi(\mathtt{I_b}, \mathtt{I_a}, m) \in \mathbb{X}, \tag{1}$$

holds $\forall\ \mathtt{I_b}, \mathtt{I_a} \in \mathbb{X}$, $m \in (0, 1]$ and $\mathfrak{x} \in [0, 1]$.

A significant number of researchers have spent the last ten years striving to improve the definition of preinvexity in different ways. Kalsoom [19] introduced the generalized m-preinvex function, which is stated by

Definition 2 Assume that $\Pi : \mathbb{X} \times \mathbb{X} \times (0, 1] \to \mathbb{R}^n$ and $\mathbb{X} \subseteq \mathbb{R}^n$. Then $\Pi : \mathbb{X} \to \mathbb{R}$ is generalized m-preinvex w.r.t. Π if

$$\mathtt{F}(m\mathtt{I_a} + \mathfrak{x}\Pi(\mathtt{I_b}, \mathtt{I_a}, m)) \leq \mathfrak{x}\mathtt{F}(\mathtt{I_b}) + m(1-\mathfrak{x})\mathtt{F}(\mathtt{I_a}), \tag{2}$$

holds for every $\mathtt{I_b}, \mathtt{I_a} \in \mathbb{X}$, $m \in (0, 1]$ and $\mathfrak{x} \in [0, 1]$.

The following extended Condition C involving m-preinvexity was also discussed by Du et al. in [20].

Extended Condition *C*: Assume that $\mathbb{X} \subset \mathbb{R}$ an open invex subset w.r.t. $\Pi : \mathbb{X} \times \mathbb{X} \times (0, 1] \to \mathbb{R}$. We say that the function Π satisfied the Extended Condition C, if for any $\mathtt{I_b}, \mathtt{I_a} \in \mathbb{X}$, $\mathfrak{x} \in [0, 1]$, we obtain

$$\Pi(\mathtt{I_a}, m\mathtt{I_a} + \mathfrak{x}\,\Pi(\mathtt{I_b}, \mathtt{I_a}, m), m) = -\mathfrak{x}\,\Pi(\mathtt{I_b}, \mathtt{I_a}, m) \tag{3}$$

$$\Pi(\mathtt{I_b}, m\mathtt{I_a} + \mathfrak{x}\,\Pi(\mathtt{I_b}, \mathtt{I_a}, m), m) = (1-\mathfrak{x})\,\Pi(\mathtt{I_b}, \mathtt{I_a}, m) \tag{4}$$

$$\Pi(\mathtt{I_b}, \mathtt{I_a}, m) = -\Pi(\mathtt{I_a}, \mathtt{I_b}, m). \tag{5}$$

Definition 3 (see [21]) Let $\mathtt{F} : \mathbb{I} \to \mathbb{R}$ be the function, then $\mathtt{F}$ is called $\mathfrak{h}$-preinvex w.r.t. $\Pi : \mathbb{I} \times \mathbb{I} \neq \emptyset \to \mathbb{R}$ and $\mathfrak{h} \neq 0$, if

$$\mathtt{F}(\mathtt{I_a} + \mathfrak{x}\Pi(\mathtt{I_b}, \mathtt{I_a})) \leqq \mathfrak{h}(\mathfrak{x})\mathtt{F}(\mathtt{I_b}) + \mathfrak{h}(1-\mathfrak{x})\,\mathtt{F}(\mathtt{I_a}), \quad (\forall \mathtt{I_a}, \mathtt{I_b} \in \mathbb{X};\ \mathfrak{x} \in [0, 1]). \tag{6}$$

H-H inequality has been crucial to the formulation of the convex analysis theory. Many mathematicians have been drawn to the inequality and have been a source of numerous generalizations. The classical Hermite-Hadamard inequality expressed as (see [22]):

$$\mathtt{F}\left(\frac{\mathtt{I_a} + \mathtt{I_b}}{2}\right) \leq \frac{1}{\mathtt{I_b} - \mathtt{I_a}} \int_{\mathtt{I_a}}^{\mathtt{I_b}} \mathtt{F}(\mathrm{y})d\mathrm{y} \leq \frac{\mathtt{F}(\mathtt{I_a}) + \mathtt{F}(\mathtt{I_b})}{2}. \tag{7}$$

Fejér inequality is the most amazing and famous inequality in the literature. Many mathematicians have worked on different concepts from different angles in the field of inequalities. The first time Fejér [23] was introduce the following inequality, which is stated by: Suppose $\mathtt{F}:[\mathfrak{x}_1, \mathfrak{x}_2]\to\mathbb{R}$ is a convex function. Then, the inequality

$$\mathtt{F}\Big(\frac{\mathfrak{x}_1+\mathfrak{x}_2}{2}\Big)\int_{\mathfrak{x}_1}^{\mathfrak{x}_2}\mathtt{W}(x)dx\leq\int_{\mathfrak{x}_1}^{\mathfrak{x}_2}\mathtt{F}(x)\mathtt{W}(x)dx\leq\frac{\mathtt{F}(\mathfrak{x}_1)+\mathtt{F}(\mathfrak{x}_2)}{2}\int_{\mathfrak{x}_1}^{\mathfrak{x}_2}\mathtt{W}(x)dx, \tag{8}$$

holds, where $\mathtt{W}:[\mathfrak{x}_1, \mathfrak{x}_2]\to\mathbb{R}$ is non-negative, integrable and symmetric to $\frac{\mathfrak{x}_1+\mathfrak{x}_2}{2}$.

Pachpatte-type inequality is the most wonderful and meaningful inequality in the literature. Many mathematicians have worked on different concepts from different angles in the field of inequalities. The Pachpatte-type inequality [24] is defined as: If $\mathtt{F}_1$ and $\mathtt{F}_2$ be convex functions on $[\mathtt{I_a}, \mathtt{I_b}]$, then

$$\frac{1}{\mathtt{I_b}-\mathtt{I_a}}\int_{\mathtt{I_a}}^{\mathtt{I_b}}\mathtt{F}_1(\mathfrak{x})\mathtt{F}_2(\mathfrak{x})d\mathfrak{x}\leq\frac{1}{3}\mathtt{M}(\mathtt{I_a}, \mathtt{I_b})+\frac{1}{6}\mathtt{N}(\mathtt{I_a}, \mathtt{I_b}), \tag{9}$$

and

$$2\mathtt{F}_1\left(\frac{\mathtt{I_a}+\mathtt{I_b}}{2}\right)\mathtt{F}_2\left(\frac{\mathtt{I_a}+\mathtt{I_b}}{2}\right) \tag{10}$$

$$\leq\frac{1}{\mathtt{I_b}-\mathtt{I_a}}\int_{\mathtt{I_a}}^{\mathtt{I_b}}\mathtt{F}_1(\mathfrak{x})\mathtt{F}_2(\mathfrak{x})d\mathfrak{x}+\frac{1}{3}\mathtt{M}(\mathtt{I_a}, \mathtt{I_b})+\frac{1}{6}\mathtt{N}(\mathtt{I_a}, \mathtt{I_b}), \tag{11}$$

where

$$\begin{aligned}\mathtt{M}(\mathtt{I_a}, \mathtt{I_b})&=\mathtt{F}_1(\mathtt{I_a})\mathtt{F}_2(\mathtt{I_a})+\mathtt{F}_1(\mathtt{I_b})\mathtt{F}_2(\mathtt{I_b}),\\ \mathtt{N}(\mathtt{I_a}, \mathtt{I_b})&=\mathtt{F}_1(\mathtt{I_a})\mathtt{F}_2(\mathtt{I_b})+\mathtt{F}_1(\mathtt{I_b})\mathtt{F}_2(\mathtt{I_a}).\end{aligned} \tag{12}$$

Definition 4 (see [25, 26]) Let $\mathtt{F}\in\mathscr{L}[\mathtt{I_a}, \mathtt{I_b}]$ on $[\mathtt{I_a}, \mathtt{I_b}]$. Then, for $\alpha>0$ Left Riemann-Liouville Fractional Integral Operators (L-RLFIO) and Right Riemann-Liouville Fractional Integral Operators (R-RLFIO) are stated as follows:

$$\mathtt{I}^{\alpha}_{\mathtt{I_a}^+}\mathtt{F}(x):=\frac{1}{\Gamma(\alpha)}\int_{\mathtt{I_a}}^{x}(x-\mathfrak{x})^{\alpha-1}\ \mathtt{F}(\mathfrak{x})d\mathfrak{x}\qquad(0\leqq\mathtt{I_a}<x<\mathtt{I_b}) \tag{13}$$

and

$$\mathtt{I}^{\alpha}_{\mathtt{I_b}^-}\mathtt{F}(x):=\frac{1}{\Gamma(\alpha)}\int_{x}^{\mathtt{I_b}}(\mathfrak{x}-x)^{\alpha-1}\ \mathtt{F}(\mathfrak{x})d\mathfrak{x}\qquad(0\leqq\mathtt{I_b}<x<\mathtt{I_b}), \tag{14}$$

respectively, where $\Gamma(\alpha)=\int_0^\infty\mathfrak{x}^{\alpha-1}e^{-\mathfrak{x}}d\mathfrak{x}$ is the Euler gamma function.

2.1 *Center-radius order relation*

Let $\mathrm{k} = [\underline{\mathrm{k}}, \bar{\mathrm{k}}] \in \mathbb{R}_I$, then the interval radius and center are stated as $\mathrm{k_r} = \dfrac{\bar{\mathrm{k}} - \underline{\mathrm{k}}}{2}$ and $\mathrm{k_c} = \dfrac{\bar{\mathrm{k}} + \underline{\mathrm{k}}}{2}$ respectively. Collectively, they are stated as:

$$\mathrm{k} = \langle \mathrm{k_c}, \mathrm{k_r} \rangle = \left\langle \frac{\bar{\mathrm{k}} + \underline{\mathrm{k}}}{2}, \frac{\bar{\mathrm{k}} - \underline{\mathrm{k}}}{2} \right\rangle. \tag{15}$$

Definition 5 Suppose $\mathrm{k} = [\underline{\mathrm{k}}, \bar{\mathrm{k}}] = \langle \mathrm{k_c}, \mathrm{k_r} \rangle$, $\mathrm{f} = [\underline{\mathrm{f}}, \bar{\mathrm{f}}] = \langle \mathrm{f_c}, \mathrm{f_r} \rangle \in \mathbb{R}_I$, then the above center-radius order relation is stated as:

$$\mathrm{k} \preceq_{\mathrm{cr}} \mathrm{f} \Longleftrightarrow \begin{cases} \mathrm{k_c} < \mathrm{f_c}, & \text{if } \mathrm{k_c} \neq \mathrm{f_c}; \\ \mathrm{k_r} \leq \mathrm{f_r}, & \text{if } \mathrm{k_c} = \mathrm{f_c}. \end{cases} \tag{16}$$

We have either $\mathrm{k} \preceq_{\mathrm{cr}} \mathrm{f}$ or $\mathrm{f} \preceq_{\mathrm{cr}} \mathrm{k}$ for any two intervals $\mathrm{k}, \mathrm{f} \in \mathbb{R}_I$.

Definition 6 [27] Suppose interval-valued function $\mathtt{F} : [\mathtt{I_a}, \mathtt{I_b}] \subset \mathbb{R}$ is stated as

$$\mathtt{F} = [\underline{\mathtt{F}}, \overline{\mathtt{F}}]. \tag{17}$$

Then, $\mathtt{F}$ is Riemann integrable over $[\mathtt{I_a}, \mathtt{I_b}]$, if and only if $\underline{\mathtt{F}}$ and $\overline{\mathtt{F}}$ are Riemann integrable over $[\mathtt{I_a}, \mathtt{I_b}]$, i.e.,

$$(IR) \int_{\mathtt{I_a}}^{\mathtt{I_b}} \mathtt{F}(\mathfrak{x}) d\mathfrak{x} = \left[(R) \int_{\mathtt{I_a}}^{\mathtt{I_b}} \underline{\mathtt{F}}(\mathfrak{x}) d\mathfrak{x}, \, (R) \int_{\mathtt{I_a}}^{\mathtt{I_b}} \overline{\mathtt{F}}(\mathfrak{x}) d \right]. \tag{18}$$

Corollary 1 Suppose $\mathtt{F} : [\mathtt{I_a}, \mathtt{I_b}]$ be an interval-valued function such that $\mathtt{F} = \left[\underline{\mathtt{F}}, \overline{\mathtt{F}}\right]$ with $\underline{\mathtt{F}}, \overline{\mathtt{F}} \in \mathbb{R}_{[\mathtt{I_a}, \mathtt{I_b}]}$. Then

$$\mathrm{I}^{\alpha}_{\mathtt{I_a}+} \mathtt{F}(x) = \left[\mathrm{I}^{\alpha}_{\mathtt{I_a}+} \underline{\mathtt{F}}(x), \mathrm{I}^{\alpha}_{\mathtt{I_a}+} \overline{\mathtt{F}}(x) \right] \tag{19}$$

and

$$\mathrm{I}^{\alpha}_{\mathtt{I_b}-} \mathtt{F}(x) = \left[\mathrm{I}^{\alpha}_{\mathtt{I_b}-} \underline{\mathtt{F}}(x), \mathrm{I}^{\alpha}_{\mathtt{I_b}-} \overline{\mathtt{F}}(x) \right]. \tag{20}$$

The features of integration were enhanced by Shi et al. (see [28]), who also explored that it maintains order with regard to the center-radius order.

Theorem 1 Suppose $\mathtt{F}, \mathtt{E} : [\mathtt{I_a}, \mathtt{I_b}] \subset \mathbb{R}$ be the two interval-valued functions, which are defined as

$$\mathtt{F} = [\underline{\mathtt{F}}, \overline{\mathtt{F}}] \text{ and } \mathtt{E} = [\underline{\mathtt{E}}, \overline{\mathtt{E}}]. \text{ Then } \int_{\mathtt{I_a}}^{\mathtt{I_b}} \mathtt{F}(\mathfrak{x}) d\mathfrak{x} \preceq_{\mathrm{cr}} \int_{\mathtt{I_a}}^{\mathtt{I_b}} \mathtt{E}(\mathfrak{x}) d\mathfrak{x}, \tag{21}$$

holds if $\mathtt{F}(\mathfrak{x}) \preceq_{\mathrm{cr}} \mathtt{E}(\mathfrak{x})$ for all $\mathfrak{x} \in [\mathtt{I_a}, \mathtt{I_b}]$.

Example 1 Suppose $\mathtt{F} = [\mathfrak{x}, \mathfrak{x}+1]$ and $\mathtt{E} = [\mathfrak{x}^2+1, 2\mathfrak{x}+1]$. Then for $\mathfrak{x} \in [0, 1]$

$$\mathtt{F}_c = \mathfrak{x} + \frac{1}{2}, \quad \mathtt{F}_r = \frac{1}{2}, \quad \mathtt{E}_c = \frac{\mathfrak{x}^2+2\mathfrak{x}+2}{2} \quad \text{and} \quad \mathtt{E}_r = \frac{2\mathfrak{x}-\mathfrak{x}^2}{2}. \tag{22}$$

Utilizing Definition 5, we have $\mathtt{F}(\mathfrak{x}) \preceq_{cr} \mathtt{E}(\mathfrak{x})$, $\mathfrak{x} \in [0, 1]$ since

$$\int_0^1 [\mathfrak{x}, \mathfrak{x}+1] d\mathfrak{x} = \left[\frac{1}{2}, \frac{3}{2}\right] \quad \text{and} \quad \int_0^1 [\mathfrak{x}^2+1, 2\mathfrak{x}+1] d\mathfrak{x} = \left[\frac{4}{3}, 2\right]. \tag{23}$$

Now, again using Definition 5, we have

$$\int_0^1 \mathtt{F}(\mathfrak{x}) d\mathfrak{x} \preceq_{cr} \int_0^1 \mathtt{E}(\mathfrak{x}) d\mathfrak{x}. \tag{24}$$

The following Figures 1 and 2 represent the graphical validation of Theorem 1.

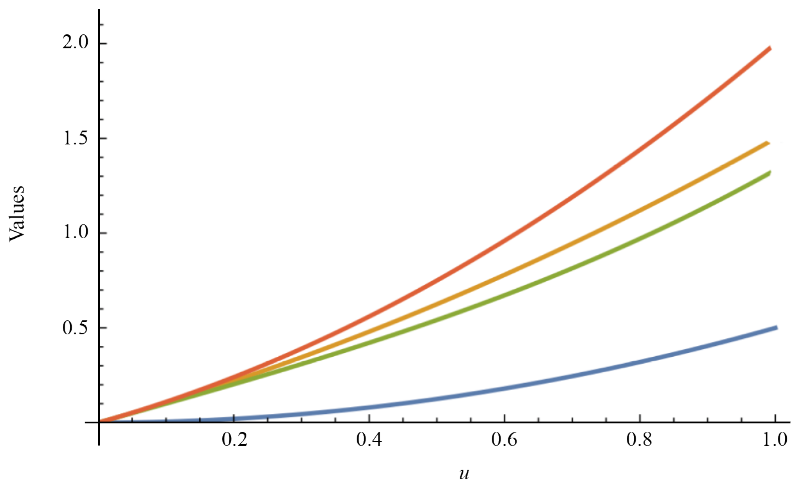

Figure 1. A red color represents $2\mathfrak{x}+1$; an orange color represents $\mathfrak{x}+1$; a green color represents $\mathfrak{x}^2+1$; and a blue color represents $\mathfrak{x}$

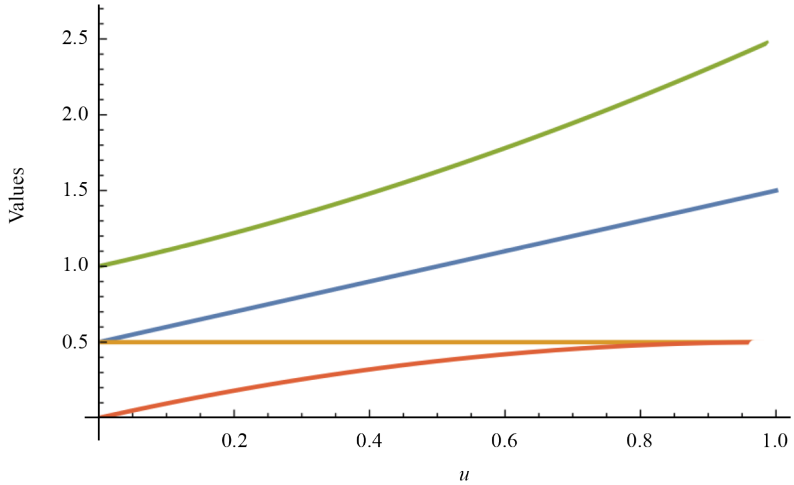

Figure 2. A green color represents $\frac{\mathfrak{x}^2+2\mathfrak{x}+2}{2}$, a blue color represents $\mathfrak{x}+\frac{1}{2}$, an orange color represents $\frac{1}{2}$; a red color represents $\frac{2\mathfrak{x}-\mathfrak{x}^2}{2}$

Example 2 Considering the assumptions of Theorem 1 and let $\mathtt{F} = [\mathfrak{x}, 2\mathfrak{x}]$ and $\mathtt{E} = [\mathfrak{x}^2, \mathfrak{x}^2+2]$, then for $\mathfrak{x} \in [0, 1]$, we have

$$\mathtt{F}_{\mathbf{C}} = \frac{3\mathfrak{x}}{2}, \quad \mathtt{F}_{\mathbf{R}} = \frac{\mathfrak{x}}{2}, \quad \mathtt{E}_{\mathbf{C}} = \mathfrak{x}^2+1 \quad \text{and} \quad \mathtt{E}_{\mathbf{R}} = 1. \tag{25}$$

From Definition 5, it follows $\mathtt{F}(\mathfrak{x}) \preceq_{\mathbf{CR}} \mathtt{E}(\mathfrak{x})$ for $\mathfrak{x} \in [0, 1]$.
Since,

$$\int_0^1 [\mathfrak{x}, 2\mathfrak{x}]d\mathfrak{x} = \left[\frac{1}{2}, 1\right] \tag{26}$$

and

$$\int_0^1 [\mathfrak{x}^2, \mathfrak{x}^2+2]d\mathfrak{x} = \left[\frac{1}{3}, \frac{7}{3}\right]. \tag{27}$$

Also, from Theorem 1, we have

$$\int_0^1 \mathtt{F}(\mathfrak{x})d\mathfrak{x} \preceq_{\mathbf{CR}} \int_0^1 \mathtt{E}(\mathfrak{x})d\mathfrak{x}. \tag{28}$$

The following Figures 3 and 4 represent the graphical validation of Theorem 1.

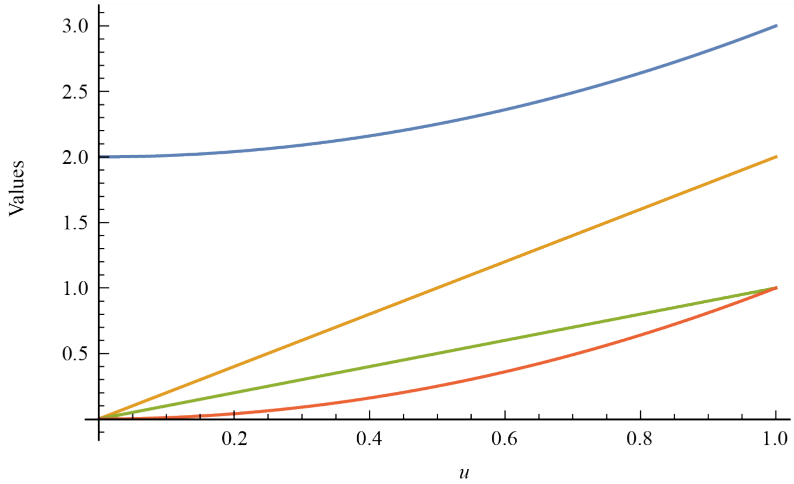

Figure 3. A blue color represents $\mathfrak{r}^2+2$; an orange color represents $2\mathfrak{r}$; a green color represents $\mathfrak{r}$; and a red color represents $\mathfrak{r}^2$. The graph above clearly explores the validity of the $\preceq_{\mathbf{CR}}$-order relationship

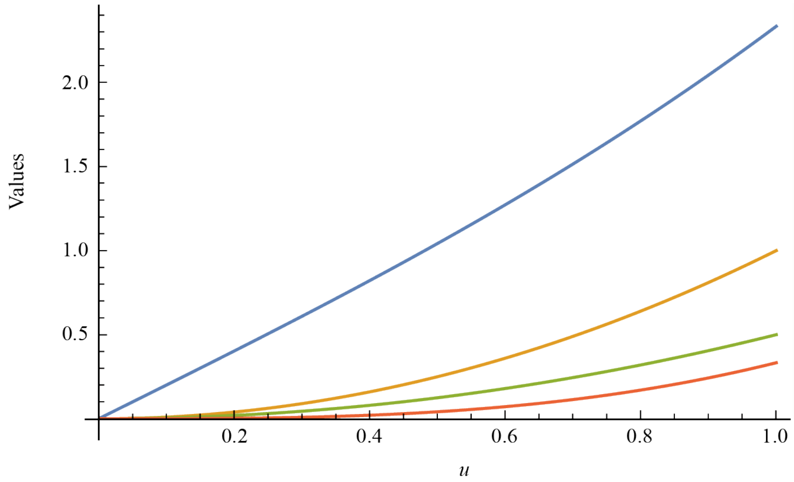

Figure 4. A blue color represents $2\mathfrak{r}+\dfrac{\mathfrak{r}^3}{3}$; an orange color represents $\mathfrak{r}^2$; a green color irepresents $\dfrac{\mathfrak{r}^2}{2}$; and a red color represents $\dfrac{\mathfrak{r}^3}{3}$

Among the numerous definitions and interpretations of interval-valued analysis provided by different writers (see [29–32]), we select the analysis related to interval-valued center-radius order defined by Bhunia [33]. Each stated notion and its definitions are completely different, even though they seem to be the same at first glance. On the basis of interval-valued analysis, numerous scholars have linked different convex functions with integral inequalities, producing a plenty of significant findings. Roman-Flores [34] explored the Minkowski-type inequalities, Chalco-Cano [35] studied the Ostrowski-type inequalities, and (see [36]) looked into the Opial-type inequalities. Zhao [37] presented the h-convexity pertaining to interval-valued and studied a refinements of the H-H inequality. In order to verify Jensen's inequalities, Zhang [31] and Costa [32] introduced fuzzy interval-valued function. The initial description of $(h1,\ h2)$-convexity in the frame of interval analysis was given by An [38]. Zhao [39] investigated the H-H inequalities via interval-valued coordinated convexity. Additionally, this idea was employed to investigate the Fejér-type inequalities for preinvex functions [40, 41] and n-polynomial convex interval-valued functions [42]. The idea of preinvexity in the frame of interval analysis was recently extended to encompass interval-valued involving coordinated preinvex functions by Lai et al. [43].

3. Interval-valued center-radius order preinvex functions and relevant results

To advance the pertinent work, first we introduce the new definition namely interval-valued center-radius order $(\mathfrak{h}, m)$-preinvex function. Further, we present novel versions of the fractional H-H, Fejér and Pachpatte-type inequalities for the newly introduced definition via RLFIO.

Definition 7 Let $\mathfrak{h} : [0, 1] \to \mathbb{R}^+$ be a real function and $\mathtt{F} : [\mathtt{I_a}, \mathtt{I_a} + \Pi(\mathtt{I_b}, \mathtt{I_a})]$ be an interval-valued function stated by $\mathtt{F} = [\underline{\mathtt{F}}, \overline{\mathtt{F}}]$. Then, $\mathtt{F}$ is said to be interval-valued center-radius order $(\mathfrak{h}, m)$-preinvex w.r.t. Π iff

$$\mathtt{F}\left(m\mathtt{I_a} + \mathfrak{x}\Pi\left(\mathtt{I_b}, \mathtt{I_a}, m\right)\right) \preceq_{cr} \mathfrak{h}(\mathfrak{x})\mathtt{F}(\mathtt{I_b}) + m\mathfrak{h}(1-\mathfrak{x})\, \mathtt{F}(\mathtt{I_a}) \quad \left(\forall \mathtt{I_a}, \mathtt{I_b} \in \mathbb{X};\ m, \mathfrak{x} \in [0, 1]\right). \tag{29}$$

Proposition 1 Suppose that the interval-valued function $\mathtt{F} : [\mathtt{I_a}, \mathtt{I_a} + \Pi(\mathtt{I_b}, \mathtt{I_a})] \to \mathbb{R}_I$ is stated as $\mathtt{F} = [\underline{\mathtt{F}}, \overline{\mathtt{F}}] = \langle \mathtt{F_c}, \mathtt{F_r} \rangle$. Let $\mathtt{F_c}$ and $\mathtt{F_r}$ are $(\mathfrak{h}, m)$-preinvex functions, then $\mathtt{F}$ is an interval-valued center-radius order $(\mathfrak{h}, m)$-preinvex functions.

4. Hermite-Hadamard type integral inequality involving interval valued center-radius order perinvex function pertaining to Riemann-Liouville fractional integral operator

The concept of inequality is the most significant property in the category of pure and applied mathematics, with various and notable applications. With this aim, the most familiar and renowned inequality in literature is the H-H inequality. This is more significant in literature; there is no doubt about its properties and applications. Several researchers have worked on various concepts under the topic of inequalities. Because of its universal outlook in the arena of science, this inequality has emerged as a dynamic subject of interest.

In this portion, we explore the H-H inequality of the involving interval valued center-radius order preinvex function pertaining to Riemann-Liouville fractional operator.

Theorem 2 Suppose $\mathtt{F} : [m\mathtt{I_a}, m\mathtt{I_a} + \Pi(\mathtt{I_b}, \mathtt{I_a}, m)] \to \mathbb{R}$ is an interval-valued function that is given by

$$\mathtt{F}(\mathfrak{x}) = \left[\underline{\mathtt{F}}(\mathfrak{x}), \overline{\mathtt{F}}(\mathfrak{x})\right] \tag{30}$$

for all $\mathfrak{x} \in [m\mathtt{I_a}, \mathtt{I_b}]$. If $\mathtt{F} : [m\mathtt{I_a}, m\mathtt{I_a} + \Pi(\mathtt{I_b}, \mathtt{I_a}, m)] \to \mathbb{R}$ is an interval-valued center-radius order $(\mathfrak{h}, m)$-preinvex mapping. Then for $\mathfrak{h}\left(\frac{1}{2}\right) > 0$, we have

$$\begin{aligned}
&\frac{1}{\alpha\mathfrak{h}\left(\frac{1}{2}\right)}\mathtt{F}\left(m\mathtt{I_a} + \frac{1}{2}\Pi(\mathtt{I_b}, \mathtt{I_a}, m)\right) \\
&\preceq_{cr} \frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b}, \mathtt{I_a}, m))^{\alpha}}\left[\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b}, \mathtt{I_a}, m))^-}\mathtt{F}(m\mathtt{I_a}) + \mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{F}(m\mathtt{I_a} + \Pi(\mathtt{I_b}, \mathtt{I_a}, m))\right] \\
&\preceq_{cr} \left[\mathtt{F}(m\mathtt{I_a}) + \mathtt{F}(m\mathtt{I_a} + \Pi(\mathtt{I_b}, \mathtt{I_a}, m))\right]\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}(\mathfrak{x}) + \mathfrak{h}(1-\mathfrak{x})]d\mathfrak{x}.
\end{aligned} \tag{31}$$

Proof. Employing the interval-valued center-radius order $(\mathfrak{h}, m)$-preinvex function and extended condition C,

$$\mathtt{F}\left(mx+\frac{1}{2}\Pi(y, x, m)\right) \preceq_{cr} \mathfrak{h}\left(\frac{1}{2}\right)[m\mathtt{F}(x)+\mathtt{F}(y)].$$

For $x = m\mathtt{I_a} + \mathfrak{x}\Pi(\mathtt{I_b}, \mathtt{I_a}, m)$ and $y = m\mathtt{I_a} + (1-\mathfrak{x})\Pi(\mathtt{I_b}, \mathtt{I_a}, m)$, it is seen that

$$\mathtt{F}\left(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b}, \mathtt{I_a}, m)+\frac{1}{2}\Pi\left(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b}, \mathtt{I_a}, m), m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b}, \mathtt{I_a}, m)\right)\right)$$

$$\preceq_{cr} \mathfrak{h}\left(\frac{1}{2}\right)\left[\mathtt{F}\left(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b}, \mathtt{I_a}, m)\right)+\mathtt{F}\left(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b}, \mathtt{I_a}, m)\right)\right].$$

This implies that

$$\frac{1}{\mathfrak{h}\left(\frac{1}{2}\right)}\mathtt{F}\left(m\mathtt{I_a}+\frac{1}{2}\Pi(\mathtt{I_b}, \mathtt{I_a}, m)\right) \preceq_{cr} \left[\mathtt{F}\left(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b}, \mathtt{I_a}, m)\right)+\mathtt{F}\left(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b}, \mathtt{I_a}, m)\right)\right]. \tag{32}$$

Upon multiplication of (32) by $\mathfrak{x}^{\alpha-1}$ and then integrating over [0, 1],

$$\frac{1}{\mathfrak{h}\left(\frac{1}{2}\right)}\mathtt{F}\left(m\mathtt{I_a}+\frac{1}{2}\Pi(\mathtt{I_b}, \mathtt{I_a}, m)\right)\int_0^1 \mathfrak{x}^{\alpha-1}d\mathfrak{x}$$

$$\preceq_{cr} \left[\int_0^1 \mathfrak{x}^{\alpha-1}\mathtt{F}\left(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b}, \mathtt{I_a}), m\right)d\mathfrak{x}+\int_0^1 \mathfrak{x}^{\alpha-1}\mathtt{F}\left(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b}, \mathtt{I_a}, m)\right)d\mathfrak{x}\right]$$

$$= \left[\int_0^1 \mathfrak{x}^{\alpha-1}\left(\underline{\mathtt{F}}\left(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b}, \mathtt{I_a}, m)\right)+\underline{\mathtt{F}}\left(\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b}, \mathtt{I_a}, m)\right)\right)d\mathfrak{x}\,,\right.$$

$$\left.\int_0^1 \mathfrak{x}^{\alpha-1}\left(\overline{\mathtt{F}}\left(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b}, \mathtt{I_a}, m)\right)+\overline{\mathtt{F}}\left(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b}, \mathtt{I_a}, m)\right)\right)d\mathfrak{x}\right]$$

$$= \left[\int_{m\mathtt{I_a}}^{m\mathtt{I_a}+\Pi(\mathtt{I_b}, \mathtt{I_a}, m)}\left(\frac{\mathfrak{x}-m\mathtt{I_a}}{\Pi(\mathtt{I_b}, \mathtt{I_a}, m)}\right)^{\alpha-1}\underline{\mathtt{F}}(\mathfrak{x})\frac{d\mathfrak{x}}{\Pi(\mathtt{I_b}, \mathtt{I_a}, m)}\right.$$

$$+\int_{m\mathtt{I_a}}^{m\mathtt{I_a}+\Pi(\mathtt{I_b}, \mathtt{I_a}, m)}\left(\frac{m\mathtt{I_a}+\Pi(\mathtt{I_b}, \mathtt{I_a}, m)-\mathfrak{x}}{\Pi(\mathtt{I_b}, \mathtt{I_a}, m)}\right)^{\alpha-1}\underline{\mathtt{F}}(\mathfrak{x})\frac{d\mathfrak{x}}{\Pi(\mathtt{I_b}, \mathtt{I_a}, m)},$$

$$\int_{m\mathtt{I_a}}^{m\mathtt{I_a}+\Pi(\mathtt{I_b}, \mathtt{I_a}, m)}\left(\frac{\mathfrak{x}-m\mathtt{I_a}}{\Pi(\mathtt{I_b}, \mathtt{I_a}, m)}\right)^{\alpha-1}\overline{\mathtt{F}}(\mathfrak{x})\frac{d\mathfrak{x}}{\Pi(\mathtt{I_b}, \mathtt{I_a}, m)}$$

$$+\int_{m\mathtt{I_a}}^{m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)}\left(\frac{m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)-\mathfrak{x}}{\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)}\right)^{\alpha-1}\overline{\mathtt{F}}(\mathfrak{x})\frac{d\mathfrak{x}}{\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)}\right]$$

$$=\left[\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\left[\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^-}\underline{\mathtt{F}}(\mathtt{I_a})+\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\underline{\mathtt{F}}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\right],\right.$$

$$\left.\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\left[\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^-}\overline{\mathtt{F}}(m\mathtt{I_a})+\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\overline{\mathtt{F}}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\right]\right]$$

$$=\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\left[\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^-}\mathtt{F}(m\mathtt{I_a})+\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{F}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\right].$$

Now, it can be concluded that

$$\begin{aligned}&\frac{1}{\alpha\mathfrak{h}\left(\frac{1}{2}\right)}\mathtt{F}\left(m\mathtt{I_a}+\frac{1}{2}\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)\right)\\&\preceq_{\mathrm{cr}}\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\left[\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^-}\mathtt{F}(m\mathtt{I_a})+\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{F}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\right].\end{aligned}\tag{33}$$

Hence, the first inequality is established. For the next part, we have

$$\mathtt{F}\left(m\mathtt{I_a}+\mathfrak{x}\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)\preceq_{\mathrm{cr}}\mathfrak{h}(\mathfrak{x})\mathtt{F}(\mathtt{I_b})+m\mathfrak{h}(1-\mathfrak{x})\,\mathtt{F}(m\mathtt{I_a})$$

and

$$\mathtt{F}\left(m\mathtt{I_a}+(1-\mathfrak{x})\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)\preceq_{\mathrm{cr}}m\mathfrak{h}(\mathfrak{x})\mathtt{F}(\mathtt{I_a})+\mathfrak{h}(1-\mathfrak{x})\,\mathtt{F}(\mathtt{I_b}).$$

Consequently, we have

$$\begin{aligned}&\int_0^1\mathfrak{x}^{\alpha-1}\mathtt{F}\left(m\mathtt{I_a}+\mathfrak{x}\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)d\mathfrak{x}+\int_0^1\mathfrak{x}^{\alpha-1}\mathtt{F}\left(m\mathtt{I_a}+(1-\mathfrak{x})\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)d\mathfrak{x}\\&\preceq_{\mathrm{cr}}\left[\mathtt{F}(m\mathtt{I_a})+\mathtt{F}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\right]\int_0^1\mathfrak{x}^{\alpha-1}[\mathfrak{h}(\mathfrak{x})+\mathfrak{h}(1-\mathfrak{x})]d\mathfrak{x}.\end{aligned}$$

We conclude,

$$
\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\left[\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^-}\mathtt{F}(m\mathtt{I_a})+\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{F}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\right]
$$

$$
\preceq_{\mathrm{cr}}[\mathtt{F}(m\mathtt{I_a})+\mathtt{F}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))]\int_0^1\mathfrak{x}^{\alpha-1}[\mathfrak{h}(\mathfrak{x})+\mathfrak{h}(1-\mathfrak{x})]d\mathfrak{x}. \tag{34}
$$

We attained the desired inequality by combining (33) and (34), i.e.,

$$
\frac{1}{\alpha\mathfrak{h}\left(\frac{1}{2}\right)}\mathtt{F}\left(m\mathtt{I_a}+\frac{1}{2}\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)\right)
$$

$$
\preceq_{\mathrm{cr}}\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\left[\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^-}\mathtt{F}(m\mathtt{I_a})+\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{F}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\right]
$$

$$
\preceq_{\mathrm{cr}}[\mathtt{F}(m\mathtt{I_a})+\mathtt{F}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))]\int_0^1\mathfrak{x}^{\alpha-1}[\mathfrak{h}(\mathfrak{x})+\mathfrak{h}(1-\mathfrak{x})]d\mathfrak{x}.
$$

□

Corollary 2 Choosing $\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)=\mathtt{I_b}-m\mathtt{I_a}$, then we get the new findings for interval-valued center-radius order $(\mathfrak{h},\,m)$-convex functions:

$$
\frac{1}{\alpha\mathfrak{h}\left(\frac{1}{2}\right)}\mathtt{F}\left(\frac{m\mathtt{I_a}+\mathtt{I_b}}{2}\right)\preceq_{\mathrm{cr}}\frac{\Gamma(\alpha)}{(\mathtt{I_b}-m\mathtt{I_a})^{\alpha}}\left[\mathrm{I}^{\alpha}_{\mathtt{I_b}^-}\mathtt{F}(m\mathtt{I_a})+\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{F}(\mathtt{I_b})\right]
$$

$$
\preceq_{\mathrm{cr}}[\mathtt{F}(m\mathtt{I_a})+\mathtt{F}(\mathtt{I_b})]\int_0^1\mathfrak{x}^{\alpha-1}[\mathfrak{h}(\mathfrak{x})+\mathfrak{h}(1-\mathfrak{x})]d\mathfrak{x}. \tag{35}
$$

Corollary 3 Choosing $\mathfrak{h}(\mathfrak{x})=\mathfrak{x}$ and $\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)=\mathtt{I_b}-m\mathtt{I_a}$, then we get the new findings for interval-valued center-radius order m-convex functions:

$$
\mathtt{F}\left(\frac{m\mathtt{I_a}+\mathtt{I_b}}{2}\right)\preceq_{\mathrm{cr}}\frac{\Gamma(\alpha+1)}{2(\mathtt{I_b}-m\mathtt{I_a})^{\alpha}}\left[\mathrm{I}^{\alpha}_{\mathtt{I_b}^-}\mathtt{F}(m\mathtt{I_a})+\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{F}(\mathtt{I_b})\right]\preceq_{\mathrm{cr}}\frac{\mathtt{F}(m\mathtt{I_a})+\mathtt{F}(\mathtt{I_b})}{2}, \tag{36}
$$

where $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$.

Corollary 4 Choosing $\mathfrak{h}(\mathfrak{x})=\mathfrak{x}$, $m=1$ and $\Pi(\mathtt{I_b},\,\mathtt{I_a})=\mathtt{I_b}-\mathtt{I_a}$ then we get the new findings for interval-valued center-radius order convex functions:

$$
\mathtt{F}\left(\frac{\mathtt{I_a}+\mathtt{I_b}}{2}\right)\preceq_{\mathrm{cr}}\frac{\Gamma(\alpha+1)}{2(\mathtt{I_b}-\mathtt{I_a})^{\alpha}}\left[\mathrm{I}^{\alpha}_{\mathtt{I_b}^-}\mathtt{F}(\mathtt{I_a})+\mathrm{I}^{\alpha}_{\mathtt{I_a}^+}\mathtt{F}(\mathtt{I_b})\right]\preceq_{\mathrm{cr}}\frac{\mathtt{F}(\mathtt{I_a})+\mathtt{F}(\mathtt{I_b})}{2}. \tag{37}
$$

Example 3 Let $\mathtt{F}(\mathfrak{x}) = [-\mathfrak{x}^2+1, \mathfrak{x}^2+2]$, $\Pi(\mathtt{I_b}, \mathtt{I_a}, m) = \mathtt{I_b} - m\mathtt{I_a}$, $\mathtt{I_a} = 0$, $m = 1$ and $\mathtt{I_b} = 1$. Then, for $\alpha = \frac{1}{2}$ and $\mathfrak{h}(\mathfrak{x}) = \mathfrak{x}$, all conditions of Theorem 2 are examined. We have

$$\frac{1}{\alpha\mathfrak{h}\left(\frac{1}{2}\right)}\mathtt{F}\left(m\mathtt{I_a} + \frac{1}{2}\Pi(\mathtt{I_b}, \mathtt{I_a}, m)\right) \approx [3, 9], \tag{38}$$

$$\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b}, \mathtt{I_a}, m))^{\alpha}}\left[\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b}, \mathtt{I_a}, m))^-}\mathtt{F}(m\mathtt{I_a}) + \mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{F}(\mathtt{I_a}+\Pi(\mathtt{I_b}, \mathtt{I_a}, m))\right] \approx \left[\frac{4.75}{1.875}, \frac{17.75}{1.875}\right] \tag{39}$$

and

$$[\mathtt{F}(m\mathtt{I_a}) + \mathtt{F}(m\mathtt{I_a}+\Pi(\mathtt{I_b}, \mathtt{I_a}, m))]\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}(\mathfrak{x}) + \mathfrak{h}(1-\mathfrak{x})]d\mathfrak{x} \approx [2, 10]. \tag{40}$$

As a result,

$$[3, 9] \preceq_{\mathrm{cr}} \left[\frac{4.75}{1.875}, \frac{17.75}{1.875}\right] \preceq_{\mathrm{cr}} [2, 10]. \tag{41}$$

$\Longrightarrow$ The above Example 3 indicates the numerical validity of Theorem 2.
The following Figures 5 and 6 represent the graphical validation of Theorem 2.

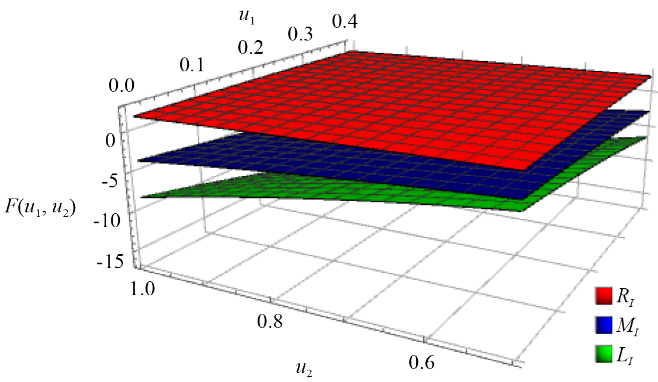

Figure 5. The graph validation of Theorem 2 for the function $\mathtt{F}(\mathfrak{x}) = -\mathfrak{x}^2 + 1$

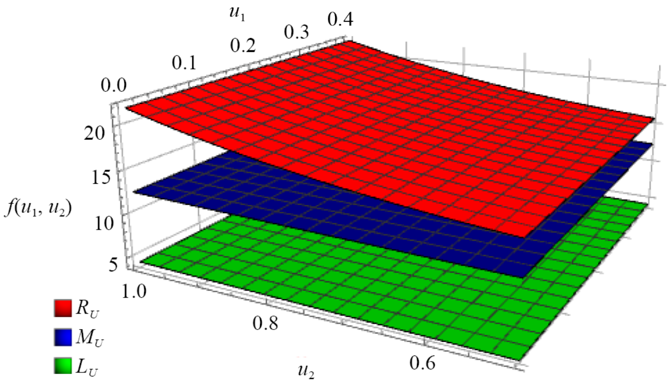

Figure 6. The graph validation of Theorem 2 for the function $\mathtt{F}(\mathfrak{x}) = \mathfrak{x}^2 + 2$

5. Fejér type integral inequality of the first and second kind involving interval valued center-radius order perinvex function pertaining to Riemann-Liouville fractional integral operator

The Fejér inequality is the widely utilized inequality in the literature. This inequality is an extended form of the famous H-H inequality and holds for convexity under a symmetric, non-negative weight function. This inequality offers estimates for the weighted integral of a convex function with a symmetric and non-negative weight function. It is especially applied in approximation theory and integral inequalities and has implications in fractional calculus. This inequality has gained a lot of attention in the scientific community because of its pervasive viewpoint.

In this section, we explore the Fejér inequality of the first and second kind involving interval valued center-radius order preinvex function pertaining to Riemann-Liouville fractional operator.

Theorem 3 (Fejér-type inequality of the first kind)

Let an interval-valued function $\mathtt{F} : [m\mathtt{I_a},\, m\mathtt{I_a} + \Pi(\mathtt{I_b},\, \mathtt{I_a},\, m)] \to \mathbb{R}$ is stated as

$$\mathtt{F}(\mathfrak{x}) = \left[\underline{\mathtt{F}}(\mathfrak{x}),\, \overline{\mathtt{F}}(\mathfrak{x})\right] \tag{42}$$

for all $\mathfrak{x} \in [\mathtt{I_a},\, \mathtt{I_b}]$. If $\mathtt{F} : [m\mathtt{I_a},\, m\mathtt{I_a} + \Pi(\mathtt{I_b},\, \mathtt{I_a},\, m)] \to \mathbb{R}$ is an interval-valued center-radius order $(\mathfrak{h}, m)$-preinvex function and $\mathtt{W} : [m\mathtt{I_a},\, m\mathtt{I_a} + \Pi(\mathtt{I_b},\, \mathtt{I_a},\, m)] \to \mathbb{R}$, $\mathtt{W} > 0$ is symmetric w.r.t. $m\mathtt{I_a} + \dfrac{1}{2}\Pi(\mathtt{I_b},\, \mathtt{I_a},\, m)$, then assuming

$$\int_{m\mathtt{I_a}}^{m\mathtt{I_a}+\Pi(\mathtt{I_b},\, \mathtt{I_a},\, m)} \mathtt{W}(\mathfrak{x})d\mathfrak{x} > 0, \tag{43}$$

the following inequalities hold true:

$$\mathtt{F}\left(m\mathtt{I_a} + \frac{1}{2}\Pi(\mathtt{I_b},\, \mathtt{I_a},\, m)\right) \frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\, \mathtt{I_a},\, m))^{\alpha}}$$

$$\left[\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{W}(m\mathtt{I_a} + \Pi(\mathtt{I_b},\, \mathtt{I_a},\, m)) + \mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\, \mathtt{I_a},\, m))^-}\mathtt{W}(\mathtt{I_a},\, m)\right]$$

$$\preceq_{cr} 2\mathfrak{h}\left(\frac{1}{2}\right)\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\times\Big[\mathrm{I}^{\alpha}_{m\mathtt{I_a}+}\mathtt{F}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))$$

$$\mathtt{W}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))+\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))-}\mathtt{F}(m\mathtt{I_a})\mathtt{W}(m\mathtt{I_a})\Big]. \tag{44}$$

Proof. Employing the concept of interval-valued center-radius $(\mathfrak{h},\, m)$-preinvexity and extended condition C, we have

$$\mathtt{F}\left(m\mathtt{I_a}+\frac{1}{2}\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)\right)\preceq_{cr}\mathfrak{h}\left(\frac{1}{2}\right)\left[\mathtt{F}\left(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)\right)d\mathfrak{x}+\mathtt{F}\left(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)\right)d\mathfrak{x}\right].$$

Multiplying by $\mathfrak{x}^{\alpha-1}\mathtt{W}\left(m\mathtt{I_a}+\mathfrak{x}\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)=\mathfrak{x}^{\alpha-1}\mathtt{W}\left(m\mathtt{I_a}+(1-\mathfrak{x})\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)$ and integrating over [0, 1],

$$\mathtt{F}\left(m\mathtt{I_a}+\frac{1}{2}\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)\right)\int_0^1\mathfrak{x}^{\alpha-1}\mathtt{W}\left(m\mathtt{I_a}+\mathfrak{x}\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)d\mathfrak{x}$$

$$\preceq_{cr}\mathfrak{h}\left(\frac{1}{2}\right)\left[\int_0^1\mathfrak{x}^{\alpha-1}\mathtt{F}\left(m\mathtt{I_a}+\mathfrak{x}\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)\cdot\mathtt{W}\left(m\mathtt{I_a}+\mathfrak{x}\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)d\mathfrak{x}\right.$$

$$\left.+\int_0^1\mathfrak{x}^{\alpha-1}\mathtt{F}\left(m\mathtt{I_a}+(1-\mathfrak{x})\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)\cdot\mathtt{W}\left(m\mathtt{I_a}+(1-\mathfrak{x})\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)d\mathfrak{x}\right] \tag{45}$$

$$=\mathfrak{h}\left(\frac{1}{2}\right)\left[\int_0^1\mathfrak{x}^{\alpha-1}\mathtt{F}\left(m\mathtt{I_a}+\mathfrak{x}\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)\cdot\mathtt{W}\left(m\mathtt{I_a}+\mathfrak{x}\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)d\mathfrak{x}\right.$$

$$\left.+\int_0^1\mathfrak{x}^{\alpha-1}\mathtt{F}\left(m\mathtt{I_a}+(1-\mathfrak{x})\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)\cdot\mathtt{W}\left(m\mathtt{I_a}+\mathfrak{x}\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)d\mathfrak{x}\right],$$

since

$$\int_0^1\mathfrak{x}^{\alpha-1}\mathtt{F}\left(m\mathtt{I_a}+\mathfrak{x}\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)\cdot\mathtt{W}\left(m\mathtt{I_a}+\mathfrak{x}\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)d\mathfrak{x}$$

$$=\frac{1}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\int_{m\mathtt{I_a}}^{m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)}(\mathfrak{x}-m\mathtt{I_a})^{\alpha-1}\mathtt{F}(\mathfrak{x})\mathtt{W}(\mathfrak{x})d\mathfrak{x} \tag{46}$$

$$=\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\left[\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))-}\mathtt{F}(m\mathtt{I_a})\mathtt{W}(m\mathtt{I_a})\right].$$

Additionally,

$$\int_0^1 \mathfrak{x}^{\alpha-1}\mathtt{F}\left(m\mathtt{I_a}+(1-\mathfrak{x})\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)\cdot\mathtt{W}\left(m\mathtt{I_a}+\mathfrak{x}\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)d\mathfrak{x}$$

$$= \frac{1}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\int_{m\mathtt{I_a}}^{m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)}(\mathfrak{x}-\mathtt{I_a})^{\alpha-1}\mathtt{F}(2m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)-\mathfrak{x})\mathtt{W}(\mathfrak{x})d\mathfrak{x}$$

$$= \frac{1}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\int_{m\mathtt{I_a}}^{m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)}((m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))-\mathfrak{x})^{\alpha-1}\mathtt{F}(\mathfrak{x})\mathtt{W}(\mathtt{I_a}+\mathtt{I_b}-\mathfrak{x})d\mathfrak{x} \tag{47}$$

$$= \frac{1}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\int_{m\mathtt{I_a}}^{m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)}((m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))-\mathfrak{x})^{\alpha-1}\mathtt{F}(\mathfrak{x})\mathtt{W}(\mathfrak{x})d\mathfrak{x}$$

$$= \frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\left[\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{F}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\mathtt{W}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\right],$$

and

$$\int_0^1 \mathfrak{x}^{\alpha-1}\mathtt{W}\left(m\mathtt{I_a}+\mathfrak{x}\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)d\mathfrak{x} = \frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\left[\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^-}\mathtt{W}(m\mathtt{I_a})\right]. \tag{48}$$

Using (46), (47), and (48) in (45), we have

$$\begin{aligned}
&\frac{1}{2}\mathtt{F}\left(m\mathtt{I_a}+\frac{1}{2}\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)\right)\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\left[\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{W}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))+\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^-}\mathtt{W}(m\mathtt{I_a})\right]\\
&\preceq_{\mathrm{cr}} \mathfrak{h}\left(\frac{1}{2}\right)\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\\
&\times\left[\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{F}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\mathtt{W}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))+\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^-}\mathtt{F}(m\mathtt{I_a})\mathtt{W}(m\mathtt{I_a})\right].
\end{aligned} \tag{49}$$

From (49), we have

$$\begin{aligned}
&\mathtt{F}\left(m\mathtt{I_a}+\frac{1}{2}\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)\right)\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\left[\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{W}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))+\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^-}\mathtt{W}(m\mathtt{I_a})\right]\\
&\preceq_{\mathrm{cr}} 2\mathfrak{h}\left(\frac{1}{2}\right)\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\\
&\times\left[\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{F}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\mathtt{W}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))+\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^-}\mathtt{F}(m\mathtt{I_a})\mathtt{W}(m\mathtt{I_a})\right].
\end{aligned}$$

This is the required result. $\square$

Corollary 5 Choosing $\Pi(\mathtt{I_b},\ \mathtt{I_a},\ m) = \mathtt{I_b} - m\mathtt{I_a}$, then we get new finding for interval-valued center-radius order $(\mathfrak{h},\ m)$-convex functions, i.e.,

$$\begin{aligned} &\mathtt{F}\left(\frac{m\mathtt{I_a}+\mathtt{I_b}}{2}\right)\frac{\Gamma(\alpha)}{(\mathtt{I_b}-m\mathtt{I_a})^{\alpha}}\left[\mathrm{I}^{\alpha}_{m\mathtt{I_a}+}\mathtt{W}(\mathtt{I_b})+\mathrm{I}^{\alpha}_{\mathtt{I_b}-}\mathtt{W}(m\mathtt{I_a})\right] \\ &\preceq_{\mathrm{cr}} 2\mathfrak{h}\left(\frac{1}{2}\right)\frac{\Gamma(\alpha)}{(\mathtt{I_b}-m\mathtt{I_a})^{\alpha}}\left[\mathrm{I}^{\alpha}_{\mathtt{I_a}+}\mathtt{F}(\mathtt{I_b})\mathtt{W}(\mathtt{I_b})+\mathrm{I}^{\alpha}_{\mathtt{I_b}-}\mathtt{F}(m\mathtt{I_a})\mathtt{W}(m\mathtt{I_a})\right]. \end{aligned} \tag{50}$$

Corollary 6 Choosing $\Pi(\mathtt{I_b},\ \mathtt{I_a},\ m) = \mathtt{I_b} - m\mathtt{I_a}$ and $m = 1$, then we get new finding for interval-valued center-radius order $\mathfrak{h}$-convex functions, i.e.,

$$\begin{aligned} &\mathtt{F}\left(\frac{\mathtt{I_a}+\mathtt{I_b}}{2}\right)\frac{\Gamma(\alpha)}{(\mathtt{I_b}-\mathtt{I_a})^{\alpha}}\left[\mathrm{I}^{\alpha}_{\mathtt{I_a}+}\mathtt{W}(\mathtt{I_b})+\mathrm{I}^{\alpha}_{\mathtt{I_b}-}\mathtt{W}(\mathtt{I_a})\right] \\ &\preceq_{\mathrm{cr}} 2\mathfrak{h}\left(\frac{1}{2}\right)\frac{\Gamma(\alpha)}{(\mathtt{I_b}-\mathtt{I_a})^{\alpha}}\left[\mathrm{I}^{\alpha}_{\mathtt{I_a}+}\mathtt{F}(\mathtt{I_b})\mathtt{W}(\mathtt{I_b})+\mathrm{I}^{\alpha}_{\mathtt{I_b}-}\mathtt{F}(\mathtt{I_a})\mathtt{W}(\mathtt{I_a})\right]. \end{aligned} \tag{51}$$

Theorem 4 (Fejér inequality of the second kind) Suppose an interval-valued function $\mathtt{F} : [m\mathtt{I_a}, m\mathtt{I_a} + \Pi(\mathtt{I_b}, \mathtt{I_a}, m)] \to \mathbb{R}$ is stated as

$$\mathtt{F}(\mathfrak{x}) = \left[\underline{\mathtt{F}}(\mathfrak{x}),\ \overline{\mathtt{F}}(\mathfrak{x})\right] \tag{52}$$

for all $\mathfrak{x} \in [m\mathtt{I_a},\ \mathtt{I_b}]$. If $\mathtt{F} : [m\mathtt{I_a},\ m\mathtt{I_a} + \Pi(\mathtt{I_b},\ \mathtt{I_a},\ m)] \to \mathbb{R}$ is an interval-valued center-radius order $(\mathfrak{h},\ m)$-preinvex function and $\mathtt{W} : [m\mathtt{I_a},\ m\mathtt{I_a} + \Pi(\mathtt{I_b},\ \mathtt{I_a},\ m)] \to \mathbb{R}$, $\mathtt{W} > 0$ is symmetric w.r.t. $m\mathtt{I_a} + \dfrac{1}{2}\Pi(\mathtt{I_b},\ \mathtt{I_a},\ m)$, then

$$\begin{aligned} &\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\ \mathtt{I_a},\ m))^{\alpha}}\left[\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\ \mathtt{I_a},\ m))-}\mathtt{F}(m\mathtt{I_a})\mathtt{W}(m\mathtt{I_a})+\mathrm{I}^{\alpha}_{m\mathtt{I_a}+}\mathtt{F}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\ \mathtt{I_a},\ m))\mathtt{W}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\ \mathtt{I_a},\ m))\right] \\ &\preceq_{\mathrm{cr}} [\mathtt{F}(m\mathtt{I_a})+\mathtt{F}(\mathtt{I_b})]\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}(\mathfrak{x})+\mathfrak{h}(1-\mathfrak{x})]\mathtt{W}(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\ \mathtt{I_a},\ m))\,d\mathfrak{x}. \end{aligned} \tag{53}$$

Proof. Utilizing Definition 7 and $\mathtt{W}$ as symmetric w.r.t. $m\mathtt{I_a} + \dfrac{1}{2}\Pi(\mathtt{I_b},\ \mathtt{I_a},\ m)$, we have

$$\begin{aligned} &\mathtt{F}(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\ \mathtt{I_a},\ m))\cdot\mathtt{W}(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\ \mathtt{I_a},\ m)) \\ &\preceq_{\mathrm{cr}} [\mathfrak{h}(\mathfrak{x})\mathtt{F}(\mathtt{I_b})+\mathfrak{h}(1-\mathfrak{x})\ \mathtt{F}(m\mathtt{I_a})]\cdot\mathtt{W}(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\ \mathtt{I_a},\ m)) \end{aligned}$$

and

$$\mathtt{F}(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\mathtt{W}(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b},\mathtt{I_a},m))$$

$$\preceq_{cr}[\mathfrak{h}(\mathfrak{x})\mathtt{F}(m\mathtt{I_a})+\mathfrak{h}(1-\mathfrak{x})\,\mathtt{F}(\mathtt{I_b})]\cdot\mathtt{W}(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b},\mathtt{I_a},m)).$$

Upon addition of the above inequalities, we have

$$\begin{aligned}&\mathtt{F}(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\mathtt{W}(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\mathtt{I_a},m))\\&+\mathtt{F}(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\mathtt{W}(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b},\mathtt{I_a},m))\\&\preceq_{cr}[\mathtt{F}(m\mathtt{I_a})(\mathfrak{h}(1-\mathfrak{x})\mathtt{W}(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\mathtt{I_a},m))+\mathfrak{h}(\mathfrak{x})\mathtt{W}(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b},\mathtt{I_a},m)))\\&+\mathtt{F}(\mathtt{I_b})(\mathfrak{h}(\mathfrak{x})\mathtt{W}(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\mathtt{I_a},m))+\mathfrak{h}(1-\mathfrak{x})\mathtt{W}(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b},\mathtt{I_a},m)))].\end{aligned}$$

Employing the symmetric property of $\mathtt{W}$, we have

$$\begin{aligned}&[\mathtt{F}(m\mathtt{I_a})(\mathfrak{h}(1-\mathfrak{x})\mathtt{W}(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\mathtt{I_a},m))+\mathfrak{h}(\mathfrak{x})\mathtt{W}(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b},\mathtt{I_a},m)))\\&+\mathtt{F}(\mathtt{I_b})(\mathfrak{h}(\mathfrak{x})\mathtt{W}(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\mathtt{I_a},m))+\mathfrak{h}(1-\mathfrak{x})\mathtt{W}(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b},\mathtt{I_a},m)))]\\&=\mathtt{F}(m\mathtt{I_a})[\mathfrak{h}(\mathfrak{x})+\mathfrak{h}(1-\mathfrak{x})]\mathtt{W}(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\mathtt{I_a},m))+\mathtt{F}(\mathtt{I_b})[\mathfrak{h}(\mathfrak{x})+\mathfrak{h}(1-\mathfrak{x})]\mathtt{W}(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\mathtt{I_a},m)).\\&=[\mathtt{F}(m\mathtt{I_a})+\mathtt{F}(\mathtt{I_b})][\mathfrak{h}(\mathfrak{x})+\mathfrak{h}(1-\mathfrak{x})]\mathtt{W}(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\mathtt{I_a},m)).\end{aligned}\tag{54}$$

Multiplying by $\mathfrak{x}^{\alpha-1}$ and then integrating over [0, 1], we obtain

$$\begin{aligned}&\int_0^1\mathfrak{x}^{\alpha-1}\mathtt{F}(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\mathtt{W}(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\mathtt{I_a},m))\,d\mathfrak{x}\\&+\int_0^1\mathfrak{x}^{\alpha-1}\mathtt{F}(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\mathtt{W}(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b},\mathtt{I_a},m))\,d\mathfrak{x}\\&\preceq_{cr}[\mathtt{F}(m\mathtt{I_a})+\mathtt{F}(\mathtt{I_b})]\int_0^1\mathfrak{x}^{\alpha-1}[\mathfrak{h}(\mathfrak{x})+\mathfrak{h}(1-\mathfrak{x})]\mathtt{W}(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\mathtt{I_a},m))\,d\mathfrak{x}.\end{aligned}\tag{55}$$

From Definition 6 and 4, we have

$$\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\left[\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^-}\mathtt{F}(m\mathtt{I_a})\mathtt{W}(m\mathtt{I_a})+\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{F}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\mathtt{W}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\right]$$

$$\preceq_{\mathrm{cr}}[\mathtt{F}(m\mathtt{I_a})+\mathtt{F}(\mathtt{I_b})]\int_0^1\mathfrak{r}^{\alpha-1}[\mathfrak{h}(\mathfrak{r})+\mathfrak{h}(1-\mathfrak{r})]\mathtt{W}(m\mathtt{I_a}+\mathfrak{r}\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\,d\mathfrak{r}.$$

This is the required proof. □

Corollary 7 Choosing $\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)=\mathtt{I_b}-m\mathtt{I_a}$, then we get a new finding involving interval-valued center-radius order $(\mathfrak{h},\,m)$-convex functions, i.e.,

$$\begin{aligned}&\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\left[\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^-}\mathtt{F}(m\mathtt{I_a})\mathtt{W}(m\mathtt{I_a})+\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{F}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\mathtt{W}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\right]\\&\preceq_{\mathrm{cr}}[\mathtt{F}(m\mathtt{I_a})+\mathtt{F}(\mathtt{I_b})]\int_0^1\mathfrak{r}^{\alpha-1}[\mathfrak{h}(\mathfrak{r})+\mathfrak{h}(1-\mathfrak{r})]\mathtt{W}(m\mathtt{I_a}+\mathfrak{r}\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\,d\mathfrak{r}.\end{aligned}\tag{56}$$

Corollary 8 Choosing $\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)=\mathtt{I_b}-m\mathtt{I_a}$ and $m=1$, then we get a new finding involving interval-valued center-radius order $\mathfrak{h}$-convex functions, i.e.,

$$\begin{aligned}&\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a}))^{\alpha}}\left[\mathrm{I}^{\alpha}_{(\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a}))^-}\mathtt{F}(\mathtt{I_a})\mathtt{W}(\mathtt{I_a})+\mathrm{I}^{\alpha}_{\mathtt{I_a}^+}\mathtt{F}(\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a}))\mathtt{W}(\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a}))\right]\\&\preceq_{\mathrm{cr}}[\mathtt{F}(\mathtt{I_a})+\mathtt{F}(\mathtt{I_b})]\int_0^1\mathfrak{r}^{\alpha-1}[\mathfrak{h}(\mathfrak{r})+\mathfrak{h}(1-\mathfrak{r})]\mathtt{W}(\mathtt{I_a}+\mathfrak{r}\Pi(\mathtt{I_b},\,\mathtt{I_a}))\,d\mathfrak{r}.\end{aligned}\tag{57}$$

Remark 1 Combining Theorems 3 and 4 for $\mathtt{W}(\mathfrak{r})=1$, attain Theorem 2.

Example 4 Let $m=1$, $\mathtt{F}(\mathfrak{r})=[\mathfrak{r}^2,\,\mathfrak{r}^3+1]$, $\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)=\mathtt{I_b}-m\mathtt{I_a}$, $\mathtt{I_a}=0$, and $\mathtt{I_b}=1$. Then, for $\mathfrak{h}(\mathfrak{r})=\mathfrak{r}$ and $\mathtt{W}(\mathfrak{r})=\left(\frac{1}{2}-\mathfrak{r}\right)^2$, we obtain

$$\mathtt{F}\left(m\mathtt{I_a}+\frac{1}{2}\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)\right)\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}$$

$$\left[\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{W}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))+\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^-}\mathtt{W}(m\mathtt{I_a})\right]\approx\left[\frac{7}{60},\,\frac{21}{40}\right],$$

$$2\mathfrak{h}\left(\frac{1}{2}\right)\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}$$

$$\left[\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{F}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\mathtt{W}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))+\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^-}\mathtt{F}(m\mathtt{I_a})\mathtt{W}(m\mathtt{I_a})\right]\approx[0.20,\,0.65].\tag{58}$$

As a result,

$$\left[\frac{7}{60}, \frac{21}{40}\right] \preceq_{cr} [0.20, 0.65]. \tag{59}$$

$\Longrightarrow$ Example 4 indicates the numerical validity of Theorem 3.

Example 5 Similarly, considering Example 4, then all the assumptions in Theorem 4 are satisfied. The following Figures 7 and 8 represent the graphical validation of Theorem 3.

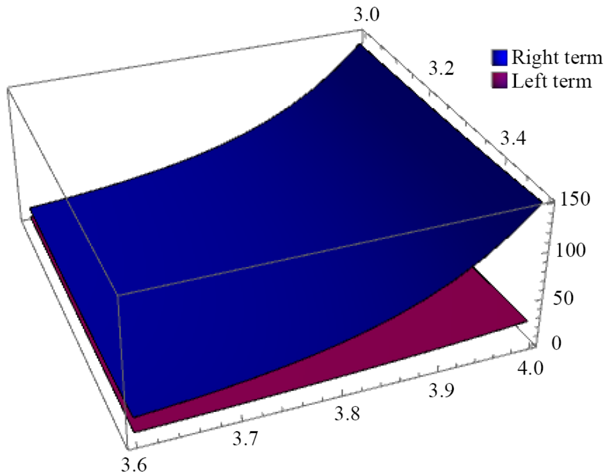

Figure 7. The graph validation of Theorem 3 for the function $F(\mathfrak{x}) = \mathfrak{x}^2$

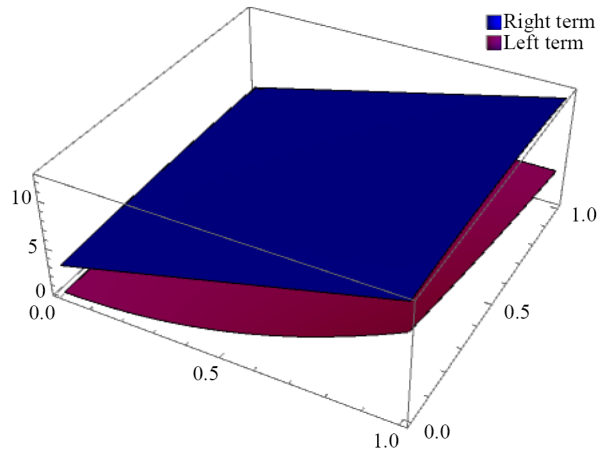

Figure 8. The graph validation of Theorem 3 for the function $F(\mathfrak{x}) = \mathfrak{x}^3 + 1$

6. Pachpatte-type integral inequality involving interval valued center-radius order perinvex function pertaining to Riemann-Liouville fractional integral operator

Pachpatte-type inequalities are widely recognized and frequently used for a variety of convex functions in various contexts. Research on this kind of integral inequality in the area of mathematical interpretation is both fascinating and alluring. The study of convexity and the notion of Pachpatte-type inequality are closely related.

In this section, we explore the Pachpatte-type inequality involving interval-valued center-radius order preinvex function pertaining to the Riemann-Liouville fractional operator.

Theorem 5 Let $\mathtt{F}, \mathtt{E} : [m\mathtt{I_a}, m\mathtt{I_a} + \Pi(\mathtt{I_b}, \mathtt{I_a}, m)] \to \mathbb{R}$ be interval-valued functions, given as:

$$\mathtt{F}(\mathfrak{r}) = \left[\underline{\mathtt{F}}(\mathfrak{r}), \overline{\mathtt{F}}(\mathfrak{r})\right] \text{ and } \mathtt{E}(\mathfrak{r}) = \left[\underline{\mathtt{E}}(\mathfrak{r}), \overline{\mathtt{E}}(\mathfrak{r})\right] \tag{60}$$

for all $\mathfrak{r} \in [m\mathtt{I_a}, \mathtt{I_b}]$ and $\mathtt{F}, \mathtt{E} \in \mathscr{I}\mathscr{R}_{([m\mathtt{I_a}, \mathtt{I_b}])}$. If $\mathtt{F} : [m\mathtt{I_a}, m\mathtt{I_a} + \Pi(\mathtt{I_b}, \mathtt{I_a}, m)] \to \mathbb{R}$ is an interval valued center-radius order $(\mathfrak{h}_1, m)$-preinvex function and $\mathtt{E} : [m\mathtt{I_a}, m\mathtt{I_a} + \Pi(\mathtt{I_b}, \mathtt{I_a}, m)] \to \mathbb{R}$ is an interval valued center-radius order $(\mathfrak{h}_2, m)$-preinvex function, then

$$\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b}, \mathtt{I_a}, m))^{\alpha}} \left[I^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b}, \mathtt{I_a}, m))^-} \mathtt{F}(m\mathtt{I_a})\mathtt{E}(m\mathtt{I_a}) + I^{\alpha}_{m\mathtt{I_a}^+} \mathtt{F}(m\mathtt{I_a} + \Pi(\mathtt{I_b}, \mathtt{I_a}, m))\mathtt{E}(m\mathtt{I_a} + \Pi(\mathtt{I_b}, \mathtt{I_a}, m)) \right]$$

$$\preceq_{\mathrm{cr}} \mathtt{M}(m\mathtt{I_a}, m\mathtt{I_a} + \Pi(\mathtt{I_b}, \mathtt{I_a}, m)) \int_0^1 \mathfrak{r}^{\alpha-1} [\mathfrak{h}_2(\mathfrak{r})\mathfrak{h}_1(\mathfrak{r}) + \mathfrak{h}_2(1-\mathfrak{r})\mathfrak{h}_1(1-\mathfrak{r})] d\mathfrak{r} \tag{61}$$

$$+ \mathtt{N}(m\mathtt{I_a}, m\mathtt{I_a} + \Pi(\mathtt{I_b}, \mathtt{I_a}, m)) \int_0^1 \mathfrak{r}^{\alpha-1} [\mathfrak{h}_2(1-\mathfrak{r})\mathfrak{h}_1(\mathfrak{r}) + \mathfrak{h}_2(\mathfrak{r})\mathfrak{h}_1(1-\mathfrak{r})] d\mathfrak{r}$$

holds true, where

$$\mathtt{M}(m\mathtt{I_a}, m\mathtt{I_a} + \Pi(\mathtt{I_b}, \mathtt{I_a}, m)) = \mathtt{F}(m\mathtt{I_a})\mathtt{E}(m\mathtt{I_a}) + \mathtt{F}(\mathtt{I_b})\mathtt{E}(\mathtt{I_b}) \tag{62}$$

and

$$\mathtt{N}(m\mathtt{I_a}, m\mathtt{I_a} + \Pi(\mathtt{I_b}, \mathtt{I_a}, m)) = \mathtt{F}(m\mathtt{I_a})\mathtt{E}(\mathtt{I_b}) + \mathtt{F}(\mathtt{I_b})\mathtt{E}(m\mathtt{I_a}). \tag{63}$$

Proof. Using the interval-valued center-radius order-$(\mathfrak{h}, m)$-preinvexity, we have

$$\mathtt{F}(m\mathtt{I_a} + \mathfrak{r}\Pi(\mathtt{I_b}, \mathtt{I_a}, m)) \preceq_{\mathrm{cr}} \mathfrak{h}_1(\mathfrak{r})\mathtt{F}(\mathtt{I_b}) + m\mathfrak{h}_1(1-\mathfrak{r})\,\mathtt{F}(\mathtt{I_a})$$

and

$$\mathtt{E}(m\mathtt{I_a} + \mathfrak{r}\Pi(\mathtt{I_b}, \mathtt{I_a}, m)) \preceq_{\mathrm{cr}} \mathfrak{h}_2(\mathfrak{r})\mathtt{E}(\mathtt{I_b}) + m\mathfrak{h}_2(1-\mathfrak{r})\,\mathtt{E}(\mathtt{I_a}).$$

Consequently, upon multiplication, it follows

$$\begin{aligned}
&\mathtt{F}(m\mathtt{I}_{\mathtt{a}}+\mathfrak{x}\Pi(\mathtt{I}_{\mathtt{b}},\,\mathtt{I}_{\mathtt{a}},\,m))\cdot\mathtt{E}(m\mathtt{I}_{\mathtt{a}}+\mathfrak{x}\Pi(\mathtt{I}_{\mathtt{b}},\,\mathtt{I}_{\mathtt{a}},\,m))\\
&\preceq_{\mathrm{cr}}[\mathfrak{h}_1(\mathfrak{x})\mathtt{F}(\mathtt{I}_{\mathtt{b}})+m\mathfrak{h}_1(1-\mathfrak{x})\,\mathtt{F}(\mathtt{I}_{\mathtt{a}})]\cdot[\mathfrak{h}_2(\mathfrak{x})\mathtt{E}(\mathtt{I}_{\mathtt{b}})+m\mathfrak{h}_2(1-\mathfrak{x})\,\mathtt{E}(\mathtt{I}_{\mathtt{a}})]\\
&=\mathfrak{h}_1(\mathfrak{x})\mathfrak{h}_2(\mathfrak{x})[\mathtt{F}(\mathtt{I}_{\mathtt{b}})\mathtt{E}(\mathtt{I}_{\mathtt{b}})]+m^2\mathfrak{h}_1(1-\mathfrak{x})\mathfrak{h}_2(1-\mathfrak{x})[\mathtt{F}(\mathtt{I}_{\mathtt{a}})\mathtt{E}(\mathtt{I}_{\mathtt{a}})]\\
&\quad+\mathfrak{h}_1(\mathfrak{x})\mathfrak{h}_2(1-\mathfrak{x})[\mathtt{F}(\mathtt{I}_{\mathtt{b}})\mathtt{E}(\mathtt{I}_{\mathtt{a}})]+m\mathfrak{h}_1(1-\mathfrak{x})\mathfrak{h}_2(\mathfrak{x})[\mathtt{F}(\mathtt{I}_{\mathtt{a}})\mathtt{E}(\mathtt{I}_{\mathtt{b}})].
\end{aligned}\tag{64}$$

Similarly,

$$\begin{aligned}
&\mathtt{F}(m\mathtt{I}_{\mathtt{a}}+(1-\mathfrak{x})\Pi(\mathtt{I}_{\mathtt{b}},\,\mathtt{I}_{\mathtt{a}},\,m))\cdot\mathtt{E}(m\mathtt{I}_{\mathtt{a}}+(1-\mathfrak{x})\Pi(\mathtt{I}_{\mathtt{b}},\,\mathtt{I}_{\mathtt{a}},\,m))\\
&\preceq_{\mathrm{cr}}[m\mathfrak{h}_1(\mathfrak{x})\mathtt{F}(\mathtt{I}_{\mathtt{a}})+\mathfrak{h}_1(1-\mathfrak{x})\,\mathtt{F}(\mathtt{I}_{\mathtt{b}})]\cdot[m\mathfrak{h}_2(\mathfrak{x})\mathtt{E}(\mathtt{I}_{\mathtt{a}})+\mathfrak{h}_2(1-\mathfrak{x})\,\mathtt{E}(\mathtt{I}_{\mathtt{b}})]\\
&=m^2\mathfrak{h}_1(\mathfrak{x})\mathfrak{h}_2(\mathfrak{x})[\mathtt{F}(\mathtt{I}_{\mathtt{a}})\mathtt{E}(\mathtt{I}_{\mathtt{a}})]+\mathfrak{h}_1(1-\mathfrak{x})\mathfrak{h}_2(1-\mathfrak{x})[\mathtt{F}(\mathtt{I}_{\mathtt{b}})\mathtt{E}(\mathtt{I}_{\mathtt{b}})]\\
&\quad+m\mathfrak{h}_1(\mathfrak{x})\mathfrak{h}_2(1-\mathfrak{x})[\mathtt{F}(\mathtt{I}_{\mathtt{a}})\mathtt{E}(\mathtt{I}_{\mathtt{b}})]+m\mathfrak{h}_1(1-\mathfrak{x})\mathfrak{h}_2(\mathfrak{x})[\mathtt{F}(\mathtt{I}_{\mathtt{b}})\mathtt{E}(\mathtt{I}_{\mathtt{a}})].
\end{aligned}\tag{65}$$

Adding (64) and (65), then multiplication by $\mathfrak{x}^{\alpha-1}$, and finally integrating over [0, 1] results in

$$\begin{aligned}
&\int_0^1\mathfrak{x}^{\alpha-1}\mathtt{F}(m\mathtt{I}_{\mathtt{a}}+\mathfrak{x}\Pi(\mathtt{I}_{\mathtt{b}},\,\mathtt{I}_{\mathtt{a}},\,m))\cdot\mathtt{E}(m\mathtt{I}_{\mathtt{a}}+\mathfrak{x}\Pi(\mathtt{I}_{\mathtt{b}},\,\mathtt{I}_{\mathtt{a}},\,m))\,d\mathfrak{x}\\
&+\int_0^1\mathfrak{x}^{\alpha-1}\mathtt{F}(m\mathtt{I}_{\mathtt{a}}+(1-\mathfrak{x})\Pi(\mathtt{I}_{\mathtt{b}},\,\mathtt{I}_{\mathtt{a}},\,m))\cdot\mathtt{E}(m\mathtt{I}_{\mathtt{a}}+(1-\mathfrak{x})\Pi(\mathtt{I}_{\mathtt{b}},\,\mathtt{I}_{\mathtt{a}},\,m))\,d\mathfrak{x}\\
=&\Bigg[\int_0^1\mathfrak{x}^{\alpha-1}\underline{\mathtt{F}}(m\mathtt{I}_{\mathtt{a}}+\mathfrak{x}\Pi(\mathtt{I}_{\mathtt{b}},\,\mathtt{I}_{\mathtt{a}},\,m))\cdot\underline{\mathtt{E}}(m\mathtt{I}_{\mathtt{a}}+\mathfrak{x}\Pi(\mathtt{I}_{\mathtt{b}},\,\mathtt{I}_{\mathtt{a}},\,m))\,d\mathfrak{x}\\
&+\int_0^1\mathfrak{x}^{\alpha-1}\underline{\mathtt{F}}(m\mathtt{I}_{\mathtt{a}}+(1-\mathfrak{x})\Pi(\mathtt{I}_{\mathtt{b}},\,\mathtt{I}_{\mathtt{a}},\,m))\cdot\underline{\mathtt{E}}(m\mathtt{I}_{\mathtt{a}}+(1-\mathfrak{x})\Pi(\mathtt{I}_{\mathtt{b}},\,\mathtt{I}_{\mathtt{a}},\,m))\,d\mathfrak{x},\\
&\int_0^1\mathfrak{x}^{\alpha-1}\overline{\mathtt{F}}(m\mathtt{I}_{\mathtt{a}}+\mathfrak{x}\Pi(\mathtt{I}_{\mathtt{b}},\,\mathtt{I}_{\mathtt{a}},\,m))\cdot\overline{\mathtt{E}}(m\mathtt{I}_{\mathtt{a}}+\mathfrak{x}\Pi(\mathtt{I}_{\mathtt{b}},\,\mathtt{I}_{\mathtt{a}},\,m))\,d\mathfrak{x}\\
&+\int_0^1\mathfrak{x}^{\alpha-1}\overline{\mathtt{F}}(m\mathtt{I}_{\mathtt{a}}+(1-\mathfrak{x})\Pi(\mathtt{I}_{\mathtt{b}},\,\mathtt{I}_{\mathtt{a}},\,m))\cdot\overline{\mathtt{E}}(m\mathtt{I}_{\mathtt{a}}+(1-\mathfrak{x})\Pi(\mathtt{I}_{\mathtt{b}},\,\mathtt{I}_{\mathtt{a}},\,m))\,d\mathfrak{x}\Bigg]
\end{aligned}$$

$$\preceq_{\mathrm{cr}} [\mathtt{F}(\mathtt{I_b})\mathtt{E}(\mathtt{I_b})+m^2\mathtt{F}(\mathtt{I_a})\mathtt{E}(\mathtt{I_a})]\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})+\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})]d\mathfrak{x}$$

$$+[\mathtt{F}(\mathtt{I_a})\mathtt{E}(\mathtt{I_b})+m\mathtt{F}(\mathtt{I_b})\mathtt{E}(\mathtt{I_a})]\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_2(\mathfrak{x})+\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})]d\mathfrak{x}$$

$$=\Bigg[[\underline{\mathtt{F}}(\mathtt{I_b})\underline{\mathtt{E}}(\mathtt{I_b})+m^2\underline{\mathtt{F}}(\mathtt{I_a})\underline{\mathtt{E}}(\mathtt{I_a})]\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})+\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})]d\mathfrak{x}$$

$$+[\underline{\mathtt{F}}(\mathtt{I_a})\underline{\mathtt{E}}(\mathtt{I_b})+m\underline{\mathtt{F}}(\mathtt{I_b})\underline{\mathtt{E}}(\mathtt{I_a})]\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})+\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})]d\mathfrak{x},$$

$$[\overline{\mathtt{F}}(\mathtt{I_b})\overline{\mathtt{E}}(\mathtt{I_b})+m^2\overline{\mathtt{F}}(\mathtt{I_a})\overline{\mathtt{E}}(\mathtt{I_a})]\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})+\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})]d\mathfrak{x}$$

$$+\ [m\overline{\mathtt{F}}(\mathtt{I_a})\overline{\mathtt{E}}(\mathtt{I_b})+m\overline{\mathtt{F}}(\mathtt{I_b})\overline{\mathtt{E}}(\mathtt{I_a})]\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})+\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})]d\mathfrak{x}\Bigg].$$

From Definition 6, it follows

$$\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^{\alpha}}\left[\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))^-}\mathtt{F}(m\mathtt{I_a})\mathtt{E}(m\mathtt{I_a})+\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{F}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\mathtt{E}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\right]$$

$$\preceq_{\mathrm{cr}} [\mathtt{F}(m\mathtt{I_a})\mathtt{E}(m\mathtt{I_a})+\mathtt{F}(\mathtt{I_b})\mathtt{E}(\mathtt{I_b})]\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})+\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})]d\mathfrak{x}$$

$$+[\mathtt{F}(m\mathtt{I_a})\mathtt{E}(\mathtt{I_b})+\mathtt{F}(\mathtt{I_b})\mathtt{E}(m\mathtt{I_a})]\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})+\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})]d\mathfrak{x}$$

$$=\mathtt{M}(m\mathtt{I_a},\,m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})+\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})]d\mathfrak{x}$$

$$+\mathtt{N}(m\mathtt{I_a},\,m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})+\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})]d\mathfrak{x}.$$

This completes the proof. □

Corollary 9 Choosing $\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)=\mathtt{I_b}-m\mathtt{I_a}$ and $m=1$, then we get new findings involving interval-valued center-radius order convexity, i.e.,

$$\frac{\Gamma(\alpha)}{(\mathtt{I_b}-\mathtt{I_a})^{\alpha}}\left[\mathrm{I}^{\alpha}_{\mathtt{I_b}^-}\mathtt{F}(\mathtt{I_a})\mathtt{E}(\mathtt{I_a})+\mathrm{I}^{\alpha}_{\mathtt{I_a}^+}\mathtt{F}(\mathtt{I_b})\mathtt{E}(\mathtt{I_b})\right]$$

$$\preceq_{cr} \mathtt{M}(\mathtt{I_a},\, \mathtt{I_b}) \int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(\mathfrak{x}) + \mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})]d\mathfrak{x}$$

$$+\mathtt{N}(\mathtt{I_a},\, \mathtt{I_b}) \int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(\mathfrak{x}) + \mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})]d\mathfrak{x}, \tag{66}$$

Theorem 6 Let $\mathtt{F},\, \mathtt{E} : [\mathtt{I_b} + \Pi(\mathtt{I_a},\, \mathtt{I_b}),\, \mathtt{I_b}] \to \mathbb{R}$ be interval-valued functions, given a

$$\mathtt{F}(\mathfrak{x}) = \left[\underline{\mathtt{F}}(\mathfrak{x})\, , \overline{\mathtt{F}}(\mathfrak{x})\right] \quad \text{and} \quad \mathtt{E}(\mathfrak{x}) = \left[\underline{\mathtt{E}}(\mathfrak{x})\, , \overline{\mathtt{E}}(\mathfrak{x})\right] \tag{67}$$

for all $\mathfrak{x} \in [\mathtt{I_a},\, \mathtt{I_b}]$ and $\mathtt{F},\, \mathtt{E} \in \mathscr{I}\mathscr{R}_{([\mathtt{I_a},\, \mathtt{I_b}])}$. If $\mathtt{F} : [\mathtt{I_b} + \Pi(\mathtt{I_a},\, \mathtt{I_b}),\, \mathtt{I_b}] \to \mathbb{R}$ is an interval-valued center-radius order $(\mathfrak{h}_1,\, m)$-preinvex function and $\mathtt{E} : [\mathtt{I_b} + \Pi(\mathtt{I_a},\, \mathtt{I_b}),\, \mathtt{I_b}] \to \mathbb{R}$ is an interval-valued center-radius order $(\mathfrak{h}_2,\, m)$-preinvex function, then

$$\begin{aligned}
&\frac{1}{\alpha\mathfrak{h}_1\left(\frac{1}{2}\right)\mathfrak{h}_2\left(\frac{1}{2}\right)}\mathtt{F}\left(m\mathtt{I_a} + \frac{1}{2}\Pi(\mathtt{I_b},\, \mathtt{I_a},\, m)\right)\cdot \mathtt{E}\left(m\mathtt{I_a} + \frac{1}{2}\Pi(\mathtt{I_b},\, \mathtt{I_a},\, m)\right)\\
&\preceq_{cr} \frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\, \mathtt{I_a},\, m))^{\alpha}}\\
&\left[\mathtt{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\, \mathtt{I_a},\, m))^-}\mathtt{F}(m\mathtt{I_a})\mathtt{E}(m\mathtt{I_a}) + \mathtt{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{F}(m\mathtt{I_a} + \Pi(\mathtt{I_b},\, \mathtt{I_a},\, m))\mathtt{E}(m\mathtt{I_a} + \Pi(\mathtt{I_b},\, \mathtt{I_a},\, m))\right]\\
&+\mathtt{M}(m\mathtt{I_a},\, m\mathtt{I_a} + \Pi(\mathtt{I_b},\, \mathtt{I_a},\, m)) \int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x}) + \mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})]d\mathfrak{x}\\
&+\mathtt{N}(m\mathtt{I_a},\, m\mathtt{I_a} + \Pi(\mathtt{I_b},\, \mathtt{I_a},\, m)) \int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(\mathfrak{x}) + \mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})]d\mathfrak{x}
\end{aligned} \tag{68}$$

holds true, where $\mathtt{M}(\mathtt{I_a},\, \mathtt{I_a} + \Pi(\mathtt{I_b},\, \mathtt{I_a}))$ and $\mathtt{N}(\mathtt{I_a},\, \mathtt{I_a} + \Pi(\mathtt{I_b},\, \mathtt{I_a}))$.

Proof. Utilizing the idea of interval-valued center-radius order $(\mathfrak{h} - m)$-preinvexity and extended condition C,

$$\frac{1}{\mathfrak{h}\left(\frac{1}{2}\right)}\mathtt{F}\left(m\mathtt{I_a} + \frac{1}{2}\Pi(\mathtt{I_b},\, \mathtt{I_a},\, m)\right)$$

$$\preceq_{cr} [\mathtt{F}(m\mathtt{I_a} + \mathfrak{x}\Pi(\mathtt{I_b},\, \mathtt{I_a},\, m)) + \mathtt{F}(m\mathtt{I_a} + (1-\mathfrak{x})\Pi(\mathtt{I_b},\, \mathtt{I_a},\, m))].$$

It follows that

$$\begin{aligned}
&\mathtt{F}\left(m\mathtt{I}_\mathtt{a}+\frac{1}{2}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)\\
&=\mathtt{F}\left(m\mathtt{I}_\mathtt{a}+\mathfrak{x}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)+\frac{1}{2}\Pi\left(m\mathtt{I}_\mathtt{a}+(1-\mathfrak{x})\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m),\,m\mathtt{I}_\mathtt{a}+\mathfrak{x}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)\right)\\
&\preceq_{\mathrm{cr}}\mathfrak{h}_1\left(\frac{1}{2}\right)\left[\mathtt{F}\left(m\mathtt{I}_\mathtt{a}+\mathfrak{x}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)+\mathtt{F}\left(m\mathtt{I}_\mathtt{a}+(1-\mathfrak{x})\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)\right]
\end{aligned} \tag{69}$$

and

$$\begin{aligned}
&\mathtt{E}\left(m\mathtt{I}_\mathtt{a}+\frac{1}{2}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)\\
&=\mathtt{E}\left(m\mathtt{I}_\mathtt{a}+\mathfrak{x}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)+\frac{1}{2}\Pi\left(m\mathtt{I}_\mathtt{a}+(1-\mathfrak{x})\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m),\,m\mathtt{I}_\mathtt{a}+\mathfrak{x}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)\right)\\
&\preceq_{\mathrm{cr}}\mathfrak{h}_2\left(\frac{1}{2}\right)\left[\mathtt{E}\left(m\mathtt{I}_\mathtt{a}+\mathfrak{x}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)+\mathtt{E}\left(m\mathtt{I}_\mathtt{a}+(1-\mathfrak{x})\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)\right].
\end{aligned} \tag{70}$$

Upon multiplying (69) and (70), one gets

$$\begin{aligned}
&\mathtt{F}\left(m\mathtt{I}_\mathtt{a}+\frac{1}{2}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)\cdot\mathtt{E}\left(m\mathtt{I}_\mathtt{a}+\frac{1}{2}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)\\
&\preceq_{\mathrm{cr}}\mathfrak{h}_1\left(\frac{1}{2}\right)\left[\mathtt{F}\left(m\mathtt{I}_\mathtt{a}+\mathfrak{x}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)+\mathtt{F}\left(m\mathtt{I}_\mathtt{a}+(1-\mathfrak{x})\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)\right]\\
&\quad\cdot\mathfrak{h}_2\left(\frac{1}{2}\right)\left[\mathtt{E}\left(m\mathtt{I}_\mathtt{a}+\mathfrak{x}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)+\mathtt{E}\left(m\mathtt{I}_\mathtt{a}+(1-\mathfrak{x})\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)\right]\\
&=\mathfrak{h}_1\left(\frac{1}{2}\right)\mathfrak{h}_2\left(\frac{1}{2}\right)\times\Big[\underline{\mathtt{F}}\left(m\mathtt{I}_\mathtt{a}+\mathfrak{x}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)\cdot\underline{\mathtt{E}}\left(m\mathtt{I}_\mathtt{a}+\mathfrak{x}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)\\
&\quad+\underline{\mathtt{F}}\left(m\mathtt{I}_\mathtt{a}+(1-\mathfrak{x})\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)\cdot\underline{\mathtt{E}}\left(m\mathtt{I}_\mathtt{a}+(1-\mathfrak{x})\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)\\
&\quad+\underline{\mathtt{F}}\left(m\mathtt{I}_\mathtt{a}+\mathfrak{x}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)\cdot\underline{\mathtt{E}}\left(m\mathtt{I}_\mathtt{a}+(1-\mathfrak{x})\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)\\
&\quad+\underline{\mathtt{F}}\left(m\mathtt{I}_\mathtt{a}+(1-\mathfrak{x})\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)\cdot\underline{\mathtt{E}}\left(m\mathtt{I}_\mathtt{a}+\mathfrak{x}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right),
\end{aligned}$$

$$\overline{\mathtt{F}}(m\mathtt{I_a}+\mathfrak{r}\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\overline{\mathtt{E}}(m\mathtt{I_a}+\mathfrak{r}\Pi(\mathtt{I_b},\mathtt{I_a},m))$$

$$+\overline{\mathtt{F}}(m\mathtt{I_a}+(1-\mathfrak{r})\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\overline{\mathtt{E}}(m\mathtt{I_a}+(1-\mathfrak{r})\Pi(\mathtt{I_b},\mathtt{I_a},m))$$

$$+\overline{\mathtt{F}}(m\mathtt{I_a}+\mathfrak{r}\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\overline{\mathtt{E}}(m\mathtt{I_a}+(1-\mathfrak{r})\Pi(\mathtt{I_b},\mathtt{I_a},m))$$

$$+\overline{\mathtt{F}}(m\mathtt{I_a}+(1-\mathfrak{r})\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\overline{\mathtt{E}}(m\mathtt{I_a}+\mathfrak{r}\Pi(\mathtt{I_b},\mathtt{I_a},m))\Big]$$

$$=\mathfrak{h}_1\left(\frac{1}{2}\right)\mathfrak{h}_2\left(\frac{1}{2}\right)\Big\{\Big[\underline{\mathtt{F}}(m\mathtt{I_a}+\mathfrak{r}\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\underline{\mathtt{E}}(m\mathtt{I_a}+\mathfrak{r}\Pi(\mathtt{I_b},\mathtt{I_a},m)),$$

$$\overline{\mathtt{F}}(m\mathtt{I_a}+\mathfrak{r}\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\overline{\mathtt{E}}(m\mathtt{I_a}+\mathfrak{r}\Pi(\mathtt{I_b},\mathtt{I_a},m))\Big]$$

$$+\Big[\underline{\mathtt{F}}(m\mathtt{I_a}+(1-\mathfrak{r})\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\underline{\mathtt{E}}(m\mathtt{I_a}+(1-\mathfrak{r})\Pi(\mathtt{I_b},\mathtt{I_a},m)),$$

$$\overline{\mathtt{F}}(m\mathtt{I_a}+(1-\mathfrak{r})\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\overline{\mathtt{E}}(m\mathtt{I_a}+(1-\mathfrak{r})\Pi(\mathtt{I_b},\mathtt{I_a},m))\Big]$$

$$+\Big[\underline{\mathtt{F}}(m\mathtt{I_a}+\mathfrak{r}\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\underline{\mathtt{E}}(m\mathtt{I_a}+(1-\mathfrak{r})\Pi(\mathtt{I_b},\mathtt{I_a},m)),$$

$$\overline{\mathtt{F}}(m\mathtt{I_a}+\mathfrak{r}\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\overline{\mathtt{E}}(m\mathtt{I_a}+(1-\mathfrak{r})\Pi(\mathtt{I_b},\mathtt{I_a},m))\Big]$$

$$+\Big[\underline{\mathtt{F}}(m\mathtt{I_a}+(1-\mathfrak{r})\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\underline{\mathtt{E}}(m\mathtt{I_a}+\mathfrak{r}\Pi(\mathtt{I_b},\mathtt{I_a},m)),$$

$$\overline{\mathtt{F}}(m\mathtt{I_a}+(1-\mathfrak{r})\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\overline{\mathtt{E}}(m\mathtt{I_a}+\mathfrak{r}\Pi(\mathtt{I_b},\mathtt{I_a},m))\Big]\Big\}$$

$$=\mathfrak{h}_1\left(\frac{1}{2}\right)\mathfrak{h}_2\left(\frac{1}{2}\right)\times\Big[\mathtt{F}(m\mathtt{I_a}+\mathfrak{r}\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\mathtt{E}(m\mathtt{I_a}+\mathfrak{r}\Pi(\mathtt{I_b},\mathtt{I_a},m))$$

$$+\mathtt{F}(m\mathtt{I_a}+(1-\mathfrak{r})\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\mathtt{E}(m\mathtt{I_a}+(1-\mathfrak{r})\Pi(\mathtt{I_b},\mathtt{I_a},m))\Big]$$

$$+\mathfrak{h}_1\left(\frac{1}{2}\right)\mathfrak{h}_2\left(\frac{1}{2}\right)\times\Big[\mathtt{F}(m\mathtt{I_a}+\mathfrak{r}\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\mathtt{E}(m\mathtt{I_a}+(1-\mathfrak{r})\Pi(\mathtt{I_b},\mathtt{I_a},m))$$

$$+\mathtt{F}(m\mathtt{I_a}+(1-\mathfrak{r})\Pi(\mathtt{I_b},\mathtt{I_a},m))\cdot\mathtt{E}(m\mathtt{I_a}+\mathfrak{r}\Pi(\mathtt{I_b},\mathtt{I_a},m))\Big] \tag{71}$$

Utilizing the definition of interval-valued function,

$$\preceq_{\mathrm{cr}} \mathfrak{h}_1\left(\frac{1}{2}\right)\mathfrak{h}_2\left(\frac{1}{2}\right)\times\Big[\mathtt{F}\left(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)\right)\cdot\mathtt{E}\left(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)\right)$$

$$+\mathtt{F}\left(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)\right)\cdot\mathtt{E}\left(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)\right)\Big]$$

$$+\mathfrak{h}_1\left(\frac{1}{2}\right)\mathfrak{h}_2\left(\frac{1}{2}\right)\Big[\left(\mathfrak{h}_1(\mathfrak{x})\mathtt{F}(\mathtt{I_b})+\mathfrak{h}_1(1-\mathfrak{x})\,\mathtt{F}(m\mathtt{I_a})\right)\cdot\left(\mathfrak{h}_2(\mathfrak{x})\mathtt{E}(m\mathtt{I_a})+\mathfrak{h}_2(1-\mathfrak{x})\,\mathtt{E}(\mathtt{I_b})\right)$$

$$+\,\left(\mathfrak{h}_1(\mathfrak{x})\mathtt{F}(m\mathtt{I_a})+\mathfrak{h}_1(1-\mathfrak{x})\,\mathtt{F}(\mathtt{I_b})\right)\cdot\mathfrak{h}_2(\mathfrak{x})\mathtt{E}(\mathtt{I_b})+\mathfrak{h}_2(1-\mathfrak{x})\,\mathtt{E}(m\mathtt{I_a})\Big]$$

$$=\mathfrak{h}_1\left(\frac{1}{2}\right)\mathfrak{h}_2\left(\frac{1}{2}\right)\times\Big\{\mathtt{F}\left(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)\right)\cdot\mathtt{E}\left(m\mathtt{I_a}+\mathfrak{x}\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)\right)$$

$$+\mathtt{F}\left(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)\right)\cdot\mathtt{E}\left(m\mathtt{I_a}+(1-\mathfrak{x})\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)\right)$$

$$+\,\mathtt{M}(m\mathtt{I_a},\,m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\Big[\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})+\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})\Big]$$

$$+\mathtt{N}(m\mathtt{I_a},\,m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\Big[\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})+\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})\Big]\Big\}.$$

Upon multiplication by $\mathfrak{x}^{\alpha-1}$ and integrating over [0, 1], one has

$$\frac{1}{\alpha}\mathtt{F}\left(m\mathtt{I_a}+\frac{1}{2}\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)\right)\cdot\mathtt{E}\left(m\mathtt{I_a}+\frac{1}{2}\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m)\right)$$

$$\preceq_{\mathrm{cr}} \mathfrak{h}_1\left(\frac{1}{2}\right)\mathfrak{h}_2\left(\frac{1}{2}\right)\Big\{\int_0^1\mathfrak{x}^{\alpha-1}\mathtt{F}\left(m\mathtt{I_a}+\mathfrak{x}\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)$$

$$\cdot\,\mathtt{E}\left(m\mathtt{I_a}+\mathfrak{x}\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)d\mathfrak{x}+\int_0^1\mathfrak{x}^{\alpha-1}\mathtt{F}\left(m\mathtt{I_a}+(1-\mathfrak{x})\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)$$

$$\cdot\,\mathtt{E}\left(m\mathtt{I_a}+(1-\mathfrak{x})\Pi\left(\mathtt{I_b},\,\mathtt{I_a},\,m\right)\right)d\mathfrak{x}$$

$$+\mathtt{M}(m\mathtt{I_a},\,m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\int_0^1\mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})+\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})]d\mathfrak{x}$$

$$+\mathtt{N}(m\mathtt{I_a},\,m\mathtt{I_a}+\Pi(\mathtt{I_b},\,\mathtt{I_a},\,m))\int_0^1\mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})+\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})]d\mathfrak{x}\Big\}$$

$$= \mathfrak{h}_1\left(\frac{1}{2}\right)\mathfrak{h}_2\left(\frac{1}{2}\right)\left\{\left[\int_0^1 \mathfrak{x}^{\alpha-1}\underline{\mathtt{F}}(m\mathtt{I}_\mathtt{a}+\mathfrak{x}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m))\cdot\underline{\mathtt{E}}(m\mathtt{I}_\mathtt{a}+\mathfrak{x}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m))\,d\mathfrak{x}\right.\right.$$

$$+\int_0^1 \mathfrak{x}^{\alpha-1}\underline{\mathtt{F}}(m\mathtt{I}_\mathtt{a}+(1-\mathfrak{x})\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m))\cdot\underline{\mathtt{E}}(m\mathtt{I}_\mathtt{a}+(1-\mathfrak{x})\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m))\,d\mathfrak{x},$$

$$\int_0^1 \mathfrak{x}^{\alpha-1}\overline{\mathtt{F}}(m\mathtt{I}_\mathtt{a}+\mathfrak{x}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m))\cdot\overline{\mathtt{E}}(m\mathtt{I}_\mathtt{a}+\mathfrak{x}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m))\,d\mathfrak{x}$$

$$\left.+\int_0^1 \mathfrak{x}^{\alpha-1}\overline{\mathtt{F}}(m\mathtt{I}_\mathtt{a}+(1-\mathfrak{x})\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m))\cdot\overline{\mathtt{E}}(m\mathtt{I}_\mathtt{a}+(1-\mathfrak{x})\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m))\,d\mathfrak{x}\right]$$

$$+\mathtt{M}(m\mathtt{I}_\mathtt{a},\,m\mathtt{I}_\mathtt{a}+\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m))\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})+\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})]d\mathfrak{x}$$

$$\left.+\,\mathtt{N}(m\mathtt{I}_\mathtt{a},\,m\mathtt{I}_\mathtt{a}+\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m))\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})+\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})]d\mathfrak{x}\right\}.$$

Consequently,

$$\frac{1}{\alpha\mathfrak{h}_1\left(\frac{1}{2}\right)\mathfrak{h}_2\left(\frac{1}{2}\right)}\mathtt{F}\left(m\mathtt{I}_\mathtt{a}+\frac{1}{2}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)\cdot\mathtt{E}\left(m\mathtt{I}_\mathtt{a}+\frac{1}{2}\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)\right)$$

$$\preceq_{\mathrm{cr}}\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m))^\alpha}$$

$$\left[\mathrm{I}^\alpha_{(m\mathtt{I}_\mathtt{a}+\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m))^-}\mathtt{F}(m\mathtt{I}_\mathtt{a})\mathtt{E}(m\mathtt{I}_\mathtt{a})+\mathrm{I}^\alpha_{m\mathtt{I}_\mathtt{a}^+}\mathtt{F}(m\mathtt{I}_\mathtt{a}+\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m))\mathtt{E}(m\mathtt{I}_\mathtt{a}+\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m))\right]$$

$$+\mathtt{M}(m\mathtt{I}_\mathtt{a},\,m\mathtt{I}_\mathtt{a}+\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m))\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})+\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})]d\mathfrak{x}$$

$$+\mathtt{N}(m\mathtt{I}_\mathtt{a},\,m\mathtt{I}_\mathtt{a}+\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m))\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})+\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})]d\mathfrak{x}.$$

This completes the proof. □

Corollary 10 Choosing $\Pi(\mathtt{I}_\mathtt{b},\,\mathtt{I}_\mathtt{a},\,m)=\mathtt{I}_\mathtt{b}-m\mathtt{I}_\mathtt{a}$ and $m=1$ in Theorem 6, one gets

$$\frac{1}{\alpha\mathfrak{h}_1\left(\frac{1}{2}\right)\mathfrak{h}_2\left(\frac{1}{2}\right)}\mathtt{F}\left(\frac{\mathtt{I}_\mathtt{a}+\mathtt{I}_\mathtt{b}}{2}\right)\cdot\mathtt{E}\left(\frac{\mathtt{I}_\mathtt{a}+\mathtt{I}_\mathtt{b}}{2}\right)$$

$$\preceq_{cr} \frac{\Gamma(\alpha)}{(\mathtt{I_b}-\mathtt{I_a})^{\alpha}}\left[\mathrm{I}^{\alpha}_{\mathtt{I_b}^-}\mathtt{F}(\mathtt{I_a})\mathtt{E}(\mathtt{I_a})+\mathrm{I}^{\alpha}_{\mathtt{I_a}^+}\mathtt{F}(\mathtt{I_b})\mathtt{E}(\mathtt{I_b})\right]$$

$$+\mathtt{M}(\mathtt{I_a},\, \mathtt{I_b})\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})+\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})]d\mathfrak{x}$$

$$+\mathtt{N}(\mathtt{I_a},\, \mathtt{I_b})\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})+\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})]d\mathfrak{x}. \tag{72}$$

Example 6 Let $\mathtt{F}(\mathfrak{x}) = [\mathfrak{x}^2,\, \mathfrak{x}^3+1]$, $\mathtt{E}(\mathfrak{x}) = [-\mathfrak{x}^2+1,\, \mathfrak{x}^2+2]$, $\Pi(\mathtt{I_b},\, \mathtt{I_a},\, m) = \mathtt{I_b}-m\mathtt{I_a}$, $\mathtt{I_a} = 0$, $m = 1$ and $\mathtt{I_b} = 1$. Then, for $\mathfrak{h}(\mathfrak{x}) = \mathfrak{x}$ and $\alpha = \frac{1}{2}$ all the conditions of Theorem 5 are examined. Let us denote

$$\frac{\Gamma(\alpha)}{(\Pi(\mathtt{I_b},\, \mathtt{I_a},\, m))^{\alpha}}\left[\mathrm{I}^{\alpha}_{(m\mathtt{I_a}+\Pi(\mathtt{I_b},\, \mathtt{I_a},\, m))^-}\mathtt{F}(m\mathtt{I_a})\mathtt{E}(m\mathtt{I_a})+\mathrm{I}^{\alpha}_{m\mathtt{I_a}^+}\mathtt{F}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\, \mathtt{I_a},\, m))\mathtt{E}(m\mathtt{I_a}+\Pi(\mathtt{I_b},\, \mathtt{I_a},\, m))\right]$$

$$\approx [0.43,\, 12.78],$$

$$\mathtt{M}(m\mathtt{I_a},\, m\mathtt{I_a}+\Pi(\mathtt{I_b},\, \mathtt{I_a},\, m))\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})+\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})]d\mathfrak{x} \tag{73}$$

$$+\mathtt{N}(m\mathtt{I_a},\, m\mathtt{I_a}+\Pi(\mathtt{I_b},\, \mathtt{I_a},\, m))\int_0^1 \mathfrak{x}^{\alpha-1}[\mathfrak{h}_2(1-\mathfrak{x})\mathfrak{h}_1(\mathfrak{x})+\mathfrak{h}_2(\mathfrak{x})\mathfrak{h}_1(1-\mathfrak{x})]d\mathfrak{x} \approx [0.53,\, 15.47].$$

Consequently, it is evident that

$$[0.43,\, 12.78] \preceq_{cr} [0.53,\, 15.47].$$

$\Longrightarrow$ Example 6 shows the numerical validity of Theorem 5.
Example 7 Similarly, considering Example 6, then all the assumptions in Theorem 6 are satisfied.
The Figures 9 and 10 represent the graphical validation of Theorem 5.

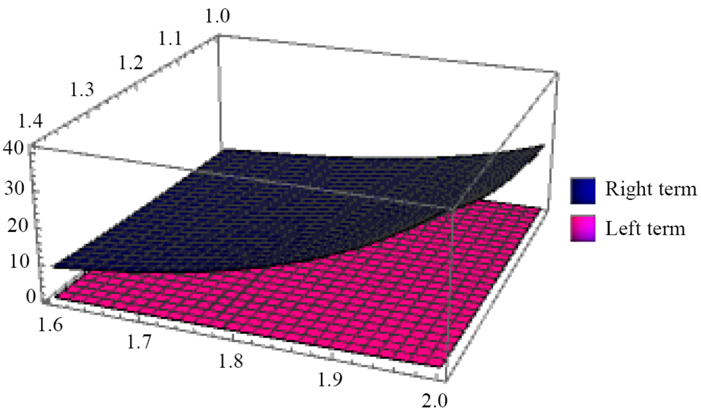

Figure 9. The graph validation of Theorem 5 for the function $\mathtt{F}(\mathfrak{r}) = \mathfrak{r}^2$, and $\mathtt{E}(\mathfrak{r}) = -\mathfrak{r}^2 + 1$

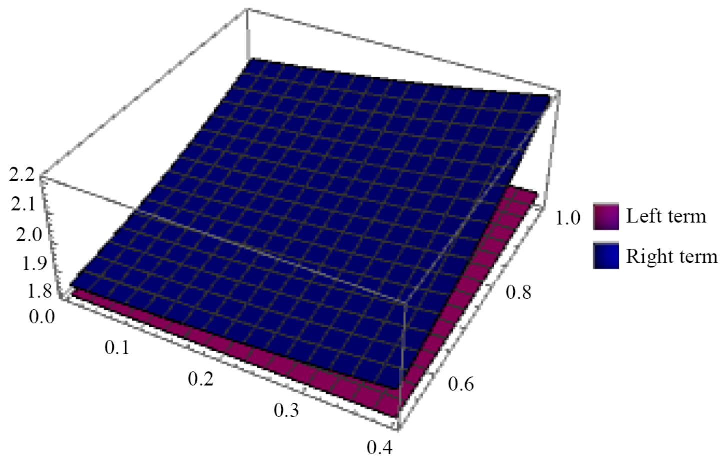

Figure 10. The graph validation of Theorem 5 for the function $\mathtt{F}(\mathfrak{r}) = \mathfrak{r}^3 + 1$, and $\mathtt{E}(\mathfrak{r}) = \mathfrak{r}^2 + 2$

7. Conclusions

The subject interval analysis in the context of fractional calculus provides an effective method to investigate systems with uncertainty and memory effects. This method helps us in obtaining more precise results and deeper insight into intricate mathematical problems. In this paper, first we introduced the interval-valued center-radius order preinvex function. Further, we proved the Hermite-Hadamard, Pachpatte and Fejér inequalities via the interval-valued center-radius order $(\mathfrak{h}, m)$-preinvex function pertaining to the RLFIO. For verification of the results, some graphical, numerical, and literature validations were estimated as well. The methodology of this paper can also be extended to other fractional operators like tempered, Caputo-Fabrizio and Atangana-Baleanu operator, etc.

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Conflict of interest

The authors declare no competing financial interest.

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