Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Approximation of the Jensen Gap for First Order Differentiable Functions

Tareq Saeed[1], Muhammad Adil Khan[2], Asma Bibi[2], Shahid Khan[3], Yu-Ming Chu[4,5*]

[1]Financial Mathematics and Actuarial Science (FMAS)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
[2]Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan
[3]Higher Education Department, Directorate General of Commerce Education and Management Sciences, Peshawar 25000, Khyber Pakhtunkhwa, Pakistan
[4]Institute for Advanced Study Honoring Chen Jian Gong, Hangzhou Normal University, Hangzhou 311121 China
[5]Department of Mathematics, Huzhou University, Huzhou 313000, China
E-mail: chuyuming@zjhu.edu.cn

Received: 25 June 2025; **Revised:** 13 July 2025; **Accepted:** 25 July 2025

Abstract: This article proposes two upper bounds for the gap of discrete and integral Jensen inequality for the functions $f \in C[a, b]$. First bound is obtained when $|f'|^q,\ q > 1$ is convex and second is obtained when this function is concave. In the convex case, an example is discussed to authenticate the bound when $|f'|$ is not convex. Accordingly and consequently, bounds for the gap of celebrated Hermite-Hadamard and well-known Hölder inequalities are deduced. Also, some estimates for the quasi-arithmetic and power means, and for some basic divergences are obtained from the main results.

***Keywords*:** convex function, Jensen's inequality, information theory

MSC: 26A51, 26D15, 68P30

1. Introduction

Mathematical inequalities for convex functions are proved to be very useful tools in various aspects [1–6]. Among these tools, Jensen's inequality is the most powerful and it is described as [7, p. 43]:

Theorem 1 If $f\colon [k_1, k_2] \to \mathbb{R}$ is a convex function and $y_a \in [k_1, k_2]$, $\ell_a \geq 0$ for $a = 1, 2, \ldots, n$ with $0 < \ell_n = \sum_{a=1}^{n} \ell_a$, and $\bar{y} = \frac{1}{\ell_n} \sum_{a=1}^{n} \ell_a y_a$. Then

$$f(\bar{y}) \leq \frac{1}{\ell_n} \sum_{a=1}^{n} \ell_a f(y_a). \tag{1}$$

This can be remarkably noted that the inequality under consideration has been generalized in different senses. One of their generalizations is in the Riemann integral sense. This generalization is described as [8]:

DOI: https://doi.org/10.37256/cm.6520257588

Theorem 2 Let $[k_1, k_2] \subset \mathbb{R}$ and $\varepsilon_1, \varepsilon_2$: $[a_1, a_2] \to \mathbb{R}$ be two measurable functions such that $\varepsilon_1(x) \in [k_1, k_2], \ \forall x \in [a_1, a_2]$. Further, suppose that the functions $\varepsilon_2, \ \varepsilon_1\varepsilon_2$ and $(f \circ \varepsilon_1).\varepsilon_2$ are integrable on $[a_1, a_2]$ for f: $[k_1, k_2] \to \mathbb{R}$ as convex function. Also, suppose that $\varepsilon_2(x) \geq 0$ on $[a_1, a_2], \ \int_{a_1}^{a_2} \varepsilon_2(x)dx = \varepsilon_n > 0, $ and $\bar{\varepsilon} = \frac{1}{\varepsilon_n}\int_{a_1}^{a_2} \varepsilon_1(x)\varepsilon_2(x)dx,$ then

$$f(\bar{\varepsilon}) \leq \frac{1}{\varepsilon_n}\int_{a_1}^{a_2}(f \circ \varepsilon_1)(x)\varepsilon_2(x)dx. \tag{2}$$

Some useful applications of this inequality can be found in statistics [9], optimization [10], economics [11], finance [12] and information theory [13–15] etc. For some particular convex functions and substitutions, this inequality provides some other inequalities such as AM-GM, the Hermite-Hadamard, Ky Fan's, Hölder's, and Levinson's inequalities etc. The literature on various results around Jensen's inequality can be found in [16–19].

2. Bounds for the gap of Jensen's inequality with applications

Noteworthy that the discrepancy of both sides of Jensen's inequality is known as the Jensen gap. The following main result is about a bound for this gap from the right. The bound is acquired, using basic concept of convexity.

Theorem 3 Let f: $[k_1, k_2] \to \mathbb{R}$ be a first order differentiable function, $|f'|^q$ is convex for $q > 1$, and $y_a \in [k_1, k_2]$, $\ell_a \in \mathbb{R}$ for each $a \in \{1, ..., n\}$. Further, assume that $\sum_{a=1}^{n} \ell_a = \ell_n \neq 0$ and $\bar{y} = \frac{1}{\ell_n}\sum_{a=1}^{n} \ell_a y_a \in [k_1, k_2]$. Then

$$\left|f(\bar{y}) - \frac{1}{\ell_n}\sum_{a=1}^{n}\ell_a f(y_a)\right| \leq \frac{1}{|\ell_n|}\sum_{a=1}^{n}|\ell_a(\bar{y}-y_a)|\left(\frac{|f'(\bar{y})|^q + |f'(y_a)|^q}{2}\right)^{\frac{1}{q}}. \tag{3}$$

Proof. The well-known integration gives the following result keeping in mind the condition: if $\bar{y} \neq y_a$ for $a = 1, 2, ..., n$ without loss of generality

$$\sum_{a=1}^{n}\frac{\ell_a(\bar{y}-y_a)}{\ell_n}\int_0^1 f'((1-t)y_a + t\bar{y})dt = \sum_{a=1}^{n}\frac{\ell_a(\bar{y}-y_a)}{\ell_n}\left[\frac{f((1-t)y_a+t\bar{y})}{\bar{y}-y_a}\bigg|_0^1\right]. \tag{4}$$

Implies that

$$\sum_{a=1}^{n}\frac{\ell_a(\bar{y}-y_a)}{\ell_n}\int_0^1 f'((1-t)y_a + t\bar{y})dt = f(\bar{y}) - \frac{1}{\ell_n}\sum_{a=1}^{n}\ell_a f(y_a). \tag{5}$$

Upon applying the absolute value and then the triangle inequality to the above, we deduce

$$\left|f(\bar{y}) - \frac{1}{\ell_n}\sum_{a=1}^{n}\ell_a f(y_a)\right| \leq \sum_{a=1}^{n}\left|\frac{\ell_a(\bar{y}-y_a)}{\ell_n}\right|\int_0^1 \left|f'((1-t)y_a + t\bar{y})\right|dt. \tag{6}$$

Application of the Hölder inequality in above inequality gives the following inequality

$$\left|f(\bar{y})-\frac{1}{\ell_n}\sum_{a=1}^{n}\ell_a f(y_a)\right| \leq \sum_{a=1}^{n}\left|\frac{\ell_a(\bar{y}-y_a)}{\ell_n}\right|\left(\int_0^1\left|f'((1-t)y_a+t\bar{y})\right|^q dt\right)^{\frac{1}{q}}. \tag{7}$$

From the statement, $|f'|^q$ is convex therefore the above inequality becomes

$$\left|f(\bar{y})-\frac{1}{\ell_n}\sum_{a=1}^{n}\ell_a f(y_a)\right| \tag{8}$$

$$\leq \sum_{a=1}^{n}\left|\frac{\ell_a(\bar{y}-y_a)}{\ell_n}\right|\left(\int_0^1\left((1-t)\left|f'(y_a)\right|^q+t\left|f'(\bar{y})\right|^q\right)dt\right)^{\frac{1}{q}}. \tag{9}$$

Calculating the integral in the above inequality, we obtain (3). □

The following theorem is about the second main result, which also proposes an upper bound for the gap of Jensen's inequality but this time using concavity.

Theorem 4 Assume a function $f\colon [k_1, k_2]\to\mathbb{R}$ and impose the condition that $|f'|^q$ is concave for $q>1$, and assume that $y_a\in[k_1, k_2],\ \ell_a\in\mathbb{R}$ for each $a\in\{1, 2, \ldots, n\}$ with $\ell_a\colon=\sum_{a=1}^{n}\ell_a\neq 0$. Also, let $\bar{y}\colon=\frac{1}{\ell_a}\sum_{a=1}^{n}\ell_a y_a\in[k_1, k_2],$ then

$$\left|f(\bar{y})-\frac{1}{\ell_a}\sum_{a=1}^{n}\ell_a f(y_a)\right| \leq \sum_{a=1}^{n}\left|\frac{\ell_a(\bar{y}-y_a)}{\ell_a}\right|\left|f'\left(\frac{\bar{y}+y_a}{2}\right)\right|. \tag{10}$$

Proof. Application of the generalized Jensen's inequality for concave functions in (7) acquires

$$\begin{aligned}
\left|f(\bar{y})-\frac{1}{\ell_a}\sum_{a=1}^{n}\ell_a f(y_a)\right| &\leq \sum_{a=1}^{n}\left|\frac{\ell_a(\bar{y}-y_a)}{\ell_n}\right|\left(\left|f'\left(\int_0^1\left(t\bar{y}+(1-t)y_a\right)dt\right)\right|^q\right)^{\frac{1}{q}}\\
&= \sum_{a=1}^{n}\left|\frac{\ell_a(\bar{y}-y_a)}{\ell_n}\right|\left|f'\left(\int_0^1\left(t\bar{y}+(1-t)y_a\right)dt\right)\right| \qquad (11)\\
&= \sum_{a=1}^{n}\left|\frac{\ell_a(\bar{y}-y_a)}{\ell_n}\right|\left|f'\left(\frac{\bar{y}+y_a}{2}\right)\right|,
\end{aligned}$$

which is the required inequality. □

Theorem 3 directly acquires a remarkably true variant of the Hölder inequality as:

Proposition 5 Taking two positive tuples $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n),$ for $q>1$ $t_2\notin\left(1, 1+\frac{1}{q}\right),$ $t_1>1,$ and $\frac{1}{t_2}+\frac{1}{t_1}=1$. Then

$$\left(\sum_{a=1}^{n} p_a^{t_1}\right)^{\frac{1}{t_1}}\left(\sum_{a=1}^{n} q_a^{t_2}\right)^{\frac{1}{t_2}}-\sum_{a=1}^{n} q_a p_a$$

$$\leq\left[\frac{t_2}{2^{\frac{1}{q}}}\sum_{a=1}^{n}\left|p_a^{t_1}\left(\frac{\sum_{a=1}^{n} p_a q_a}{\sum_{a=1}^{n} p_a^{t_1}}-q_a p_a^{-\frac{t_1}{t_2}}\right)\right|\left(\left(\frac{\sum_{a=1}^{n} p_a q_a}{\sum_{a=1}^{n} p_a^{t_1}}\right)^{q(t_2-1)}+\frac{q_a^{q(t_2-1)}}{p_a^{q}}\right)^{\frac{1}{q}}\right]^{\frac{1}{t_2}}\left(\sum_{a=1}^{n} p_a^{t_1}\right)^{\frac{1}{t_1}} \tag{12}$$

Proof. Let $f(k)=k^{t_2}$ for $k>0,$ then one can readily verify that $|f'|^q$ and f are convex functions, and therefore (3) can be applied for $f(k)=k^{t_2},\ \ell_a=p_a^{t_1}$ and $y_a=q_a p_a^{-\frac{t_1}{t_2}},$ the following result can be obtained

$$\left(\left(\sum_{a=1}^{n} p_a^{t_1}\right)^{t_2-1}\left(\sum_{a=1}^{n} q_a^{t_2}\right)-\left(\sum_{a=1}^{n} q_a p_a\right)^{t_2}\right)^{\frac{1}{t_2}}$$

$$\leq\left[\frac{t_2}{2^{\frac{1}{q}}}\sum_{a=1}^{n}\left|p_a^{t_1}\left(\frac{\sum_{a=1}^{n} p_a q_a}{\sum_{a=1}^{n} p_a^{t_1}}-q_a p_a^{-\frac{t_1}{t_2}}\right)\right|\left(\left(\frac{\sum_{a=1}^{n} p_a q_a}{\sum_{a=1}^{n} p_a^{t_1}}\right)^{q(t_2-1)}+\frac{q_a^{q(t_2-1)}}{p_a^{q}}\right)^{\frac{1}{q}}\right]^{\frac{1}{t_2}}\times\left(\sum_{a=1}^{n} p_a^{t_1}\right)^{\frac{1}{t_1}}. \tag{13}$$

Since the inequality $x_1^{\theta}-x_2^{\theta}\leq(x_1-x_2)^{\theta}$ holds for $\theta\in[0,\ 1]$ and $0\leq x_2\leq x_1,$ therefore by putting $\theta=\frac{1}{t_2},\ x_1=\left(\sum_{a=1}^{n} q_a^{t_2}\right)\left(\sum_{a=1}^{n} p_a^{t_1}\right)^{t_2-1},\ x_2=\left(\sum_{a=1}^{n} q_a p_a\right)^{t_2}$, it follows that

$$\left(\sum_{a=1}^{n} p_a^{t_1}\right)^{\frac{1}{t_1}}\left(\sum_{a=1}^{n} q_a^{t_2}\right)^{\frac{1}{t_2}}-\sum_{a=1}^{n} q_a p_a\leq\left(\left(\sum_{a=1}^{n} q_a^{t_2}\right)\left(\sum_{a=1}^{n} p_a^{t_1}\right)^{t_2-1}-\left(\sum_{a=1}^{n} q_a p_a\right)^{t_2}\right)^{\frac{1}{t_2}}. \tag{14}$$

Now (13) and (14) give (12). □

Theorem 4 directly gives a remarkable variant of Hölder's inequality as:

Proposition 6 Taking two positive tuples $(\eta_1,\ \eta_2,\ .....\eta_n)$ and $(\sigma_1,\ \sigma_2....\sigma_n)$ and further targeting $\xi\in(1,\ 1+\frac{1}{q})$, $\varsigma>1$ for $q>1$, imposing $\frac{1}{\varsigma}+\frac{1}{\xi}=1$, then

$$\left(\sum_{a=1}^{n}\eta_a^{\varsigma}\right)^{\frac{1}{\varsigma}}\left(\sum_{a=1}^{n}\sigma_a^{\xi}\right)^{\frac{1}{\xi}}-\left(\sum_{a=1}^{n}\sigma_a\eta_a\right)$$

$$\leq\left[\xi\sum_{a=1}^{n}\left|\sigma_a^{\xi}\left(\frac{\sum_{a=1}^{n}\sigma_a\eta_a}{\sum_{a=1}^{n}\sigma_a^{\xi}}-\eta_a\sigma_a^{-\frac{\xi}{\varsigma}}\right)\right|\left(\frac{\sum_{a=1}^{n}\sigma_a\eta_a+\eta_a\sigma_a^{-\frac{\xi}{\varsigma}}\sum_{a=1}^{n}\sigma_a^{\xi}}{2}\right)^{\xi-1}\right]^{\frac{1}{\xi}}. \tag{15}$$

Proof. Consider $f(k)=k^{\xi}$. It is evident that f is convex for $k>0$, while $|f'|^q$ is concave. Hence, by applying (10) with $f(k)=k^{\xi}$, $\ell_a=\sigma_a^{\xi}$, and $y_a=\eta_a\sigma_a^{-\frac{\xi}{\varsigma}}$, and simplifying, we arrive at

$$
\begin{aligned}
&\left(\left(\sum_{a=1}^{n}\eta_a^{\varsigma}\right)^{\xi-1}\left(\sum_{a=1}^{n}\sigma_a^{\xi}\right)-\left(\sum_{a=1}^{n}\sigma_a\eta_a\right)^{\xi}\right)^{\frac{1}{\xi}}\\
&\le\left[\xi\sum_{a=1}^{n}\left|\sigma_a^{\xi}\left(\frac{\sum_{a=1}^{n}\sigma_a\eta_a}{\sum_{a=1}^{n}\sigma_a^{\xi}}-\eta_a\sigma_a^{-\frac{\xi}{\varsigma}}\right)\right|\left(\frac{\sum_{a=1}^{n}\sigma_a\eta_a+\eta_a\sigma_a^{-\frac{\xi}{\varsigma}}\sum_{a=1}^{n}\sigma_a^{\xi}}{2}\right)^{\xi-1}\right]^{\frac{1}{\xi}}.
\end{aligned}
\tag{16}
$$

since the inequality $c^l-d^l\le(c-d)^l$ holds for $0\le l\le 1$ and $0\le d\le c$, therefore by putting $c=\left(\sum_{a=1}^{n}\eta_a^{\varsigma}\right)^{\xi-1}\left(\sum_{a=1}^{n}\sigma_a^{\xi}\right)$, $d=\left(\sum_{a=1}^{n}\sigma_a\eta_a\right)^{\xi}$, and $l=\frac{1}{\xi}$ we obtain

$$
\begin{aligned}
&\left(\sum_{a=1}^{n}\sigma_a^{\xi}\right)^{\frac{1}{\xi}}\left(\sum_{a=1}^{n}\eta_a^{\varsigma}\right)^{\frac{1}{\varsigma}}-\left(\sum_{a=1}^{n}\sigma_a\eta_a\right)\\
&\le\left[\left(\sum_{a=1}^{n}\sigma_a^{\xi}\right)\left(\sum_{a=1}^{n}\eta_a^{\varsigma}\right)^{\xi-1}-\left(\sum_{a=1}^{n}\sigma_a\eta_a\right)^{\xi}\right]^{\frac{1}{\xi}}.
\end{aligned}
\tag{17}
$$

Thus (16) and (17) gives (15) □

Theorem 4 gives another remarkably true variant of the Hölder's inequality as:

Corollary 7 Taking positive tuples $(\eta_1,\ \eta_2,\ \eta_3,\ \ldots,\ \eta_n)=\eta$ and $(\sigma_1,\ \sigma_2,\ \sigma_3,\ \ldots,\ \sigma_n)=\sigma$ and further taking $\varsigma\in(0,\ 1)$ with $\frac{1}{\varsigma}\in(1,\ 1+\frac{1}{q})$ and $\xi=\frac{\varsigma}{\varsigma-1}$ for $q>1$. Then

$$
\begin{aligned}
&\left(\sum_{a=1}^{n}\eta_a^{\varsigma}\right)^{\frac{1}{\varsigma}}\left(\sum_{a=1}^{n}\sigma_a^{\xi}\right)^{\frac{1}{\xi}}-\sum_{a=1}^{n}\eta_a\sigma_a\\
&\le\frac{1}{\varsigma}\sum_{a=1}^{k}\left|\sigma_a^{\xi}\left(\frac{\sum_{a=1}^{n}\eta_a^{\varsigma}}{\sum_{a=1}^{n}\sigma_a^{\xi}}-\eta_a^{\varsigma}\sigma_a^{-\xi}\right)\right|\times\left(\frac{\frac{\sum_{a=1}^{n}\eta_a^{\varsigma}}{\sum_{a=1}^{n}\sigma_a^{\xi}}+\eta_a^{\varsigma}\sigma_a^{-\xi}}{2}\right)^{\frac{1}{\varsigma}-1}.
\end{aligned}
\tag{18}
$$

Proof. Let $f(k)=k^{\frac{1}{\varsigma}}$ for $k>0$. It is clear that $|f'|^q$ is concave and f is convex on $(0,\infty)$ when $\frac{1}{\varsigma}\in(1,\ 1+\frac{1}{q})$ and $q>1$. Therefore, applying (10) with $f(k)=k^{\frac{1}{\varsigma}}$, $\ell_a=\sigma^{\xi}$, and $y_a=\eta_a^{\varsigma}\sigma_a^{-\xi}$, it follows that

$$\left(\frac{1}{\sum_{a=1}^{n}\sigma_a^{\xi}}\sum_{a=1}^{n}\eta_a^{\varsigma}\right)^{\frac{1}{\varsigma}}-\frac{1}{\sum_{a=1}^{n}\sigma_a^{\xi}}\sum_{a=1}^{n}\sigma_a\eta_a$$

$$\leq\frac{1}{\sum_{a=1}^{n}\sigma_a^{\xi}}\sum_{a=1}^{n}\left|\sigma_a^{\xi}\left(\frac{\sum_{a=1}^{n}\eta_a^{\varsigma}}{\sum_{a=1}^{n}\sigma_a^{\xi}}-\eta_a^{\varsigma}\sigma_a^{-\xi}\right)\right|\times\frac{1}{\varsigma}\left(\frac{\left(\frac{\sum_{a=1}^{n}\eta_a^{\varsigma}}{\sum_{a=1}^{n}}+\eta_a^{\varsigma}\sigma_a^{-\xi}\right)}{2}\right)^{\frac{1}{\varsigma}-1}. \tag{19}$$

Multiplication of the term $\sum_{a=1}^{n}\sigma_a^{\xi}$ with both sides of the above inequality gives

$$\left(\sum_{a=1}^{n}\eta_a^{\varsigma}\right)^{\frac{1}{\varsigma}}\left(\sum_{a=1}^{n}\sigma_a^{\xi}\right)^{\frac{1}{\xi}}-\sum_{a=1}^{n}\eta_a\sigma_a$$

$$\leq\frac{1}{\varsigma}\sum_{a=1}^{k}\left|\sigma_a^{\xi}\left(\frac{\sum_{a=1}^{n}\eta_a^{\varsigma}}{\sum_{a=1}^{n}\sigma_a^{\xi}}-\eta_a^{\varsigma}\sigma_a^{-\xi}\right)\right|\times\left(\frac{\frac{\sum_{a=1}^{n}\eta_a^{\varsigma}}{\sum_{a=1}^{n}\sigma_a^{\xi}}+\eta_a^{\varsigma}\sigma_a^{-\xi}}{2}\right)^{\frac{1}{\varsigma}-1}. \tag{20}$$

□

Definition 8 Mathematical formula of the power mean is [14]

$$\mathscr{M}_r(\ell,y)=\begin{cases}\left(\frac{1}{\ell_n}\sum_{a=1}^{n}\ell_a y_a^r\right)^{\frac{1}{r}}, & r\neq 0,\\ \left(\prod_{a=1}^{n}y_a^{\ell_a}\right)^{\frac{1}{\ell_n}}, & r=0.\end{cases} \tag{21}$$

Where $r\in(-\infty,\infty)$ is its order and $(\ell_1,\ell_2,\ldots,\ell_n)=\ell,\ (y_1,y_2,\ldots,y_n)=y$ are two tuples with $\ell_n=\sum_{a=1}^{n}\ell_a$.

From Theorem 3, an inequality for the power mean is established as follows:

Corollary 9 Let $y=(y_1,y_2,\ldots,y_n)$ and $\ell=(\ell_1,\ell_2,\ldots,\ell_n)$ be some positive tuples with $\sum_{a=1}^{n}\ell_a=\ell_n$. Also, assume that $r,t\in(-\infty,\infty)\setminus\{0\},\ r>t,$ and $q>1$,

1. if $t,r>0,$ then

$$\mathscr{M}_r^t(\ell,y)-\mathscr{M}_t^t(\ell,y)\leq\frac{t}{2^{\frac{1}{q}}r\ell_n}\sum_{a=1}^{n}\left|\ell_a\left(\mathscr{M}_r^r(\ell,y)-y_a^r\right)\right|$$

$$\times\left(\frac{y_a^{qr}\left(\mathscr{M}_r^r(\ell,y)\right)^{\frac{qt}{r}}+y_a^{qt}\left(\mathscr{M}_r^r(\ell,y)\right)^q}{y_a^{qr}\left(\mathscr{M}_r^r(\ell,y)\right)^q}\right)^{\frac{1}{q}}. \tag{22}$$

2. If $t,r<0$ with $\frac{t}{r}\notin\left(1,1+\frac{1}{q}\right),$ then

$$\mathscr{M}_t^t(\ell, y) - \mathscr{M}_r^t(\ell, y) \leq \frac{t}{2^{\frac{1}{q}} r\ell_n} \sum_{a=1}^{n} |\ell_a (\mathscr{M}_r^r(\ell, y) - y_a^r)| \times \left(\frac{y_a^{qr} (\mathscr{M}_r^r(\ell, y))^{\frac{qt}{r}} + y_a^{qt} (\mathscr{M}_r^r(\ell, y))^q}{y_a^{qr} (\mathscr{M}_r^r(\ell, y))^q} \right)^{\frac{1}{q}}. \tag{23}$$

3. If $t < 0$ and $r > 0$, then

$$\mathscr{M}_t^t(\ell, y) - \mathscr{M}_r^t(\ell, y) \leq - \frac{t}{2^{\frac{1}{q}} r\ell_n} \sum_{a=1}^{n} |\ell_a (\mathscr{M}_r^r(\ell, y) - y_a^r)| \times \left(\frac{y_a^{qr} (\mathscr{M}_r^r(\ell, y))^{\frac{qt}{r}} + y_a^{qt} (\mathscr{M}_r^r(\ell, y))^q}{y_a^{qr} (\mathscr{M}_r^r(\ell, y))^q} \right)^{\frac{1}{q}}. \tag{24}$$

Proof. 1. Assume $f(\xi) = \xi^{\frac{t}{r}}$ with $\xi > 0$. For the specified values of t, r: $|f'|^q$ is convex, and f is concave. Using (3) for $f(\xi) = \xi^{\frac{t}{r}}$, then substitution of y_a by y_a^r provides (22).

2. Let $f(\xi) = \xi^{\frac{t}{r}}$, $\xi > 0$. The functions $|f'|^q$ and f are convex, adopting the steps of part **1**, (23) can be obtained.

3. By taking the proposed values of r, t, it can be observed that the functions $|f'|^q$ and $f(\xi) = \xi^{\frac{t}{r}}$ are convex for $\xi > 0$. Thus, following the procedure of part **2**, (24) can be established. □

Theorem 3 directly establishes a remarkable relation between different means as:

Corollary 10 Provided that $q > 1$, $y = (y_1, y_2, \ldots, y_n)$ and $\ell = (\ell_1, \ell_2, \ldots, \ell_n)$ are two positive n-tuples, and $\sum_{a=1}^{n} \ell_a = \ell_n$. Then

$$\mathbf{1}.\ \frac{\mathscr{M}_1(\ell, y)}{\mathscr{M}_0(\ell, y)} \leq \exp\left(\frac{1}{\ell_n} \sum_{a=1}^{n} |\ell_a (\mathscr{M}_1(\ell, y) - y_a)| \left(\frac{\mathscr{M}_1^q(\ell, y) + y_a^q}{2 y_a^q \mathscr{M}_1^q(\ell, y)} \right)^{\frac{1}{q}} \right). \tag{25}$$

$$\mathbf{2}.\ \mathscr{M}_1(\ell, y) - \mathscr{M}_0(\ell, y) \leq \frac{1}{\ell_n} \sum_{a=1}^{n} \left| \ell_a \left(\ln \frac{\mathscr{M}_0(\ell, y)}{y_a} \right) \right| \left(\frac{\mathscr{M}_0^q(\ell, y) + y_a^q}{2} \right)^{\frac{1}{q}}. \tag{26}$$

Proof. 1. Let $f(\xi) = -\ln \xi$ for $\xi > 0$, then $|f'|^q$ is a convex function. So, (25) can be obtained using (3) for such functions.

2. Let $f(\xi) = e^{\xi}$ for all $\xi \in \mathbb{R}$, then the function $|f'|^q$ is convex. (26) can be obtained using (3) for such functions along with $y_a = \ln y_a$. □

Theorem 4 provides a platform to establish an inequality in terms of power mean as:

Corollary 11 Assuming that $\ell_a = \sum_{a=1}^{n} \eta_a$ for a positive tuple $\eta = (\eta_1, \eta_2, \ldots, \eta_a)$. Further, assuming another positive tuple $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_a)$, $-\infty < t < r < 0, q > 1$, and $1 < \frac{t}{r} < 1 + \frac{1}{q}$, then

$$\mathscr{M}_t^t(\eta, \sigma) - \mathscr{M}_r^t(\eta, \sigma) \leq \frac{t}{r\ell_a} \sum_{a=1}^{n} |\eta_a (\mathscr{M}_r^r(\eta, \sigma) - \sigma_a^r)| \left(\frac{\mathscr{M}_r^r(\eta, \sigma) + \sigma_a^r}{2} \right)^{\frac{t}{r} - 1}. \tag{27}$$

Proof. Assume $f(k)=k^{\frac{t}{r}}$ for $k>0$. It is evident that, for the specified values of q, r, and t the function $|f'|^q$ is concave and f is convex. Thus, consequent upon the use of (4) for $f(k)=k^{\frac{t}{r}}$ and $y_a \to \sigma_a^r$, we have

$$\left|f\left(\frac{\sum_{a=1}^n \eta_a\sigma_a^r}{\sum_{a=1}^n \eta_a}\right)-\left(\frac{\sum_{a=1}^n \eta_a f(\sigma_a)}{\sum_{a=1}^n \eta_a}\right)\right| \le \sum_{a=1}^n \left|\eta_a\left(\frac{\sum_{a=1}^n \eta_a\sigma_a^r}{\sum_a^n \eta_a}-\sigma_a^r\right)\right|\frac{t}{r}\left(\frac{\bar{\sigma}+\sigma_a}{2}\right)^{\frac{t}{r}-1}.$$

$$\left|\left(\frac{\sum_{a=1}^n \eta_a\sigma_a^r}{\sum_{a=1}^n \eta_a}\right)^{\frac{t}{r}}-\left(\frac{\sum_{a=1}^n \eta_a\sigma_a^t}{\sum_{a=1}^n \eta_a}\right)\right| \le \sum_{a=1}^n |\eta_a(\mathscr{M}_r^r(\eta,\sigma)-\sigma_a^r)|\frac{t}{r}\left(\frac{\frac{\sum_{a=1}^n \eta_a\sigma_a^r}{\ell_a}+\sigma_a^r}{2}\right)^{\frac{t}{r}-1}. \tag{28}$$

Simple calculations give the desired result. □

Definition 12 Let $(y_1, y_2, \ldots, y_n)=y$ and $(\ell_1, \ell_2, \ldots, \ell_n)=\ell$ be two n-tuples of positive numbers, and $\sum_{a=1}^n \ell_a=\ell_n$. Then, the quasi-arithmetic (q-a) mean $\mathscr{M}$ with respect to a continuous and strictly monotone function φ is [14]

$$\mathscr{M}_\varphi(\ell, y)=\varphi^{-1}\left(\frac{1}{\ell_n}\sum_{a=1}^n \ell_a\varphi(y_a)\right). \tag{29}$$

Theorem 3 yields the following bound for the well-known q-a mean as a straightforward application:

Corollary 13 Let $q>1$ and suppose two positive n-tuples $y=(y_1, y_2, \ldots, y_n)$, $\ell=(\ell_1, \ell_2, \ldots, \ell_n)$ with $\sum_{a=1}^n \ell_a=\ell_n$. Further, assume a continuous and strictly monotone function φ. Then if $|(\chi\circ\varphi^{-1})'|^q$ is convex for a function $\chi\circ\varphi^{-1}$, the following inequality holds

$$\begin{aligned}\left|\chi\left(\mathscr{M}_\varphi(\ell, y)\right)-\frac{1}{\ell_n}\sum_{a=1}^n \ell_a\chi(y_a)\right| \le& \frac{1}{|\ell_n|}\sum_{a=1}^n \left|\ell_a\left(\varphi(\mathscr{M}_\varphi(\ell, y))-\varphi(y_a)\right)\right| \\ &\times\left(\frac{|(\chi\circ\varphi^{-1})'(\varphi(\mathscr{M}_\varphi(\ell, y)))|^q+|(\chi\circ\varphi^{-1})'(\varphi(y_a))|^q}{2}\right)^{\frac{1}{q}}.\end{aligned} \tag{30}$$

Proof. Inequality (30) follows from inequality (3) by taking $f\to\chi\circ\varphi^{-1}$ and $y_a\to\varphi(y_a)$. □

Theorem 4 gives remarkably true inequality for q-a mean as:

Corollary 14 Let $\mathbf{y}=(y_1, y_2, y_3, \ldots, y_n)$, $\mathbf{w}=(\ell_1, \ell_2, \ell_3, \ldots, \ell_n)$ are positive tuples, and $\sum_{a=1}^n \ell_a=\ell_n$. Also, $g\circ\varphi^{-1}$, and φ are functions such that the later one is continuous and strictly monotone. Further, impose the condition that $|(g\circ\varphi^{-1})'|^q$ is concave function. Then

$$\begin{aligned}\left|g(\mathscr{M}_\varphi(\mathbf{w}, \mathbf{y}))-\frac{1}{\ell_n}\sum_{a=1}^n \ell_a g(y_a)\right| \le& \frac{1}{|\ell_n|}\sum_{a=1}^n \left|\ell_a\left(\varphi(\mathscr{M}_\varphi(\mathbf{w}, \mathbf{y})-\varphi(y_a)\right)\right| \\ &\times\left|(g\circ\varphi^{-1})'\left(\frac{\varphi(\mathscr{M}_\varphi(\mathbf{w}, \mathbf{y}))+\varphi(y_a)}{2}\right)\right|.\end{aligned} \tag{31}$$

Proof. The setting of $y_a\to\varphi(y_a)$ and $f\to g\circ\varphi^{-1}$ in (4), (31) can be obtained. □

The generalized form of Theorem 3 is:

Theorem 15 Assume that: f: $[k_1, k_2] \to \mathbb{R}$ is a function, $q > 1$, and the function $|f'|^q$ is convex. Further: ε_1, ε_2 are integrable functions defined on the interval $[a_1, a_2]$, and for all $x \in [a_1, a_2]$ the values of ε_1 lie in $[k_1, k_2]$. Furthermore: $\int_{a_1}^{a_2} \varepsilon_2(x)dx = \varepsilon_n \neq 0$ and $\bar{\varepsilon} = \frac{1}{\varepsilon_n}\int_{a_1}^{a_2} \varepsilon_1(x)\varepsilon_2(x)dx \in [k_1, k_2]$. Then

$$\left| f(\bar{\varepsilon}) - \frac{1}{\varepsilon_n}\int_{a_1}^{a_2} (f\circ\varepsilon_1)(x)\varepsilon_2(x)dx \right| \leq \frac{1}{|\varepsilon_n|}\int_{a_1}^{a_2} |\varepsilon_2(x)\left(\bar{\varepsilon} - \varepsilon_1(x)\right)| \left(\frac{|f'(\bar{\varepsilon})|^q + |f'(\varepsilon_1(x))|^q}{2} \right)^{\frac{1}{q}} dx. \tag{32}$$

Example 1 Let $f(x) = \frac{2}{3}x^{\frac{3}{2}}$, $x \in [\frac{1}{2}, 1]$, then $|f'(x)| = \sqrt{x}$. Now $|f'(x)|$ is not convex but $|f'(x)|^2 = x$ is convex. Also, let $\varepsilon_1(x) = x$, $\varepsilon_2(x) = 1$, then $\varepsilon_n = \int_{a_1}^{a_2} \varepsilon_2(x)dx = \int_{\frac{1}{2}}^{1} 1dx = 0.5$ and $\bar{\varepsilon} = \frac{1}{\varepsilon_n}\int_{a_1}^{a_2} \varepsilon_1(x)\varepsilon_2(x)dx = 2\int_{\frac{1}{2}}^{1} xdx = 0.75$. Using (32) for these values we get the following result

$$0.0061 < 0.108 \tag{33}$$

The result calculated in (33) shows that the upper bound for the Jensen gap proposed in (32) is remarkably close to the true discrepancy.

Remark 16 Theorem 15 immediately yields the integral form of Proposition 5, as well as Corollaries 9 through 10 and 13.

As a direct application of Theorem 15, a remarkable bound for the discrepancy of Hermite-Hadamard inequality is proposed as:

Corollary 17 Let $|f'|^q$ for $q > 1$ be a convex function, when $f \in C[z_1, z_2]$. Then

$$\left| \frac{1}{z_2 - z_1}\int_{z_1}^{z_2} f(x)dx - f\left(\frac{z_1 + z_2}{2}\right) \right| \leq \frac{1}{2 \times 2^{\frac{1}{q}}(z_2 - z_1)}\int_{z_1}^{z_2} |(z_1 + z_2 - 2x)| \left(\left| f'\left(\frac{z_1 + z_2}{2}\right) \right|^q + |f'(x)|^q \right)^{\frac{1}{q}} dx. \tag{34}$$

Proof. Let $[k_1, k_2] = [z_1, z_2]$, $\varepsilon_2(x) = 1$ and $\varepsilon_1(x) = x$ for all $x \in [z_1, z_2]$. Then inequality (34) follows easily from (32). □

Remark 18 For $q = 1$, the result (3) and its connected results give some results which are presented in [14].

The generalized form of Theorem 4 is:

Theorem 19 Assume that f: $[k_1, k_2] \to \mathbb{R}$ is a function and for $q > 1$ the function $|f'|^q$ is concave. Further, assume that ε_1, ε_2 are integrable functions defined on the interval $[a_1, a_2]$. Here, also assume that for all $x \in [a_1, a_2]$ the values of the function ε_1 lie in $[k_1, k_2]$. Now, if $\int_{a_1}^{a_2} \varepsilon_2(x)dx = \varepsilon_n \neq 0$ and $\bar{\varepsilon} = \frac{1}{\varepsilon_n}\int_{a_1}^{a_2} \varepsilon_1(x)\varepsilon_2(x)dx \in [k_1, k_2]$, then

$$\left| f(\bar{\varepsilon}) - \frac{1}{\varepsilon_n}\int_{a_1}^{a_2} (f\circ\varepsilon_1)(x)\varepsilon_2(x)dx \right| \leq \int_{a_1}^{a_2} \left| \frac{\varepsilon_2(x)\left(\bar{\varepsilon} - \varepsilon_1(x)\right)}{\varepsilon_n} \right| \left| f'\left(\frac{\bar{\varepsilon} + \varepsilon_1(x)}{2}\right) \right| dx. \tag{35}$$

As a direct application of Theorem 19, another remarkable bound for the discrepancy of Hermite-Hadamard inequality is proposed as:

Corollary 20 For $q > 1$ and a function $f \in C[z_1, z_2]$, assuming $|f'|^q$ as a concave function. Then

$$\left| \frac{1}{z_2 - z_1}\int_{z_1}^{z_2} f(x)dx - f\left(\frac{z_1 + z_2}{2}\right) \right| \leq \int_{z_1}^{z_2} \left| \frac{z_1 + z_2 - 2x}{2(z_2 - z_1)} \right| \left| f'\left(\frac{z_1 + z_2 + 2x}{4}\right) \right| dx. \tag{36}$$

Proof. Let $[k_1, k_2] = [z_1, z_2]$, $\varepsilon_2(x) = 1,$ and $\varepsilon_1(x) = x$ for all $x \in [z_1, z_2]$. Then their substitutions in (35) gives (36). □

3. Applications to divergences and entropies

Definition 21 The Csiszár Γ-divergence for positive tuples $(l_1, l_2, \ldots, l_n) = \mathbf{l}$, and $(m_1, m_2, \ldots, m_n) = \mathbf{m}$ and a function $\Gamma\colon (0, \infty) \to \mathbb{R}$ is provided with the following mathematical formula [20]

$$M_\Gamma(\mathbf{l}, \mathbf{m}) = \sum_{a=1}^{n} m_a \Gamma\left(\frac{l_a}{m_a}\right). \tag{37}$$

Theorem 22 Let the function $|\Gamma'|^q$ for $q > 1$ is convex for $\Gamma\colon (0, \infty) \to \mathbb{R}$ as an arbitrary function. Further, assume that $\mathbf{l} = (l_1, l_2, \ldots, l_n)$, $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ are positive n-tuples. Then

$$\left|\Gamma\left(\frac{\sum_{a=1}^{n} l_a}{\sum_{a=1}^{n} m_a}\right)\sum_{a=1}^{n} m_a - M_\Gamma(\mathbf{l}, \mathbf{m})\right| \le \frac{1}{2^{\frac{1}{q}}} \sum_{a=1}^{n} \left|m_a\left(\frac{\sum_{a=1}^{n} l_a}{\sum_{a=1}^{n} m_a} - \frac{l_a}{m_a}\right)\right| \left(\left|\Gamma'\left(\frac{\sum_{a=1}^{n} l_a}{\sum_{a=1}^{n} m_a}\right)\right|^q + \left|\Gamma'\left(\frac{l_a}{m_a}\right)\right|^q\right)^{\frac{1}{q}}. \tag{38}$$

Proof. Applying Theorem 3 with $f = \Gamma$, $\ell_a = m_a$, and $y_a = \frac{l_a}{m_a}$ for $a \in 1, 2, \ldots, n$, (38) follows. □

Definition 23 Let $(m_1, m_2, \ldots, m_n) = \mathbf{m}$ be a positive probability distribution (ppd). The mathematical formula for Shannon entropy is

$$SE(\mathbf{m}) = -\sum_{a=1}^{n} m_a \log m_a. \tag{39}$$

Corollary 24 The following inequality holds provided that $q > 1$ and $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ is a ppd.

$$\log n - SE(\mathbf{m}) \le \frac{1}{2^{\frac{1}{q}}} \sum_{a=1}^{n} |nm_a - 1| \left(\frac{1}{n^q} + m_a^q\right)^{\frac{1}{q}}. \tag{40}$$

Proof. The inequality (38) gives directly (40), when we use the convex function $\Gamma(\xi) = -\log \xi$ $(\xi \in (0, \infty))$, and $l_a = 1$ therein. □

Definition 25 The mathematical formula of Kullback-Leibler divergence (KL-divergence) for two ppds, $(m_1, m_2, \ldots, m_n) = \mathbf{m}$ and $(l_1, l_2, \ldots, l_n) = \mathbf{l}$, is given by

$$K_d(\mathbf{l}, \mathbf{m}) = \sum_{a=1}^{n} l_a \log\left(\frac{l_a}{m_a}\right). \tag{41}$$

Corollary 26 Let $(l_1, l_2, \ldots, l_n) = \mathbf{l}$ and $(m_1, m_2, \ldots, m_n) = \mathbf{m}$ be two ppds. Further with $q > 1$, the following inequality holds:

$$K_{\mathtt{d}}(\mathbf{m}, \mathbf{l}) \leq \frac{1}{2^{\frac{1}{q}}} \sum_{a=1}^{n} \left|m_a - l_a\right| \left(1 + \left(\frac{m_a}{l_a}\right)^q\right)^{\frac{1}{q}}. \tag{42}$$

Proof. Let $\Gamma(\xi) = -\log \xi$, $\xi \in (0, \infty)$, then clearly $|\Gamma'|^q = \xi^{-q}$. Here Γ and $|\Gamma'|^q$ for $q > 1$ are convex functions. Using $\Gamma(\xi) = -\log \xi$, $\xi \in (0, \infty)$, in (38), we deduce (42). □

Definition 27 The mathematical form of Bhattacharyya coefficient for two ppds $(l_1, l_2, \ldots, l_n) = \mathbf{l}$ and $(m_1, m_2, \ldots, m_n) = \mathbf{m}$ is given by

$$\mathtt{B}_{\mathtt{c}}(\mathbf{l}, \mathbf{m}) = \sum_{a=1}^{n} \sqrt{l_a m_a}.$$

Corollary 28 The following inequality holds, provided two ppds $(m_1, m_2, \ldots, m_n) = \mathbf{m}$, $(l_1, l_2, \ldots, l_n) = \mathbf{l}$, and $q > 1$:

$$1 - \mathtt{B}_{\mathtt{c}}(\mathbf{l}, \mathbf{m}) \leq \frac{1}{2^{1+\frac{1}{q}}} \sum_{a=1}^{n} \left|m_a - l_a\right| \left(1 + \left(\sqrt{\frac{m_a}{l_a}}\right)^q\right)^{\frac{1}{q}}. \tag{43}$$

Proof. Let $\Gamma(\xi) = -\sqrt{\xi}$, $\xi \in (0, \infty)$, then Γ and $|\Gamma'(\xi)|^q = \frac{1}{2^q} \xi^{-\frac{q}{2}}$ are convex for $q > 1$. Using this function in (38), we get (43). □

Remark 29 The integral form of the above inequalities which are obtained for various divergences, can be presented as direct applications of Theorem 15.

3.1 *Inequalities for the Zipf-Mandelbrot entropy*

The Zipf-Mandelbrot (Z-M) entropy is one of the most prominent entropies in information theory. This entropy has been estimated through various results around Jensen's inequality, for some suitable results see [21]. Now to present the formal mathematical form of this entropy, first we give probability mass function for (Z-M) law by assuming the values as $a \in \{1, 2, \ldots, n\}$, $s > 0$, $\Lambda \geq 0$, with a generalized harmonic number $G_{n, \Lambda, s} := \sum_{a=1}^{n} \frac{1}{(a+\Lambda)^s}$, as below:

$$\Gamma_{(a, n, \Lambda, s)} = \frac{1/(a+\Lambda)^s}{G_{n, \Lambda, s}}, \tag{44}$$

Now the Zipf-Mandelbrot entropy can be followed as:

$$Z(G, \Lambda, s) = \frac{s}{G_{n, \Lambda, s}} \sum_{a=1}^{n} \frac{\log(a+\Lambda)}{(a+\Lambda)^s} + \log G_{n, \Lambda, s}. \tag{45}$$

Corollary 30 Let $s > 0$, $\Lambda \geq 0$ and $l_a \geq 0$ for $a = 1, 2, \ldots, n$ with $\sum_{a=1}^{n} l_a = 1$. Then

$$-Z(G, \Lambda, s) - \frac{1}{G_{n, \Lambda, s}} \sum_{a=1}^{n} \frac{\log l_a}{(a+\Lambda)^s} \leq \frac{1}{2^{\frac{1}{q}}} \sum_{a=1}^{n} \left|\frac{1}{(a+\Lambda)^s G_{n, \Lambda, s}} - l_a\right| \left(1 + \frac{1}{l_a^q (a+\Lambda)^{qs} G_{n, \Lambda, s}^q}\right)^{\frac{1}{q}}. \tag{46}$$

Proof. For $m_a = \frac{1}{(a+\Lambda)^s G_{n,\,\Lambda,\,s}},\ a = 1, 2, \ldots, n,$ we have

$$\begin{aligned}\sum_{a=1}^{n} m_a \log \frac{m_a}{l_a} &= \sum_{a=1}^{n} \frac{1}{(a+\Lambda)^s G_{n,\,\Lambda,\,s}} \left(-s \log(a+\Lambda) - \log G_{n,\,\Lambda,\,s} - \log l_a\right) \\ &= -Z(G,\,\Lambda,\,s) - \frac{1}{G_{n,\,\Lambda,\,s}} \sum_{a=1}^{n} \frac{\log l_a}{(a+\Lambda)^s},\end{aligned} \tag{47}$$

and

$$\frac{1}{2^{\frac{1}{q}}} \sum_{a=1}^{n} \left|m_a - l_a\right| \left(1 + \left(\frac{m_a}{l_a}\right)^q\right)^{\frac{1}{q}} = \frac{1}{2^{\frac{1}{q}}} \sum_{a=1}^{n} \left|\frac{1}{(a+\Lambda)^s G_{n,\,\Lambda,\,s}} - l_a\right| \left(1 + \frac{1}{l_a^q (a+\Lambda)^{qs} G_{n,\,\Lambda,\,s}^q}\right)^{\frac{1}{q}}. \tag{48}$$

Now using (47) and (48) in (42), we get (46). □

Corollary 31 Let $\Lambda_1,\, \Lambda_2 \geq 0,\ s_1,\, s_2 > 0$. Then

$$\begin{aligned}-Z(G,\,\Lambda_1,\,s_1) + \sum_{a=1}^{n} \frac{\log(a+\Lambda_2)^{s_2} G_{n,\,\Lambda_2,\,s_2}}{(a+\Lambda_1)^{s_1} G_{n,\,\Lambda_1,\,s_1}} \leq & \frac{1}{2^{\frac{1}{q}}} \sum_{a=1}^{n} \left|\frac{1}{(a+\Lambda_1)^{s_1} G_{n,\,\Lambda_1,\,s_1}} - \frac{1}{(a+\Lambda_2)^{s_2} G_{n,\,\Lambda_2,\,s_2}}\right| \\ & \times \left(1 + \left(\frac{(a+\Lambda_2)^{s_2} G_{n,\,\Lambda_2,\,s_2}}{(a+\Lambda_1)^{s_1} G_{n,\,\Lambda_1,\,s_1}}\right)^q\right)^{\frac{1}{q}}.\end{aligned} \tag{49}$$

Proof. For $m_a = \frac{1}{(a+\Lambda_1)^{s_1} G_{n,\,\Lambda_1,\,s_1}},\ l_a = \frac{1}{(a+\Lambda_2)^{s_2} G_{n,\,\Lambda_2,\,s_2}},\ a = 1, 2, \ldots, n,$ we have

$$\begin{aligned}\sum_{a=1}^{n} m_a \log \frac{m_a}{l_a} &= \sum_{a=1}^{n} \frac{1}{(a+\Lambda_1)^{s_1} G_{n,\,\Lambda_1,\,s_1}} \\ &\quad \times \left(\log(a+\Lambda_2)^{s_2} G_{n,\,\Lambda_2,\,s_2} - \log(a+\Lambda_1)^{s_1} G_{n,\,\Lambda_1,\,s_1}\right) \\ &= -Z(G,\,\Lambda_1,\,s_1) + \sum_{a=1}^{n} \frac{\log(a+\Lambda_2)^{s_2} G_{n,\,\Lambda_2,\,s_2}}{(a+\Lambda_1)^{s_1} G_{n,\,\Lambda_1,\,s_1}}.\end{aligned} \tag{50}$$

Also,

$$\begin{aligned}\frac{1}{2^{\frac{1}{q}}} \sum_{a=1}^{n} \left|m_a - l_a\right| \left(1 + \left(\frac{m_a}{l_a}\right)^q\right)^{\frac{1}{q}} = & \frac{1}{2^{\frac{1}{q}}} \sum_{a=1}^{n} \left|\frac{1}{(a+\Lambda_1)^{s_1} G_{n,\,\Lambda_1,\,s_1}} - \frac{1}{(a+\Lambda_2)^{s_2} G_{n,\,\Lambda_2,\,s_2}}\right| \\ & \times \left(1 + \left(\frac{(a+\Lambda_2)^{s_2} G_{n,\,\Lambda_2,\,s_2}}{(a+\Lambda_1)^{s_1} G_{n,\,\Lambda_1,\,s_1}}\right)^q\right)^{\frac{1}{q}}.\end{aligned} \tag{51}$$

Now utilizing (50) and (51) in (42), we get (49). □

4. Conclusion

The Jensen gap can be utilized for obtaining error bounds while approximating some parameters specially in optimization. In this article, two upper bounds are proposed for the discrepancies of discrete as well as generalized Jensen inequality for the functions $f \in C$ such that $|f'|^q$ for $q > 1$ is either convex or concave. The first bound which is obtained for the convex function under consideration is useful for the functions when $|f'|$ is not convex and in this regard an example is discussed. In the example, numerically the Jensen gap is calculated as true discrepancy which is 0.0061, while the upper bound proposed in inequality (32) for this gap is calculated as 0.108 for $q = 2$, which shows that the proposed upper bound is remarkably close to the true discrepancy.

New bounds for the discrepancy of the Hölder and Hermite-Hadamard inequalities are deduced from the main results. Further, some inequalities for geometric, power, and quasi-arithmetic means, Csiszár, and Kullback-Leibler divergences, Bhattacharya coefficient and Shannon, and Zipf-Mandelbrot entropies are obtained.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Adil Khan M, Husain Z, Chu YM. New estimates for Csiszár divergence and Zipf-Mandelbrot entropy via Jensen-Mercer's inequality. *Complexity*. 2020; 2020(1): 8928691.

[2] Agarwal P, Dragomir SS, Jleli M, Samet B. *Advances in Mathematical Inequalities and Applications*. Berlin: Springer; 2018.

[3] Cloud MJ, Drachman BC, Lebedev LP. *Inequalities with Applications to Engineering*. Cham Heidelberg New York Dordrecht London: Springer; 2014.

[4] Jain S, Mehrez K, Baleanu D, Agarwal P. Certain Hermite-Hadamard inequalities for logarithmically convex functions with applications. *Mathematics*. 2019; 7(2): 163.

[5] Khan KA, Niaz T, Pečarić D, Pečarić J. Refinement of Jensen's inequality and estimation of f-and Rényi divergence via Montgomery identity. *Journal of Inequalities and Applications*. 2018; 2018: 318.

[6] Lakshmikantham V, Vatsala AS. *Theory of Differential and Integral Inequalities with Initial Time Difference and Applications*. Berlin: Springer; 1999.

[7] Pečarić J, Proschan F, Tong YL. *Convex Functions, Partial Orderings and Statistical Applications*. Inc: Academic Press; 1992.

[8] Dragomir SS, Pearce CEM, Pečarić J. Interpolations of Jensen's inequality. *Tamkang Journal of Mathematics*. 2003; 34(2): 175-188.

[9] Mukhopadhyay N. On sharp Jensen's inequality and some unusual applications. *Communications in Statistics–Theory and Methods*. 2011; 40: 1283-1297.

[10] White CC, Harrington DP. Application of Jensen's inequality to adaptive suboptimal design. *Journal of Optimization Theory and Applications*. 1980; 32: 89-99.

[11] Lin Q. Jensen inequality for superlinear expectations. *Statistics and Probability Letters*. 2019; 151: 79-83.

[12] Azar SA. Jensen's inequality in finance. *International Advances in Economic Research*. 2008; 14: 433-440.

[13] Khan S, Adil Khan M, Chu YM. New converses of Jensen inequality via Green functions with applications. *Journal of the Royal Academy of Exact, Physical and Natural Sciences Series A: Mathematics*. 2020; 114: 114.

[14] Adil Khan M, Khan S, Erden S, Samraiz M. A new approach for the derivation of bounds for the Jensen difference. *Mathematical Methods in the Applied Sciences*. 2022; 45: 36-48.

[15] Tapus N, Popescu PG. A new entropy upper bound. *Applied Mathematics Letters*. 2012; 25: 1887-1890.

[16] Goodrich CS, Lyons B. Nonlocal difference equations with sign-changing coefficients. *Applied Mathematics Letters*. 2020; 106: 106371.

[17] Lovričević N, Pečarić D, Pečarić J. Zipf-Mandelbrot law, f-divergences and the Jensen-type interpolating inequalities. *Journal of Inequalities and Applications*. 2018; 2018: 36.

[18] Pasari SN, Barani A, Abbasi N. Generalized integral Jensen inequality. *Journal of Inequalities and Applications*. 2024; 2024(1): 25.

[19] Horváth L. Refining the integral Jensen inequality for finite signed measures using majorization. *Journal of the Royal Academy of Exact, Physical and Natural Sciences Series A: Mathematics*. 2024; 118(3): 129.

[20] Bradanović SI, Lovričević N. Generalized Jensen and Jensen-Mercer inequalities for strongly convex functions with applications. *Journal of Inequalities and Applications*. 2024; 2024(1): 112.

[21] Mehmood N, Butt SI, Pečarić D, Pečarić J. Generalizations of cyclic refinements of Jensen's inequality by Lidstone's polynomial with applications in information theory. *Journal of Mathematical Inequalities*. 2020; 14(1): 249-271.