Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

UNIVERSAL WISER PUBLISHER

Research Article

Fractional Derivative Effects on Exploration of Soliton Solutions of (3 + 1)-D Kadomtsev-Petviashvili-Sawada-Kotera-Ramani Model Using Modified Extended Direct Algebraic Approach

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Received: 26 June 2025; **Revised:** 10 August 2025; **Accepted:** 12 August 2025

Abstract: This investigation studies the newly created (3+1)-D Kadomtsev-Petviashvili-Sawada-Kotera-Ramani equation with the effect of the conformable fractional derivative. The modified extended direct algebraic approach is used to investigate novel solitons and various other exact solutions. Furthermore, several kinds of analytical solutions are created, including bright, dark, and singular solitons. Additionally, singular periodic, hyperbolic solutions, Weierstrass elliptic doubly periodic solutions, and exponential solutions are derived. This study provides a framework for explaining various nonlinear phenomena that emerge in a variety of scientific fields, including fluid mechanics, ocean physics, and marine physics. Different types of acquired solutions are visually displayed to assist them with a physical understanding of the results in the sense of the conformable fractional derivative. Our findings shed light on the intricate dynamics of fluid waves and give vital new insights into the behavior of traveling waves and their many forms.

Keywords: conformable fractional derivative, analytical wave solutions, nonlinear dispersive systems, advanced integrability methods

MSC: 35C07, 35C08, 35C09

1. Introduction

Due to their relevance and importance in solitary wave theory, higher-dimensional integrable differential equations have garnered a lot of interest in recent decades [1, 2]. Akinyemi et al. [3] explored novel soliton solutions for generalized (2 + 1)-dimensional Boussinesq-Kadomtsev-Petviashvili-like equations, contributing to the understanding of nonlinear wave dynamics in higher dimensions. Similarly, Tao et al. [4] investigated some soliton structures for the (2 + 1)-dimensional non-linear transmission line, highlighting their significance in telecommunications systems. The advancement of scientific fields has been significantly aided by these equations. Nonlinear wave types are naturally

DOI: https://doi.org/10.37256/cm.6420257594

occurring phenomena that merit particular consideration as solutions to nonlinear wave equations, taking into account the solitons (see [5–7]). Many domains depend on these waves, such as nonlinear optics, plasma [8], and hydrodynamics [9].

A broad range of natural disciplines, including biology, optical fibres, water waves, fluid mechanics, and many more, use Nonlinear Partial Differential Equations (NLPDEs) to represent complex processes [10–13]. We may learn about the physical properties of these occurrences by looking at NLPDEs and attempting to identify their numerous solutions. The Non-Linear Schrödinger Equation (NLSE) is among the most significant NLPDEs and has drawn a lot of attention from academics because of its complex mathematical structure and wide range of applications. The NLSE is especially noteworthy for its use in simulating wave propagation in multidisciplinary contexts, such as quantum physics, dynamics of fluids, and optics. A fundamental component of NLPDE research, soliton theory, has been crucial in developing analytical techniques for resolving these kinds of equations [14, 15]. In order to investigate and resolve NLPDEs, several scholars have created a variety of analytical tools, including variational methods, inverse scattering transform, and perturbation methods, greatly advancing both theory and practice. The auxiliary equation method [16], the extended tanh-function technique [17], the generalized Arnous method [18], the Lie symmetry approach [19], Kudryashov approaches [20], and the generalized exponential function method [21], are some of the techniques that have been developed recently to obtain soliton solutions for nonlinear Schrödinger equations and many others [22–26].

Additional examples of integrating the conventional Korteweg-De Vries (KdV) equation with any of the 5th-order KdV equations were provided in [3, 4, 6] and a few of their citations. An improvement and update of this equation was created as a linear combination in [27], as follows:

$$\mathscr{R}_t + \phi\left(3\mathscr{R}^2 + \mathscr{R}_{xx}\right)_x + \rho\left(15\mathscr{R}^3 + 15\mathscr{R}\mathscr{R}_{xx} + \mathscr{R}_{xxxx}\right)_x = 0, \tag{1}$$

referred to as the KdV-SK-R equation. This equation was used by Hirota and Ito [28] to describe the solitons' resonances in one dimension. Equation (1) recovers the KdV equation for $\rho = 0$. Nonetheless, the 5th-order Sawada-Kotera equation is restored when $\phi = 0$.

In Ref. [6], the authors created a new integrable equation that is stated as follows, implementing the same concept of the KdV-SK-R Eq. (1).

$$\mathscr{R}_{xt} + \left(3\mathscr{R}^2 + \mathscr{R}_{xx}\right)_x + \left(15\mathscr{R}^3 + 15\mathscr{R}\mathscr{R}_{xx} + \mathscr{R}_{xxxx}\right)_{xx} + \eta\mathscr{R}_{yy} = 0, \tag{2}$$

and this is going to be known as the Kadomtsev-Petviashvili-Sawada-Kotera-Ramani (KP-SK-R) equation.

For $\eta = 0$, Eq. (2) can also be simplified to be Eq. (1). In order to verify its Lax integrability, the Lax pair was also created in Ref. [6]. An unlimited number of conservation rules are established by adding the potential function. The two-dimensional solitons' resonances were also described by this equation. The examination of the integrability of a sixth-order (3+1)-dimensional extension of the KP-SK-R Equation (2) was presented in [29], driven by the aforementioned scientific applications and several more. It looks like this:

$$\mathscr{R}_{xt} + \left(3\mathscr{R}^2 + \mathscr{R}_{xx}\right)_{xx} + \left(15\mathscr{R}^3 + 15\mathscr{R}\mathscr{R}_{xx} + \mathscr{R}_{xxxx}\right)_{xx} + \gimel_1\mathscr{R}_{yy} + \gimel_2\mathscr{R}_{zz} + \mathbf{b}_1\mathscr{R}_{xy} + \mathbf{b}_2\mathscr{R}_{yz} + \mathbf{b}_3\mathscr{R}_{xz} = 0, \tag{3}$$

where $\gimel_i$, $i = 1, 2$, and $\mathbf{b}_j$; $j = 1, 2, 3$ are non-zero arbitrary constants; and $\mathscr{R}$ denotes a certain function of the space variables x, y, z, and time variable t. Currently, Fractional Order Differential Equations (FODEs) are employed by mathematicians, engineers, and physicists in many methodologies [30, 31]. It is clear from the above how those accurate models might reveal events in such systems that were previously unknown. Accurate travelling wave solutions to nonlinear equations have garnered more attention recently, especially for Non-Linear Fractional Differential Equations (NFDEs) [32, 33]. The growing interest in FDEs is largely driven by their ability to provide more accurate and flexible models for real-world processes compared to their integer order counterparts. For example, they have been successfully applied in modeling viscoelastic materials, fluid flow in porous media, electrical circuits with fractal structures, control systems with memory effects, and wave phenomena in complex environments. Furthermore, recent studies have extended their applications to engineering processes, such as the fractional model of a falling object with linear and quadratic frictional forces and liquid vortex formation in a swirling container, considering the fractional time derivative of Caputo [34, 35]. Recently, Khalil et al. [36] introduced a novel definition named conformable fractional derivative, which is an extension of the classical limit definition of the derivative and obeys the classical properties, including the linearity property, product rule, quotient rule, Rolle's theorem and the mean value theorem coincide with the classical definition of Riemann-Liouville and Caputo on polynomials up to a constant multiple. In a short time, many studies and discussions related to conformable fractional derivative have appeared in several areas of applications, for example, Won Sang [37] discussed the fractional Newton mechanics with conformable. Ahmed [38] studied the conformable fractional stochastic differential equations with control function.

In this article, our aim is to make a novel study of this equation with the influence of the conformable fractional derivative, which will take the following form:

$$D_t^{\alpha}\mathscr{R}_x + \left(3\mathscr{R}^2 + \mathscr{R}_{xx}\right)_{xx} + \left(15\mathscr{R}^3 + 15\mathscr{R}\mathscr{R}_{xx} + \mathscr{R}_{xxxx}\right)_{xx} + \daleth_1\mathscr{R}_{yy} + \daleth_2\mathscr{R}_{zz} + \mathbf{b}_1\mathscr{R}_{xy} + \mathbf{b}_2\mathscr{R}_{yz} + \mathbf{b}_3\mathscr{R}_{xz} = 0, \tag{4}$$

where,

$$D_t^{\alpha}\mathscr{R}(x,\, y,\, z,\, t) = \lim_{\delta \to 0} \frac{\mathscr{R}(x,\, y,\, z,\, t + \delta t^{1-\alpha}) - \mathscr{R}(x,\, y,\, z,\, t)}{\delta},\; t > 0,$$

is the conformable fractional derivative of order α, $0 < \alpha \leq 1$ [36]. When putting $\alpha = 1$ in the previous equation, it will return to its original form as in [29].

The motivation of this study is to extract numerous and various kinds of solutions of the proposed model, which have not been reported before in the literature. In this study, the Modified Extended (ME) direct algebraic approach is implemented for the first time to extract these solutions. These solutions include bright, dark, and singular solitons. Moreover, singular periodic, hyperbolic solutions, Weierstrass elliptic doubly periodic solutions, and exponential solutions are also acquired.

The structure of this manuscript can be summarized as follows: The proposed model and its theoretical background are given in Section 1 as an introduction. The general algorithmic steps of the ME direct algebraic approach are presented in section 2. In section 3, the symbolic computations are completed, and the results are summarized using the Wolfram Mathematica program. Multiple dynamic wave patterns of various solutions are visually presented in section 4 through the use of 3-D, contour, and 2-D simulations. A comparison with some papers from the literature is presented in section 5. The findings of the investigation are presented in section 6. In section 7, we present some future directions for the studied model.

2. The ME direct algebraic approach in summary

An analytical method for obtaining precise and explicit solutions to NLPDEs is the ME Direct Algebraic Approach. By adding more parameters and transformation structures, this method improves on the conventional straight algebraic approach and makes it possible to derive a larger class of soliton, periodic, and rational solutions [39, 40].

By considering the following NLPDE:

$$\mathscr{P}\left(\mathscr{R},\, \frac{\partial^{\alpha}\mathscr{R}}{\partial t^{\alpha}},\, \mathscr{R}_x,\, \mathscr{R}_y,\, \mathscr{R}_z,\, \mathscr{R}_{xx},\, \mathscr{R}_{yy},\, \mathscr{R}_{zz},\, \frac{\partial^{\alpha}\mathscr{R}_x}{\partial t^{\alpha}},\, \frac{\partial^{\alpha}\mathscr{R}_{xx}}{\partial t^{\alpha}},\, ...\right) = 0, \tag{5}$$

where $\mathscr{P}$ represents a function of $\mathscr{R}$ and its partial derivatives with respect to the independent variables, namely $x,\ y,\ z,$ and t. The following procedures briefly describe the suggested approach.

Procedure-(1): Assuming the travelling wave transformation below, we shall get the solution of Eq. (5):

$$\mathscr{R} = \Upsilon(\xi), \qquad \xi = ax + by + cz - \frac{\kappa t^{\alpha}}{\alpha},\ 0 < \alpha \leq 1, \tag{6}$$

where $a,\ b$, and c represent some unknown constants that could be determined later in the article.

It is possible to generate the following Non-Linear Ordinary Differential Equation (NLODE) by converting Eq. (6) into Eq. (5):

$$\mathscr{Q}(\Upsilon,\, \Upsilon',\, \Upsilon'',\, \Upsilon''',\, \ldots) = 0. \tag{7}$$

Procedure-(2): In this manner, we may write the solution to Eq. (7) as follows:

$$\Upsilon(\xi) = \sum_{j=-\mathbb{M}}^{\mathbb{M}} \mho_j \rho^j(\xi), \tag{8}$$

where $\mho_j$ are real-valued parameters to be evaluated such that $\mho_{\mathbb{M}}$ and $\mho_{-\mathbb{M}}$ can't be zero together in one set of results. Furthermore, $\rho(\xi)$ satisfies the following auxiliary equation requirements:

$$\rho'(\xi) = \sqrt{\pi_0 + \pi_1\rho(\xi) + \pi_2\rho^2(\xi) + \pi_3\rho^3(\xi) + \pi_4\rho^4(\xi) + \pi_6\rho^6(\xi)}. \tag{9}$$

The previous equation has the following cases for its solution:

Case 1: When $\pi_0 = \pi_1 = \pi_3 = \pi_6 = 0$, the following solutions are brought up:

$$\rho(\xi) = \sqrt{-\frac{\pi_2}{\pi_4}}\operatorname{sech}\left[\xi\sqrt{\pi_2}\right], \quad \pi_2 > 0,\ \pi_4 < 0,$$

$$\rho(\xi) = \sqrt{\frac{\pi_2}{\pi_4}}\operatorname{csch}\left[\xi\sqrt{\pi_2}\right], \quad \pi_2 > 0,\ \pi_4 > 0,$$

$$\rho(\xi) = \sqrt{-\frac{\pi_2}{\pi_4}}\sec\left[\xi\sqrt{-\pi_2}\right], \quad \pi_2 < 0,\ \pi_4 > 0.$$

$$\rho(\xi) = \sqrt{-\frac{\pi_2}{\pi_4}}\operatorname{csch}\left[\theta - \xi\sqrt{\pi_2}\right], \quad \pi_2 = 1,\ \pi_4 = \frac{1}{2}.$$

Case 2: When $\pi_1 = \pi_3 = \pi_6 = 0,\ \pi_0 = \dfrac{\pi_2^2}{4\pi_4}$, the following solutions are brought up:

$$\rho(\xi) = \sqrt{\frac{-\pi_2}{2\ \pi_4}}\tanh\left[\xi\sqrt{\frac{-\pi_2}{2}}\right], \quad \pi_2 < 0, \pi_4 > 0,$$

$$\rho(\xi) = \sqrt{\frac{\pi_2}{2\ \pi_4}}\tan\left[\xi\sqrt{\frac{\pi_2}{2}}\right], \quad \pi_2 > 0,\ \pi_4 > 0.$$

Case 3: When $\pi_3 = \ \pi_4 = \ \pi_6 = 0$, the following solution is brought up:

$$\rho(\xi) = e^{\xi\sqrt{\pi_2}} - \frac{\pi_1}{2\ \pi_2}, \quad \pi_2 > 0, \pi_0 = \frac{\pi_1^2}{4\pi_2}.$$

$$\rho(\xi) = \frac{\pi_1 \sinh(2\sqrt{\pi_2}\xi)}{2\pi_2} - \frac{\pi_1}{2\pi_2}, \quad \pi_2 > 0,\ \pi_0 = 0.$$

$$\rho(\xi) = \frac{\pi_1 \sin(\sqrt{-r_2}\xi)}{2\pi_2} - \frac{\pi_1}{2\pi_2}, \quad \pi_2 < 0,\ \pi_0 = 0.$$

Case 4: When $\pi_0 = \pi_1 = \pi_6 = 0$, the following solutions are brought up:

$$\rho(\xi) = -\frac{\pi_2}{\pi_3}\left(\tanh\left[\frac{1}{2}\xi\sqrt{\pi_2}\right] + 1\right), \quad \pi_2 > 0,\ \pi_3^2 = 4\pi_2\pi_4,$$

$$\rho(\xi) = -\frac{\pi_2}{\pi_3}\left(\coth\left[\frac{1}{2}\xi\sqrt{\pi_2}\right] + 1\right), \quad \pi_2 > 0,\ \pi_3^2 = 4\pi_2\pi_4,$$

$$\rho(\xi) = \frac{\pi_2 \operatorname{sech}^2\left[\frac{1}{2}\xi\sqrt{\pi_2}\right]}{2\sqrt{\pi_2\pi_4}\tanh\left[\frac{1}{2}\xi\sqrt{\pi_2}\right] - \pi_3}, \quad \pi_2 > 0, \pi_4 > 0,\ \pi_3^2 = 4\pi_2\pi_4,$$

$$\rho(\xi) = \frac{\pi_2 \sec^2\left[\frac{1}{2}\xi\sqrt{-\pi_2}\right]}{2\sqrt{-\pi_2\pi_4}\tan\left[\frac{1}{2}\xi\sqrt{-\pi_2}\right] + \pi_3}, \quad \pi_2 < 0, \pi_4 > 0,\ \pi_3^2 = 4\pi_2\pi_4.$$

Case 5: When $\pi_2 = \pi_4 = \pi_6 = 0$, the following solution is brought up:

$$\rho(\xi) = \wp\left(\frac{\sqrt{\pi_3}}{2}\xi;\ -\frac{4\pi_1}{\pi_3},\ -\frac{4\pi_0}{\pi_3}\right), \quad \pi_3 > 0.$$

Case 6: When $\pi_1 = \pi_3 = 0$, the following solutions are brought up:

$$\rho(\xi) = \sqrt{\frac{2\pi_2 \operatorname{sech}^2(\xi\sqrt{\pi_2})}{2\sqrt{\pi_4^2 - 4\pi_2\pi_6} - \left(\sqrt{\pi_4^2 - 4\pi_2\pi_6} + \pi_4\right)\operatorname{sech}^2(\xi\sqrt{\pi_2})}}, \quad \pi_2 > 0,$$

$$\rho(\xi) = \sqrt{\frac{2\pi_2 \sec^2(\xi\sqrt{-\pi_2})}{2\sqrt{\pi_4^2 - 4\pi_2\pi_6} - \left(\sqrt{\pi_4^2 - 4\pi_2\pi_6} - \pi_4\right)\sec^2(\xi\sqrt{-\pi_2})}}, \quad \pi_2 < 0.$$

Procedure-(3): By using Eq. (7) and the homogeneous balancing principle, the integer $\mathbb{M}$ may be computed.

Procedure-(4): The substitution of Eq. (8) along Eq. (9) in Eq. (7) yields a polynomial function in $\rho(\xi)$. After that, we will use Wolfram Mathematica software (version 13.2) to solve the resulting system of nonlinear equations by setting the sum of all terms with matching powers to zero, allowing us to determine the values of the unknowns.

3. Novel conformable analytical solutions

By implementing the aforementioned transformation described in Eq. (6) to Eq. (4), where $\Upsilon(\xi)$ acts as the solution form, it yields the following:

$$\begin{aligned} &a^6\Upsilon^{(6)} + a^4(15\Upsilon+1)\Upsilon^{(4)} + 15a^4\left(\Upsilon''\right)^2 + 30a^4\Upsilon^{(3)}\Upsilon' + 6a^2(15\Upsilon+1)\left(\Upsilon'\right)^2 \\ &+\Upsilon''\left(a\mathbf{b}_1 b + a\mathbf{b}_3 c + 3a^2\Upsilon(15\Upsilon+2) - a\kappa + b^2\daleth_1 + \mathbf{b}_2 bc + c^2\daleth_2\right) = 0. \end{aligned} \tag{10}$$

We may obtain the following resulted equation by integrating Eq. (10) twice and assuming the zero integration constant:

$$a^6\Upsilon^{(4)} + a^4\Upsilon'' + \Upsilon\left(15a^4\Upsilon'' + ab\mathbf{b}_1 + a\mathbf{b}_3 c - a\kappa + b^2\daleth_1 + b\mathbf{b}_2 c + c^2\daleth_2\right) + 15a^2\Upsilon^3 + 3a^2\Upsilon^2 = 0. \tag{11}$$

Therefore, using the homogeneous balancing principle described in the previous section between $\Upsilon^{(4)}$ and Υ^3, getting $\mathbb{M} = 2$ and the solution of Eq. (5) as follows:

$$\Upsilon(\xi) = \mho_0 + \mho_1\rho(\xi) + \mho_2\rho^2(\xi) + \frac{\mho_{-1}}{\rho(\xi)} + \frac{\mho_{-2}}{\rho^2(\xi)}, \tag{12}$$

using Eq. (12) with the aid of the auxiliary (9), by inserting them into Eq. (11), the following results can be obtained by using Mathematica software to solve a series of non-linear algebraic equations that are produced when the coefficients of the same powers are grouped and set to zero:

Case (1): When $\pi_0 = \pi_1 = \pi_3 = \pi_6 = 0$, one can acquire the upcoming sets of results:

(1.1) $\mho_{-2} = \mho_{-1} = \mho_0 = \mho_1 = 0$, $\mho_2 = -2\pi_4 a^2$,

$$b=-\frac{\left(\mathbf{b}_2c+\sqrt{(a\mathbf{b}_1+\mathbf{b}_2c)^2-4\gimel_1\left(16\pi_2^2a^6+4\pi_2a^4+a\mathbf{b}_3c-a\kappa+c^2\gimel_2\right)}\right)+a\mathbf{b}_1}{2\gimel_1}.$$

(1.2) $\mho_{-2}=\mho_{-1}=\mho_1=0, \mho_0=-\frac{1}{60}\left(3+60\pi_2a^2+\sqrt{120\pi_2\left(14\pi_2a^2-1\right)a^2+9}\right)$, $\mho_2=-2\pi_4a^2$,

$$b=-\frac{10a\mathbf{b}_1+10\mathbf{b}_2c+\sqrt{\begin{array}{l}10a^2\gimel_1\sqrt{120\pi_2\left(14\pi_2a^2-1\right)a^2+9}+30a^2\gimel_1+\mathscr{H}_1-400c\gimel_1\left(a\mathbf{b}_3+c\gimel_2\right)\\+100\left(a\mathbf{b}_1+\mathbf{b}_2c\right)^2+400a\kappa\gimel_1\end{array}}}{20\gimel_1},$$

where $\mathscr{H}_1=200\pi_2a^4\gimel_1\left(-44\pi_2a^2-\sqrt{120\pi_2\left(14\pi_2a^2-1\right)a^2+9}+4\right)$.

Using the solution set (1.1), (1.1.1) a bright soliton solution is acquired when $\pi_2>0$ and $\pi_4<0$ as follows:

$$\mathscr{R}_{1.1.1}=2\pi_2a^2\operatorname{sech}^2\left[\left(ax+by+cz-\frac{\kappa t^\alpha}{\alpha}\right)\sqrt{\pi_2}\right]. \tag{13}$$

(1.1.2) A singular soliton solution is acquired when $\pi_2>0$ and $\pi_4>0$ as follows:

$$\mathscr{R}_{1.1.2}=-2\pi_2a^2\operatorname{csch}^2\left[\left(ax+by+cz-\frac{\kappa t^\alpha}{\alpha}\right)\sqrt{\pi_2}\right]. \tag{14}$$

(1.1.3) A singular periodic solution is acquired when $\pi_2<0$ and $\pi_4>0$ as follows:

$$\mathscr{R}_{1.1.3}=2\pi_2a^2\sec^2\left[\left(ax+by+cz-\frac{\kappa t^\alpha}{\alpha}\right)\sqrt{-\pi_2}\right]. \tag{15}$$

(1.1.4) A singular soliton solution is acquired when $\pi_2=1$ and $\pi_4=\frac{1}{2}$ as follows:

$$\mathscr{R}_{1.1.4}=-2a^2\operatorname{csch}^2\left[\theta-\left(ax+by+cz-\frac{\kappa t^\alpha}{\alpha}\right)\right], \tag{16}$$

where θ is a constant.

Using solution set (1.2), (1.2.1) a bright soliton solution is acquired when $\pi_2>0$ and $\pi_4<0$ as follows:

$$\mathscr{R}_{1.2.1}=-\frac{1}{20}-\frac{1}{60}\sqrt{120a^2\pi_2\left(14\pi_2a^2-1\right)+9}-a^2\pi_2\left(1-2\operatorname{sech}^2\left[\left(ax+by+cz-\frac{\kappa t^\alpha}{\alpha}\right)\sqrt{\pi_2}\right]\right). \tag{17}$$

(1.2.2) A singular soliton solution is acquired when $\pi_2>0$ and $\pi_4>0$ as follows:

$$\mathscr{R}_{1.2.2}=-\frac{1}{20}-\frac{1}{60}\sqrt{120a^2\pi_2\left(14\pi_2a^2-1\right)+9}-a^2\pi_2\left(1+2\operatorname{csch}^2\left[\left(ax+by+cz-\frac{\kappa t^\alpha}{\alpha}\right)\sqrt{\pi_2}\right]\right). \tag{18}$$

(1.2.3) A singular periodic solution is acquired when $\pi_2<0$ and $\pi_4>0$ as follows:

$$\mathscr{R}_{1.2.3}=-\frac{1}{20}-\frac{1}{60}\sqrt{120a^2\pi_2\left(14\pi_2a^2-1\right)+9}-a^2\pi_2\left(1-2\sec^2\left[\left(ax+by+cz-\frac{\kappa t^\alpha}{\alpha}\right)\sqrt{-\pi_2}\right]\right). \tag{19}$$

(1.2.4) A singular soliton solution is acquired when $\pi_2=1$ and $\pi_4=\frac{1}{2}$ as follows:

$$\mathscr{R}_{1.2.4}=-\frac{1}{20}-\frac{1}{60}\sqrt{120a^2\left(14a^2-1\right)+9}-a^2\left(1+2\operatorname{csch}^2\left[\theta-\left(ax+by+cz-\frac{\kappa t^\alpha}{\alpha}\right)\right]\right), \tag{20}$$

where θ is a constant.

Case (2): When $\pi_1=\pi_3=\pi_6=0,\ \pi_0=\frac{\pi_2^2}{4\pi_4}$, one can acquire the upcoming sets of results:

(2.1) $\mho_{-2}=\mho_{-1}=\mho_1=0,\ \mho_0=-\pi_2a^2,\ \mho_2=-2\pi_4a^2,$

$$b=-\frac{a\mathbf{b}_1+\mathbf{b}_2c+\sqrt{(a\mathbf{b}_1+\mathbf{b}_2c)^2-4\beth_1\left(16\pi_2^2a^6+4\pi_2a^4+a\mathbf{b}_3c-a\kappa+c^2\beth_2\right)}}{2\beth_1}.$$

(2.2) $\mho_{-2}=\mho_{-1}=\mho_1=0,\ \mho_0=-\frac{1}{60}\left(3+30\pi_2a^2+\sqrt{60\pi_2\left(7\pi_2a^2+1\right)a^2+9}\right),\ \mho_2=-2\pi_4a^2,$

$$b=-\frac{10a\mathbf{b}_1+10\mathbf{b}_2c+\sqrt{\begin{array}{l}10a^2\beth_1\sqrt{120\pi_2(14\pi_2a^2-1)a^2+9}+30a^2\beth_1+\mathscr{H}_1-400c\beth_1(a\mathbf{b}_3+c\beth_2)\\+100(a\mathbf{b}_1+\mathbf{b}_2c)^2+400a\kappa\beth_1\end{array}}}{20\beth_1}$$

where $\mathscr{H}_1=200\pi_2a^4\beth_1\left(-44\pi_2a^2-\sqrt{120\pi_2\left(14\pi_2a^2-1\right)a^2+9}+4\right)$.

(2.3) $\mho_2=\mho_{-1}=\mho_1=0,\ \mho_0=-\pi_2a^2,\ \mho_{-2}=-\frac{a^2\pi_2^2}{2\pi_4},$

$$b=-\frac{a\mathbf{b}_1+\mathbf{b}_2c+\sqrt{(a\mathbf{b}_1+\mathbf{b}_2c)^2-4\beth_1\left(16\pi_2^2a^6+4\pi_2a^4+a\mathbf{b}_3c-a\kappa+c^2\beth_2\right)}}{2\beth_1}.$$

(2.4) $\mho_2=\mho_{-1}=\mho_1=0,\ \mho_0=-\frac{1}{60}\left(3+30\pi_2a^2+\sqrt{60\pi_2a^2\left(7\pi_2a^2+1\right)+9}\right),\ \mho_{-2}=-\frac{a^2\pi_2^2}{2\pi_4},$

$$b=-\frac{10a\mathbf{b}_1+10\mathbf{b}_2c+\sqrt{10a^2\beth_1\sqrt{\begin{array}{l}120\pi_2\left(14\pi_2a^2-1\right)a^2+9+30a^2\beth_1+\mathscr{H}_1-400c\beth_1\left(a\mathbf{b}_3+c\beth_2\right)\\+100\left(a\mathbf{b}_1+\mathbf{b}_2c\right)^2+400a\kappa\beth_1\end{array}}}}{20\beth_1},$$

where $\mathscr{H}_1 = 200\pi_2 a^4 \gimel_1 \left(-44\pi_2 a^2 - \sqrt{120\pi_2\left(14\pi_2 a^2 - 1\right)a^2 + 9} + 4\right)$.

(2.5) $\mho_{-1} = \mho_1 = 0,\ \mho_0 = -2\pi_2 a^2,\ \mho_{-2} = -\dfrac{a^2\pi_2^2}{2\pi_4},\ \mho_2 = -2\pi_4 a^2,$

$$b = -\frac{a\mathbf{b}_1 + \mathbf{b}_2 c + \sqrt{\left(a\mathbf{b}_1 + \mathbf{b}_2 c\right)^2 - 4\gimel_1\left(16\pi_2^2 a^6 + 4\pi_2 a^4 + a\mathbf{b}_3 c - a\kappa + c^2\gimel_2\right)}}{2\gimel_1}.$$

(2.6) $\mho_{-1} = \mho_1 = 0,\ \mho_0 = -\dfrac{1}{60}\left(3 + \sqrt{240\pi_2 a^2\left(28\pi_2 a^2 + 1\right) + 9}\right),\ \mho_{-2} = -\dfrac{a^2\pi_2^2}{2\pi_4},\ \mho_2 = -2\pi_4 a^2,$

$$b = -\frac{10a\mathbf{b}_1 + 10\mathbf{b}_2 c + \sqrt{10a^2\gimel_1\sqrt{\begin{array}{l}120\pi_2\left(14\pi_2 a^2 - 1\right)a^2 + 9 + 30a^2\gimel_1 + \mathscr{H}_1 - 400c\gimel_1\left(a\mathbf{b}_3 + c\gimel_2\right)\\ +100\left(a\mathbf{b}_1 + \mathbf{b}_2 c\right)^2 + 400a\kappa\gimel_1\end{array}}}}{20\gimel_1},$$

where $\mathscr{H}_1 = 200\pi_2 a^4 \gimel_1 \left(-44\pi_2 a^2 - \sqrt{120\pi_2\left(14\pi_2 a^2 - 1\right)a^2 + 9} + 4\right)$.

Using solution set (2.1), (2.1.1) a bright soliton solution is acquired when $\pi_2 < 0$ and $\pi_4 > 0$ as follows:

$$\mathscr{R}_{2.1,\ 1} = -a^2\pi_2 \operatorname{sech}^2\left[\frac{\sqrt{-\pi_2}\left(ax + by + cz - \dfrac{\kappa t^\alpha}{\alpha}\right)}{\sqrt{2}}\right]. \tag{21}$$

(2.1.2) A singular periodic solution is acquired when $\pi_2 > 0$ and $\pi_4 > 0$ as follows:

$$\mathscr{R}_{2.1,\ 2} = -a^2\pi_2 \sec^2\left[\frac{\sqrt{\pi_2}\left(ax + by + cz - \dfrac{\kappa t^\alpha}{\alpha}\right)}{\sqrt{2}}\right]. \tag{22}$$

Using solution set (2.2), (2.2.1) a dark soliton solution is acquired when $\pi_2 < 0$ and $\pi_4 > 0$ as follows:

$$\mathscr{R}_{2.2,\ 1} = -\frac{1}{60}\left(3 + 30\pi_2 a^2 + \sqrt{9 + 60\pi_2\left(7\pi_2 a^2 + 1\right)a^2}\right) + a^2\pi_2 \tanh^2\left[\frac{\sqrt{-\pi_2}\left(ax + by + cz - \dfrac{\kappa t^\alpha}{\alpha}\right)}{\sqrt{2}}\right]. \tag{23}$$

(2.2.2) A singular periodic solution is acquired when $\pi_2 > 0$ and $\pi_4 > 0$ as follows:

$$\mathscr{R}_{2.2,\ 2} = -\frac{1}{60}\left(3 + 30\pi_2 a^2 + \sqrt{9 + 60\pi_2\left(7\pi_2 a^2 + 1\right)a^2}\right) - a^2\pi_2 \tan^2\left[\frac{\sqrt{\pi_2}\left(ax + by + cz - \dfrac{\kappa t^\alpha}{\alpha}\right)}{\sqrt{2}}\right]. \tag{24}$$

Using solution set (2.3), (2.3.1) a singular soliton solution is acquired when $\pi_2 < 0$ and $\pi_4 > 0$ as follows:

$$\mathscr{R}_{2.3.1} = a^2\pi_2 \operatorname{csch}^2\left[\frac{\sqrt{-\pi_2}\left(ax+by+cz-\frac{\kappa t^{\alpha}}{\alpha}\right)}{\sqrt{2}}\right]. \tag{25}$$

(2.3.2) A singular periodic solution is acquired when $\pi_2 > 0$ and $\pi_4 > 0$ as follows:

$$\mathscr{R}_{2.3.2} = a^2\pi_2 \csc^2\left[\frac{\sqrt{-\pi_2}\left(ax+by+cz-\frac{\kappa t^{\alpha}}{\alpha}\right)}{\sqrt{2}}\right]. \tag{26}$$

Using solution set (2.4), the solutions to Eq. (4) are expressed in the following way:
(2.4.1) A singular soliton solution is acquired when $\pi_2 < 0$ and $\pi_4 > 0$ as follows:

$$\mathscr{R}_{2.4.1} = -\frac{1}{60}\left(3+30\pi_2 a^2+\sqrt{9+60a^2\pi_2\left(7\pi_2 a^2+1\right)}\right)+a^2\pi_2 \coth^2\left[\frac{\sqrt{-\pi_2}\left(ax+by+cz-\frac{\kappa t^{\alpha}}{\alpha}\right)}{\sqrt{2}}\right]. \tag{27}$$

(2.4.2) A singular periodic solution is acquired when $\pi_2 > 0$ and $\pi_4 > 0$ as follows:

$$\mathscr{R}_{2.4.2} = -\frac{1}{60}\left(3+30\pi_2 a^2+\sqrt{9+60a^2\pi_2\left(7\pi_2 a^2+1\right)}\right)-a^2\pi_2 \cot^2\left[\frac{\sqrt{\pi_2}\left(ax+by+cz-\frac{\kappa t^{\alpha}}{\alpha}\right)}{\sqrt{2}}\right]. \tag{28}$$

Using solution set (2.5), (2.5.1) a singular soliton solution is acquired when $\pi_2 < 0$ and $\pi_4 > 0$ as follows:

$$\mathscr{R}_{2.5.1} = 4a^2\pi_2 \operatorname{csch}^2\left[\sqrt{2}\left(ax+by+cz-\frac{\kappa t^{\alpha}}{\alpha}\right)\sqrt{-\pi_2}\right]. \tag{29}$$

(2.5.2) A singular periodic solution is acquired when $\pi_2 > 0$ and $\pi_4 > 0$ as follows:

$$\mathscr{R}_{2.5.2} = -4a^2\pi_2 \csc^2\left[\sqrt{2}\left(ax+by+cz-\frac{\kappa t^{\alpha}}{\alpha}\right)\sqrt{\pi_2}\right]. \tag{30}$$

Using solution set (2.6), (2.6.1) a singular soliton solution is acquired when $\pi_2 < 0$ and $\pi_4 > 0$ as follows:

$$\mathscr{R}_{2.6.1} = -\frac{1}{60}\left(3+\sqrt{9+240a^2\pi_2\left(28\pi_2 a^2+1\right)}\right)+a^2\pi_2\left(1+2\operatorname{csch}^2\left[\left(ax+by+cz-\frac{\kappa t^{\alpha}}{\alpha}\right)\sqrt{-2\pi_2}\right]\right). \tag{31}$$

(2.6.2) A singular periodic solution is acquired when $\pi_2 > 0$ and $\pi_4 > 0$ as follows:

$$\mathscr{R}_{2.6.2} = -\frac{1}{60}\left(3+\sqrt{9+240a^2\pi_2\left(28\pi_2 a^2+1\right)}\right) - a^2\pi_2\left(1+2\csc^2\left[\left(ax+by+cz-\frac{\kappa t^\alpha}{\alpha}\right)\sqrt{2\pi_2}\right]\right). \tag{32}$$

Case (3): When $\pi_3 = \pi_4 = \pi_6 = 0$, one can acquire the upcoming sets of results:

(3.1) $\mho_2 = \mho_1 = \mho_0 = 0,\ \mho_{-1} = -a^2\pi_1,\ \mho_{-2} = -2\pi_0 a^2,\ \pi_2 = \dfrac{\pi_1^2}{4\pi_0}$,

$$b = -\frac{2\pi_0\left(a\mathbf{b}_1+\mathbf{b}_2 c\right)+\frac{1}{8}\sqrt{256\pi_0^2\left(a\mathbf{b}_1+\mathbf{b}_2 c\right)^2-64\gimel_1\left(\pi_1^4a^6+4\pi_0\pi_1^2a^4+16\pi_0^2c\left(a\mathbf{b}_3+c\gimel_2\right)-16\pi_0^2a\kappa\right)}}{4\pi_0\gimel_1}.$$

(3.2) $\mho_2 = \mho_1 = 0,\ \mho_0 = -\dfrac{3\left(5\pi_1^2a^2+4\pi_0\right)+\sqrt{3}\sqrt{35\pi_1^4a^4-40\pi_0\pi_1^2a^2+48\pi_0^2}}{240\pi_0},\ \mho_{-1} = -a^2\pi_1,\ \mho_{-2} = -2\pi_0 a^2$,

$\pi_2 = \dfrac{\pi_1^2}{4\pi_0}$,

$$b = -\frac{40\pi_0^2\left(a\mathbf{b}_1+\mathbf{b}_2 c\right)+\sqrt{10\pi_0^3\gimel_1\left(-55\pi_1^4a^6+80\pi_0\pi_1^2a^4-640\pi_0^2c\left(a\mathbf{b}_3+c\gimel_2\right)+16\pi_0^2a(3a+40\kappa)\right)+\mathscr{H}_2}}{80\pi_0^2\gimel_1},$$

where $\mathscr{H}_2 = 1{,}600\pi_0^4\left(a\mathbf{b}_1+\mathbf{b}_2 c\right)^2+\left(40\sqrt{3}\pi_0^3a^2\gimel_1-50\gimel_1\sqrt{3}\pi_0^2\pi_1^2a^4\right)\sqrt{35\pi_1^4a^4-40\pi_0\pi_1^2a^2+48\pi_0^2}$.

Using solution set (3.1), we may create the solutions to Eq. (4) as follows:

(3.1) If $\pi_2 > 0$ and $\pi_0 = \dfrac{\pi_1^2}{4\pi_2}$, then an exponential solution is given as next:

$$\mathscr{R}_{3.1} = -\frac{\pi_1^3a^2e^{\sqrt{\pi_2}\left(ax+by+cz-\frac{\kappa t^\alpha}{\alpha}\right)}}{\left(\pi_1 e^{\sqrt{\pi_2}\left(ax+by+cz-\frac{\kappa t^\alpha}{\alpha}\right)}-2\pi_0\right)^2}, \tag{33}$$

such that $\pi_1 e^{\sqrt{\pi_2}\left(ax+by+cz-\frac{\kappa t^\alpha}{\alpha}\right)}-2\pi_0 \neq 0$.

Using solution set (3.2), we may create the solutions to Eq. (4) as follows:

(3.2) If $\pi_2 > 0$ and $\pi_0 = \dfrac{\pi_1^2}{4\pi_2}$, then an exponential solution is given as next:

$$\mathscr{R}_{3.2} = -\frac{3\left(5\pi_1^2a^2+4\pi_0\right)+\sqrt{105\pi_1^4a^4-120\pi_0\pi_1^2a^2+146\pi_0^2}}{240\pi_0}-\frac{\pi_1^3a^2e^{\sqrt{\pi_2}\left(ax+by+cz-\frac{\kappa t^\alpha}{\alpha}\right)}}{\left(\pi_1 e^{\sqrt{\pi_2}\left(ax+by+cz-\frac{\kappa t^\alpha}{\alpha}\right)}-2\pi_0\right)^2}. \tag{34}$$

Case (4): When $\pi_0 = \pi_1 = \pi_6 = 0$, one can acquire the upcoming sets of results:

(4.1) $\mho_{-2} = \mho_{-1} = \mho_1 = \mho_0 = 0,\ \mho_2 = -2\pi_4 a^2$,

$$b=-\frac{a\mathbf{b}_1+\mathbf{b}_2c+\sqrt{(a\mathbf{b}_1+\mathbf{b}_2c)^2-4\gimel_1\left(16\pi_2^2a^6+4\pi_2a^4+a\mathbf{b}_3c-a\kappa+c^2\gimel_2\right)}}{2\gimel_1}.$$

(4.2) $\mho_{-2}=\mho_{-1}=\mho_1=0,\ \mho_0=-\frac{1}{60}\left(3+60\pi_2a^2+\sqrt{9+120\pi_2\left(14\pi_2a^2-1\right)a^2}\right),\ \mho_2=-2\pi_4a^2,\ \pi_3=0,$

$$b=-\frac{\sqrt{100a^2\mathbf{b}_1^2+\mathscr{H}_4+200a\mathbf{b}_2\mathbf{b}_1c+400a\kappa\gimel_1+100\mathbf{b}_2^2c^2}+10a\mathbf{b}_1-400c\gimel_1\left(a\mathbf{b}_3+c\gimel_2\right)+10\mathbf{b}_2c}{20\gimel_1},$$

where $\mathscr{H}_3=300a^2\gimel_1+10a^2\gimel_1\sqrt{120\pi_2\left(14\pi_2a^2-1\right)a^2+9}-200\pi_2a^4\gimel_1\left(44\pi_2a^2+\sqrt{120\pi_2\left(14\pi_2a^2-1\right)a^2+9}-4\right).$

(4.3) $\mho_{-2}=\mho_{-1}=\mho_0=0,\ \mho_1=-2\sqrt{\pi_2\pi_4}a^2,\ \mho_2=-2\pi_4a^2,\ \pi_3=+2\sqrt{\pi_2\pi_4},$

$$b=-\frac{a\mathbf{b}_1+c\mathbf{b}_2+\sqrt{a^2\mathbf{b}_1^2-4\gimel_1\left(\pi_2^2a^6+\pi_2a^4+a\mathbf{b}_3c-a\kappa+c^2\gimel_2\right)+2a\mathbf{b}_2\mathbf{b}_1c+\mathbf{b}_2^2c^2}}{2\gimel_1}.$$

(4.4) $\mho_{-2}=\mho_{-1}=0,\ \mho_0=-\frac{1}{60}\left(3+15\pi_2a^2+\sqrt{9+15\pi_2a^2\left(7\pi_2a^2-2\right)}\right),\ \mho_1=-2a^2\sqrt{\pi_2\pi_4},\ \pi_3=+2\sqrt{\pi_2\pi_4},$ $\mho_2=-2\pi_4a^2,$

$$b=-\frac{\sqrt{100a^2\mathbf{b}_1^2+\mathscr{H}_3+200a\mathbf{b}_2\mathbf{b}_1c+400a\kappa\gimel_1+100\mathbf{b}_2^2c^2}+10a\mathbf{b}_1-400c\gimel_1\left(a\mathbf{b}_3+c\gimel_2\right)+10\mathbf{b}_2c}{20\gimel_1}.$$

where $\mathscr{H}_3=300a^2\gimel_1+10a^2\gimel_1\sqrt{120\pi_2\left(14\pi_2a^2-1\right)a^2+9}-200\pi_2a^4\gimel_1\left(44\pi_2a^2+\sqrt{120\pi_2\left(14\pi_2a^2-1\right)a^2+9}-4\right).$

Using solution set (4.1), we may create the solutions to Eq. (4) as follows:

(4.1.1) A singular soliton solution is found when $\pi_2>0$, $\pi_4>0$ and $\pi_3^2\neq4\pi_2\pi_4$ as follows:

$$\mathscr{R}_{4.1.1}=-2\pi_2a^2\operatorname{csch}^2\left[\sqrt{\pi_2}\left(ax+by+cz-\frac{\kappa t^\alpha}{\alpha}\right)\right]. \tag{35}$$

(4.1.2) A singular periodic solution is found when $\pi_2<0$, $\pi_4>0$ and $\pi_3^2\neq4\pi_2\pi_4$ as follows:

$$\mathscr{R}_{4.1.2}=-2\pi_2a^2\csc^2\left[\sqrt{-\pi_2}\left(ax+by+cz-\frac{\kappa t^\alpha}{\alpha}\right)\right]. \tag{36}$$

Using solution set (4.2), we may create the solutions to Eq. (4) as follows:

(4.2.1) A singular soliton solution is found when $\pi_2>0$, $\pi_4>0$ and $\pi_3^2\neq4\pi_2\pi_4$ as follows:

$$\mathscr{R}_{4.2.1}=-\frac{1}{60}\left(3+60\pi_2a^2+\sqrt{9+120\pi_2\left(14\pi_2a^2-1\right)a^2}\right)-2\pi_2a^2\operatorname{csch}^2\left[\sqrt{\pi_2}\left(ax+by+cz-\frac{\kappa t^\alpha}{\alpha}\right)\right]. \tag{37}$$

(4.2.2) A singular periodic solution is found when $\pi_2 < 0$, $\pi_4 > 0$ and $\pi_3^2 \neq 4\pi_2\pi_4$ as follows:

$$\mathscr{R}_{4.2.2} = -\frac{1}{60}\left(3+60\pi_2 a^2+\sqrt{9+120\pi_2\left(14\pi_2 a^2-1\right)a^2}\right)-2\pi_2 a^2 \csc^2\left[\sqrt{-\pi_2}\left(ax+by+cz-\frac{\kappa t^{\alpha}}{\alpha}\right)\right]. \tag{38}$$

Using solution set (4.3), we may create the solutions to Eq. (4) as follows:
(4.3) If $\pi_2 > 0$, $\pi_4 > 0$ and $\pi_3^2 = 4\pi_2\pi_4$, then bright soliton solution are as follows:

$$\mathscr{R}_{4.3.1} = \frac{1}{2}\pi_2 a^2 \operatorname{sech}^2\left[\frac{1}{2}\sqrt{\pi_2}\left(ax+by+cz-\frac{\kappa t^{\alpha}}{\alpha}\right)\right], \tag{39}$$

or singular soliton as follows:

$$\mathscr{R}_{4.3.2} = -\frac{1}{2}\pi_2 a^2 \operatorname{csch}^2\left[\frac{1}{2}\sqrt{\pi_2}\left(ax+by+cz-\frac{\kappa t^{\alpha}}{\alpha}\right)\right]. \tag{40}$$

Using solution set (4.4), we may create the solutions to Eq. (4) as follows:
(4.4) If $\pi_2 > 0$, $\pi_4 > 0$ and $\pi_3^2 = 4\pi_2\pi_4$, then a dark soliton solution is acquired as:

$$\mathscr{R}_{4.4.1} = -\frac{1}{60}\left(3+\sqrt{15\pi_2\left(7\pi_2 a^2-2\right)a^2+9}-15\pi_2 a^2\left(1-2\tanh^2\left[\frac{1}{2}\sqrt{\pi_2}\left(ax+by+cz-\frac{\kappa t^{\alpha}}{\alpha}\right)\right]\right)\right), \tag{41}$$

or singular soliton as follows:

$$\mathscr{R}_{4.4.2} = -\frac{1}{60}\left(3+\sqrt{15\pi_2\left(7\pi_2 a^2-2\right)a^2+9}-15\pi_2 a^2\left(1-2\coth^2\left[\frac{1}{2}\sqrt{\pi_2}\left(ax+by+cz-\frac{\kappa t^{\alpha}}{\alpha}\right)\right]\right)\right). \tag{42}$$

Case (5): When $\pi_2 = \pi_4 = \pi_6 = 0$, one can acquire the upcoming set of results:
$\mho_{-2} = \frac{90\pi_1\pi_3 a^4+2}{675\pi_3^2 a^4}$, $\mho_{-1} = \pi_1 - a^2$, $\mho_0 = -\frac{1}{15}$, $\mho_1 = 0$, $\mho_2 = 0$, $\pi_0 = -\frac{45\pi_1\pi_3 a^4+1}{675\pi_3^2 a^6}$,

$$b = -\frac{5a\mathbf{b}_1+5\mathbf{b}_2 c+\sqrt{20a\daleth_1\left(15\pi_1\pi_3 a^5+a+5\kappa\right)-100a\mathbf{b}_3 c\daleth_1+25\left(a\mathbf{b}_1+\mathbf{b}_2 c\right)^2-100c^2\daleth_1\daleth_2}}{10\daleth_1}.$$

Then, the Weierstrass elliptic periodic solution is obtained as below:

$$\mathscr{R}_5 = -\frac{1}{15}+\frac{2\left(45\pi_1\pi_3 a^4+1\right)}{675\pi_3^2 a^4 \wp\left(\frac{\xi\sqrt{\pi_3}}{2}; -\frac{4\pi_1}{\pi_3}, \frac{4\left(45\pi_1\pi_3 a^4+1\right)}{675a^6\pi_3^3}\right)^2}-\frac{\pi_1 a^2}{\wp\left(\frac{\xi\sqrt{\pi_3}}{2}; -\frac{4\pi_1}{\pi_3}, \frac{4\left(45\pi_1\pi_3 a^4+1\right)}{675a^6\pi_3^3}\right)}, \tag{43}$$

where $\pi_3 > 0$.

Case (6): When $\pi_1 = \pi_3 = 0$, one can acquire the upcoming set of results:

$$\mho_{-2}=\frac{-3,600\pi_2\pi_4a^3-60\sqrt{9\pi_4^2a^2\left(1-20\pi_2a^2\right)^2-3\pi_6\left(3,200\pi_2^3a^6-240\pi_2^2a^4+1\right)}+180\pi_4a}{5,400\pi_6a},\ \mho_{-1}=\mho_1=\mho_2=0,\ \mho_0=-\frac{1}{3}4\pi_2a^2-\frac{1}{15},$$

$$\pi_0=\frac{60\pi_2\pi_4a^3+\sqrt{-1,200\pi_2^2\left(8\pi_2\pi_6-3\pi_4^2\right)a^6+360\pi_2\left(2\pi_2\pi_6-\pi_4^2\right)a^4+9\pi_4^2a^2-3\pi_6}-3\pi_4a}{360\pi_6a^3},$$

$$b=-\frac{5a\mathbf{b}_1}{10\beth_1+5\mathbf{b}_2c+\sqrt{\left(5a\mathbf{b}_1+5\mathbf{b}_2c\right)^2-20\beth_1\left(80\pi_2^2a^6-a^2+\frac{\pi_4a^2}{90\pi_6}\mathscr{H}_4\right)+5a\mathbf{b}_3c-5a\kappa+5c^2\beth_2}},$$

where $\mathscr{H}_4=-3,600\pi_2\pi_4a^4+180\pi_4a^2-60\sqrt{9\pi_4^2a^4\left(1-20\pi_2a^2\right)^2-3\pi_6a^2\left(3,200\pi_2^3a^6-240\pi_2^2a^4+1\right)}$.

Through the above solution set, the hyperbolic solution can be obtained for Eq. (4) when $\pi_2>0$ and $\pi_4^2>4\pi_2\pi_6$ as follows:

$$\mathscr{R}_6=-\frac{1}{45}\left(60\pi_2a^2+3\right)+\frac{3\pi_4a\left(20\pi_2a^2-1\right)+\sqrt{9\pi_4^2\left(1-20\pi_2a^2\right)^2a^2-240\pi_6\pi_2^2\left(40\pi_2a^2-3\right)a^4-3\pi_6}}{180\pi_2\pi_6a}$$
$$\times\left(\pi_4-\sqrt{\pi_4^2-4\pi_2\pi_6}\cosh\left[2\sqrt{\pi_2}\left(ax+by+cz-\frac{\kappa t^\alpha}{\alpha}\right)\right]\right).\tag{44}$$

4. Results and discussion

The current study successfully derived new solutions for (3+1)-dimensional KP-SK-R equation with the aid of the ME direct algebraic approach. These solutions include the Jacobi Elliptic (JE) function, (bright, dark, singular) soliton, Weierstrass elliptic functions, hyperbolic functions, exponential functions, and singular periodic solutions. Numerical simulations of some gained solutions are presented in this section in 3D, contour, and 2D form to show their physical implications with three different values of α. Figure 1 depicts the bright soliton solution of Eq. (13) in 3-D and contour plots when assuming $a=1.1,\ c=1,\ \kappa=1.4,\ \pi_2=0.6,\ \mathbf{b}_1=1.95,\ \mathbf{b}_2=1.9,\ \mathbf{b}_3=-1.25,\ \beth_1=0.35,\ \beth_2=0.4$ and $-10\le x\le 10$. This solution is a localized wave packet that maintains its shape due to a balance be- tween dispersion and nonlinearity in a nonlinear medium. It manifests as a steady, self-contained pulse that doesn't disperse with time. Figure 2 shows a collective 2D plot of what is drawn in Figure 1. Figure 3 shows the singular soliton solution of Eq. (14) in 3-D and contour forms when assuming $a=1.2,\ c=1,\ \kappa=1.5,\ \pi_2=0.45,\ \mathbf{b}_1=1.8,\ \mathbf{b}_2=1.85,\ \mathbf{b}_3=-1.75,\ \beth_1=0.4,\ \beth_2=0.45$ and $-10\le x\le 10$. This solution combines singularities, which are sharp, localized peaks that travel without altering their general profile but have blow-ups at finite sites, with soliton-like propagation. Extreme waves and concentrated energy concentrations in nonlinear systems are two examples of phenomena that are frequently connected to it. Figure 4 shows a collective 2D plot of what is drawn in Figure 3. Figure 5 depicts the singular periodic solution of Eq. (15) in 3-D and contour forms when assuming $a=1.15,\ c=0.8,\ \kappa=1.55,\ \pi_2=-0.6,\ \mathbf{b}_1=1.75,\ \mathbf{b}_2=1.7,\ \mathbf{b}_3=-1.8,\ \beth_1=0.45,\ \beth_2=0.5$ and x from -10 to 10. This solution exhibits periodic oscillations but contains singularities at specific points, representing wave profiles with repeating patterns interrupted by discontinuities or blow-ups. This kind of behavior frequently simulates localized instabilities in nonlinear media or breaking waves. Figure 6 displays a collective 2D plot of what is drawn in Figure 5. Figure 7 displays the dark soliton solution of Eq. (23) in 3-D and contour forms by assuming $a = 1.1,\ c = 0.85,\ \kappa = 1.99,\ \pi_2 = -0.55,\ \mathbf{b}_1 = 1.7,\ \mathbf{b}_2 = 1.8,\ \mathbf{b}_3 = -1.85,\ \gimel_1 = 0.4,\ \gimel_2 = 0.55$ and x from -10 to 10. This kind is localized waveforms that appear as dips or voids within a continuous wave background. The dark solitons show phase shifts during interactions and retain their form during propagation. They are frequently linked to systems, like optical fibers or Bose-Einstein condensates, where energy or density is locally depleted. Figure 8 displays a collective 2D plot of what is drawn in Figure 7. Figure 9 displays the singular periodic solution also, but in the form of a tan function associated with Eq. (24) in 3-D and contour forms by assuming $a = 1.05,\ c = 0.75,\ \kappa = 1.99,\ \pi_2 = 0.5,\ \mathbf{b}_1 = 1.7,\ \mathbf{b}_2 = 1.8,\ \mathbf{b}_3 = -1.85,\ \gimel_1 = 0.4,\ \gimel_2 = 0.5$ and x from -20 to 20. Figure 10 displays a collective 2D plot of what is drawn in Figure 9.

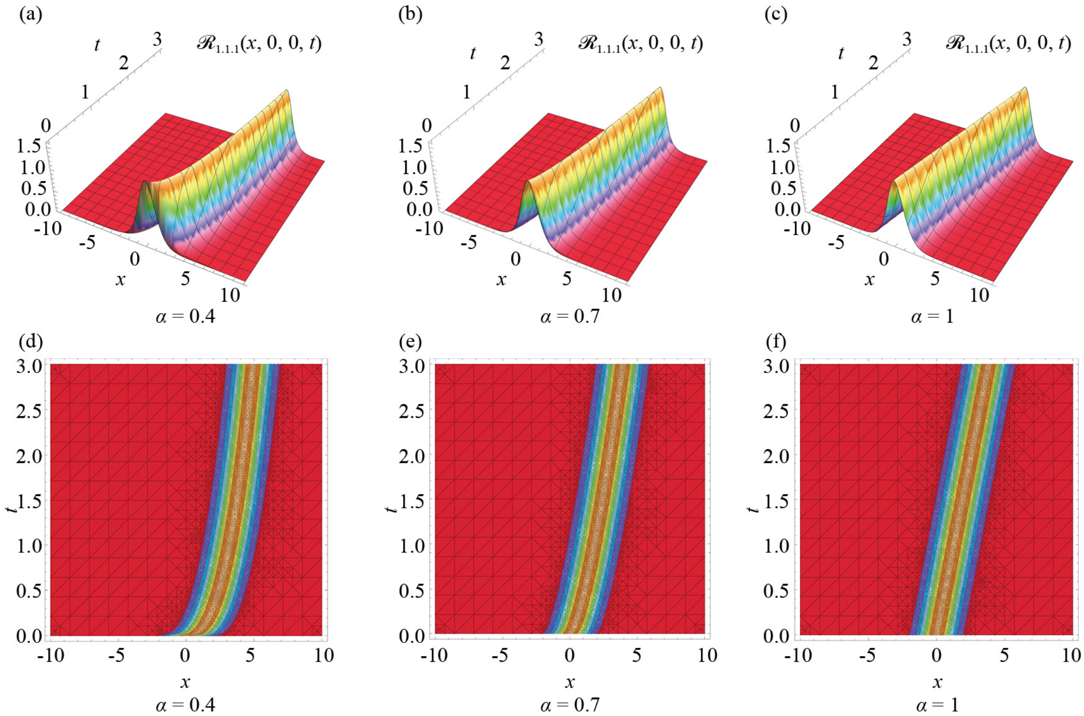


Figure 1. 3D and contour visual plots for the solution in Eq. (13)

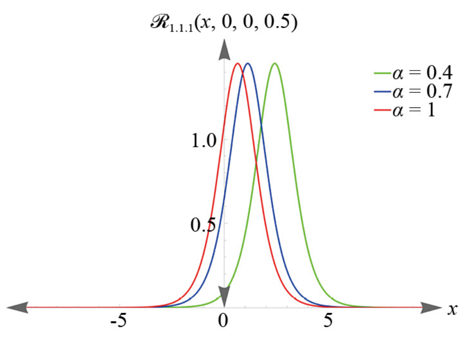


Figure 2. 2D graph of Figure 1

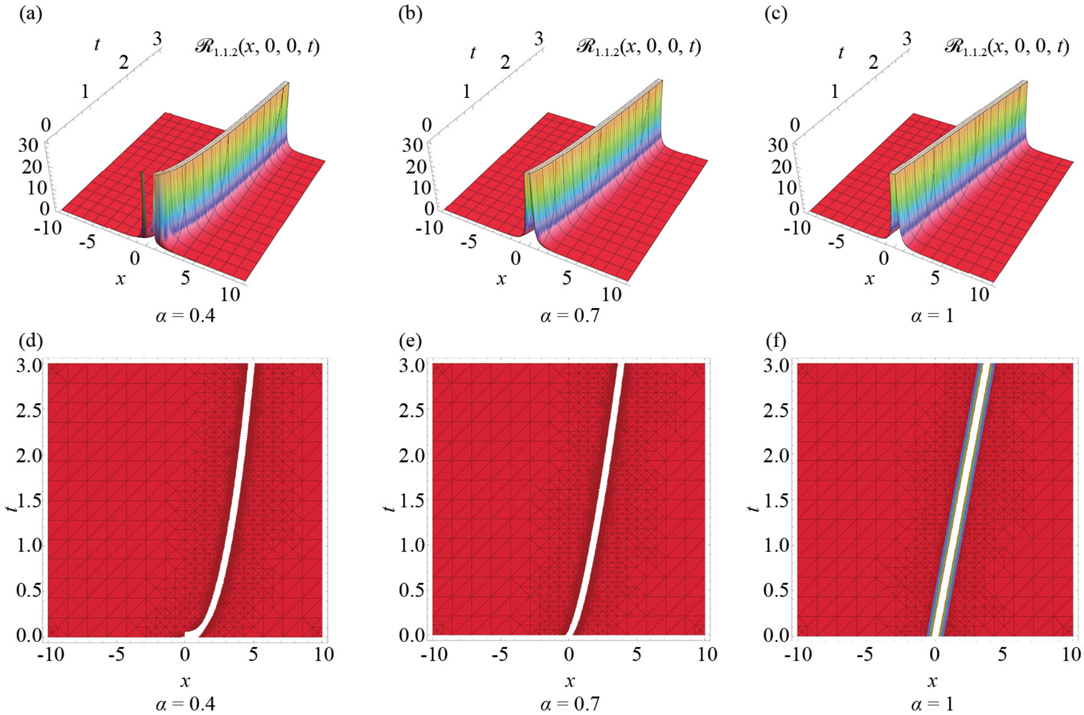


Figure 3. 3D and contour graphs for singular soliton solution of Eq. (14)

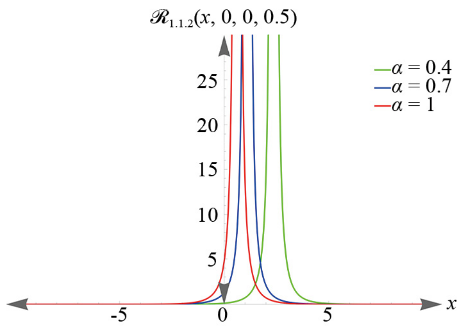
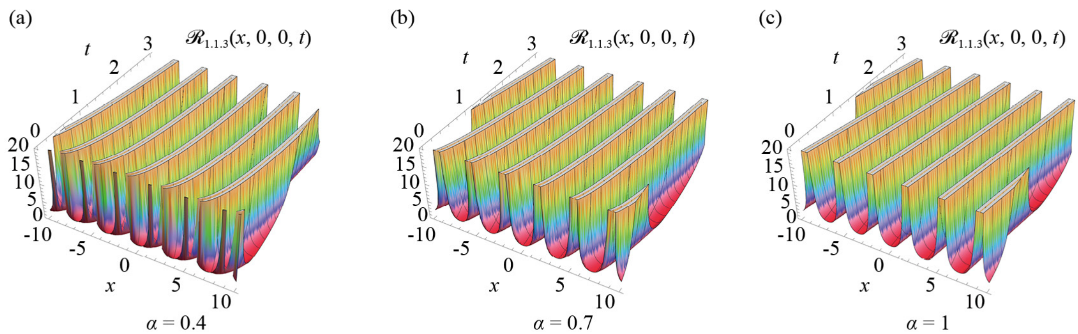


Figure 4. 2D graph of Figure 3



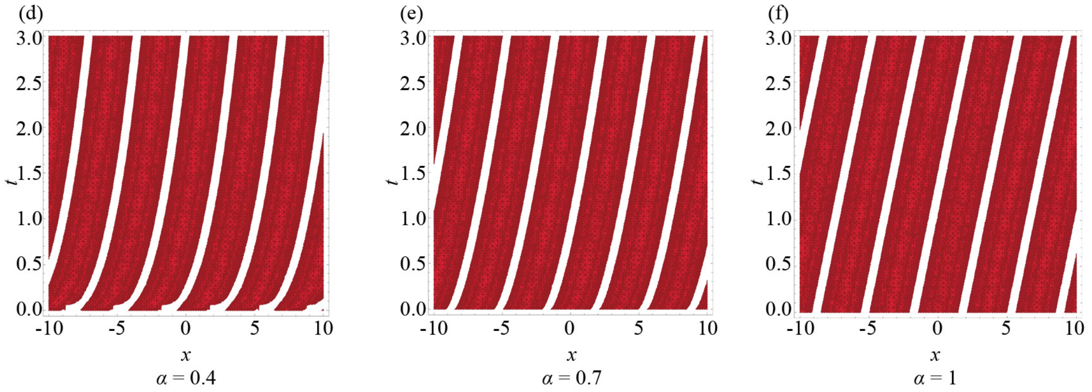


Figure 5. 3D and contour visual plots for the solution in Eq. (15)

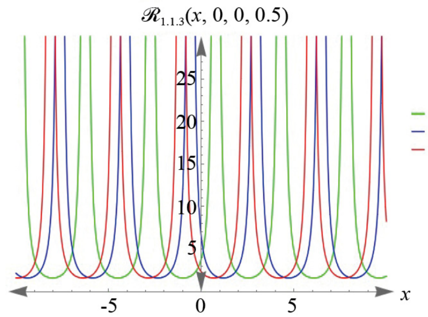


Figure 6. 2D graph of Figure 5

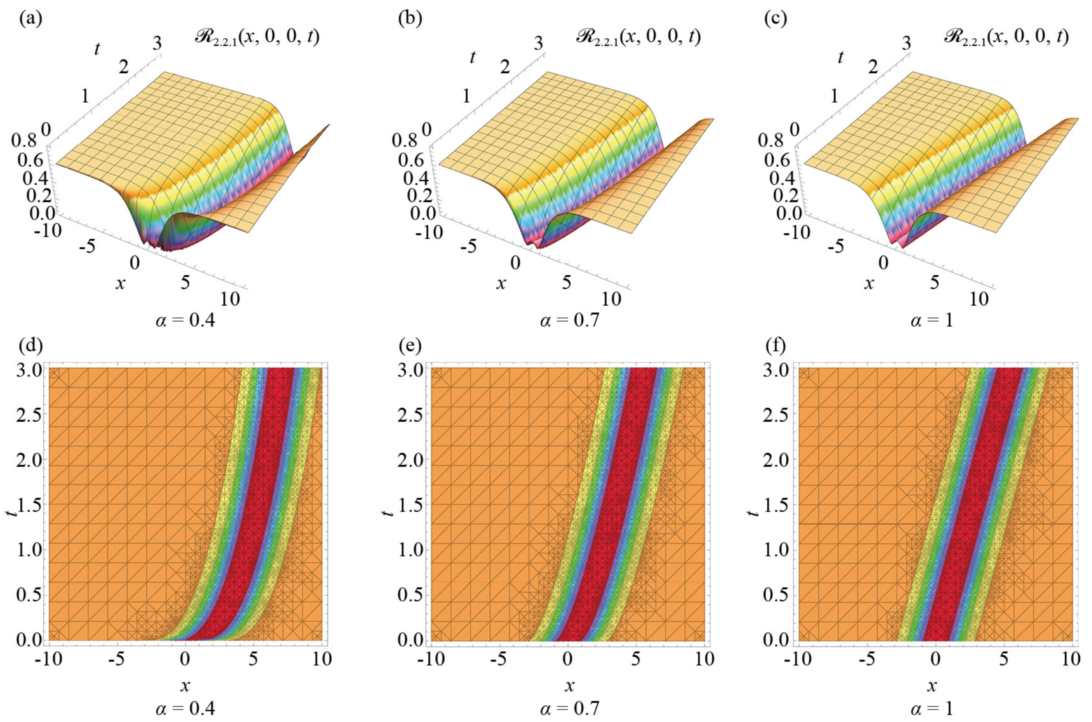


Figure 7. 3D and contour visual plots for the solution in Eq. (23)

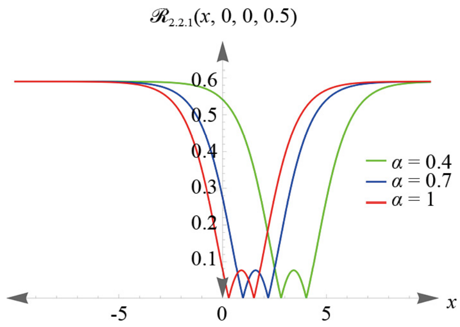


Figure 8. 2D graph of Figure 7

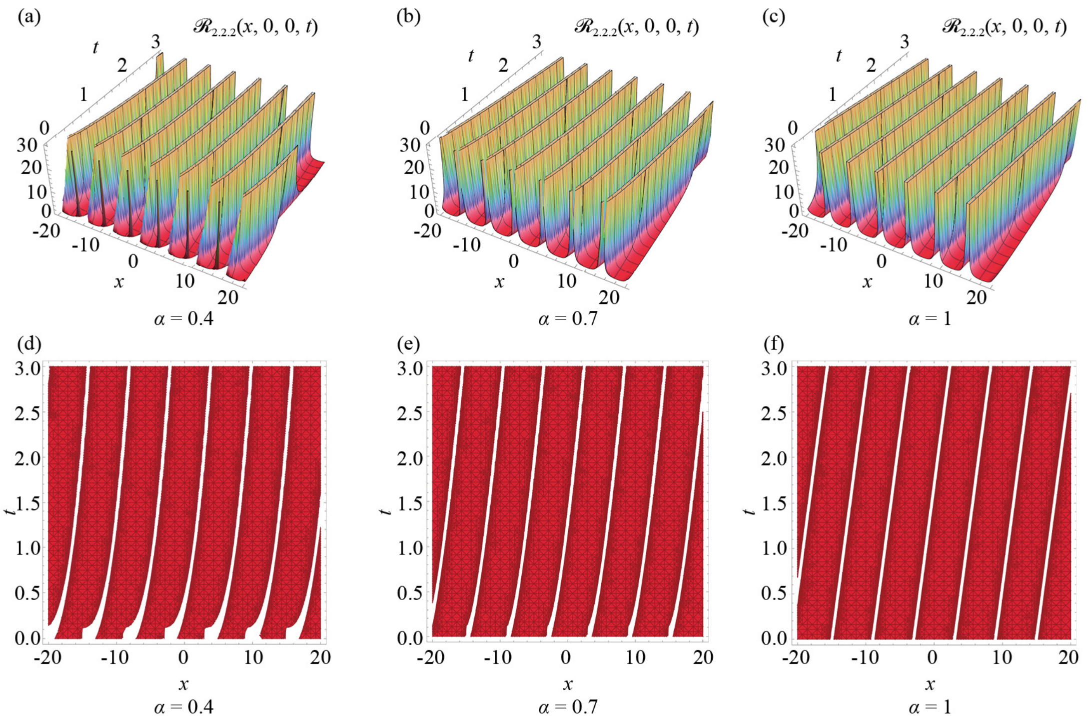


Figure 9. 3D and contour visual plots for the solution in Eq. (24)

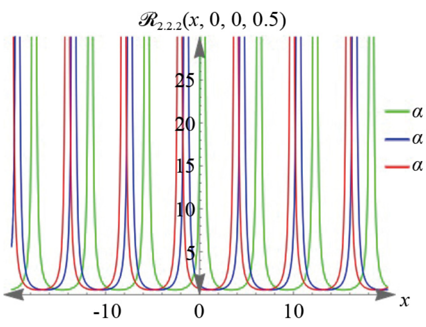


Figure 10. 2D graph of Figure 9

5. Comparison with literature

For the confirmation of the accuracy and relevance of the present results, a detailed comparison with the previously reported methods has been carried out. The overview of significant similarities and dissimilarities of our method with those in the literature is given in Table 1. This comparative review not only situates the current work in the broader research landscape but also points out the advances realized in the fields of methodology, accuracy, and scope of application.

Table 1. Literature comparison of methods used to solve the $(3+1)$-dimensional KP-SK-R model

Method	Type of solutions obtained	Main contributions
Modified extended tanh-function & $\exp(-\Phi(\xi))$-expansion method.	Bright, dark, kink, breather, and hybrid wave solutions.	Finite-series expansions provide solitary, periodic, and hybrid wave structures; effective for conformable/fractional forms of the model.
Kudryashov/Kudryashov *R*-function method.	Exact soliton families including bell, kink, singular types.	Algebraic ansatz that yields closed-form solitary and singular solutions; useful for higher-order or fractional KP-SK-R variants.
Jacobi elliptic function expansion, polynomial trial, and complete discrimination system.	Periodic, Jacobi elliptic, trigonometric, and hyperbolic solutions.	Generates elliptic-function families and recovers hyperbolic/trigonometric limits; useful to obtain periodic and solitary wave limits.
Hirota bilinear (direct) method.	Multiple-soliton solutions (multi-soliton).	Used to derive N-soliton solutions and check integrability via bilinear form; suitable for constructing explicit multi-soliton interactions.

6. Conclusion

In this study, various unique optical solitons and novel additional wave solutions for the established novel model in (4) were identified using the ME direct algebraic method. Several different kinds of solutions were acquired, such as bright solitons, dark solitons, and singular solitons, periodic wave solutions, singular periodic solutions, exponential solutions, hyperbolic solutions, and Weierstrass elliptic doubly periodic solutions. Graphs of various forms have been created for the obtained answers, and several physical attributes have been raised. The effect of the conformable fractional derivative was discussed graphically. The solutions presented in this paper are novel, as they were compared to similar investigations in the past.

7. Future directions

Future research on the (3+1)-dimensional KP-SK-R equation also has several promising directions. One is the application of integrable deformation techniques and nonlocal symmetries to determine more generalized classes of solutions, such as rogue waves and hybrid soliton-breather solutions. The combination of symbolic computations and machine learning could automate and accelerate the determination of novel exact solutions. In addition, numerical computations may validate and illustrate the stability and interactions of such solutions in the presence of physical perturbations. The extension of the analysis to non-integrable perturbations, fractional-order extensions, and multi-component versions can further increase the physical relevance and mathematical richness of the KP-SK-Ramani model to real applications.

Conflict of interest

The authors declare no competing financial interest.

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