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Research Article

Structural Identifiability of Systems with Multiple Nonlinearities

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Abstract: The structural identifiability (SI) problem considers for dynamical systems with multiple nonlinearities under uncertainty. It shows the widely used paradigm based on a priori parametric identifiability is not applicable in this case. The geometric framework (GF) is derivate from the system phase portrait and reflects the system nonlinear part properties under uncertainty. GF gives a conception of the system nonlinear part. The SI analysis problem interprets as a solution to the structural identification problem. The concept of S-synchronizability, which is the basis for estimation structural identifiability, introduce. Conditions of identifiability and structural identifiability are obtained. The constant excitation impact of input is studied on structural identifiability of the system. It shows that the input, which is constantly excited, can give the insignificant GF. Conditions are obtained for the existence of insignificant frameworks. Approaches are proposed to the estimation of structural identifiability systems with two nonlinearities and difficulties are noted. It is shown that a priori information is critical about the relation of variables. The approach is proposed to SI estimation based on the analysis of the influence graph.

***Keywords*:** structural identifiability, structure, geometric framework, nonlinearity, excitation constancy, importance graph, S-synchronizability, degree of non-identifiability

1. Introduction

The problem of identifying dynamic systems is one of the most relevant areas of study. Foundational results on the parametric identification of systems are obtained. At the same time, research is continuing to evaluate the parametric identifiability of dynamic systems. Identifiability is a condition for obtaining adequate models. Many studies have devoted to solving this problem [1-7]. Various approaches and methods are used to estimate identifiability: Taylor expansion [7]; series generation method [8]; similarity transformation [9] and differential algebra. The possibility of further application of parametric identification procedures is the result of applying these methods and approaches. The requirement of identifiability is reduced to estimating the non-degeneracy of the information matrix formed from the input and output data of the system. In identification theory, this requirement is equivalent to the condition of constant excitation. Call this direction parametric identification (IP).

This parametric paradigm is the main direction of research in identification theory and transformed into nonlinear systems. Many authors study parametric identification of nonlinear systems [1-2, 5-7, 10-11]. In [1], the approach based on the analysis of the system's output sensitivity is used to study identifiability. The effectiveness of this approach is shown for estimating the identifiability of the system parameter combination. In [11], local IP conditions are obtained for

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various types of experimental data. A critical analysis of the approaches used to estimate the identifiability of biological models is given in [7]. Methods for identifiability estimating nonlinear systems are based on the approaches described in the first paragraph of this section. The requirements for the data analysis used in the IP task solution is considered in [5]. The study of various types of identifiability (global, local, structural, and practical) is described in many works [1, 5]. Most studies on IP are associated with a priori identifiability.

Note that the identifiability problem of nonlinear systems has its peculiarities and relevance. To show this, consider the second-order system with hysteresis

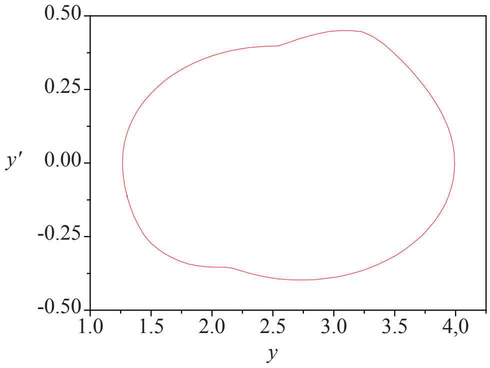
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -3 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}\varphi(y)+\begin{bmatrix}0\\ 1\end{bmatrix}u,$$

$$y=x_1, \tag{1}$$

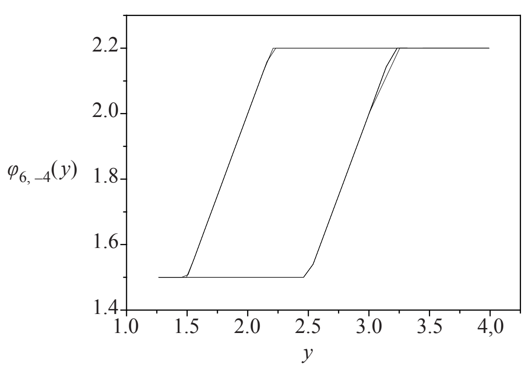
where $y\in R, u\in R$ is an output and input

$$\varphi(y)=\begin{cases}2.2, & \text{if } (y-d>2.2)\,\&\,(y'>0),\\ y-d, & \text{if } (y-d\le 2.2)\,\&\,(y'>0),\\ 1.5, & \text{if } (y-d\le 1.5)\,\&\,(y'>0),\\ 2.2, & \text{if } (y>2.2)\,\&\,(y'<0),\\ y, & \text{if } (y\le 2.2)\,\&\,(y'<0),\\ 1.5, & \text{if } (y\le 1.5)\,\&\,(y'<0),\end{cases}\qquad d=1$$

Show the impact of the input $u(t)$ on the nonlinear properties of the system (1). This influence reflects on the system of identifiability. Let $u_{6,-4}(t) = 6 - 4\sin(0.1\pi t)$. The phase portrait of the system and the hysteresis are shown in Figure 1. We see that the system (1) is identifiable.



(a) phase portrait

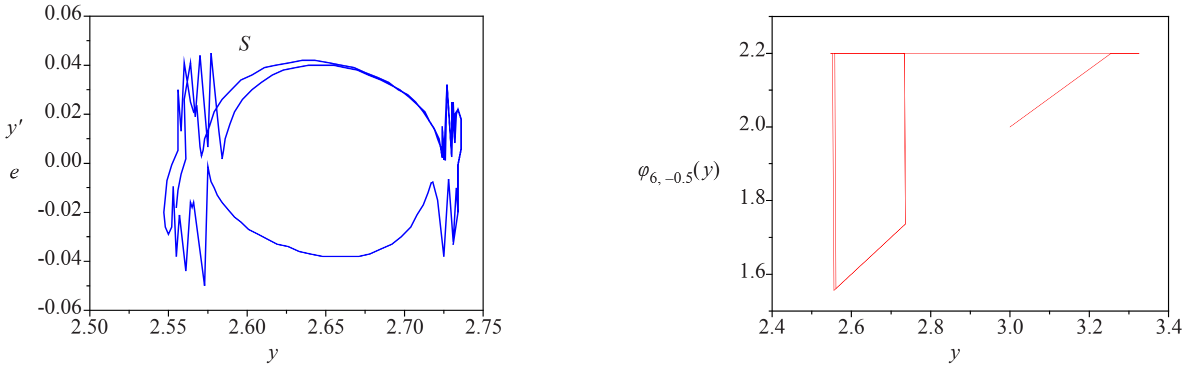


(b) nonlinearity

Figure 1. Results of structure estimation for $u_{6,-4}(t)$

Figure 2 presents the system properties for $u_{6,-0.5}(t) = 6 - 0.5\sin(0.1\pi t)$. Such input does not guarantee the identifiability of the system based on experimental data. We cannot conclude about the properties of the system from the phase portrait. Nonlinearity has a form that complicates decision-making. Presented results show that the identifiability problem of nonlinear systems is relevant. The input plays a significant role in solving the identifiability problem.

This problem is complicated when multiplicity nonlinearities are in the system. As a rule, parametric identification methods are approximation and level the influence of nonlinearity. The example shows that the structural identifiability estimation is possible only if the system structure is known. The structural identifiability problem is not solved by parametric methods under uncertainty.



(a) phase portrait

(b) nonlinearity

Figure 2. Results of structure estimation for $u_{6,\,-0.5}(t)$

Works on IP problems do not consider the estimation problem of the system structure. Therefore, the concept of structural identifiability does not reflect the essence of the identifiability problem. But this terminology is used in problems of assessing identifiability. Therefore, in this section, we adhere to this terminology to continue analyzing the results obtained.

The concept of identifiability and smooth identifiability are introduced for nonlinear systems in [12]. In [13], there is a relationship studied between the identifiability and observability of nonlinear biological systems. The structural identifiability of time series is described by nonlinear regression and autoregressive equations studied in [14]. In [15], the observability controllability and reachability joint estimate for nonlinear processes were obtained. The identification problem of parameters is considered for systems with several nonlinearities in [16-18]. The structure of nonlinearity specify a priori is not studied.

So, the system identifiability is understood as the possibility to estimate its parameters. Methods are based on the information matrix nondegeneracy estimation. Similar results are obtained in the parametric estimation theory. They are based on checking the constant excitation (CE) condition of the input and output of the system. As a rule, the model structure specifies a priori, and the essence of the structural identifiability is not always clear. The identifiability of nonlinear system is reduced to the IP problem in the following sense: how the nonlinear system structure (form, dependence) to estimate under uncertainty. The SI problem is not studied in this form. But these are the structural aspects of the system identifiability. The question: input provides the structural identifiability of the system as structural identification, is not be considered. This formulation is proposed in [19]. In [20], results presented for systems with a single nonlinearity gives the solution to this problem.

In this paper, we consider the structural identifiability problem of the dynamical system with nonlinearities under uncertainty. It is very complex problem since the methods for formalizing the system structure have not been developed. The concept of SI (h-identifiability) is introduced in [19]. The proposed approach solves the problem of estimating the nonlinear system structure. It is based on the analysis of geometric frameworks that the state of the nonlinear system reflected. Below we give the summary and generalization of the results obtained in [19-20].

2. Problem statement

Consider the system

$$\dot{X} = AX + B_\varphi \Phi(Y) + B_u U,$$

$$Y = C^T X, \tag{2}$$

where $X \in R^n$, $U \in R^p$, $Y \in R^k$ are state vector, input and output; $A \in R^{n\times n}$, $B_u \in R^{n\times p}$, $B_\varphi \in R^{n\times q}$, $C \in R^{k\times n}$ are matrices of corresponding dimensions; $\Phi(Y) \in R^q$ is nonlinear vector function. A is Hurwitz matrix.

The nonlinear function $\varphi_i(\xi) \in \Phi$ is smooth and satisfies the condition

$$\chi \in \mathcal{F}_\varphi = \left\{ \gamma_{1,i}\xi^2 \le \varphi_i(\xi)\xi \le \gamma_{2,i}\xi^2,\ \xi \ne 0,\ \varphi_i(0) = 0,\ \gamma_{1,i} \ge 0, \gamma_{2,i} < \infty \right\}, \tag{3}$$

where $\xi \in R$ is the input of a nonlinear element. ξ is a linear combination of state variables. For the system (2), the information set is known

$$\mathrm{I}_o = \left\{ U(t),\ Y(t),\ t \in J = [t_0,\ t_k] \right\}. \tag{4}$$

Problem: analyze the set I_o and estimate the structural identifiability of the system (2).

Apply the approach to structural identification proposed in [17]. It is based on the transition into a structural space and the construction of S_{ey} framework. S_{ey} reflects the properties of the nonlinear part (2). The analysis S_{ey} is related to solving the SI problem of the system. To distinguish the approach described from IP, we use the h-identifiability term (HI).

Let $q = 1$, $C = [1, 0, \ldots, 0]^T$, $B_\varphi = B_u = I = [0, 0, \ldots, 0, 1]^T$, $u \in R$, $Y = y$, $y \in R$, $\Phi(Y) = \varphi(Y)$. Denote the system (2), (3) with the specified parameters as $S_{y\varphi}$.

Next statements are given for the system $S_{y\varphi}$. $S_{y\varphi}$ is the particular case of the system (2), (3). We will denote $S_{y\varphi}$ as (2) further.

3. Method of constructing S_{ey}-framework

The construction of the S_{ey}-framework requires the formation of a set $\mathrm{I}_{N,\,g}$, containing information about the function $\varphi(y)$. S_{ey} is described by a function $f_{ey} : y \rightarrow e$, where $e \in R$ is a variable that reflects the change in the nonlinearity $\varphi(y)$ under uncertainty. Describe the method of obtaining $\mathrm{I}_{N,\,g}$ [18]. Apply the differentiation operation to $y(t)$ and denote the obtained variable as x_1. Obtain the information set $\mathrm{I}_{ent} = \{\mathrm{I}_o, x_1\}$.

Remark 1 If the variables measured with an error apply filtering or smoothing procedures.

Select the subset $\mathrm{I}_g \subset \mathrm{I}_{ent}$ corresponding to the partial solution of system (2) (steady state). Apply the mathematical model

$$\hat{x}_1^l(t) = H^T \left[1\ u(t)\ y(t)\right]^T, \tag{5}$$

to the selection of the linear component in x_1, where $H \in R^3$ is parameters vector. The variable x_1 is defined on the interval $J_g = J \setminus J_{tr}$.

We determine the vector H as the solution to the problem

$$\min_H Q(e)\Big|_{e=\hat{x}_1^l - x_1} \rightarrow H_{opt},\ Q(e) = 0.5e^2.$$

Find the forecast for the variable x_1 by applying the model (5) $\forall t \in \mathrm{I}_g$ and form the error $e(t) = \hat{x}_1^l(t) - x_1(t)$. $e(t)$ is the function of nonlinearity $\varphi(y)$. We have $\mathrm{I}_{N,g} = \{y(t), e(t)\ t \in J_g\}$. Apply the notation $y(t)$ supposing $y(t) \in \mathrm{I}_{N,g}$.

Remark 2 The choice of model (5) structure is one of the stages for structural identification of the system (2). The type model is defined by the input and system information.

The phase portrait $\mathcal{S}$ is described by the function $\Gamma_{ey} : \{y\} \to \{e\}$, $\forall t \in J_g$ does not always guarantee the decision-maker on system nonlinear properties under uncertainty. Go to the structure space $\mathcal{P}_{ye} = (y, e)$. Consider the function $\Gamma_{ey} : \{y\} \to \{e\}$, $\forall t \in J_g$ which on the plane (y, e) describes the change in the framework $\mathcal{S}_{ey}$. $\mathrm{I}_{N,g}$ contains the information about $\varphi(y)$. Therefore, $\mathcal{S}_{ey}$ describes the change in the nonlinear function in the generalized form. The identification of the form $\varphi(y)$ is based on the use of the input satisfied certain conditions. The input must have the property of constant excitation (see below). Such input gives the closed framework $\mathcal{S}_{ey}$.

Apply the model (5) and represent the system S as

$$S_y : \begin{cases} \dot{\tilde{X}} = A\tilde{X} + I\zeta, \\ \tilde{y} = C^T \tilde{X}, \end{cases}$$

$$S_\varphi : e = f\left(y, x_1\right), \tag{6}$$

where $\tilde{X} \in R^n$ is the variable describing the general solution of the system (2); $\zeta \in R$ is a bounded perturbation appearing as the analysis result of the variable e.

Consider the identifiability problem of system S_y, S_φ.

4. Structural identifiability of nonlinear system $S_{y\varphi}$

Consider the system S_φ and properties of the set $\mathrm{I}_{N,g}$, which allow us to solve the structural identification problem, and, consequently, the identifiability estimation.

Let conditions hold.

B1. The input is constantly excited at the interval J.

B2. The analysis of $\mathcal{S}_{ey}$ gives the solution to the estimation problem of the nonlinear properties of the system $S_{y\varphi}$.

We will state the basic concepts following [20].

Definition 1 If $u(t)$ satisfies B1 and B2 conditions, then the input $u(t)$ is representative.

Let the framework $\mathcal{S}_{ey}$ be closed, and the area $\mathcal{S}_{ey}$ is not zero. Denote height $\mathcal{S}_{ey}$ as $h(\mathcal{S}_{ey})$ where height is the distance between two points on the opposite sides of the framework $\mathcal{S}_{ey}$.

Theorem 1 [20]. Let (i) the linear part of the system $S_{y\varphi}$ is stable; (ii) the nonlinearity $\varphi(\cdot)$ satisfies the condition (3); (iii) the input is bounded, and constantly excited; (iv) $h(\mathcal{S}_{ey}) \geq \delta_S$, where $\delta_S > 0$. Then the framework $\mathcal{S}_{ey}$ is identified on the set $\mathrm{I}_{N,g}$.

Theorem 1 shows conditions in the framework $\mathcal{S}_{ey}$. The framework $\mathcal{S}_{ey}$ must be closed, hence, it must have the height or distance between opposite points of the framework. It ensures that $\mathcal{S}_{ey}$ has a diameter (see below).

Definition 2 The framework $\mathcal{S}_{ey}$ is called h-identifiable if theorem 1 holds for $\mathcal{S}_{ey}$.

Let $\mathcal{S}_{ey}$ be h-identifiable. h-identifiability features are considered in [19-20].

But a "bad" input existing that is constantly excited. This input gives a so-called "insignificant" $\mathcal{S}_{ey}$-framework ($\mathcal{NS}_{ey}$-framework). However, the $\mathcal{NS}_{ey}$-structure can be h-identifiable. The system (2) identification with the $\mathcal{NS}_{ey}$-framework gives results which are not typical for the system.

Conditions of an existence $\mathcal{NS}_{ey}$-structure. Consider a class of nonlinear functions to which the homothety operation is applicable [22].

Let $\mathcal{S}_{ey} = \mathcal{F}^l_{\mathcal{S}_{ey}} \cup \mathcal{F}^r_{\mathcal{S}_{ey}}$, where $\mathcal{F}^l_{\mathcal{S}_{ey}}$, $\mathcal{F}^r_{\mathcal{S}_{ey}}$ are left and right fragments $\mathcal{S}_{ey}$. Determine for $\mathcal{F}^l_{\mathcal{S}_{ey}}$, $\mathcal{F}^r_{\mathcal{S}_{ey}}$ secant

$$\gamma_S^r = a^r y,\ \gamma_S^l = a^l y, \tag{7}$$

where a^l, a^r are numbers determined using the least-squares method (LSM).

Theorem 2 [16]. Let (i) the framework $\mathcal{S}_{ey}$ is h-identifiable and has the form $\mathcal{S}_{ey} = F_{\mathcal{S}_{ey}}^l \cup F_{\mathcal{S}_{ey}}^r$, where $\mathcal{F}_{\mathcal{S}_{ey}}^l$, $\mathcal{F}_{\mathcal{S}_{ey}}^r$ is the left and right fragment of $\mathcal{S}_{ey}$; (ii) secants for $\mathcal{F}_{\mathcal{S}_{ey}}^l$, $\mathcal{F}_{\mathcal{S}_{ey}}^r$ are described by equations (7). Then $\mathcal{S}_{ey}$ is $\mathcal{NS}_{ey}$-framework if

$$\left\| a^l \right| - \left| a^r \right\| > \delta_h,\ \delta_h > 0. \tag{8}$$

Remark 3 $\mathcal{NS}_{ey}$-frameworks are often the result of inadequate application of input action.

Introduce designations: $\mathcal{D}_y = \mathrm{dom}(\mathcal{S}_{ey})$ is definition range of the framework $\mathcal{S}_{ey}$, $D_y = D_y(\mathcal{D}_y) = \max_t y(t) - \min_t y(t)$ is diameter $\mathcal{D}_y$. Let $u(t) \in \mathrm{U}$ where U is an acceptable set of inputs for the system (2). The set U contains representative inputs.

Definition 3 If $\mathcal{D}_y$ of the structure $\mathcal{S}_{ey}$ has the maximum diameter D_y, the input S-synchronizes the system (2).

Definition 4 The input $u(t) \in \mathrm{U_S} \subseteq \mathrm{U}$ is the S-synchronizer system $\mathcal{S}_{y\varphi}$ if the definition range $\mathcal{D}_y$ of the framework $\mathcal{S}_{ey}$ has the maximum diameter D_y.

Consider a reference framework $\mathcal{S}_{ey}^{ref}$. $\mathcal{S}_{ey}^{ref}$ is the framework $\mathcal{S}_{ey}$ reflecting all properties of the function $\varphi(y)$. Designate by the diameter $D_y(\mathcal{S}_{ey}^{ref})$ as D_y^{ref}. D_y^{ref} exists if the input the system (2) is S-synchronizing.

Definitions 2, 3 show if $\mathcal{S}_{ey} \cong \mathcal{S}_{ey}^{ref}$, then $\left| D_y - D_y^{ref} \right| \le \varepsilon_y$ where $\varepsilon_y \ge 0$, $\cong$ is the proximity sign. Elements of the subset $\mathrm{U_S}$ have property

$$\left| D_y\left(\mathcal{S}_{ey}\left(u(t)\big|_{u \in \mathrm{U_S}} \right) \right) - D_y^{ref} \right| \le \varepsilon_y. \tag{9}$$

Synchronization $u(t) \in \mathrm{U}$ is the choice of the input $u_h(t) \in \mathrm{U}$ such that reflects all features $\varphi(y)$ in $\mathcal{S}_{ey}$. It is true if $u(t)$ ensures $\max_{u_h} D_y$ and $\mathcal{S}_{ey} \neq \mathcal{NS}_{ey}$. We interpret the choice $u_h(t) \in \mathrm{U}$ as ensuring synchronization between structures of the model and the system. $d_{h,y} = \max_{u_h} D_y$ is the condition of h-identifiability which can represent as

$$\left| D_y\left(\mathcal{S}_{ey}\left(u(t)\big|_{u \in \mathrm{U_S}} \right) \right) - d_{h,y} \right| \le \varepsilon_y. \tag{10}$$

The condition for $\mathcal{NS}_{ey}$

$$\left| D_y\left(\mathcal{S}_{ey}\left(u(t)\big|_{u \in \mathrm{U} \backslash \mathrm{U_S}} \right) \right) - d_{h,y} \right| > \varepsilon_y. \tag{11}$$

(10) can be interpreted as proximity domain

$$Q_D = \left| \mathcal{S}_{ey}\left(u(t)\big|_{u \in \mathrm{U_S}} \right) - \mathcal{S}_{ey}^{ref} \right|, \tag{12}$$

which is understood as $\left| \dot{y}(t) - \dot{y}^{ref}(t) \right| \le \varepsilon_y$ for almost $\forall t \ge \tilde{t}$.

We will write $\delta Q_D \le \varepsilon_y$ if frameworks under consideration are close. If the condition $\delta Q_D \le \varepsilon_y$ is true for Q_D for almost $\forall t \geq t^*$ then the domain Q_D will be called the S-synchronizability area on the set of inputs $\{u_h(t)\}$ or the structural identifiability domain on the set $\{S_h(u_h(t))\}$, where S_h is the phase portrait of the system $S_{y\varphi}$.

So, two criteria (8) and (11) are presented for the existence of the insignificant framework. Structure of systems S_φ and $S_{y\varphi}$ are structurally unidentifiable in this case.

Let the input $u_h(t)$ synchronize the system $S_{y\varphi}$. If $u(t)$ is S-synchronizing, then we will write $u_h(t) \in$ S. Note that a finite set $\{u_h(t)\} \in$ S exists for the system $S_{y\varphi}$. The choice of optimal $u_h(t)$ depends on $d_{h,y}$ and (10). The hold of the condition (10) is one of the prerequisites for SI of the system $S_{y\varphi}$.

Definition 5 If framework S_{ey} is h-identified and conditions $\left\| a^r \right| - \left| a^r \right\| \leq \delta_h$, (9) are satisfied, then the framework S_{ey} or the system (2) (system $S_{y\varphi}$) is structurally identified or h_{δ_h}-identifiable.

Definition 5 shows if the system (2) is h_{δ_h}-identified then the framework S_{ey} has the maximum diameter of area $\mathcal{D}_y$.

Definition 6 The model (5) is *SM*-identifying if the framework S_{ey} is h_{δ_h}-identifiable.

The framework S_{ey} is defined on $u_h(t) \in$ S and $u_h(t)1$ satisfies condition B1. Therefore, S_{ey} corresponds to the nonlinearity $\varphi(y)$ defined on the class

$$\varphi(y) \in \mathcal{F}_\varphi = \left\{ \varphi(y) \in R \,\middle|\, \varphi(y, \mathrm{A}), \mathrm{A} \in R^{n_\mathrm{A}}, \alpha_i \in \mathrm{A}, \alpha_i \in \left[\bar{\alpha}_i, \bar{\bar{\alpha}}_i \right] \right\},$$

where $\bar{\alpha}_i, \bar{\bar{\alpha}}_i$ are some numbers.

Note that the term *SM*-identifying does not coincide with the concept proposed in [24].

Theorem 3 [19]. Let (i) the input $u(t) \in$ S is constantly excited; (ii) the system $S_{y\varphi}$ phase portrait have m features; (iii) S_{ey}-framework is h_{δ_h}-identified and contains fragments corresponding to features of the system $S_{y\varphi}$. Then the model (5) is *SM*-identifying.

The theorem 3 shows if the model (5) is not *SM*-identifying, then model (5) structure or the informational set (4) need to change.

Let c_S is the center of the framework S_{ey} on the set $J_y = \{y(t)\}$, c_{D_y} is the center of the area $\mathcal{D}_y$.

Theorem 4 [20]. Let the set $\mathrm{U_S}$ given for the system $S_{y\varphi}$ and (i) exists $\varepsilon \geq 0$ such that $\left| c_S - c_{D_y} \right| \leq \varepsilon$; (ii) $\left\| a^l \right| - \left| a^r \right\| \leq \delta_h$, where a^l, a^r are coefficients of secants (8) for $(\mathcal{F}^l_{S_{ey}}, \mathcal{F}^r_{S_{ey}}) \subset S_{ey}$. Then the system $S_{y\varphi}$ is h_{δ_h}-identifiable and the input $u_h(t) \in$ S, and the structure S_{ey} defines the class $\mathcal{F}_\varphi$.

Since $\varphi(y) \in \mathcal{F}_\varphi$ the center c_{D_y} of area $\mathcal{D}_y$ $c_{D_y} \in J_{c_{D_y}}$, where $J_{c_{D_y}}$ is some interval.

Let some subset $\{u_{h,i}(t)\} \subset \mathrm{U_S} \subseteq \mathrm{U}$ $(i \geq 1)$ whose elements have the property of S-synchronizability exists. The framework $S_{ey,i}(u_{h,i})$ with the diameter $D_{y,i}$ of area $\mathcal{D}_{y,i}$ corresponds to every $u_{h,i}(t)$. As $u_{h,i}(t) \in$ S the diameter $D_{y,i}$ has the property $d_{h,\Sigma}$-optimality. Let the hypothetical framework S_{ey} (the framework S^{ref}_{ey}) of the system $S_{y\varphi}$ have diameter $d_{h,\Sigma}$.

Definition 7 The framework $S_{ey,i}$ has $d_{h,\Sigma}$-optimality property on the set U_h if $\varepsilon_\Sigma > 0$ such that $\left| d_{h,\Sigma} - D_{y,i} \right| \leq \varepsilon_\Sigma$ $\forall i = \overline{1, \# \mathrm{U}_h}$.

Definition 8 If $(\{u_{h,i}(t)\} = \mathrm{U}_h \subset \mathrm{U})$ & $(u_{h,i}(t) \in$ S$)$, $i \geq 1$ and frameworks $S_{ey,i}(u_{h,i})$ have $d_{h,\Sigma}$-optimality property, then frameworks $S_{ey,i}(u_{h,i})$ are structurally indiscernible on sets $\{u_{h,i}(t)\}$ $J_y(u(t) = u_{h,i}(t))$.

So, the h_{δ_h}-identifiability estimate can be obtained from any input, following definitions 6, 7.

Definition 9 If frameworks $S_{ey,i}(u_{h,i})$ have $d_{h,\Sigma}$-optimality property, then $S_{ey,i}(u_{h,i})$ is locally structurally identifiable on the set U_h.

Denote the framework $S_{ey,i}(u_{h,i})$ had $d_{h,\Sigma}$-optimality property as $S^{\Sigma}_{ey,i}$, and the locally structurally identified framework $S_{ey,i}(u_{h,i})$ as $S^{LSI}_{ey,i}$.

The framework $\mathcal{S}_{ey}$ is locally structurally identifiable on the set $\mathrm{U}_h \subseteq \mathrm{U}_\mathrm{S}$ if

$$(\exists u_h \in \mathrm{S}), \text{что} \left(\mathcal{S}_{ey} \cong \mathcal{S}_{ey}^{\Sigma}\right) \rightarrow \mathcal{S}_{ey} \cong \mathcal{S}_{ey}^{LSI} \tag{13}$$

Remark 4 We consider nonlinearities satisfying condition (3). Therefore, notes made above are valid.

Definition 10 The framework $\mathcal{S}_{ey,\, i}(u_i) \notin \mathrm{U}_\mathrm{S}$ that does not have the $d_{h,\, \Sigma}$-optimality property is locally structurally unidentifiable on the set U_h.

The framework $\mathcal{S}_{ey,\, i}(u_i \notin \mathrm{U}_\mathrm{S})$ that is structurally unidentifiable on the set U_h defines the class $\mathcal{F}_\varphi^N \not\subset \mathcal{F}_\varphi$.

Remark 5 The described approach applies to the nonlinear system with the dynamic law of nonlinearity change. In this case, the multilevel analysis gives the solution to the identifiability problem.

The identifiability of system S is considered in [21].

5. On excitation constancy effect on identifiability of system

In [23], the excitation constancy influence is studied on the identifiability estimation of the system with hysteresis. It shows that not every input with the CE property guarantees the structural identifiability of the system. Below we present results that allow estimating the CE impact.

Consider the input $u \in \mathcal{PE}_\alpha$, where $\mathcal{PE}_\alpha$ is the constant excitation property

$$\mathcal{PE}_\alpha : u^2(t) \geq \alpha$$

holds for $\exists \alpha > 0$ and $\forall t \geq t_0$ on some interval $T > 0$.

Let input $u(t)$ of the system $S_{y\varphi}$ have the property $u(t) \in \mathcal{PEF}_{\alpha,\, \omega_k}$, where

$$u_k(t) : \left(u_k \in \mathcal{PE}_\alpha\right) \& \left(u_k \in \mathcal{PF}_{\omega_k}\right) \& \left(\overline{u_k \in \mathrm{S}}\right),\ \mathcal{PF}_{\omega_k} : u_k(t) = \mathcal{RF}_k\left(\Omega_k\right), \tag{14}$$

$\mathcal{RF}(\Omega_k)$ is the model for $u_k(t)$ based on the Fourier series and given on the set of frequencies $\Omega_k = \{\omega_1, \omega_2, \ldots, \omega_k\}$.

Let $u_k \in \mathrm{U}_k$, $\mathrm{U}_k = \mathrm{U} \backslash \mathrm{U}_\mathrm{S}$. Consequently $u_k \notin \mathrm{S}$. For $u_h \in \mathrm{S}$ holds

$$u_h(t) : \left(u_h \in \mathcal{PE}_\alpha\right) \& \left(u_h \in \mathcal{PF}_{\omega_h}\right) \& \left(u_h \in \mathrm{S}\right),\ \mathcal{PF}_{\omega_h} : u_h(t) = \mathcal{RF}_h\left(\Omega_h\right), \tag{15}$$

where $\Omega_h \neq \Omega_k$.

Compare (14), (15) and obtain

$$\left(\mathcal{RF}_h\left(\Omega_h\right) \neq \mathcal{RF}_k\left(\Omega_k\right)\right) \Rightarrow \mathcal{S}_{ey}^h \neq \mathcal{S}_{ey}^k \Rightarrow \mathcal{S}_{ey}^k = \mathcal{NS}_{ey}. \tag{16}$$

From (16) have

$$\left(\mathcal{D}_y\left(\mathcal{S}_{ey}^h\right) \neq \mathcal{D}_y\left(\mathcal{S}_{ey}^k\right)\right) \Rightarrow \left[D_y\left(\mathcal{S}_{ey}^h\right) \geq D_y\left(\mathcal{S}_{ey}^k\right)\right]. \tag{17}$$

The definitional domain of frameworks $\mathcal{S}_{ey}^h$, $\mathcal{S}_{ey}^k$ does not coincide, and $\mathcal{S}_{ey}^h$ is $d_{h,\, \Sigma}$-optimal on the set U_h. Therefore, the fulfillment of the condition (11) follows from inequality (17). Consequently, the structure of the system $S_{y\varphi}$ nonlinear part with u_k has indicators that do not coincide with the structurally identifiable parameters (2) with u_h.

So, the CE condition of the input affects the h_{δ_h}-identifiability of the S_φ-system, and, consequently, the system $S_{y\varphi}$. The statement is true.

Theorem 5. Let (i) the input u_k satisfies condition (14); (ii) the S_{ey}^k-framework corresponds to the input u_k; (iii) there is the input $u_h \in \mathrm{S}$ such that the condition (15) is satisfied; (iv) conditions (16), (17) hold. Then (a) the S_φ-system is structurally unidentifiable by the input u_k; (b) structural parameters of the S_φ-system do not correspond to the parameters system $S_{y\varphi}$ with the identifiable structure S_{ey}^h.

Obtain the non-identifiability degree estimate of the S_φ-system. Let the phase portrait $\mathcal{S}$ constructed for the system $S_{y\varphi}$. Definitional domains of $\mathcal{S}$ and $\mathcal{S}_{ey}$ frameworks are coincident. Therefore, the diameter $D(\mathcal{S}_{ey})$ of the framework $\mathcal{S}_{ey}$ is known. Consider the set $\{u_i(t)\}$ having the property $\mathcal{PE}_\alpha$. Determine for each $u_i(t)$ the structure $\mathcal{S}_{ey,\, i}$ and obtain $D_{y,\, i}(\mathcal{S}_{ey,\, i})$. Suppose $d_{h,\, y} = \max_{u_i}|D_y(\mathcal{D}(\mathcal{S}_{ey,\, i}))|$ and denote the corresponding input as u_h. Determine diameters $d_{y,\, j} = |D_{y,\, j}(\mathcal{D}[\mathcal{S}_{ey,\, j}(u_j \in \mathcal{V})])|$ for all inputs $\mathcal{V} = \{u_i(t)\}\backslash\{u_h\}$. Since $u_h \in \mathrm{S}$, therefore $d_{h,\, y} > D_{y,\, j}$, $\forall j \geq 1$. Then evaluate the non-identifiability degree as

$$SI_j = SI\left(\mathcal{S}_{ey,j}\right) = \frac{d_{h,y} - d_{y,j}}{d_{h,y}} \tag{18}$$

(18) shows that system $S_{y\varphi}$ is structurally identifiable if $SI_j \to 0$. The structural identifiability area Q_D is defined by the condition (10).

Remark 6 If fragments $\mathcal{F}_S^l, \mathcal{F}_S^r$ select on the phase portrait $\mathcal{S}$, then the estimate for the non-identifiability is defined as

$$SI = SI\left(\mathcal{S}\right) = \frac{d_y^l\left(\mathcal{F}_S^l\right)}{d_y^r\left(\mathcal{F}_S^r\right)},$$

where $d_y^l(\mathcal{F}_S^l)$, $d_y^r(\mathcal{F}_S^r)$ are diameters of fragments $\mathcal{F}_S^l, \mathcal{F}_S^r$. The system S_φ is structurally identifiable if $SI(\mathcal{S}) \leq O(2)$ where $O(2)$ is neighborhood 1.

The input amplitude can influence the SI of nonlinear systems. Modify conditions (14), (15)

$$u_k(t):\left(u \in \mathcal{PE}_\alpha\right)\&\left(u \in \mathcal{PF}_{\omega_k}\right)\&\left(\overline{u_h \in \mathrm{S}}\right),\ \mathcal{PF}_{\omega_k}: u_k(t) = \mathcal{RF}_k\left(G_k, \Omega_k\right), \tag{19}$$

$$u_h(t):\left(u_h \in \mathcal{PE}_\alpha\right)\&\left(u_h \in \mathcal{PF}_{\omega_h}\right)\&\left(u_h \in \mathrm{S}\right),\ \mathcal{PF}_{\omega_h}: u_h(t) = \mathcal{RF}_h\left(G_h, \Omega_h\right), \tag{20}$$

where G_k, G_h are model $\mathcal{RF}_k$, $\mathcal{RF}_h$ parameter vectors.

Present models $\mathcal{RF}_k$, $\mathcal{RF}_h$ as

$$\mathcal{RF}_h\left(G_h, \Omega_h\right) = g_h \widetilde{\mathcal{RF}_h}\left(\tilde{G}_h, \Omega_h\right),\ \mathcal{RF}_k\left(G_k, \Omega_k\right) = g_k \widetilde{\mathcal{RF}_k}\left(\tilde{G}_k, \Omega_k\right),$$

where $\widetilde{\mathcal{RF}_h}\left(\tilde{G}_h, \Omega_h\right)$, $\widetilde{\mathcal{RF}_k}\left(\tilde{G}_k, \Omega_k\right)$ are modifications of models (14), (15); $g_h = \max_i g_{h,i}$, $i = \overline{1, \#\Omega_h}$, $g_{h,\, i}$ is an element G_h; $g_k = \max_i g_{k,i}$, $i = \overline{1, \#\Omega_k}$. g_p ($p = k, h$) denotes the generalized amplitude of the input.

Condition (16) is transformed into the form

$$g_h \widetilde{\mathcal{RF}_h}\left(\tilde{G}_h, \Omega_h\right) \neq g_k \widetilde{\mathcal{RF}_k}\left(\tilde{G}_k, \Omega_k\right).$$

Since $u_h \in$ S then $g_h \geq g_k$. This conclusion follows from

$$D_h\left(\mathcal{S}\left(u_h\right)\right) \geq D_k\left(\mathcal{S}\left(u_k\right)\right) \Rightarrow \left|\widetilde{\mathcal{RF}_h}\left(\tilde{G}_h, \Omega_h\right)\right| \geq \left|\widetilde{\mathcal{RF}_k}\left(\tilde{G}_k, \Omega_k\right)\right|,$$

and the model $\widetilde{\mathcal{RF}_h}\left(\tilde{G}_h, \Omega_h\right)$ approximates the input ensuring S-synchronization of the system $S_{y\varphi}$.

Obtain $d_{h,\Sigma}$-optimality of the diameter $D_h(\mathcal{S}_{ey}^h)$ from $\mathcal{S}(u_h) \Rightarrow \mathcal{S}_{ey}^h$. The framework $\mathcal{S}_{ey}^k$ does not have this property (see (20)). Therefore, the input $u_k \notin$ S, which has a smaller generalized amplitude, gives the diameter $D_k(\mathcal{S}_{ey}^k)$.

Theorem 6 Let (i) the input u_k of the system S_φ satisfy the condition (19); (ii) the framework $\mathcal{S}_{ey}^k$ corresponds to input u_k; (iii) there is an input $u_h \in$ S such that the condition (20) holds; (iv) conditions (16), (17) hold. Then (a) the S_φ-system is structurally non-identifiable by the input u_k; (b) structural parameters of the system S_φ do not correspond to the system $S_{y\varphi}$ with an identifiable framework $\mathcal{S}_{ey}^h$ if $g_h \geq g_k$.

Example 1 Consider the nonlinear system with Bouc-Wen hysteresis (system S_{BW}) [23]

$$m\ddot{x} + c\dot{x} + F(x, z, t) = f(t), \tag{21}$$

$$F(x, z, t) = \alpha k x(t) + (1 - \alpha) k d z(t), \tag{22}$$

$$\dot{z} = d^{-1}\left(a\dot{x} - \beta|\dot{x}||z|^n \, sign(z) - \gamma \dot{x}|z|^n\right), \tag{23}$$

where $m > 0$ is weight, $c > 0$ is damping, $F(x, z, t)$ is the restoring force, $d > 0$, $n > 0$, $k > 0$, $\alpha \in (0, 1)$, $y(t) = x(t)$, $u(t) = f(t)$ is exciting force, α, β, γ are some numbers.

Let $n = 1.5$, $c = 2$, $m = 1$, $\beta = 0.5$, $\alpha = 0.7$, $k = 0.6$, $d = a = 1$. The set I_o has the form $I_o = \{u(t), y(t), t \in [0; t_e]\}$. Consider four variant inputs

$$u_0(t) = 2 - 2\sin\left(0.15\pi t\right),$$

$$u_1(t) = 2 - 2\sin\left(0.35\pi t\right),$$

$$u_2(t) = 2 - 2\sin\left(0.5\pi t\right),$$

$$u_3(t) = 2 - 2\sin\left(0.15\pi t\right) + 0.2\sin\left(0.35\pi t\right). \tag{24}$$

Denote phase portraits of the system S_{BW} with inputs (24) as $\mathcal{S}_i (i = \overline{0,3})$. Definitional domains the phase portrait and hysteresis coincide (see Figure 3 for the case $\mathcal{S}_0$).

Calculate diameters for the phase portrait definitional domain

$$D_{y,0}\left(\mathcal{S}_0\right) = 3.75,$$

$$D_{y,1}\left(\mathcal{S}_1\right) = 1.728,$$

$$D_{y,2}(\mathcal{S}_2) = 1.08,$$

$$D_{y,3}(\mathcal{S}_3) = 3.967. \tag{25}$$

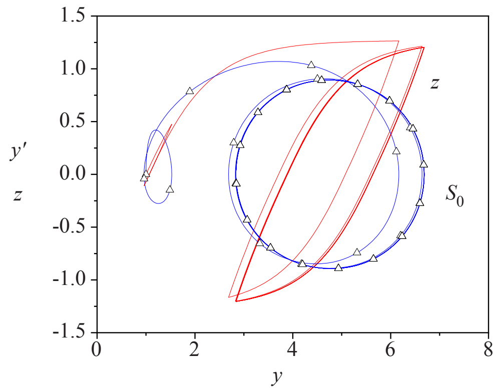


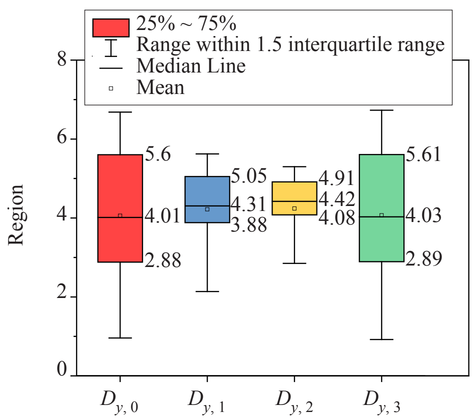
Figure 3. Phase portrait and output of the hysteresis for the system with u_0

Diameter calculation is based on the analysis of frameworks. Definitional domains of frameworks $\mathcal{S}_{ey}$, $\mathcal{S}$ and the hysteresis coincide (see Figures 5, 6). Hence, the diameter is the definitional domain $\mathcal{S}_{ey}$ and $\mathcal{S}$. Results are obtained for the system S_{BW} steady state. The analysis shows $u_0(t) \in$ S. We assume that the system S_{BW} with the phase portrait $\mathcal{S}_0$ is the standard and $d_{h,y} = D_{y,0}(\mathcal{S}_0)$. The degree of non-identifiability of the system S_{BW} for various u_i

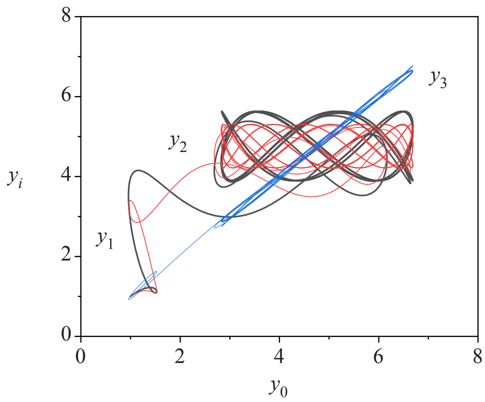
$$SI_1 = 0.549,$$

$$SI_2 = 0.718,$$

$$SI_3 = -0.035. \tag{26}$$



(a) diameter estimate framework $\mathcal{S}_i$ in ranked space



(b) adequacy estimate of framework $\mathcal{S}_i$

Figure 4. To structural identifiability assessment of system S_{BW}

We see that the S_{BW}-system with u_1, u_2 is structurally non-identifiable, and the S_{BW}-system with input u_3 is structurally indistinguishable from input u_0. So, frameworks $S_{ey,1}(u_1)$, $S_{ey,2}(u_2)$ are frameworks of class $\mathcal{NS}_{ey}$, and the framework $S_{ey,3}(u_3)$ belongs to class S_{ey}^{LSI}.

Obtained results are confirmed by Figure 4. It shows system outputs in an integrated form. Rectangular areas represent estimates of diameters within the specified limits. They confirm estimates (26).

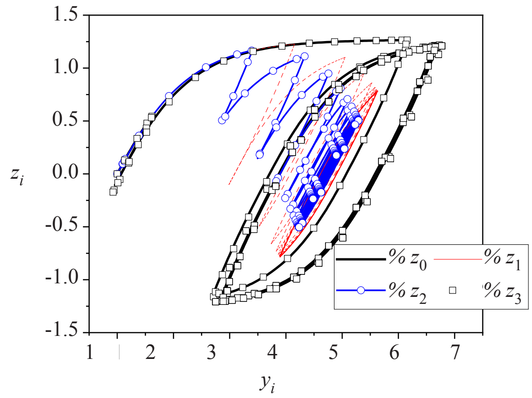


Figure 5. Comparison of S_{BW}-system hysteresis with different inputs

So, the frequency properties of the input influence the identifiability of the system significantly. It is relevant for nonlinear systems, where the minor change in input properties affects the estimation of structural parameters. This conclusion is confirmed by Figure 5, where the Bouc-Wen model (23) output is shown at different inputs. We see that $u(t)$ changes the definition domain and the actual range of the hysteresis.

The area Q_D, which confirms conclusions, is shown in Figure 6. Notation in Figure 6: 1 is S_0, 2 is S_2, 3 is S_1, 4 is S_3.

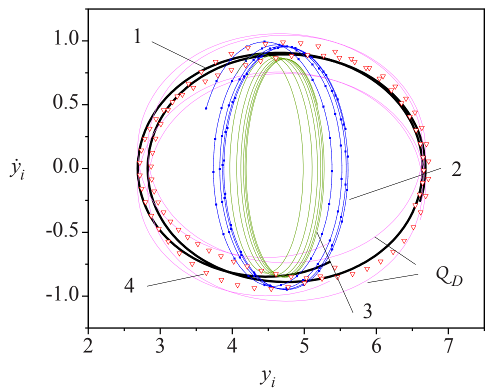


Figure 6. Structural identifiability domain

6. On structural identifiability of system with two nonlinearities

Consider the system $S_{y\varphi}$ with two nonlinearities

$$B_\varphi \varphi(y) = B_{\varphi,1}\varphi_1(y) + B_{\varphi,2}\varphi_2(y), \tag{27}$$

where $\varphi_i(y)$ satisfies the condition (3).

This case is more complex and has features. In this case, the decision-making on the structural identifiability of the S_φ-system is based on the approach described in section 4. But the analysis of S_{ey}-framework properties may be incomplete. There may be a situation where S_{ey} is partially the $\mathcal{NS}_{ey}$-structure. Consider this case.

Consider the S_{ey}-framework (Figure 7 reflects the steady-state) for the second-order system $S_{y\varphi}$, (27). Apply Theorem 2 and obtain that condition (8) does not hold. Hence, $S_{ey} = \mathcal{NS}_{ey}$.

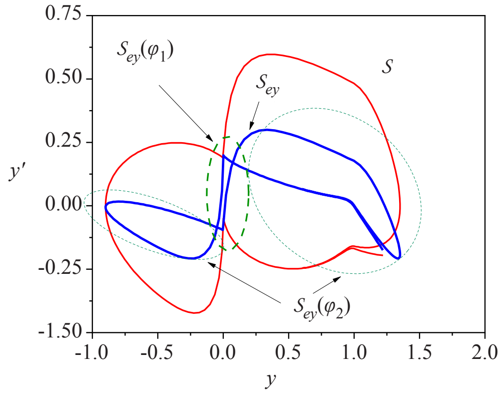


Figure 7. Example S_{ey}-framework for the system $S_{y\varphi}$, (27)

Figure 7 shows that one nonlinearity is identifiable, and the other non-linearity is not identifiable. Consider mapping $\Gamma_{yk} : \{y\} \rightarrow \{k_{ey'}\}$, $k_{ey'} = \dfrac{e}{y'}$ to the decision making. Γ_{yk} corresponds to the S_{ky}-framework (see Figure 8).

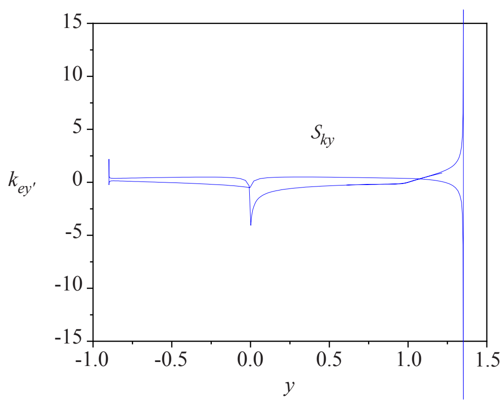


Figure 8. S_{ky}-framework

The analysis S_{ky} shows that the nonlinearity (denote it as φ_1) dominates and is identifiable, and the nonlinearity φ_2 is non-identifiable. Present the framework S_{ey} as $S_{ey} = S_{ey}^{id} \cup \mathcal{NS}_{ey}$ where $S_{ey}^{id} = S_{ey}(\varphi_1)$, $\mathcal{NS}_{ey} = S_{ey}(\varphi_2)$.

Definition 11 The system $S_{y\varphi}$, (27) is called partially structurally identifiable or identifiable on the level φ_1 under u

$\notin$ S if the fragment S_{ey}^{id} of framework S_{ey} is h-identifiable, and unidentifiable at the level φ_2 if $S_{ey}(\varphi_2) = \mathcal{NS}_{ey}$.

Definition 12 The subsystem S_φ of the system $S_{y\varphi}$, (27) is called identifiable under $u \in$ S if the fragments $S_{ey}(\varphi_1)$, $S_{ey}(\varphi_2)$ of the framework $S_{ey} = S_{ey}(\varphi_1) \cup S_{ey}(\varphi_2)$ are h-identifiable.

Definition 13 The subsystem S_φ of the system $S_{y\varphi}$, (27) called structurally identifiable under $u \in$ S if the fragments $S_{ey}(\varphi_1)$, $S_{ey}(\varphi_2)$ of the framework $S_{ey} = S_{ey}(\varphi_1) \cup S_{ey}(\varphi_2)$ are h-identifiable, and the conditions of theorem 4 are satisfied for each fragment $S_{ey}(\varphi_1)$, $S_{ey}(\varphi_2)$.

In this case, the estimation of the S_φ-system identifiability is based on the fragmentation of the framework S_{ey}. Apply the smoothing operation on the fragment $S_{ey}(\varphi_1)$. Obtain the estimate $\hat{S}_{ey}(\varphi_2)$ of the framework $S_{ey}(\varphi_2)$. The estimate of framework $S_{ey}(\varphi_1)$ formed as $\hat{S}_{ey}(\varphi_1) = S_{ey} \setminus \hat{S}_{ey}(\varphi_2)$. If the condition (27) are not satisfied for nonlinearities, then the system (2) structural identifiability analysis is the complicate problem. In this case, the system h_{δ_h}-identifiability estimate requires an extension of the approach proposed in previous sections. Demonstrate this with the example.

Example 2 Consider the system consisting of the nonlinear actuator and a controlled object. The object has linear and quadratic friction. The actuator has saturation. The system of equations has the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -c_1\varphi_1(x_2) \end{bmatrix} + \begin{bmatrix} 0 \\ c\varphi_2(u) \end{bmatrix},$$

$$y = x_1, \tag{28}$$

where $\varphi_1(x_2) = x_2^2\text{sign}(x_2)$ is the quadratic friction, $\varphi_2(u) = \text{sat}(u)$ is dry friction, $x = x_1$ is the rotation angle of the object shaft, u is current excitation winding of the actuator, y is the output, $c_1 = 2$, $c = 1$, $u(t) = 3\sin(0.1\pi t)$.

Experimental information has the form $I_o = \{u(t), y(t), t = [0, t_k]\}$, $t_k < \infty$. Construct frameworks S, S_{ey} (see Figure 9).

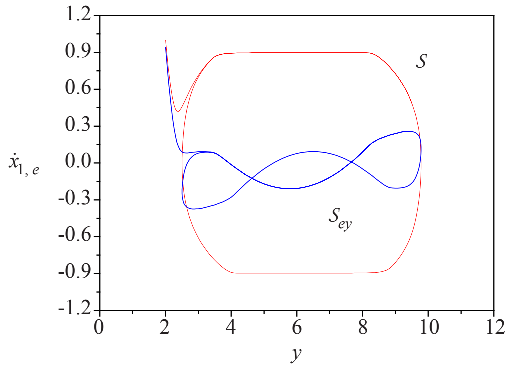


Figure 9. Frameworks S, S_{ey}

Apply the results of section 4 and obtain the system structural identifiability. The decision-making about the nonlinearity structure does not give the analysis of frameworks. It is caused by the nonlinearity of the input. The input $\varphi_2(u)$ on the interval $\bar{J}_y = [4; 8.5]$ is constant, and the condition CE is not satisfied. We can assume (see Figure 9) that $\varphi_2(u) = \text{sat}(u)$. The working interval for y is equal to $J_y = [2; 4] \vee [8.5; 10]$. The application of model (5) in this case is inefficient. Therefore, perform the analysis of the dependence $\dot{x}_2$ on available variables. The coefficient of determination between $\dot{x}_2$ and x_2, y are $r^2_{x_2\dot{x}_2} = 0.995$, $r^2_{y\dot{x}_2} = 0.916$. Therefore, there is the dependency between $\dot{x}_2$ and x_2. Apply the method of hierarchical immersion (MHI) to correction structural relationships [23]. Execute the following steps. Let

$$\hat{\varphi}_2(u) = \begin{cases} 1.8, & if\ u > 1.8, \\ u, & if\ u \le 1.8, \\ -1.8, & if\ u < -1.8. \end{cases}$$

1. The mathematical model has the form

$$\hat{\dot{x}}_2 = 0.147\hat{\varphi}_2(u) + 0.433,\ r^2_{\hat{\varphi}_2\dot{x}_2} = 0,99. \tag{29}$$

2. Exclusion of influence y. Enter the misalignment $\varepsilon = \dot{x}_2 - \hat{\dot{x}}_2$ and obtain the model

$$\hat{\varepsilon} = -0.2038y + 0.933,\ r^2_{y\varepsilon} = 0.95. \tag{30}$$

3. Determine the misalignment $\pi = \varepsilon - \hat{\varepsilon}$ and approximate it with the linear model

$$\hat{\pi}_1 = 0.424x_2 - 0.559,\ r^2_{x_2\pi} = 0.94. \tag{31}$$

So, the model (31) is adequate. Apply the model

$$\hat{\pi}_2 = -0.37|x_2|x_2 - 0.45,\ r^2_{|x_2|x_2,\pi} = 0.97. \tag{32}$$

to increase the accuracy of the approximation $\pi(t)$.

Remark 7 The implementation of MHI is based on checking the SI of the framework at each stage. Next, the mathematical model design. This stage is the prerequisite for obtaining the adequate model.

Thus, the analysis confirms the structural identification possibility of the nonlinear system (28) and its identifiability on interval J_y. It is shown that the model (5) application depends on the structure of the system. Therefore, it is not possible to propose a general method to the choice of the model structure for the system with multiple nonlinearities. The approach depends on the specifics of the system under study. This conclusion illustrates considered examples, and confirms the versatility and complexity of the SI problem under consideration.

The system (2) identifiability depends on structural relationships. As a result, the indirect influence of one variable on another can occur. This case is typical for sequentially connected parts of the system. Such systems are typical objects for the identification [16, 18]. In this case, the influence graph between the system variables is used. It is a polyhedron whose cross-section over some variable reflects the relationships between all variables of the system. Next, apply MHI; construct geometric frameworks to estimate the identifiability of each subsystem.

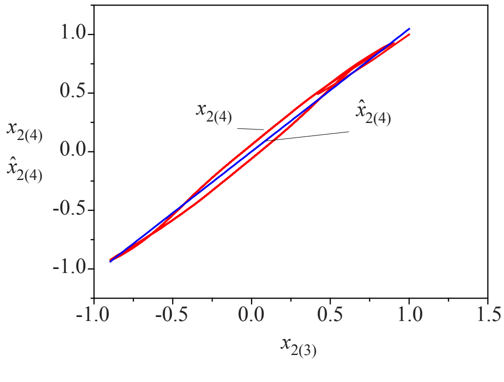


Figure 10. Confirmation of system (28) structural identifiability with cubic nonlinearity

Remark 8 The friction introduction with cubic nonlinearity does not break the system (28) work. This conclusion is valid for the system (28) with a higher degree of friction the system (28) structural identifiability with $\varphi_1(x_2) = \varphi_{1(4)}(x_2) = x_2^3$ following from Figure 10. It reflects the functional dependency for the system (28), where $x_{2(3)}$, $x_{2(4)}$ are the change rotation rate of the object shaft angle with quadratic and cubic nonlinearity. The SI check is based on the analysis of the model properties $\hat{x}_{2(4)} = 1.0472x_{2(3)} + 0.0026$, $r^2_{2(4)} = 0.998$ (see Figure 10). The system (28) with $\varphi_1(x_2) = \varphi_{1(3)}(x_2) = x_2^2\mathrm{sign}(x_2)$ is SI. Therefore, the system (28) with $\varphi_{1(4)}(x_2)$ based the analysis $\hat{x}_{2(4)} = 1.0472x_{2(3)} + 0.0026$ and Figure 10 are also structurally identifiable.

Example 3 Consider the system for generating self-oscillations containing the object (variables y_1, y_2), nonlinear (variable y_3) and linear (variable y_4) meters, and the linear amplifier-converter with the nonlinear actuator (variable y_5)

$$\dot{Y} = AY + DF(Y)$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -g & 0 & 0 & k_0 \\ 0 & 0 & -\frac{1}{T_1} & 0 & 0 \\ 0 & \frac{k_2}{T_2} & 0 & -\frac{1}{T_2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{T_3} \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{T_1} & 0 \\ 0 & 0 \\ 0 & -\frac{1}{T_3} \end{bmatrix}, F(Y) = \begin{bmatrix} f(y_1) \\ f_2(y_4) \end{bmatrix}. \tag{33}$$

The input of the system is the variable $u = y_5$. $f_i(x)$ $(i = 1, 3)$ is the saturation function with the dead zone

$$f_i(x) = \begin{cases} c, & if\ x \geq d_{2,i}, \\ 2(x - d_{1,i}), & if\ d_{1,i} < x < d_{2,i}, \\ 0, & if\ -d_{1,i} \leq x \leq d_{1,i}, \\ 2(x + d_{1,i}), & if\ -d_{1,i} < x, \\ -c, & if\ x < -d_{2,i}, \end{cases} \tag{34}$$

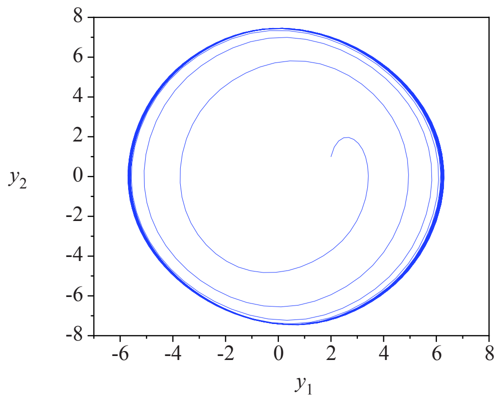


Figure 11. Object phase portrait

The object phase portrait is shown in Figure 11. It confirms the availability of self-oscillations in the system.

Apply the approach described in section 4 and obtain the object linearity. Build the influence graph (IG) to perform further analysis. The graph is shown for the nonlinear meter (NM) with the output y_3 in Figure 12. The derivative of the variable y_i denoted as dyi in Figure 12. We consider connections that exceed 75%. We have relations $\dot{y}_4$, y_1, y_5 for y_3, for $\dot{y}_3$ we have $\dot{y}_5$, y_2, y_4. Apply MHI and exclude variables that influence $\dot{y}_3$. We choose y_5 since y_2 has the indirect influence. Construct the framework $S_{\dot{y}_3 \dot{y}_5}$ described by the function $\gamma_{\dot{y}_3 \dot{y}_5}: \dot{y}_5 \to \dot{y}_3$, and determine the secant

$$\hat{\dot{y}}_3 = -1.241\dot{y}_5 + 0.0094. \tag{35}$$

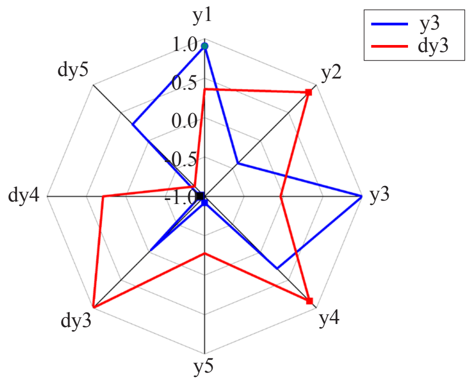


Figure 12. Influence graph for the nonlinear meter

Introduce the misalignment $\varepsilon_{\dot{y}_3} = \dot{y}_3 - \hat{\dot{y}}_3$ and construct the framework $S_{\varepsilon_{\dot{y}_3} y_1}$ described by the function $\gamma_{\varepsilon_{\dot{y}_3} y_1}: y_1 \to \varepsilon_{\dot{y}_3}$ (see Figure 13).

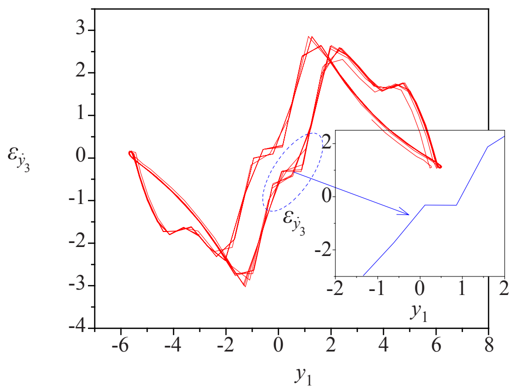


Figure 13. Estimation of nonlinearity structure for NM

Figure 13 shows that NM contains the nonlinearity described by the saturation function with the dead zone. The dead zone width depends on properties y_5. The model (35) structure is based on the analysis of the influence graph for y_3. Variables $y_i \in \mathcal{PE}_{\alpha_i}$, $i = \overline{1,3}$. Analysis of the framework $S_{\varepsilon_{\dot{y}_3} y_1}$ shows that it satisfies conditions of theorem 4. Therefore, the object and the nonlinear meter are structurally identifiable.

Next, we perform the analysis of the linear meter (LM) structure. y_1, $\dot{y}_3$ influence y_4, and y_2, y_3, y_5 influence $\dot{y}_4$. Most of the relationship is between $\dot{y}_4$ and y_3. The secant structure $S_{\dot{y}_4 y_3}$ has the form

$$\hat{\dot{y}}_4 = -1.1737 y_3 + 0.0689,\ r^2_{\dot{y}_4, y_3} = 0.88. \tag{36}$$

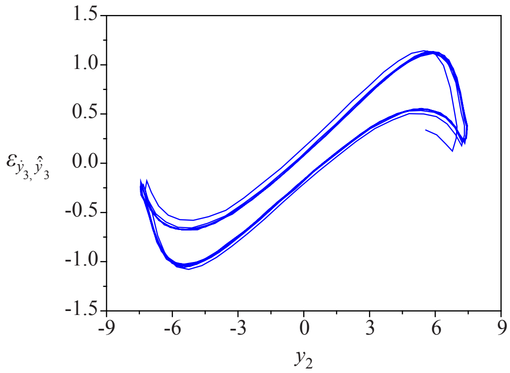


Figure 14. Linearity estimation of LM

The linear meter is h-identifiable. Perform the next MHI stage. Enter the residual $\varepsilon_{\dot{y}_3,\hat{\dot{y}}_3} = \dot{y}_3 - \hat{\dot{y}}_3$ and estimate the influence of variables y_2, y_5. Obtain $r^2_{\varepsilon_{\dot{y}_3,\hat{\dot{y}}_3}, y_5} = 0.23$, $r^2_{\varepsilon_{\dot{y}_3,\hat{\dot{y}}_3}, y_2} = 0.73$. So, $\dot{y}_4$ depends on y_2 (see Figure 14) linearly.

Remark 9 We consider indirect relationships reflected the influence of system previous elements.

Consider the linear amplifier-converter with the nonlinear actuator (LACNA). The influence graph gives $\dot{y}_5$ variables y_2, $\dot{y}_3$, y_4, $\dot{y}_4$. The LACNA phase portrait is shown in Figure 15. We see that the processes are nonlinear in LACUNA.

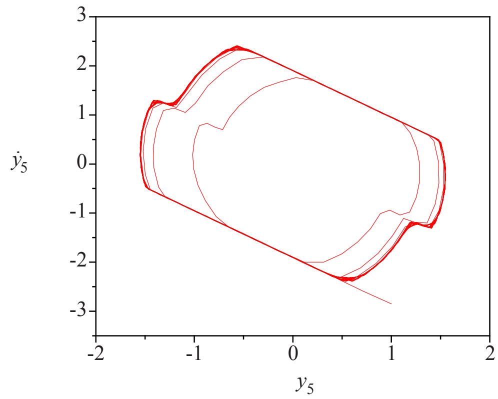


Figure 15. LACUNA phase portrait

Remark 10 The SI estimation of the $S_{\dot{y}_5 \dot{y}_3}$ framework is based on the analysis of the values range.

Consider frameworks $S_{\dot{y}_5 \dot{y}_3}$, $S_{\dot{y}_5 y_2}$ and construct secants

$$\hat{\dot{y}}_{5,2} = -0,2593 y_2 + 0,0281, \; r^2_{\dot{y}_5, y_2} = 0.90, \tag{37}$$

$$\hat{\dot{y}}_{5,3} = -0.5019 \dot{y}_3 - 0,0239, \; r^2_{\dot{y}_5, \dot{y}_3} = 0.77, \tag{38}$$

to exclude the influence y_2 and $\dot{y}_3$. In Figure 16, frameworks $S_{\dot{y}_5 \dot{y}_3}$ and $S_{\dot{y}_5 y_2}$ are shown.

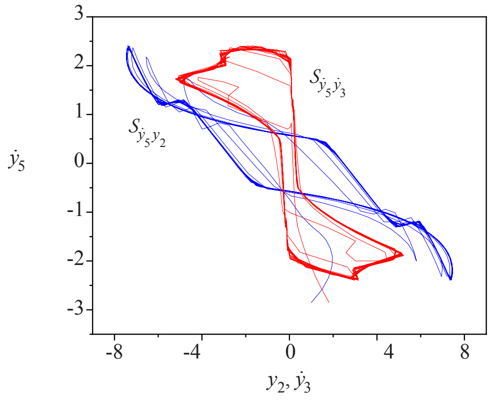


Figure 16. Frameworks $S_{\dot{y}_5 \dot{y}_3}$, $S_{\dot{y}_5 y_2}$

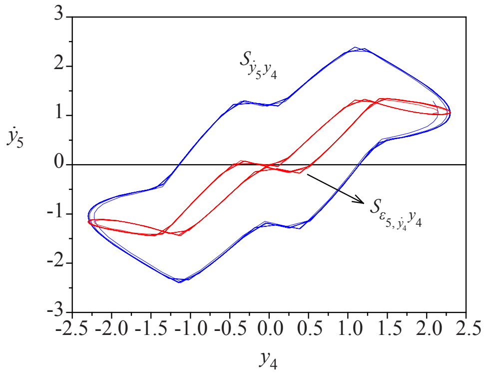


Figure 17. Frameworks $S_{\dot{y}_5 y_4}$, $S_{\varepsilon_{5,\dot{y}_4} y_4}$

Frameworks analysis shows that the processes are nonlinear in LACUNA. The final decision cannot be made on the LACUNA nonlinearity.

Introduce residuals $\varepsilon_{5,3} = \dot{y}_5 - \hat{\dot{y}}_{5,3}$, $\varepsilon_{5,2} = \dot{y}_5 - \hat{\dot{y}}_{5,2}$ and apply MHI. The analysis shows that obtained results do not allow to decide on the LACUNA structure. Consider the $S_{\dot{y}_5 y_4}$-framework (see Figure 17). Figure 16 confirms that LACUNA contains the saturation element with the dead zone. But this representation contains some indistinctness. So, consider the framework $S_{\dot{y}_5 \dot{y}_4}$ and determine the secant $\hat{\dot{y}}_{5,\dot{y}_4} = -0.411 \dot{y}_4 + 0.024$ for it, introduce the misalignment $\varepsilon_{5,\dot{y}_4} = \dot{y}_5 - \hat{\dot{y}}_{5,\dot{y}_4}$. Next, consider the framework $S_{\varepsilon_{5,\dot{y}_4} y_4}$ (see Figure 17). The $S_{\varepsilon_{5,\dot{y}_4} y_4}$-framework gives the complete representation of the nonlinearity $f_3(y_4)$ type (form). The framework $S_{\varepsilon_{5,\dot{y}_4} y_4}$ satisfies the conditions of theorem 4. Therefore, LACUNA is structurally identifiable.

Let the measurement information be known for the system (33)

$$\mathrm{I}_o = \{y_3(t), y_4(t), y_5(t), t \in [0, t_k]\},\ t_k < \infty. \tag{39}$$

Construct the influence graph to estimate SI of the system (33). Graph components are presented above.

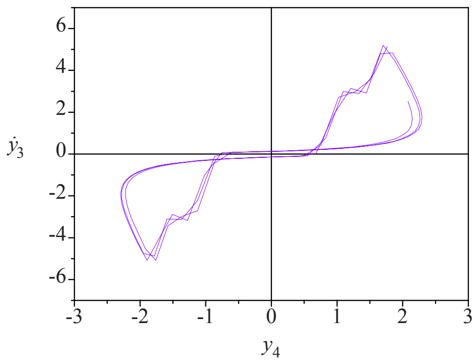


Figure 18. Framework $\mathcal{S}_{\dot{y}_3 y_4}$

We have $\dot{y}_3 \rightleftarrows (y_4, \dot{y}_5)$, $\dot{y}_4 \rightleftarrows (y_3, y_5)$, $\dot{y}_5 \rightleftarrows (\dot{y}_3, y_4)$, where $\rightleftarrows$ is the connection symbol of graph elements. The SI estimate presented for $\dot{y}_5$ in Figure 16. It confirms the identifiability of the system at this level. The relationship $\dot{y}_5 = v(\dot{y}_3)$ allows estimating the dead zone of the function $f_3(y_4)$. The analysis of y_5 relationship $\dot{y}_4 \rightleftarrows (y_3, y_5)$ confirms the linearity of this subsystem. The framework $\mathcal{S}_{\dot{y}_3 y_4}$ is shown in Figure 18, and confirms the existence of the saturation class with the variable dead zone.

$\mathcal{S}_{\dot{y}_3 y_4}$ is structurally identifiable and confirms the above conclusion. The variable y_4 is a function of y_2 (see (33)). So, obtain the identifiability of the function $f_1(y_2)$.

Denote the graph defined on I_o and contains connections in the system (33) as $\mathcal{G}$.

Definition 14 The relationship between the variables $(y_i, y_j) \in \mathcal{G}$ of the system (33) is significant if the framework $\mathcal{S}_{y_i y_j}$ ensures h-identifiability of the system (33) and corresponds to this relationship.

The subset of graph $\mathcal{G}$ significant for obtaining connections at the k-element analysis level is denoted as $\mathcal{G}_{s,k} \subset \mathcal{G}$.

Theorem 7 If the system (33) input y_5 is S-synchronizing, and frameworks $\mathcal{S}_{y_i y_j}$ are defined on graph $\mathcal{G}_{s,k} \subset \mathcal{G}$, and the conditions of theorem 4 are satisfied, then system (33) is h_{δ_h}-identifiable.

The proof of theorem 7 follows the condition $y_5 \in \mathrm{S}$ fulfillment and the subgraph $\mathcal{G}_{s,k}$ existence which the frameworks $\mathcal{S}_{y_i y_j}$ had the $d_{h,\Sigma}$-optimality property.

Continue the LACNA analysis. Consider the framework $\mathcal{S}_{\dot{y}_5 y_2, y_1}$, select fragments $\mathcal{F}_s^l$, $\mathcal{F}_s^r$, and approximate $\mathcal{F}_s^l$, $\mathcal{F}_s^r$ with secants (7) on y_1. Have $a^r = -0.485$, $a^l = -0.347$. Let $\delta_h = 0.04$. Calculate $||a^l|-|a^r|| = 0.081$, apply theorem 1 and obtain $\mathcal{S}_{\dot{y}_5 y_2, y_1} = \mathcal{NS}$. Next, apply theorem 7 and stop the hierarchical immersion method.

So, it shows the structural identifiability problem complexity for the system with multiple nonlinearities. Internal connections can influence the SI problem solution. These connections can influence the subsystem for studying their identification indirectly. In this case, considering the relationships influence level has great importance. It is a new problem that appears in the SI study of systems with multiple nonlinearities. A priori information is critical in this case. The SM-model synthesis of the analogous model (5) depends on the influence graph.

7. Conclusion

Structural identifiability conditions for the nonlinear system are obtained. They are founded on the property analysis of geometric frameworks. The role of S-synchronizability and the constant excitation is shown in the structural identifiability analysis. Conditions of structural non-identifiability and structural indistinguishability are obtained. Systems with two nonlinearities are considered. Difficulties appearing in the analysis of structural identifiability are noted. The internal organization influence of system elements is noted on the structural identifiability possibility. The SI estimating problem on the set of available dimensions is considered. The influence graph is essential in the structural identifiability analysis. It allows selecting significant variables for SI analysis and synthesize the model for geometric framework design.

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