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Research Article

Novel Results on Applications of Differential Subordination on Analytic Functions Associated with Zeta-Riemann Fractional Differential Operator

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Abstract: The findings of this study are connected with geometric function theory and were acquired by using subordination-based techniques in conjunction with Zeta-Riemann fractional differential operator information, we used the Zeta-Riemann fractional differential operator to investigate a certain classes of analytic functions. It is also shown that for particular choice of parameters for the new generalized classes, the classes of starlike, close-to-convex and α-convex functions emerges. Many classes of univalent functions were investigated with the use of convolution and subordination principle. Further, some classes are defined and developed by using the Zeta-Riemann fractional differential operator. The connections between the classes are given in the definition or in associated remarks and characterization properties are also proved, including combinations of functions belonging to those classes and inclusion relations.

***Keywords*:** analytic function, differential subordination, fractional operator

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1. Introduction and prelimiaries

Geometric Functions Theory (GFT) is an amalgamation of geometry along with analysis. The theory of analytic univalent and multivalent functions are two of the most important aspects of the GFT. Investigations in these areas constitute an ancient topic in mathematics, especially in Complex analysis, that has drawn quite a few scholars due to the absolute elegance of their geometrical properties as well as numerous study opportunities. Inarguably, the most significant field of complex analysis to feed just one or many variables is the investigation of univalent functions. Univalent functions are covered in detail in the standard works of Duren [1] and Goodman [2]. The idea of analytic function subordination was first introduced by Littlewood [3], and Rogosinski [4] developed the phrase and established the fundamental results utilizing subordination. The concept of subordination was recently employed by Srivastava and Owa [5] to explore a number of fascinating properties of the generalized hypergeometric function. A generalization of differential inequalities, known as differential subordinations, was the subject of an article by Miller and Mocanu [6] . There have been numerous fascinating properties of GFT that were studied and investigated by several authors, see [7–16].

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Let $\mathscr{A}$ denote the class of function satisfying $f(0) = f^{'}(0) - 1 = 0$ written as:

$$f(\xi) = \xi + \sum_{\kappa=2}^{\infty} a_\kappa \xi^\kappa, \tag{1}$$

which are analytic and univalent in the open unit disc $\mathscr{U} = \{\xi : |\xi| < 1\}$. If f and g are analytic in $\mathscr{U}$, f is subordinate to g, denoted $f(\xi) \prec g(\xi)$, if there exists an analytic function ϖ, with $\varpi(0) = 0$ and $|\varpi(\xi)| < 1$ for all $\xi \in \mathscr{U}$, such that $f(\xi) = g(\varpi(\xi))$, $\xi \in \mathscr{U}$. If the function g is univalent in $\mathscr{U}$, $f(\xi) \prec g(\xi)$ is given as (see [17–19]):

$$f(0) = g(0) \text{ and } f(\mathscr{U}) \subset g(\mathscr{U}).$$

For two functions $f_\iota(\xi) \in \mathscr{A} (\iota = 1,\ 2)$ are given by

$$f_\iota(\xi) = \xi + \sum_{\kappa=2}^{\infty} a_{\kappa,\ \iota} \xi^\kappa,$$

we define the convolution of $f_1(\xi)$ and $f_2(\xi)$ as (see, for example [20], p.246)

$$(f_1 * f_2)(\xi) = \xi + \sum_{\kappa=2}^{\infty} a_{\kappa,\ 1} a_{\kappa,\ 2} \xi^\kappa = (f_2 * f_1)(\xi).$$

The Zeta-Riemann fractional differential operator is an intriguing extension of classical differential operators, derived from the fusion of the ideas embedded in the Riemann zeta function and the principles of fractional calculus. It emerges from the need to analyze non-local phenomena in mathematical physics and applied mathematics through the lens of fractional derivatives. The fractional calculus, as a mathematical field, explores derivatives and integrals of arbitrary orders, allowing for a more nuanced understanding of dynamical systems that exhibit non-integer order behavior. By incorporating the zeta function, which encodes rich information about number theory and complex analysis, the Zeta-Riemann fractional differential operator captures additional complexity and characteristics from both domains, creating a powerful analytical tool.

Tayyah and Atshan [21] investigate the following fractional differential operator

$$\mathfrak{D}^{\alpha}_{(s,\ k,\ w)} f(\xi) = \sum_{\kappa=1}^{\infty} \frac{(w+1)^{\frac{s(\alpha-1)+1}{k}} \Gamma\left(\frac{\kappa}{w+1}+1\right) \Gamma\left(\frac{\delta+\eta+1}{2}\right)}{k^{\frac{s(\alpha-1)+1}{k}} \Gamma\left(\frac{\kappa}{w+1} - \frac{s(\alpha-1)+1}{k} + 1\right)} a_\kappa \xi^{(w+1)(1-\frac{s(\alpha-1)+1}{k})+\kappa-1}, \tag{2}$$

$$\left(s \in \mathbb{N},\ k > 0,\ w \geq 0,\ 0 \leq \frac{s(\alpha-1)+1}{k} < 1\right).$$

In [22] the Hurwitz-Lerch Zeta function is defined as following

$$\Phi(\xi,\ s,\ \alpha) = \sum_{\kappa=0}^{\infty} \frac{\xi^{\kappa}}{(\kappa+\alpha)^{s}}, \tag{3}$$

$s \in \mathbb{C}, \alpha \in \mathbb{C}\backslash\mathbb{Z}^{-},\ Re(s) > 1$, when $|\xi| = 1$.

Several interesting properties and characteristics of the above defined Hurwitz-Lerech Zeta function was studied and investigated by several authors see [23–26].

We define the new Hadamard product fractional differential operator.

$$\begin{aligned}\mathfrak{D}_{s}^{(\alpha,\ \beta)} f(\xi) &= \left[D^{\alpha}_{(s,\ 1,\ 0)} f(\xi)\right] * \left[\xi^{-(s(\alpha-1)+1)}(\Phi(\xi,\ s-1,\ \alpha+\beta) - \alpha^{-s}\right] \\ &= \sum_{\kappa=1}^{\infty} \frac{\kappa!}{\Gamma(\kappa - s(\alpha-1))\,(\kappa+\alpha+\beta)^{s-1}} a_{\kappa} \xi^{\kappa-[s(\alpha-1)+1]}.\end{aligned} \tag{4}$$

We note that if $s = 1$, then we have Srivastava fractional differential operator in [27] as:

$$\mathfrak{D}_{1}^{(\alpha,\ \beta)} f(\xi) = \sum_{\kappa=1}^{\infty} \frac{\kappa!}{\Gamma(\kappa-\alpha+1)} a_{\kappa} \xi^{\kappa-\alpha}. \tag{5}$$

Shexo et al. [28] introduced a new operator (Zeta-Riemann Fractional differential operator) $\mathscr{D}_{s}^{(\alpha,\ \beta)} : \mathscr{A} \to \mathscr{A}$ as follows:

$$\begin{aligned}\mathscr{D}_{s}^{(\alpha,\ \beta)} f(\xi) =& (1+\alpha+\beta)^{s-1}\Gamma(1-s(\alpha-1))\,\xi^{[s(\alpha-1)+1]}\mathfrak{D}_{s}^{(\alpha,\ \beta)} f(\xi) \\ =& \xi + \sum_{\kappa=2}^{\infty} \frac{\Gamma(1-s(\alpha-1))\,(1+\alpha+\beta)^{s-1}\kappa!}{\Gamma(\kappa-s(\alpha-1))\,(\kappa+\alpha+\beta)^{s-1}} a_{\kappa}\xi^{\kappa}.\end{aligned} \tag{6}$$

The operator $\mathscr{D}_{s}^{(\alpha,\ \beta)} f(\xi)$ is linear and bounded in the unit disk $\mathscr{U}$ and satisfying

$$\xi\left(\mathscr{D}_{s}^{(\alpha,\ \beta)} f(\xi)\right)' = s(1-\alpha)\mathscr{D}_{s}^{(\alpha+\frac{1}{s},\ \beta-\frac{1}{s})} f(\xi) - [s(1-\alpha)-1]\mathscr{D}_{s}^{(\alpha,\ \beta)} f(\xi). \tag{7}$$

The study of the Zeta-Riemann fractional differential operator is ongoing, with researchers continually uncovering its implications in both pure and applied mathematics. By delving into properties such as stability, existence of solutions, and numerical approximations, mathematicians and scientists alike can leverage this operator to develop innovative models that reflect the intricate dynamics of real-world systems. The interplay between fractional calculus and the analytical properties of the Riemann zeta function also paves the way for future investigations into connections with other areas of mathematics, including chaos theory, fractals, and complex networks.

The primary objective of this paper is to study the dependence concepts with an additional subset for univalent functions alongside a different operator which uses subordination-based techniques in conjunction with Zeta-Riemann fractional differential operator information. We will introduce many classes of univalent functions by using the convolution and subordination principle. Further, we develop some classes by using the Hurwitz-Lerch Zeta operator and we illustrate the connections between the classes in the definition or in associated remarks and characterization properties are also, we prove including combinations of functions belonging to those classes and inclusion relations.

Let Ψ be the class of analytic univalent convex functions in $\mathscr{U}$, with $\mu(0)=1$ and $Re\mu(\xi)>0$.

The next lemmas are used to demonstrate our results:

Lemma 1 [29] Let $\beta,\ \sigma\in\mathbb{C}$. Also let $\mu\in\mathscr{A}$ be convex univalent in $\mathscr{U}$ with $Re[\beta\mu(\xi)+\sigma]>0\ (\xi\in\mathscr{U}),\ \mu(0)=1$ and $p(\xi)\in\mathscr{A}$ with $p(\xi)=1+p_1\xi+p_2\xi^2+....$ is analytic in $\mathscr{U}$. If

$$p(\xi)+\frac{\xi p^{'}(\xi)}{\beta p(\xi)+\sigma}\prec\mu(\xi)\ (\xi\in\mathscr{U})$$

then $p(\xi)\prec\mu(\xi)$.

Lemma 2 [30] Let $\beta,\ \sigma\in\mathbb{C}$. Also suppose $\mu\in\mathscr{A}$ be convex univalent in $\mathscr{U}$ with $\mu(0)=1$ and $Re[\beta\mu(\xi)+\sigma]>0\,(\xi\in\mathscr{U})$, and $q(\xi)\in\mathscr{A}$ with $q(0)=1$ and $q(\xi)\prec\mu(\xi)\,(\xi\in\mathscr{U})$. If $p(\xi)=1+p_1\xi+p_2\xi^2+...$ is analytic in $\mathscr{U}$,

$$p(\xi)+\frac{\xi p^{'}(\xi)}{\beta q(\xi)+\sigma}\prec\mu(\xi)\ (\xi\in\mathscr{U}),$$

then

$$p(\xi)\prec\mu(\xi).$$

Bu using the operator $\mathscr{D}_s^{(\alpha,\ \beta)}f(\xi)$, we investigate the class $\mathfrak{R}_s^{(\alpha,\ \beta)}(\vartheta;\ \mu)$, of analytic functions $f=\{f_1,\ f_2,\ ...,\ f_\vartheta\}$ on open unit disc $\mathscr{U}$ satisfying $\frac{\xi\left(\mathscr{D}_s^{(\alpha,\ \beta)}f_i(\xi)\right)'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\ \beta)}f_j(\xi)}\prec\mu(\xi)(f_i\in\mathscr{A},\ i=1,\ 2,\ ...,\ \vartheta,\ \xi\in\mathscr{U})$, where $\xi^{-1}\sum\limits_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\ \beta)}f_j(\xi)\neq0$ and μ is convex univalent in $\mathscr{U}$ with $\mu(0)=1$. Also we define $F=\{F_1,\ F_2,\ ...,\ F_\vartheta\}$ where $F_i(\xi)=\frac{\varsigma+1}{\xi^\varsigma}\int\limits_0^\xi t^{\varsigma-1}f_i(t)dt\ (\varsigma>0;\ i=1,\ 2,\ ...,\ \vartheta)$ and proved that $F\in\mathfrak{R}_s^{(\alpha,\ \beta)}(\vartheta;\ \mu)$, whenever $f\in\mathfrak{R}_s^{(\alpha,\ \beta)}(\vartheta;\ \mu)$. Additional classes of this type are indicated by $\mathfrak{C}_s^{(\alpha,\ \beta)}(\vartheta;\ \mu)$, $\wp_s^{(\alpha,\ \beta)}(\vartheta;\ \delta,\ \mu)$ and $\mathfrak{N}_s^{(\alpha,\ \beta)}(\vartheta;\ \delta,\ \mu)$ are presented and examined here using convolutions with subordination approach.

2. Main results

Throughout this paper, unless otherwise mentioned, we set $s\geq1,\ \alpha,\ \beta\in\mathbb{R}^+$ and $\vartheta\in\mathbb{N}$.

2.1 *The class* $\mathfrak{R}_s^{(\alpha,\ \beta)}(\vartheta;\ \mu)$

Definition 1 Let $f=\{f_1,\ f_2,\ ...,\ f_\vartheta\},\ f_i\in\mathscr{A},\ 1\leq i\leq\vartheta$ be such that

$$\frac{\xi\left(\mathscr{D}_s^{(\alpha,\ \beta)}f_i(\xi)\right)'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\ \beta)}f_j(\xi)}\prec\mu(\xi)\ (\xi\in\mathscr{U};\ i=1,\ 2,\ ...,\ \vartheta),$$

where $\xi^{-1}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}f_j(\xi)\neq 0$ in $\mathscr{U}$, μ is convex univalent in $\mathscr{U}$ with $\mu(0)=1$. Then we say that $f=\{f_1,\, f_2,\, ...,\, f_\vartheta\}$ belongs to the class $\mathfrak{R}_s^{(\alpha,\,\beta)}(\vartheta;\,\mu)$.

Remark 1 (i) $\mathfrak{R}_1^{(0,\,0)}\left(1;\,\dfrac{1-\xi}{1+\xi}\right)=S^*$, the class of starlike functions introduced by Robertson [31];

(ii) $\mathfrak{R}_1^{(0,\,0)}\left(1;\,\dfrac{1+A\xi}{1+B\xi}\right)=S^*(A,\,B),\ (-1\leq B<A\leq 1)$, this class investigate by Janowski [32];

(iii) For $\vartheta=1$ and $\mu(\xi)=\dfrac{1-\xi}{1+\xi}$, we have

$$f(\xi)=\xi+\frac{\Gamma(\kappa-s(\alpha-1))\,(\kappa+\alpha+\beta)^{s-1}}{\kappa\Gamma(1-s(\alpha-1))\,(1+\alpha+\beta)^{s-1}\kappa!}\,\xi^{\kappa}\in\mathfrak{R}_s^{(\alpha,\,\beta)}\left(1;\,\frac{1-\xi}{1+\xi}\right)\ (\kappa\geq 2).$$

Theorem 1 Let $f=\{f_1,\, f_2,\, ...,\, f_\vartheta\}\in\mathfrak{R}_s^{(\alpha,\,\beta)}(\vartheta;\,\mu)$ and $F(\xi)=\dfrac{1}{\vartheta}\sum_{i=1}^{\vartheta}f_i(\xi)$. Then $F(\xi)$ satisfies the condition

$$\frac{\xi\left(\mathscr{D}_s^{(\alpha,\,\beta)}F(\xi)\right)'}{\mathscr{D}_s^{(\alpha,\,\beta)}F(\xi)}\prec\mu(\xi)(\xi\in\mathscr{U}),\tag{8}$$

where $\xi^{-1}\mathscr{D}_s^{(\alpha,\,\beta)}F(\xi)\neq 0$.

Proof. Let $f=\{f_1,\, f_2,\, ...,\, f_\vartheta\}\in\mathfrak{R}_s^{(\alpha,\,\beta)}(\vartheta;\,\mu)$. Then for any $\xi_0\in\mathscr{U}$, we have

$$\frac{\xi_0\left(\mathscr{D}_s^{(\alpha,\,\beta)}f_i(\xi_0)\right)'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}f_j(\xi_0)}\prec\mu(\mathscr{U})$$

where $\xi_0^{-1}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}f_j(\xi_0)\neq 0$ and hence equals to $\mu(w_i)$(say) for some $w_i\in\mathscr{U}$, $i=1,\,2,\,...,\,\vartheta$. So

$$\frac{\sum_{i=1}^{\vartheta}\xi_0\left(\mathscr{D}_s^{(\alpha,\,\beta)}f_i(\xi_0)\right)'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}f_j(\xi_0)}=\sum_{i=1}^{\vartheta}\mu(w_i).$$

Let $f(\xi)=\xi+\sum_{k=2}^{\infty}a_k\xi^k$. Then, from (6), we see that

$$\begin{aligned}\mathscr{D}_s^{(\alpha,\,\beta)}f(\xi)=&f(\xi)*\left\{\xi+\sum_{\kappa=2}^{\infty}\frac{\Gamma(1-s(\alpha-1))\,(1+\alpha+\beta)^{s-1}\kappa!}{\Gamma(\kappa-s(\alpha-1))\,(\kappa+\alpha+\beta)^{s-1}}\xi^{\kappa}\right\}\\ =&\left(f*\mathfrak{K}_s^{(\alpha,\,\beta)}\right)(\xi),\end{aligned}$$

where

$$\mathfrak{K}_s^{(\alpha,\,\beta)} = \xi + \sum_{\kappa=2}^{\infty} \frac{\Gamma(1-s(\alpha-1))\,(1+\alpha+\beta)^{s-1}\kappa!}{\Gamma(\kappa-s(\alpha-1))\,(\kappa+\alpha+\beta)^{s-1}}\xi^{\kappa}. \tag{9}$$

Hence

$$\frac{\xi_0\left(\mathscr{D}_s^{(\alpha,\,\beta)}F(\xi_0)\right)'}{\mathscr{D}_s^{(\alpha,\,\beta)}F(\xi_0)} = \frac{\xi_0\left[\mathfrak{K}_s^{(\alpha,\,\beta)}(\xi)*\sum\limits_{i=1}^{\vartheta} f_i(\xi_0)\right]'}{\mathfrak{K}_s^{(\alpha,\,\beta)}(\xi)*\sum\limits_{j=1}^{\vartheta} f_j(\xi_0)}.$$

Since

$$\mathscr{D}_s^{(\alpha,\,\beta)}\sum_{j=1}^{\vartheta} f_j(\xi) = \sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)} f_j(\xi),$$

we have

$$\frac{\xi_0\left(\mathscr{D}_s^{(\alpha,\,\beta)}F(\xi_0)\right)'}{\mathscr{D}_s^{(\alpha,\,\beta)}F(\xi_0)} = \frac{1}{\vartheta}\left[\frac{\xi_0\sum\limits_{i=1}^{\vartheta}\left(\mathscr{D}_s^{(\alpha,\,\beta)}f_i(\xi_0)\right)'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}f_j(\xi_0)}\right]$$

$$= \frac{1}{\vartheta}\sum_{i=1}^{\vartheta}\mu(w_i) = \mu(w_0),$$

for some $w_0 \in \mathscr{U}$, since μ is convex in $\mathscr{U}$. □

Putting $\mu(\xi) = \frac{1-\xi}{1+\xi}$ in Theorem 1 we obtain:

Corollary 1 If $f = \{f_1,\, f_2,\, \ldots,\, f_\vartheta\} \in \mathfrak{R}_s^{(\alpha,\,\beta)}\left(\vartheta;\ \frac{1-\xi}{1+\xi}\right)$, then $\mathscr{D}_s^{(\alpha,\,\beta)}f_i(\xi)$ are close-to-convex univalent functions (see Kaplan [33]).

Theorem 2 Let $f = \{f_1,\, f_2,\, \ldots,\, f_\vartheta\} \in \mathfrak{R}_s^{(\alpha,\,\beta)}(\vartheta;\ \mu)$. Define

$$F_i(\xi) = \frac{\varsigma+1}{\xi^{\varsigma}}\int_0^{\xi} t^{\varsigma-1}f_i(t)dt\ (\varsigma > 0;\ i = 1,\ 2,\ \ldots,\ \vartheta).$$

If μ is bounded in $\mathscr{U}$ and $Re(\mu(\xi)+\varsigma) > 0$, then $F = \{F_1,\, F_2,\, \ldots,\, F_\vartheta\} \in \mathfrak{R}_s^{(\alpha,\,\beta)}(\vartheta;\ \mu)$.

Proof. From the definition of $F_i(\xi)$, it follows that

$$\xi F_i^{'}(\xi)+\varsigma F_i(\xi)=(\varsigma+1)f_i(\xi),$$

and on taking convolution with $\mathfrak{D}_s^{(\alpha,\ \beta)}$ given by (9), we get

$$\xi[\mathscr{D}_s^{(\alpha,\ \beta)}F_i(\xi)]^{'}+\varsigma\mathscr{D}_s^{(\alpha,\ \beta)}F_i(\xi)=(\varsigma+1)\mathscr{D}_s^{(\alpha,\ \beta)}f_i(\xi),\ i=1,\ 2,\ ...,\ \vartheta. \tag{10}$$

Let

$$p_i(\xi)=\frac{\vartheta\xi[\mathscr{D}_s^{(\alpha,\ \beta)}F_i(\xi)]^{'}}{\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\ \beta)}F_j(\xi)}, \tag{11}$$

where $\xi^{-1}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\ \beta)}F_j(\xi)\neq 0$. From (10), we have

$$\frac{p_i(\xi)}{\vartheta}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\ \beta)}F_j(\xi)+\varsigma\mathscr{D}_s^{(\alpha,\ \beta)}F_i(\xi)=(\varsigma+1)\mathscr{D}_s^{(\alpha,\ \beta)}f_i(\xi). \tag{12}$$

Differentiating (12) with respect to ξ, we get

$$\frac{p_i^{'}(\xi)}{\vartheta}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\ \beta)}F_j(\xi)+\frac{p_i(\xi)}{\vartheta}\sum_{j=1}^{\vartheta}[\mathscr{D}_s^{(\alpha,\ \beta)}F_j(\xi)]^{'}+\varsigma[\mathscr{D}_s^{(\alpha,\ \beta)}F_i(\xi)]^{'}=(\varsigma+1)[\mathscr{D}_s^{(\alpha,\ \beta)}f_i(\xi)]^{'}.$$

From (11), we have

$$p_i^{'}(\xi)\frac{\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\ \beta)}F_j(\xi)}{\vartheta}+\frac{p_i(\xi)}{\vartheta}\frac{\sum_{i=1}^{\vartheta}p_i(\xi)\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\ \beta)}F_j(\xi)}{\vartheta\xi}+\varsigma\frac{p_i(\xi)\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\ \beta)}F_j(\xi)}{\vartheta\xi}$$

$$=(\varsigma+1)[\mathscr{D}_s^{(\alpha,\ \beta)}f_i(\xi)]^{'}.$$

Hence

$$p_i^{'}(\xi)+\frac{p_i(\xi)}{\vartheta\xi}\sum_{i=1}^{\vartheta}p_i(\xi)+\varsigma\frac{p_i(\xi)}{\xi}=\frac{(\varsigma+1)[\mathscr{D}_s^{(\alpha,\ \beta)}f_i(\xi)]^{'}}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\ \beta)}F_j(\xi)}.$$

Then

$$\frac{\xi p_i'(\xi)}{\frac{1}{\vartheta}\sum\limits_{i=1}^{\vartheta} p_i(\xi)+\varsigma}+p_i(\xi)=\frac{(\varsigma+1)\xi[\mathscr{D}_s^{(\alpha,\,\beta)}f_i(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}F_j(\xi)}\cdot\frac{1}{\frac{1}{\vartheta}\sum\limits_{i=1}^{\vartheta} p_i(\xi)+\varsigma}$$

$$=\frac{(\varsigma+1)\xi[\mathscr{D}_s^{(\alpha,\,\beta)}f_i(\xi)]'}{\frac{1}{\vartheta}\left\{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}F_j(\xi).\sum\limits_{i=1}^{\vartheta} p_i(\xi)+\varsigma\sum\limits_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}F_j(\xi)\right\}}.$$

From (12), we have

$$\frac{\xi p_i'(\xi)}{\frac{1}{\vartheta}\sum\limits_{i=1}^{\vartheta} p_i(\xi)+\varsigma}+p_i(\xi)=\frac{(\varsigma+1)\xi[\mathscr{D}_s^{(\alpha,\,\beta)}f_i(\xi)]'}{\frac{1}{\vartheta}(\varsigma+1)\sum\limits_{i=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}f_i(\xi)}\prec\mu(\xi), \tag{13}$$

since $f=\{f_1,\ f_2,\ ....,\ f_\vartheta\}\in\mathfrak{R}_s^{(\alpha,\,\beta)}(\vartheta;\ \mu)$. Now we can write for any $\xi_0\in\mathscr{U}$,

$$\frac{\frac{1}{\vartheta}\xi_0 p_i'(\xi_0)}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta} p_j(\xi_0)+\varsigma}+\frac{1}{\vartheta}p_i(\xi_0)=\frac{1}{\vartheta}\mu(w_i),$$

for some $w_i\in\mathscr{U}$. This is true for $i=1,\ 2,\ ...,\ \vartheta$. Since μ is convex, there exists a $w_0\in\mathscr{U}$ such that

$$\frac{\xi_0 Q'(\xi_0)}{Q(\xi_0)+\varsigma}+Q(\xi_0)=\mu(w_0),$$

where $Q(\xi)=\frac{1}{\vartheta}\sum\limits_{i=1}^{\vartheta} p_i(\xi)$. Hence

$$\frac{\xi Q'(\xi)}{Q(\xi)+\varsigma}+Q(\xi)\prec\mu(\xi). \tag{14}$$

Since $Re\{\mu\}$ is bounded and $Re(\mu(\xi)+\varsigma)>0$. By Lemma 1 with $Q(\xi)$ in (14) we obtain $Q(\xi)\prec\mu(\xi)$ $(\xi\in\mathscr{U})$. From (13), we have

$$\frac{\xi p_i'(\xi)}{Q(\xi)+\varsigma}+p_i(\xi)\prec\mu(\xi), \tag{15}$$

where $Q(\xi)\prec\mu(\xi)$. By Lemma 2 with $\mu(\xi)$ in (15) we obtain $p_i(\xi)\prec\mu(\xi)$ $(\xi\in\mathscr{U})$, $i=1,\ 2,\ ...,\ \vartheta$, that is

$$\frac{\xi[\mathscr{D}_s^{(\alpha,\,\beta)}F_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}F_j(\xi)} \prec \mu(\xi).$$

Now

$$F_i(\xi) = \frac{\varsigma+1}{\xi^{\varsigma}}\int_0^{\xi} t^{\varsigma-1}f_i(t)dt,\ (\varsigma > 0).$$

For every $i,\ 1 \le i \le \vartheta$,

$$\mathscr{D}_s^{(\alpha,\,\beta)}F_i(\xi) = \frac{\varsigma+1}{\xi^{\varsigma}}\int_0^{\xi} t^{\varsigma-1}\mathscr{D}_s^{(\alpha,\,\beta)}f_i(t)dt,$$

and hence

$$\sum_{i=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}F_i(\xi) = \frac{\varsigma+1}{\xi^{\varsigma}}\int_0^{\xi} t^{\varsigma-1}\sum_{i=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}f_i(t)dt$$

$$= \frac{\varsigma+1}{\xi^{\varsigma}}\int_0^{\xi} t^{\varsigma}g(t)dt,$$

where $g(t) = t^{-1}\sum_{i=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}f_i(t) \neq 0$, for $\xi \in \mathscr{U}$. Now define

$$\Omega(\xi) = \sum_{\kappa=1}^{\infty}\frac{\varsigma+1}{\varsigma+\kappa}\xi^{k-1},\ Re(\varsigma) > 0.$$

Then an easy calculations show that

$$\xi^{-1}\sum_{i=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}F_i(\xi) = (\Omega * g)(\xi) \neq 0.$$

Thus $F = \{F_1,\ F_2,\ ...,\ F_\vartheta\} \in \mathfrak{R}_s^{(\alpha,\,\beta)}(\vartheta;\ \mu)$. □

Remark 2 (i) Let $\vartheta = s = 1$, and $\alpha = \beta = 0$ in Theorem 2, we get the result obtained by Padmanabhan and Parvathem ([30], Theorem 2 with $a = 1$);

(ii) Let $\vartheta = s = \varsigma = 1,\ \alpha = \beta = 0$ and $\mu(\xi) = \dfrac{1-\xi}{1+\xi}$, in Theorem 2, we get the result obtained by Libera ([33], Theorem 1);

(iii) Let $\vartheta = s = 1,\ \alpha = \beta = 0$ and $\mu(\xi) = \dfrac{1+A\xi}{1+B\xi},\ (-1 \leq B < A \leq 1)$, in Theorem 2, we get the result obtained by Goel and Mehrok ([34], Lemma 3).

Theorem 3 If $f = \{f_1,\ f_2,\ ...,\ f_\vartheta\} \in \mathfrak{R}_s^{(\alpha+\frac{1}{s},\ \beta-\frac{1}{s})}(\vartheta;\ \mu)$, and $Re\{\mu\}$ is bounded in $\mathscr{U}$, then $f = \{f_1,\ f_2,\ ...,\ f_\vartheta\} \in \mathfrak{R}_s^{(\alpha,\ \beta)}(\vartheta;\ \mu)$ holds for $Re[\mu(\xi) + (s(1-\alpha)-1)] > 0$ in $\mathscr{U}$.

Proof. Let

$$p_i(\xi) = \frac{\vartheta\xi[\mathscr{D}_s^{(\alpha,\ \beta)} f_i(\xi)]'}{\sum_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha,\ \beta)} f_j(\xi)}\ (\xi \in \mathscr{U};\ i = 1,\ 2,\ ...,\ \vartheta), \tag{16}$$

where $\xi^{-1} \sum_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha,\ \beta)} f_j(\xi) \neq 0$. From (7) and (16), we have

$$\frac{1}{\vartheta} p_i(\xi) \sum_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha,\ \beta)} f_j(\xi) = s(1-\alpha)\mathscr{D}_s^{(\alpha+\frac{1}{s},\ \beta-\frac{1}{s})} f_i(\xi) - [s(1-\alpha)-1]\mathscr{D}_s^{(\alpha,\ \beta)} f_i(\xi). \tag{17}$$

Differentiating (17) with respect to ξ, we get

$$\frac{\xi}{\vartheta} p_i'(\xi) \sum_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha,\ \beta)} f_j(\xi) + \frac{\xi}{\vartheta} p_i(\xi) \sum_{j=1}^{m} [\mathscr{D}_s^{(\alpha,\ \beta)} f_j(\xi)]' = s(1-\alpha)\xi[\mathscr{D}_s^{(\alpha+\frac{1}{s},\ \beta-\frac{1}{s})} f_i(\xi)]'$$

$$- [s(1-\alpha)-1]\xi[\mathscr{D}_s^{(\alpha,\ \beta)} f_i(\xi)]'.$$

Using (16), we obtain

$$\frac{\xi}{\vartheta} p_i'(\xi) \sum_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha,\ \beta)} f_j(\xi) + p_i(\xi) \left[\frac{\xi}{\vartheta} \sum_{j=1}^{\vartheta} [\mathscr{D}_s^{(\alpha,\ \beta)} f_j(\xi)]' + \frac{[s(1-\alpha)-1]}{\vartheta} \sum_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha,\ \beta)} f_j(\xi) \right]$$

$$= s(1-\alpha)\xi[\mathscr{D}_s^{(\alpha+\frac{1}{s},\ \beta-\frac{1}{s})} f_i(\xi)]'.$$

Then

$$\frac{\frac{\xi}{\vartheta}p_i'(\xi)\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}f_j(\xi)}{\frac{\xi}{\vartheta}\sum_{j=1}^{\vartheta}[\mathscr{D}_s^{(\alpha,\,\beta)}f_j(\xi)]'+\frac{[s(1-\alpha)-1]}{\vartheta}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}f_j(\xi)}+p_i(\xi)$$

$$=\frac{s(1-\alpha)\xi[\mathscr{D}_s^{(\alpha+\frac{1}{s},\,\beta-\frac{1}{s})}f_i(\xi)]'}{\frac{\xi}{\vartheta}\sum_{j=1}^{\vartheta}[\mathscr{D}_s^{(\alpha,\,\beta)}f_j(\xi)]'+\frac{[s(1-\alpha)-1]}{\vartheta}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}f_j(\xi)}.$$

Using (7), we have

$$\frac{\frac{\xi}{\vartheta}p_i'(\xi)\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}f_j(\xi)}{\frac{\xi}{\vartheta}\sum_{j=1}^{\vartheta}[\mathscr{D}_s^{(\alpha,\,\beta)}f_j(\xi)]'+\frac{[s(1-\alpha)-1]}{\vartheta}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}f_j(\xi)}+p_i(\xi)=\frac{\xi[\mathscr{D}_s^{(\alpha+\frac{1}{s},\,\beta-\frac{1}{s})}f_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha+\frac{1}{s},\,\beta-\frac{1}{s})}f_j(\xi)}. \tag{18}$$

Using (16) in (18), we have

$$\frac{\xi p_i'(\xi)}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}p_j(\xi)+[s(1-\alpha)-1]}+p_i(\xi)=\frac{\xi[\mathscr{D}_s^{(\alpha+\frac{1}{s},\,\beta-\frac{1}{s})}f_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha+\frac{1}{s},\,\beta-\frac{1}{s})}f_j(\xi)}.$$

Since $f=\{f_1,\ f_2,\ ...,\ f_\vartheta\}\in\mathfrak{R}_s^{(\alpha+\frac{1}{s},\,\beta-\frac{1}{s})}(\vartheta;\ \mu)$, then we have

$$\frac{\xi p_i'(\xi)}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}p_j(\xi)+[s(1-\alpha)-1]}+p_i(\xi)=\frac{\xi[\mathscr{D}_s^{(\alpha+\frac{1}{s},\,\beta-\frac{1}{s})}f_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha+\frac{1}{s},\,\beta-\frac{1}{s})}f_j(\xi)}\prec\mu(\xi),\ i=1,\ 2,\ ...,\ \vartheta. \tag{19}$$

Therefore for any $\xi_0\in\mathscr{U}$, we have

$$\frac{\frac{1}{\vartheta}\xi_0 p_i'(\xi_0)}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}p_j(\xi_0)+[s(1-\alpha)-1]}+\frac{1}{\vartheta}p_i(\xi_0)=\frac{1}{\vartheta}\mu(w_i)$$

for some $w_0\in\mathscr{U}$. Since μ is convex, there exists a $w_i\in\mathscr{U}$, such that

$$\frac{\frac{\xi_0}{\vartheta}\sum\limits_{i=1}^{n} p_i'(\xi_0)}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta} p_j(\xi_0)+[s(1-\alpha)-1]}+\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} p_j(\xi_0)=\frac{1}{\vartheta}\sum_{i=1}^{\vartheta}\mu(w_i)=\mu(w_0).$$

Setting $Q(\xi)=\frac{1}{\vartheta}\sum\limits_{i=1}^{\vartheta} p_i(\xi)$, we have

$$\frac{\xi Q'(\xi)}{Q(\xi)+[s(1-\alpha)-1]}+Q(\xi)\prec\mu(\xi), \tag{20}$$

by Lemma 1 with $Q(\xi)$ in (20) we obtain $Q(\xi)\prec\mu(\xi)$. (19) gives us

$$\frac{\xi p_i'(\xi)}{Q(\xi)+[s(1-\alpha)-1]}+p_i(\xi)\prec\mu(\xi), \tag{21}$$

by Lemma 2 with $\mu(\xi)$ in (21) we obtain $p_i(\xi)\prec\mu(\xi)$, which implies that $f=\{f_1,\ f_2,\ \ldots,\ f_\vartheta\}\in\mathfrak{R}_s^{(\alpha,\ \beta)}(\vartheta;\ \mu)$. □

Remark 3 Let $\vartheta=s=1$, and $\alpha=\beta=0$, in Theorem 3, we get the result obtained by Padmanabhan and Parvathem ([30], Theorem 1 with $a=1$);

2.2 *The class $\mathfrak{C}_s^{(\alpha,\ \beta)}(\vartheta;\ \mu)$*

Definition 2 Let $\mathfrak{C}_s^{(\alpha,\ \beta)}(\vartheta;\ \mu)$ denote the class of functions $f\in\mathscr{A}$ which satisfies

$$\frac{\xi[\mathscr{D}_s^{(\alpha,\ \beta)}f(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\ \beta)}g_j(\xi)}\prec\mu(\xi)\ (\xi\in\mathscr{U}),$$

where $\xi^{-1}\sum\limits_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\ \beta)}g_j(\xi)\neq 0$ and $g=\{g_1,\ g_2,\ \ldots,\ g_\vartheta\}\in\mathfrak{R}_s^{(\alpha,\ \beta)}(\vartheta;\ \mu)$, μ is convex univalent in $\mathscr{U}$ with $\mu(0)=1$.

Remark 4 (i) Let $\vartheta=s=1,\ \alpha=\beta=0$ and $\mu(\xi)=\frac{1-\xi}{1+\xi}$, then $\mathfrak{C}_1^{(0,\ 0)}\left(1;\ \frac{1-\xi}{1+\xi}\right)=\mathfrak{C}$, where $\mathfrak{C}$ is the class of close-to-convex functions (see [35]);

(ii) Let $\vartheta=s=1,\ \alpha=\beta=0$ and $\mu(\xi)=\frac{1+A\xi}{1+B\xi},\ (-1\leq B<A\leq 1)$, then $\mathfrak{C}_1^{(0,\ 0)}\left(1;\ \frac{1+A\xi}{1+B\xi}\right)=\mathfrak{C}(A,\ B)$, where $\mathfrak{C}(A,\ B)$ is the generlalized class of close-to-convex functions (see [36, 37]).

Theorem 4 Let $f\in\mathfrak{C}_s^{(\alpha,\ \beta)}(\vartheta;\ \mu)$. If $Re(\mu)$ is bounded in $\mathscr{U}$ and $Re(\mu(\xi)+\tau)>0$, then

$$F(\xi)=\frac{\tau+1}{\xi^\tau}\int_0^\xi t^{\tau-1}f(t)dt\ (\xi\in\mathscr{U};\ \tau>0),$$

also belongs to $\mathfrak{C}_s^{(\alpha,\ \beta)}(\vartheta;\ \mu)$.

Proof. Since $f \in \mathfrak{C}_s^{(\alpha,\,\beta)}(\vartheta;\,\mu)$, then there exists $g = \{g_1,\, g_2,\, ...,\, g_\vartheta\} \in \mathfrak{R}_s^{(\alpha,\,\beta)}(\vartheta;\,\mu)$, such that

$$\frac{\xi[\mathscr{D}_s^{(\alpha,\,\beta)} f(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)} g_j(\xi)} \prec \mu(\xi)\ (\xi \in \mathscr{U}),$$

where $\xi^{-1}\sum\limits_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)} g_j(\xi) \neq 0$. Let

$$G_i(\xi) = \frac{\tau+1}{\xi^{\tau}}\int\limits_0^{\xi} t^{\tau-1} g_i(t)dt\ (Re\tau > 0).$$

Then by Theorem 2, we have $G = \{G_1,\, G_2,\, ...,\, G_\vartheta\} \in \mathfrak{R}_s^{(\alpha,\,\beta)}(\vartheta;\,\mu)$. Also let

$$p(\xi) = \frac{\xi[\mathscr{D}_s^{(\alpha,\,\beta)} F(\xi)]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)} G_j(\xi)}\ (\xi \in \mathscr{U}). \tag{22}$$

Now, from the definitions of G_i and F, we have

$$\xi[\mathscr{D}_s^{(\alpha,\,\beta)} G_i(\xi)]' + \tau\mathscr{D}_s^{(\alpha,\,\beta)} G_i(\xi) = (\tau+1)\mathscr{D}_s^{(\alpha,\,\beta)} g_i(\xi), \tag{23}$$

and

$$\xi[\mathscr{D}_s^{(\alpha,\,\beta)} F_i(\xi)]' + \tau\mathscr{D}_s^{(\alpha,\,\beta)} F(\xi) = (\tau+1)\mathscr{D}_s^{(\alpha,\,\beta)} f(\xi). \tag{24}$$

From (22) to (24), we have

$$\frac{1}{\vartheta}p(\xi)\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)} G_j(\xi) + \tau\mathscr{D}_s^{(\alpha,\,\beta)} F(\xi) = (\tau+1)\mathscr{D}_s^{(\alpha,\,\beta)} f(\xi). \tag{25}$$

Differentiating (25) with repect to ξ, and multiplying the resulting equation by ξ, we have

$$\frac{\xi}{\vartheta}p'(\xi)\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)} G_j(\xi) + \frac{\xi}{\vartheta}p(\xi)\sum_{j=1}^{\vartheta}[\mathscr{D}_s^{(\alpha,\,\beta)} G_j(\xi)]' + \tau\xi[\mathscr{D}_s^{(\alpha,\,\beta)} F(\xi)]' = (\tau+1)\xi[\mathscr{D}_s^{(\alpha,\,\beta)} f(\xi)]'. \tag{26}$$

From (22) into (26), we have

$$\frac{\xi}{\vartheta}p^{'}(\xi)\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}G_j(\xi)+\frac{\xi}{\vartheta}p(\xi)\sum_{j=1}^{\vartheta}[\mathscr{D}_s^{(\alpha,\,\beta)}G_j(\xi)]'+\tau\frac{p(\xi)}{\vartheta}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}G_j(\xi)=(\tau+1)\xi[\mathscr{D}_s^{(\alpha,\,\beta)}f(\xi)]'.$$

Hence, we get

$$\frac{\frac{\xi}{\vartheta}p^{'}(\xi)\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}G_j(\xi)}{\frac{\xi}{\vartheta}\sum_{j=1}^{\vartheta}[\mathscr{D}_s^{(\alpha,\,\beta)}G_j(\xi)]'+\frac{\tau}{\vartheta}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}G_j(\xi)}+p(\xi)$$

$$=\frac{(\tau+1)\xi[\mathscr{D}_s^{(\alpha,\,\beta)}f(\xi)]^{'}}{\frac{\xi}{\vartheta}\sum_{j=1}^{\vartheta}[\mathscr{D}_s^{(\alpha,\,\beta)}G_j(\xi)]'+\frac{\tau}{\vartheta}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}G_j(\xi)}=\frac{\xi[\mathscr{D}_s^{(\alpha,\,\beta)}f(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}g_j(\xi)}\ \text{(by using (23)).}$$

From the above, we have

$$\frac{\xi p^{'}(\xi)}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}Q_j(\xi)+\tau}+p(\xi)=\frac{\xi[\mathscr{D}_s^{(\alpha,\,\beta)}f(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}g_j(\xi)}\prec\mu(\xi),$$

where $Q_j(\xi)=\dfrac{\xi[\mathscr{D}_s^{(\alpha,\,\beta)}G_j(\xi)]^{'}}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}G_j(\xi)}$. Now $Q_j(\xi)\prec\mu(\xi),\ j=1,\,2,\,...,\,\vartheta$, since $G=\{G_1,\,G_2,\,...,\,G_\vartheta\}\in\mathfrak{R}_s^{(\alpha,\,\beta)}(\vartheta;\,\mu)$ and μ is a convex univalent. Since $Re(\mu(\xi)+\tau)>0$, Lemma 2 applied indicates that $p(\xi)\prec\mu(\xi)$, hence $F\in\mathfrak{C}_s^{(\alpha,\,\beta)}(\vartheta;\,\mu)$. □

Remark 5 (i) Let $\vartheta=s=1$, and $\alpha=\beta=0$ in Theorem 4, we get the result obtained by Padmanabhan and Parvathem ([30], Theorem 4 with $a=1$);

(ii) Let $\vartheta=s=1,\ \alpha=\beta=0$ and $\mu(\xi)=\dfrac{1-\xi}{1+\xi}$, in Theorem 2, we get the result obtained by Libera ([33], Theorem 1).

Theorem 5 If $f\in\mathfrak{C}_s^{(\alpha+\frac{1}{s},\,\beta-\frac{1}{s})}(\vartheta;\,\mu)$ and μ is bounded in $\mathscr{U}$, then $f\in\mathfrak{C}_s^{(\alpha,\,\beta)}(\vartheta;\,\mu)$ holds for $Re[\mu(\xi)+(s(1-\alpha)-1)]>0$ in $\mathscr{U}$.

Proof. The proof of the theorem is similar to the proof of Theorem 3. □

Remark 6 Let $\vartheta=s=1$, and $\alpha=\beta=0$ in Theorem 5, we get the result obtained by Padmanabhan and Parvathem ([30], Theorem 3 with $a=1$);

2.3 *The class* $\wp_s^{(\alpha,\,\beta)}(\vartheta;\,\delta,\,\mu)$

Definition 3 Let $\wp_s^{(\alpha,\,\beta)}(\vartheta;\,\delta,\,\mu),\ \delta\geq 0$, denote the class of function $f\in\mathscr{A}$ satisfies

$$\mathfrak{J}(\delta;\, f;\, g_1,\, g_2,\, ...,\, g_\vartheta)(\xi) = \left\{ \delta \frac{\xi\left[\mathscr{D}_s^{(\alpha+\frac{1}{s},\, \beta-\frac{1}{s})} f(\xi)\right]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha+\frac{1}{s},\, \beta-\frac{1}{s})} g_j(\xi)} + (1-\delta)\frac{\xi\left[\mathscr{D}_s^{(\alpha,\, \beta)} f(\xi)\right]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha,\, \beta)} g_j(\xi)} \right\} \prec \mu(\xi)\ (\xi \in \mathscr{U}),$$

where $g = \{g_1,\ g_{2,\ ...,}\ g_\vartheta\} \in \mathfrak{R}_s^{(\alpha,\, \beta)}(\vartheta;\ \mu)$, $\xi^{-1}\sum_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha+\frac{1}{s},\, \beta-\frac{1}{s})} g_j(\xi) \neq 0$ in $\mathscr{U}$, μ is convex univalent in $\mathscr{U}$ with $\mu(0) = 1$.

Remark 7 We note that $\wp_s^{(\alpha,\, \beta)}(\vartheta;\ 0,\ \mu) = \mathfrak{C}_s^{(\alpha,\, \beta)}(\vartheta;\ \mu)$.

Theorem 6 If $f \in \wp_s^{(\alpha,\, \beta)}(\vartheta;\ \delta,\ \mu)$ and $Re\{\mu\}$ is bounded in $\mathscr{U}$, then $f \in \wp_s^{(\alpha,\, \beta)}(\vartheta;\ 0,\ \mu) = \mathfrak{C}_s^{(\alpha,\, \beta)}(\vartheta;\ \mu)$ hold for $Re[\mu(\xi) + (s(1-\alpha)-1)] > 0$.

Proof. For $\delta = 0$, the theorem is unimportant, we can presume that $\delta \neq 0$. Let

$$p(\xi) = \frac{\xi\left[\mathscr{D}_s^{(\alpha,\, \beta)} f(\xi)\right]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha,\, \beta)} g_j(\xi)}\ (\xi \in \mathscr{U}),$$

where $\xi^{-1}\sum_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha,\, \beta)} g_j(\xi) \neq 0$. Then an easy calculations shows that

$$\frac{\xi p'(\xi)}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} q_j(\xi) + [s(1-\alpha)-1]} + p(\xi) = \frac{\xi\left[\mathscr{D}_s^{(\alpha+\frac{1}{s},\, \beta-\frac{1}{s})} f(\xi)\right]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha+\frac{1}{s},\, \beta-\frac{1}{s})} g_j(\xi)},$$

where $q_j(\xi) = \dfrac{\xi\left[\mathscr{D}_s^{(\alpha,\, \beta)} g_i(\xi)\right]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha,\, \beta)} g_j(\xi)}$. Also $\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} q_j(\xi) \prec \mu(\xi)$. Since $f \in \wp_s^{(\alpha,\, \beta)}(\vartheta;\ \delta,\ \mu)$, we have

$$\mathfrak{J}(\delta;\, f;\, g_1,\, g_2,\, ...,\, g_\vartheta)(\xi) = \frac{\delta \xi p'(\xi)}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} q_j(\xi) + [s(1-\alpha)-1]} + p(\xi) \prec \mu(\xi). \tag{27}$$

By Lemma 2 with $\mu(\xi)$ in (27) we obtain $p(\xi) \prec \mu(\xi)$ which implies $f \in \mathfrak{C}_s^{(\alpha,\, \beta)}(\vartheta;\ \mu)$. □

Remark 8 (i) Let $\vartheta = s = 1$, and $\alpha = \beta = 0$ in Theorem 6, we get the result obtained by Padmanabhan and Parvathem ([30], Theorem 5 with $a = 1$);

(ii) Let $\vartheta = s = 1,\ \alpha = \beta = 0$ and $\mu(\xi) = \dfrac{1-\xi}{1+\xi}$, in Theorem 6 we get the result obtained by Chichra ([38], Theorem 1);

(iii) Let $\vartheta = s = 1,\ \alpha = \beta = 0$ and $\mu(\xi) = \dfrac{1+(2\rho-1)\xi}{1+\xi}(0 \leq \rho < 1)$, in Theorem 6 we get the result obtained by Zmorovich and Pokhilevich ([39], Theorem 1).

Theorem 7 For $\delta > \lambda \geq 0$ and $Re\mu(\xi)$ is bounded in $\mathscr{U}$, then $\wp_s^{(\alpha,\,\beta)}(\vartheta;\,\delta,\,\mu) \subset \wp_s^{(\alpha,\,\beta)}(\vartheta;\,\lambda,\,\mu)$.

Proof. The case $\lambda = 0$ was treated in the previous theorem. Hence we assume that $\lambda \neq 0$. Suppose that $f \in \wp_s^{(\alpha,\,\beta)}(\vartheta;\,\delta,\,\mu)$. Then

$$\mathfrak{J}(\delta;\,f;\,g_1,\,g_2,\,...,\,g_\vartheta)(\xi) \prec \mu(\xi). \tag{28}$$

Let ξ_1 be any arbitrary point in $\mathscr{U}$. Then

$$\mathfrak{J}(\delta;\,f;\,g_1,\,g_2,\,...,\,g_\vartheta)(\xi_1) \prec \mu(\mathscr{U}).$$

From Theorem 6, we have

$$\frac{\xi\left[\mathscr{D}_s^{(\alpha,\,\beta)}f(\xi)\right]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}g_j(\xi)} \prec \mu(\xi). \tag{29}$$

Now

$$\mathfrak{J}(\lambda;\,f;\,g_1,\,g_2,\,...,\,g_\vartheta)(\xi) = \left(1-\frac{\lambda}{\delta}\right)\frac{\xi\left[\mathscr{D}_s^{(\alpha,\,\beta)}f(\xi)\right]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}g_j(\xi)} + \frac{\lambda}{\delta}\mathfrak{J}(\delta;\,f;\,g_1,\,g_2,\,...,\,g_\vartheta)(\xi).$$

From (28) and (29) it follows that

$$\frac{\xi_1\left[\mathscr{D}_s^{(\alpha,\,\beta)}f(\xi_1)\right]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}g_j(\xi_1)} \prec \mu(\mathscr{U})$$

and

$$\delta\frac{\xi_1\left[\mathscr{D}_s^{(\alpha+\frac{1}{s},\,\beta-\frac{1}{s})}f(\xi_1)\right]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha+\frac{1}{s},\,\beta-\frac{1}{s})}g_j(\xi_1)} + (1-\delta)\frac{\xi_1\left[\mathscr{D}_s^{(\alpha,\,\beta)}f(\xi_1)\right]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta}\mathscr{D}_s^{(\alpha,\,\beta)}g_j(\xi_1)} \prec \mu(\mathscr{U}).$$

Now $\mu(\mathscr{U})$ is convex and $\dfrac{\lambda}{\delta} < 1$, hence we have $\mathfrak{J}(\lambda;\,f;\,g_1,\,g_2,\,...,\,g_\vartheta)(\xi_1) \prec \mu(\mathscr{U})$, showing that $f \in \wp_s^{(\alpha,\,\beta)}(\vartheta;\,\lambda,\,\mu)$. □

Remark 9 (i) Let $\vartheta = s = 1$, and $\alpha = \beta = 0$ in Theorem 7, Padmanabhan and Parvathem ([30], Theorem 6 with $a = 1$);

(ii) Let $\vartheta = s = 1,\ \alpha = \beta = 0$ and $\mu(\xi) = \dfrac{1-\xi}{1+\xi}$, in Theorem 7 we obtain the result obtained by Chichra ([38], Theorem 2);

(iii) Let $\vartheta = s = 1,\ \alpha = \beta = 0$ and $\mu(\xi) = \dfrac{1+(2\rho-1)\xi}{1+\xi}$ $(0 \leq \rho < 1)$, in Theorem 7 we obtain the result obtained by Zmorovich and Pokhilevich ([39], Theorem 1).

2.4 *The class* $\mathfrak{R}_s^{(\alpha,\,\beta)}(\vartheta;\ \delta,\ \mu)$

Definition 4 Let $\mathfrak{R}_s^{(\alpha,\,\beta)}(\vartheta;\ \delta,\ \mu),\ \delta \geq 0$, denote the class of function $f \in \mathscr{A}$ satisfies

$$\mathfrak{J}(\delta;\ f;\ f_1,\ f_2,\ \ldots,\ f_\vartheta)(\xi) = \left\{ \delta \frac{\xi\left[\mathscr{D}_s^{(\alpha+\frac{1}{s},\ \beta-\frac{1}{s})} f_i(\xi)\right]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha+\frac{1}{s},\ \beta-\frac{1}{s})} f_j(\xi)} + (1-\delta) \frac{\xi\left[\mathscr{D}_s^{(\alpha,\ \beta)} f_i(\xi)\right]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha,\ \beta)} f_j(\xi)} \right\} \prec \mu(\xi)\ (\xi \in \mathscr{U}),$$

where $f = \{f_1,\ f_2,\ \ldots,\ f_\vartheta\} \in \mathfrak{R}_s^{(\alpha,\,\beta)}(\vartheta;\ \mu)$ and $\xi^{-1}\sum\limits_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha+\frac{1}{s},\ \beta-\frac{1}{s})} f_j(\xi) \neq 0$ in $\mathscr{U}$, μ is convex univalent in $\mathscr{U}$ with $\mu(0) = 1$.

We note that $\mathfrak{R}_s^{(\alpha,\,\beta)}(\vartheta;\ 0;\ \mu) = \mathfrak{R}_s^{(\alpha,\,\beta)}(\vartheta;\ \mu)$.

Theorem 8 If $f \in \mathfrak{R}_s^{(\alpha,\,\beta)}(\vartheta;\ \delta;\ \mu)$ and $Re\{\mu\}$ is bounded in $\mathscr{U}$, then $f \in \mathfrak{R}_s^{(\alpha,\,\beta)}(\vartheta;\ 0;\ \mu) = \mathfrak{R}_s^{(\alpha,\,\beta)}(\vartheta;\ \mu)$ hold for $Re[\mu(\xi) + (s(1-\alpha)-1)] > 0$.

Proof. For $\delta = 0$, the theorem is is unimportant, we can presume that $\delta \neq 0$. Let

$$p(\xi) = \frac{\xi\left[\mathscr{D}_s^{(\alpha,\ \beta)} f_i(\xi)\right]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha,\ \beta)} f_j(\xi)}\ (\xi \in \mathscr{U}),$$

where $\xi^{-1}\sum\limits_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha,\ \beta)} f_j(\xi) \neq 0$. Then an easy calculations shows that

$$\frac{\xi p'(\xi)}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta} q_j(\xi) + [s(1-\alpha)-1]} + p(\xi) = \frac{\xi\left[\mathscr{D}_s^{(\alpha+\frac{1}{s},\ \beta-\frac{1}{s})} f_i(\xi)\right]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha+\frac{1}{s},\ \beta-\frac{1}{s})} f_j(\xi)},$$

where $q_j(\xi) = \dfrac{\xi\left[\mathscr{D}_s^{(\alpha,\ \beta)} f_i(\xi)\right]'}{\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta} \mathscr{D}_s^{(\alpha,\ \beta)} f_j(\xi)}$. Also $\frac{1}{\vartheta}\sum\limits_{j=1}^{\vartheta} q_j(\xi) \prec \mu(\xi)$. Since $f(\xi) \in \mathfrak{R}_s^{(\alpha,\,\beta)}(\vartheta;\ \delta;\ \mu)$, we have

$$J(\delta;\, f;\, f_1,\, f_2,\, \dots,\, f_\vartheta)(\xi) = \frac{\delta \xi p'(\xi)}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} q_j(\xi) + [s(1-\alpha)-1]} + p(\xi) \prec \mu(\xi). \tag{30}$$

By Lemma 2 with $\mu(\xi)$ in (30) we obtain $p(\xi) \prec \mu(\xi)$ which implies $f \in \mathfrak{N}_s^{(\alpha,\, \beta)}(\vartheta;\, \mu)$. □

Theorem 9 For $\delta > \lambda \geq 0$ and $Re\{\mu\}$ is bounded in $\mathscr{U}$, then $\mathfrak{N}_s^{(\alpha,\, \beta)}(\vartheta;\, \delta;\, \mu) \subset \mathfrak{N}_s^{(\alpha,\, \beta)}(\vartheta;\, \lambda;\, \mu)$.

Proof. The proof of the theorem is similar to the proof of Theorem 7. □

Remark 10 (i) Let $\vartheta = s = 1$, and $\alpha = \beta = 0$ in Theorem 8, Padmanabhan and Parvathem [30], Theorem 7 with $a = 1$;

(ii) Let $\vartheta = s = 1,\ \alpha = \beta = 0$ and $\mu(\xi) = \dfrac{1-\xi}{1+\xi}$, in Theorem 8 we get the result obtained by Al-Amiri [40], Theorem 1;

(iii) Let $\vartheta = s = 1,\ \alpha = \beta = 0$ and $\mu(\xi) = \dfrac{1+(2\rho-1)\xi}{1+\xi}(0 \leq \rho < 1)$, in Theorem 8 we obtain the result obtained by Miller et al. ([41], Theorem 1).

3. Conclusion

As more applications and theoretical insights are discovered, the Zeta-Riemann fractional differential operator is poised to become an essential instrument in the modern mathematical toolkit. To investigate some classes of univalent functions, the theory of differential subordination is a result of the new findings in this paper. The idea for the study needed, Zeta- Riemann Fractional differential operator $\mathscr{D}_s^{(\alpha,\, \beta)} f(\xi)$, and the rationale behind the topic's investigation are all contained in Section 1's Introduction. This operator is applied in this investigation to define and investigate certain specific classes of univalent functions using the theory of differential subordination. The main findings are found in Sections 2, 3, 4, 6, where classes $\mathfrak{N}_s^{(\alpha,\, \beta)}(\vartheta;\, \mu)$, given in Definition 1, $\mathfrak{C}_s^{(\alpha,\, \beta)}(\vartheta;\, \mu)$, described in Definition 2, $\wp_s^{(\alpha,\, \beta)}(\vartheta;\, \delta,\, \mu)$ seen in Definition 3 and $\mathfrak{N}_s^{(\alpha,\, \beta)}(\vartheta;\, \delta,\, \mu)$, presented in Definition 4 are investigated, respectively. Also, we proved combinations of functions belonging to those classes and inclusion relations.

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Data availability statement

The data used to support the findings of this study are included within the article.

Conflict of interest

The authors declare no competing financial interest.

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