

Research Article

Approximate Solution of Time-Fractional Ivancevic Option Pricing Model

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Abstract: Option pricing is a vital area in financial mathematics that consists of the development of efficient models. The Ivancevic Option Pricing Model (IOPM) is a wave-based, nonlinear, adaptive version of the traditional Black-Scholes (BS) model. In this work, we use an efficient method to find approximate solutions to the time-fractional IOPM. The fractional derivative is treated using two types of kernels, namely singular and nonsingular. Convergence and the uniqueness of the proposed method are also discussed. Two test problems of adaptive market potential that are nonzero are examined to illustrate the method efficiency, simplicity, and straightforwardness in its application. Results are presented in 2D and 3D graphs to understand the behaviour of the solutions.

Keywords: Caputo derivative, Ivancevic Option Pricing Model (IOPM), Natural Transform (NT), Caputo-Fabrizio derivative, Atangana-Baleanu derivative

MSC: 35R11, 26A33, 35A22, 91G20

1. Introduction

In classical financial mathematics, the primary framework for modeling option pricing is the Black-Scholes (BS) model. Its formulation is based on assumptions such as steady volatility of the underlying asset, absence of transaction fees, log-normally distributed asset prices, and efficient markets. As an important non-linear diffusion-type stochastic partial differential equation arising in mathematical finance, the BS model adequately addresses the problem of valuing the pricing of European and American call/put options [1]. However, realistic financial markets are complex, nonlinear, and frequently exhibit volatility smiles/skews, heavy tails in asset returns, market disturbances, and memory effects [2, 3]. To tackle the limitations associated with the BS model, V. G. Ivancevic employed quantum-probability formalism and nonlinear wave theory to develop an adaptive waveform alternative for the BS model known as the Ivancevic Option Pricing Model (IOPM) [4, 5]. His approach was to consider the option price dynamics not merely as stochastic diffusion but rather as a process of nonlinear wave propagation where volatility is modeled as an adaptive wave process that evolves with market conditions rather than being fixed. The IOPM is typically expressed in terms of a nonlinear Schrödinger-type equation for the option price wave function $p(x, t)$ as [4, 5]

$$i p_t = -\frac{\sigma^2}{2} p_{xx} - \lambda |p|^2 p, \quad i = \sqrt{-1}, \quad t \geq 0, \quad (1)$$

$$p(x, 0) = \phi(x),$$

where p represents the evolution of the option price with respect to time t ($0 \leq t < T$) and the asset price x ($0 \leq x < \infty$). The parameter σ , which measures the degree of volatility in the price of the financial asset over a given time, represents the dispersion frequency coefficient of the asset price and may be taken either as a constant or a stochastic process. The term $|p|^2$ denotes the probability density function of the option price. The Landau coefficient, denoted by the adaptive market potential parameter $\lambda = \lambda(r, w)$, captures nonlinear feedback and self-organization in markets.

Many physical processes that give rise to Nonlinear Differential Equations (NLDEs) are known to exhibit complex non-locality features that cannot be completely captured within the framework of classical calculus. Numerous studies have demonstrated that these non-locality characteristics encompass a range of intricate global attributes, including non-Markovian processes, fractal behaviours, random walks, stochastic and anomalous diffusion processes, along with memory and hereditary effects. In contrast to the local attributes of integer order NLDEs, the non-local behaviours of Nonlinear Fractional Differential Equations (NLFDEs) permit that the future states of any physical system are influenced not only by the present state but also by its prior states. Consequently, fractional calculus, a subdivision of calculus, addresses the concepts, characteristics, analysis, and applications of fractional order differential and integral operators, thereby extending the principles of integer-order differential and integral operations to fractional orders [6, 7].

By employing tools from fractional calculus, it is convenient to formulate and study many real-world problems whose intricate dynamical features can not adequately modeled by integer-order calculus. This has paved the way for many studies on both new and existing mathematical models arising in physics [8, 9], biology [10–12], finance [13], and chemistry [14] within the context of NLFDEs. In this direction, many types of fractional differential operators, such as the Hadamard, Riesz, Caputo, Riemann-Liouville, Grunwald-Letnikov, Jumarie, Caputo-Fabrizio (CF), and Atangana-Baleanu (ABC) derivatives have been developed and studied for different types of NLFDEs [15–20]. However, the Caputo, CF, and ABC fractional derivatives have received considerable research attention as they have been widely applied in many recent research investigations to model numerous real-world processes. In terms of the types of kernels incorporated in their definitions, the Caputo derivative possesses the singular power-type kernel. In contrast, ABC and the CF fractional derivatives admit the non-singular Mittag-Leffler and exponential kernels, respectively. Moreover, due to the complex evolutionary dynamics inherent in many NLFDEs, there are no known methods for obtaining their exact solutions. However, a variety of approximation techniques have been developed and employed with reliable accuracy to extract analytical or numerical solutions for NLFDEs. Some of the notable methods include the Reduced Differential Transform Technique (RDTM) [21], q -Homotopy Analysis Transform Method (q -HATM) [22, 23], Natural Transform Decomposition Method (NTDM) [24, 25], and Residual Power Series Method (RPSM) [26].

In realistic financial markets, stock prices, volatility, and interest rates often exhibit features such as memory, long-range dependence, heavy tails, market crashes, and jumps that classical calculus cannot effectively capture. However, NLFDEs describing financial models are known to naturally incorporate these features along with others such as anomalous diffusion and rough volatility, making them more versatile and realistic than their integer-order counterparts. They are also able to generate distributions that are closer to empirically observed market situations, with Lévy-stable laws and power-law behavior. Furthermore, they allow for better risk assessment and more realistic option prices, and they account for market microstructure noise and memory. This motivation has allowed many mathematicians to reformulate and study the IOPM (1) as well as a wide range of other partial differential equations arising in diverse scientific and engineering domains within the context of fractional calculus. This work aims to utilise NTDM to obtain analytical solutions for the Time-Fractional Ivancevic Option Pricing Model (TFIOPM).

$${}_0^{\Theta(\mu)} D_t^\mu p = -\frac{\sigma^2}{2} p_{xx} - \lambda |p|^2 p, \quad 0 < \mu \leq 1, \quad (2)$$

$$p(x, 0) = \phi(x).$$

Here, ${}_0^{\Theta(\mu)} D_t^\mu$ denotes the time-fractional derivative in the Caputo, CF or ABC sense. In existing literature, several solution approaches have been applied to obtain various types of solutions for TFIOPM (2) as well as its integer-order counterpart (1). For instance, exact and approximate series-type solutions were obtained using the RDTM [21], q -HATM [27], Elsakki-Adomian decomposition technique [28], RPSM, and invariant subspace technique [29]. Various types of solitary wave solutions have also been obtained for the TFIOPM (2) via different methods [30, 31]. However, the NTDM is applied here for the first time to obtain analytical solutions of the TFIOPM (2) within the framework of the aforementioned three types of fractional derivatives. The NTDM is a hybrid integral transform method that elegantly combines the Adomian Decomposition method [32] with the Natural Transform (NT) method [33]. The NT bears some similarities with the Fourier and Laplace transforms, but it has some advantages over them. It has been used to solve a variety of differential, integral, and functional equations. Some advantages of the NTDM over other similar methods include its freedom from round-off errors and absence of linearization, perturbation, or discretization requirements. Apart from being straightforward, efficient, and easy to implement, the NTDM method has been recently employed to obtain convergent series-type solutions for many linear and non-linear time-fractional differential equations, such as the Noyes Field model [14], coupled Burger equations [34], and Fisher's equation [35].

The rest of this work is organized as follows: Fundamental definitions essential to this manuscript are outlined in Section 2. Section 3 presents the solution procedure of the NTDM for NLFDEs with either singular or non-singular kernels. Convergence and uniqueness analysis of the NTDM is provided in Section 4. In Section 5, the NTDM is employed to study some versions of the TFIOPM equations, and the numerical solutions are presented through various types of graphs. Section 6 provides a brief conclusion of this paper based on the results obtained.

2. Preliminaries

This section presents definitions concerning the Caputo, CF, and ABC fractional derivatives. We also introduce necessary preliminaries on NT.

Definition 1 [18] The Caputo fractional derivative of a function $g \in C_v^m$, $v \geq -1$, is given by

$${}_0^C D_t^\mu p(t) = \frac{1}{\Gamma(m-\mu)} \int_0^t (t-\varsigma)^{m-\mu-1} p^{(m)}(\varsigma) d\varsigma, \quad m-1 < \mu \leq m \in \mathbb{N}, \quad t > 0. \quad (3)$$

Definition 2 [19] The CF fractional derivative of the function $p \in H^1(a, b)$ is given by

$${}_a^{CF} D_t^\mu p(t) = \frac{M(\mu)}{1-\mu} \int_0^t \exp\left(\frac{-(t-\varsigma)\mu}{1-\mu}\right) p'(\varsigma) d\varsigma, \quad (4)$$

where $M(\mu)$ denotes the normalization function.

Definition 3 [20] The ABC fractional derivative of a function $p \in H^1(a, b)$ is given by

$${}^{\text{ABC}}D_t^\mu \mathfrak{p}(t) = \frac{M(\mu)}{1-\mu} \int_a^t E_\mu \left(\frac{-(t-\zeta)\mu}{1-\mu} \right) \mathfrak{p}'(\zeta) d\zeta, \quad (5)$$

where $E_\mu(\cdot)$ represents the Mittag-Leffler function and $M(\mu)$ is the normalization function.

Definition 4 [14, 36] The natural transform of $\mathfrak{p}(t)$ is defined as

$$\mathcal{N}[\mathfrak{p}(t)] = \mathfrak{p}(s, v) = \int_0^\infty e^{-st} \mathfrak{p}(vt) dt, \quad v, s \in (0, \infty). \quad (6)$$

Definition 5 [14, 36] The inverse natural transform of $\mathfrak{p}(s, v)$ is provided by

$$\mathcal{N}^{-1}[\mathfrak{p}(s, v)] = \mathfrak{p}(t), \quad \forall t \geq 0. \quad (7)$$

Definition 6 [14, 36] The natural transform of $D_t^\mu \mathfrak{p}(t)$ in the Caputo manner is given as

$$\mathcal{N} [{}^{\text{C}}D_t^\mu \mathfrak{p}(t)] = \left(\frac{s}{v}\right)^\mu \left(\mathcal{N}[\mathfrak{p}(t)] - \left(\frac{1}{s}\right) \mathfrak{p}(0) \right). \quad (8)$$

Definition 7 [14, 36] The natural transform of $D_t^\mu \mathfrak{p}(t)$ using the CF derivative is defined as

$$\mathcal{N} [{}^{\text{CF}}D_t^\mu \mathfrak{p}(t)] = \frac{1}{f(\mu, v, s)} \left(\mathcal{N}[\mathfrak{p}(t)] - \left(\frac{1}{s}\right) \mathfrak{p}(0) \right), \quad (9)$$

where $f(\mu, v, s) = 1 + \mu \left(\frac{v}{s}\right) - \mu$.

Definition 8 [14, 36] The natural transform of $D_t^\mu \mathfrak{p}(t)$ in ABC derivative is given by

$$\mathcal{N} [{}^{\text{ABC}}D_t^\mu \mathfrak{p}(t)] = \frac{1}{\psi(\mu, v, s)} \left(\mathcal{N}[\mathfrak{p}(t)] - \left(\frac{1}{s}\right) \mathfrak{p}(0) \right), \quad (10)$$

where $\psi(\mu, v, s) = 1 + \mu \left(\frac{v}{s}\right)^\mu - \mu$.

3. Methodology

In this section, we briefly discuss the solution methodology of the NTDM by considering the following general NLFDE:

$${}^{\Theta(\mu)}D_t^\mu \mathfrak{p} = \mathcal{M}\mathfrak{p} + \mathcal{R}\mathfrak{p} + \phi, \quad (11)$$

$$\mathfrak{p}(x, 0) = q(x).$$

Here, ${}^{\Theta(\mu)}D_t^\mu$ denotes the fractional differential operator either in the Caputo, CF or ABC sense, \mathcal{R} and \mathcal{M} represent nonlinear and linear partial differential operators, respectively, while ϕ denotes the non-homogeneous term. An application of the NT as well as its differentiation property on (11) yields

$$\frac{1}{g(\mu, v, s)} \left(\mathcal{N}[\mathbf{p}] - \frac{q(x)}{s} \right) = \mathcal{N}[\Xi], \quad (12)$$

where $\Xi = \mathcal{M}\mathbf{p} + \mathcal{R}\mathbf{p} + \phi$ and

$$g(\mu, v, s) = \begin{cases} \left(\frac{v}{s}\right)^\mu, & \text{if } \Theta(\mu) = C, \\ f(\mu, v, s), & \text{if } \Theta(\mu) = \text{CF}, \\ \psi(\mu, v, s), & \text{if } \Theta(\mu) = \text{ABC}. \end{cases} \quad (13)$$

An application of the inverse NT on (12) yields

$$\mathbf{p} = \mathcal{N}^{-1} \left[\frac{q(x)}{s} + g(\mu, v, s) \mathcal{N}[\Xi] \right]. \quad (14)$$

The NTDM assumes the solution of (11) in the form

$$\mathbf{p}(x, t) = \sum_{l=0}^{\infty} \mathbf{p}_l, \quad (15)$$

while the nonlinear component is expressed as

$$\mathcal{N}[\mathbf{p}] = \sum_{l=0}^{\infty} A_l, \quad (16)$$

with A_l denoting the Adomian polynomials [32]. Inserting (15) and (16) into (14) yields

$$\sum_{l=0}^{\infty} \mathbf{p}_l = \mathcal{N}^{-1} \left[\frac{q(x)}{s} + g(\mu, v, s) \mathcal{N}[\phi] \right] + \mathcal{N}^{-1} \left[g(\mu, v, s) \mathcal{N} \left[\mathcal{M} \sum_{l=0}^{\infty} \mathbf{p}_n + \sum_{l=0}^{\infty} A_l \right] \right]. \quad (17)$$

By equating terms on either sides of (17), we obtain the following recurrence relations

$$\mathbf{p}_0^{\Theta(\mu)}(x, t) = \mathcal{N}^{-1} \left[\frac{q(x)}{s} + g(\mu, v, s) \mathcal{N}[\phi] \right], \quad (18)$$

$$\mathbf{p}_{l+1}^{\Theta(\mu)}(x, t) = \mathcal{N}^{-1} \left[g(\mu, v, s) \mathcal{N}[\mathcal{M}\mathbf{p}_l + A_l] \right], \quad l = 1, 2, \dots$$

Finally, we insert (18) into (15), to get the analytical solution of (11) as

$$\mathbf{p}^{\Theta(\mu)}(x, t) = \mathbf{p}_0^{\Theta(\mu)}(x, t) + \mathbf{p}_1^{\Theta(\mu)}(x, t) + \mathbf{p}_2^{\Theta(\mu)}(x, t) + \dots \quad (19)$$

Accordingly, the successive approximates $\mathbf{p}_l(x, t)$ of the solution $\mathbf{p}(x, t)$ of (11) is given as

$$\mathbf{p} = \lim_{l \rightarrow \infty} \mathbf{p}_l, \quad l \geq 1.$$

4. Convergence analysis

The convergence analysis of the NTDM is discussed in this section. However, we will only restrict ourselves to the case where the NLFDE (11) is considered within the Caputo framework. We note that the idea can also be generally extended to the case where the NLFDE (11) is considered either within the CF or the ABC framework.

Theorem 1 The solution (19) obtained for (11) in the Caputo sense is unique if $0 < (\varepsilon_1 + \varepsilon_2) \frac{t^\mu}{\Gamma(1 + \mu)} < 1$ for some positive constants $\varepsilon_1, \varepsilon_2$.

Proof. Let $F = (C[J], \|\cdot\|)$ be a Banach space of all continuous functions on $J = [0, T]$, with norm $\|\phi(t)\| = \max_{t \in J} |\phi(t)|$ and $G : F \rightarrow F$ a nonlinear mapping defined by

$$\mathbf{p}_{l+1}^C(X, t) = \mathbf{p}_0^C + \mathcal{N}^{-1} \left[\left(\frac{v}{s} \right)^\mu \mathcal{N}[\mathcal{M}\mathbf{p}_l + \mathcal{R}\mathbf{p}_l] \right], \quad l \geq 0.$$

Suppose that $|\mathcal{M}\mathbf{p} - \mathcal{M}\mathbf{p}^*| < \varepsilon_1 |\mathbf{p} - \mathbf{p}^*|$ and $|\mathcal{R}\mathbf{p} - \mathcal{R}\mathbf{p}^*| < \varepsilon_2 |\mathbf{p} - \mathbf{p}^*|$, where ε_1 and ε_2 denote Lipschitz constants and $\mathbf{p} := \mathbf{p}(x, t)$ and $\mathbf{p}^* := \mathbf{p}^*(x, t)$ are distinct solutions of (11). Then we have

$$\begin{aligned} \|\mathbf{G}\mathbf{p} - \mathbf{G}\mathbf{p}^*\| &= \max_{t \in J} \left| \mathcal{N}^{-1} \left[\left(\frac{v}{s} \right)^\mu \mathcal{N}[\mathcal{M}\mathbf{p} + \mathcal{R}\mathbf{p}] \right] - \mathcal{N}^{-1} \left[\left(\frac{v}{s} \right)^\mu \mathcal{N}[\mathcal{M}\mathbf{p}^* + \mathcal{R}\mathbf{p}^*] \right] \right| \\ &\leq \max_{t \in J} \left| \mathcal{N}^{-1} \left[\left(\frac{v}{s} \right)^\mu \mathcal{N}[\mathcal{M}\mathbf{p} - \mathcal{M}\mathbf{p}^*] + \left(\frac{v}{s} \right)^\mu \mathcal{N}[\mathcal{R}\mathbf{p} - \mathcal{R}\mathbf{p}^*] \right] \right| \\ &\leq \max_{t \in J} (\varepsilon_1 + \varepsilon_2) \left[\mathcal{N}^{-1} \left[\left(\frac{v}{s} \right)^\mu \mathcal{N} \|\mathbf{p} - \mathbf{p}^*\| \right] \right] \\ &\leq (\varepsilon_1 + \varepsilon_2) \left[\mathcal{N}^{-1} \left[\left(\frac{v}{s} \right)^\mu \mathcal{N} \|\mathbf{p} - \mathbf{p}^*\| \right] \right] \end{aligned}$$

$$= (\varepsilon_1 + \varepsilon_2) \frac{t^\mu}{\Gamma(1 + \mu)} \|\mathbf{p} - \mathbf{p}^*\|.$$

This implies that G is a contraction mapping as $0 < (\varepsilon_1 + \varepsilon_2) \frac{t^\mu}{\Gamma(1 + \mu)} < 1$. Therefore, (11) admits a unique solution based on the Banach fixed-point theorem. \square

Theorem 2 The solution of (11) in the Caputo sense is convergent.

Proof. Let $\mathbf{p}_m(x, t) = \sum_{r=0}^m \mathbf{p}_r(x, t)$, we only need to demonstrate that \mathbf{p}_m is a Cauchy sequence in F . Let $q = 1, 2, 3, \dots$, then we have,

$$\begin{aligned} \|\mathbf{p}_m - \mathbf{p}_q\| &= \max_{t \in I} \left| \sum_{r=q+1}^m \mathbf{p}_r \right| \leq \max_{t \in I} \left| \mathcal{N}^{-1} \left[\left(\frac{v}{s} \right)^\mu \mathcal{N} \left[\sum_{r=q+1}^m (\mathcal{M} \mathbf{p}_{r-1} + \mathcal{R} \mathbf{p}_{r-1}) \right] \right] \right| \\ &= \max_{t \in I} \left| \mathcal{N}^{-1} \left[\left(\frac{v}{s} \right)^\mu \mathcal{N} \left[\sum_{r=q}^{m-1} (\mathcal{M} \mathbf{p}_r + \mathcal{R} \mathbf{p}_r) \right] \right] \right| \\ &\leq \max_{t \in I} \left| \mathcal{N}^{-1} \left[\left(\frac{v}{s} \right)^\mu \mathcal{N} [(\mathcal{M} \mathbf{p}_{m-1} - \mathcal{M} \mathbf{p}_{q-1} + \mathcal{R} \mathbf{p}_{m-1} - \mathcal{R} \mathbf{p}_{q-1})] \right] \right| \\ &\leq \varepsilon_1 \max_{t \in I} \left| \mathcal{N}^{-1} \left[\left(\frac{v}{s} \right)^\mu \mathcal{N} [(\mathcal{M} \mathbf{p}_{m-1} - \mathcal{M} \mathbf{p}_{q-1})] \right] \right| \\ &\quad + \varepsilon_2 \max_{t \in I} \left| \mathcal{N}^{-1} \left[\left(\frac{v}{s} \right)^\mu \mathcal{N} [(\mathcal{R} \mathbf{p}_{m-1} - \mathcal{R} \mathbf{p}_{q-1})] \right] \right| \\ &= (\varepsilon_1 + \varepsilon_2) \frac{t^\mu}{\Gamma(\mu + 1)} \|\mathbf{p}_{m-1} - \mathbf{p}_{q-1}\|. \end{aligned}$$

Assuming $m = q + 1$,

$$\|\mathbf{p}_{q+1} - \mathbf{p}_q\| \leq \varepsilon \|\mathbf{p}_q - \mathbf{p}_{q-1}\| \leq \varepsilon^2 \|\mathbf{p}_{q-1} - \mathbf{p}_{q-2}\| \leq \dots \leq \varepsilon^n \|\mathbf{p}_1 - \mathbf{p}_0\|,$$

here $\varepsilon = (\varepsilon_1 + \varepsilon_2) \frac{t^\mu}{\Gamma(\mu + 1)}$. Similarly, we have

$$\begin{aligned} \|\mathbf{p}_m - \mathbf{p}_q\| &\leq \|\mathbf{p}_{q+1} - \mathbf{p}_q\| + \|\mathbf{p}_{q+2} - \mathbf{p}_{q+1}\| + \dots + \|\mathbf{p}_m - \mathbf{p}_{m-1}\|, \\ &\leq (\varepsilon^n + \varepsilon^{n+1} + \dots + \varepsilon^{m-1}) \|\mathbf{p}_1 - \mathbf{p}_0\| \\ &\leq \varepsilon^n \left(\frac{1 - \varepsilon^{m-q}}{1 - \varepsilon} \right) \|\mathbf{p}_1\|, \end{aligned}$$

we obtain $1 - \varepsilon^{m-q} < 1$, since $0 < \varepsilon < 1$. Consequently,

$$\|p_m - p_q\| \leq \frac{\varepsilon^q}{1 - \varepsilon} \max_{t \in I} \|p_1\|.$$

As $\|p_1\| < \infty$, $\|p_m - p_q\| \rightarrow 0$ for $q \rightarrow \infty$. p_m is a Cauchy sequence in F , implies that p_m is convergent. \square

5. Results and discussion

Analytical approximations for a few TFIOPM examples are discussed in this section. These equations have been chosen because precise solutions are known and well-established techniques exist in the literature to analyze them.

Example 1 Considering the time fractional IOPM (2), let $\sigma^2 = 2$, $\lambda = 6$ and $p(x, 0) = \exp(2ix)$, its exact solution is $p(x, t) = e^{2i(x+t)}$ for $\mu = 1$.

NTDM_C: In the Caputo sense, we take

$$g(\mu, v, s) = \left(\frac{v}{s}\right)^\mu.$$

Then, in view of the NTDM solution methodology leading to (19), the following iterative solutions are derived.

$$p_0^C(x, t) = e^{2ix},$$

$$p_1^C(x, t) = \frac{2ie^{2ix}t^\mu}{\Gamma(\mu + 1)},$$

$$p_2^C(x, t) = -\frac{4e^{2ix}t^{2\mu}}{\Gamma(2\mu + 1)},$$

\vdots

Inserting $p_0^C(x, t)$, $p_1^C(x, t)$, \dots into (19), we arrive at the series solution

$$p^C(x, t) \approx e^{2ix} + \frac{2ie^{2ix}t^\mu}{\Gamma(\mu + 1)} - \frac{4e^{2ix}t^{2\mu}}{\Gamma(2\mu + 1)} + \dots \quad (20)$$

NTDM_{CF}: In the CF sense, we take

$$g(\mu, v, s) = 1 + \mu \left(\frac{v}{s}\right) - \mu.$$

Then, in view of the NTDM solution methodology leading to (19), the iterative solutions are as follows:

$$p_0^{CF}(x, t) = e^{2ix},$$

$$p_1^{CF}(x, t) = 2ie^{2ix}(\mu(t-1) + 1),$$

$$p_2^{CF}(x, t) = -2e^{2ix}(\mu^2(t^2 - 4t + 2) + 4\mu(t-1) + 2),$$

⋮

Inserting these iterates into (19) gives the series solution

$$p^{CF}(x, t) \approx e^{2ix} + 2ie^{2ix}(\mu(t-1) + 1) - 2e^{2ix}(\mu^2(t^2 - 4t + 2) + 4\mu(t-1) + 2) + \dots \quad (21)$$

NTDM_{ABC}: In the ABC sense, we take

$$g(\mu, v, s) = 1 + \mu \left(\frac{v}{s}\right)^\mu - \mu.$$

Then, in view of the NTDM solution methodology leading to (19), the iterative solutions are derived.

$$p_0^{ABC}(x, t) = e^{2ix},$$

$$p_1^{ABC}(x, t) = 2ie^{2ix} \left(1 + \frac{\mu t^\mu}{\Gamma(\mu + 1)} - \mu\right),$$

$$p_2^{ABC}(x, t) = -4e^{2ix} \left((\mu - 1)^2 + \frac{\mu^2 t^{2\mu}}{\Gamma(2\mu + 1)}\right) + \frac{2(1 - \mu)^\mu}{\Gamma(\mu + 1)},$$

⋮

Inserting these iterates into (19) gives the series solution

$$p^{ABC}(x, t) \approx e^{2ix} + 2ie^{2ix} \left(1 - \mu + \frac{\mu t^\mu}{\Gamma(\mu + 1)}\right) - 4e^{2ix} \left((\mu - 1)^2 + \frac{\mu^2}{\Gamma(2\mu + 1)}\right) - \frac{2(1 - \mu)^\mu}{\Gamma(\mu + 1)} + \dots \quad (22)$$

Table 1. NTDM_C, NTDM_{CF}, NTDM_{ABC}, and exact solution of Example 1 for $\mu = 1$ and $x = \pi - 1$

t	Real part				Imaginary part			
	NTDM _C	NTDM _{CF}	NTDM _{ABC}	Exact	NTDM _C	NTDM _{CF}	NTDM _{ABC}	Exact
0.2	-0.0292016	-0.0292016	-0.0292016	-0.0291995	-0.999579	-0.999579	-0.999579	-0.999574
0.4	0.362245	0.362245	0.362245	0.362358	-0.932384	-0.932384	-0.932384	-0.932039
0.6	0.695658	0.695658	0.695658	0.696707	-0.721322	-0.721322	-0.721322	-0.717356
0.8	0.91647	0.91647	0.91647	0.921061	-0.411804	-0.411804	-0.411804	-0.389418
1	0.987393	0.987393	0.987393	1	-0.0853046	-0.0853046	-0.0853046	0
1.2	0.897733	0.897733	0.897733	0.921061	0.136381	0.136381	0.136381	0.389418
1.4	0.672698	0.672698	0.672698	0.696707	0.0868719	0.0868719	0.0868719	0.717356
1.6	0.382707	0.382707	0.382707	0.362358	-0.44906	-0.44906	-0.44906	0.932039

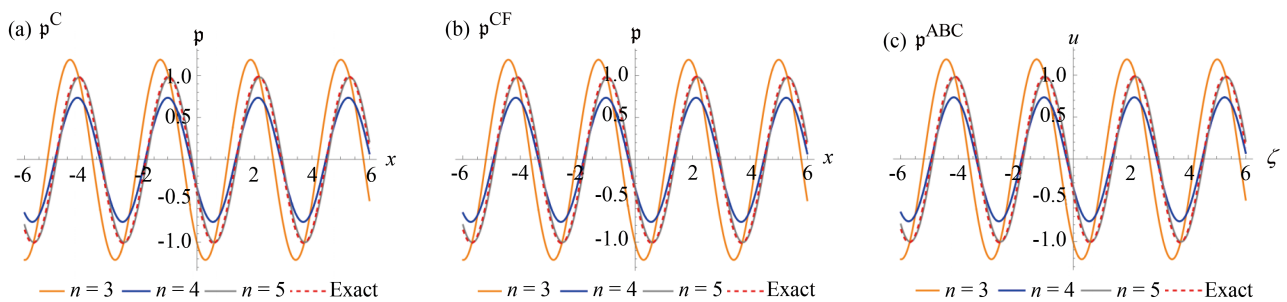


Figure 1. 2D representations of approximate solutions of the real part of the TFIOPM in Example 1 at $\mu = 1$ and $t = 1$

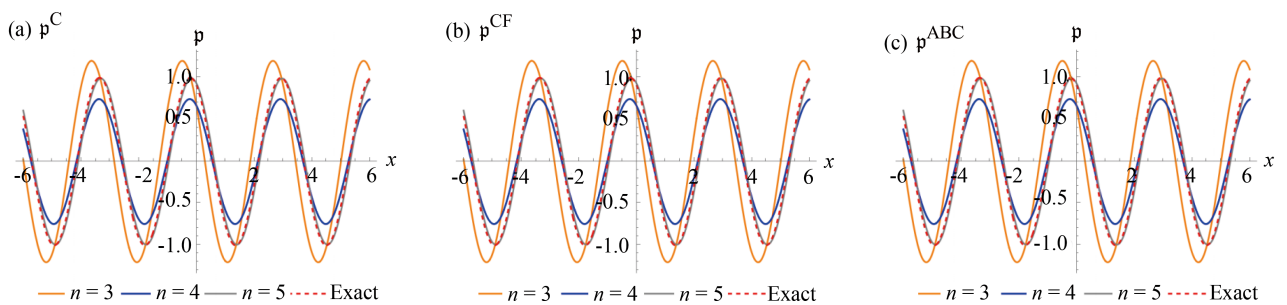


Figure 2. 2D representations of approximate solutions of the real part of the TFIOPM in Example 1 at $\mu = 1$ and $t = 1$

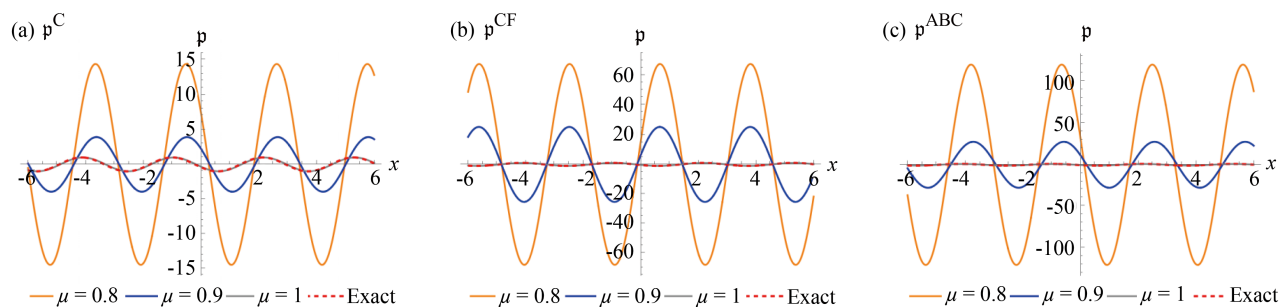


Figure 3. Real part of the TFIOPM for Example 1 at $t = 1$

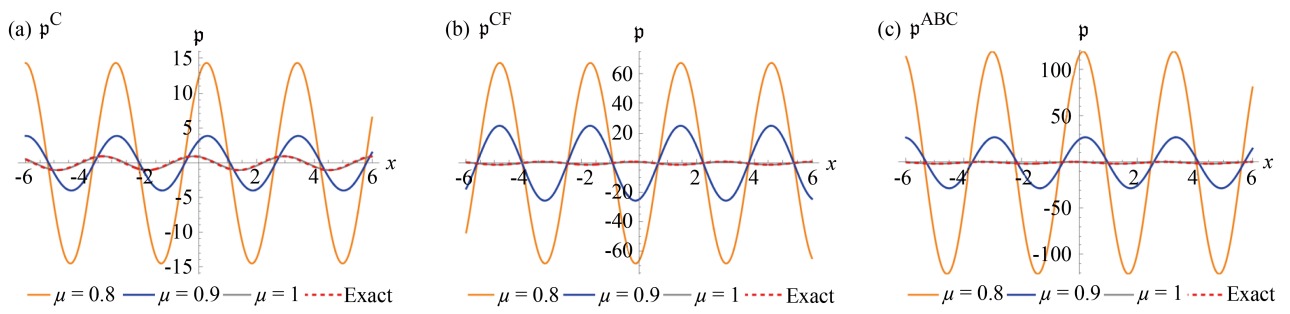


Figure 4. Imaginary part of the TFIOPM for Example 1 at $t = 1$

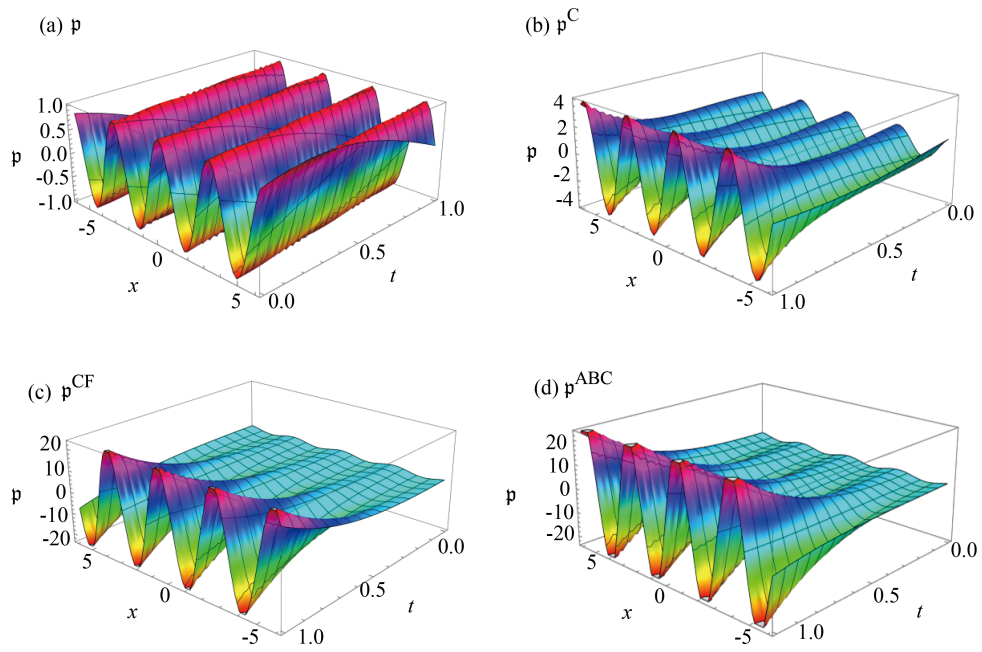
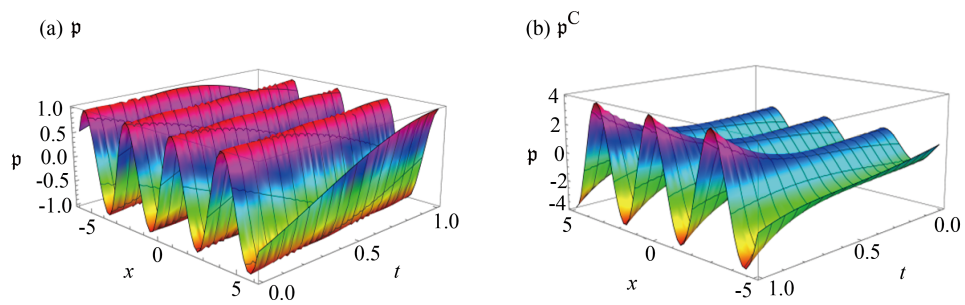


Figure 5. Real part of the approximate solution for Example 1



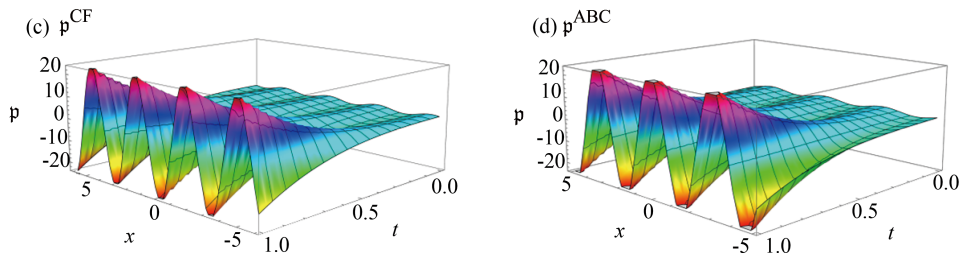


Figure 6. Imaginary portion of the estimated solution for Example 1

Example 2 TFIOPM equation (2) is considered with the following parameters. Let $\sigma^2 = 2$, $\lambda = 6$ and $p(x, 0) = 1$. For $\mu = 1$, precise solution takes the form $p(x, t) = e^{6it}$.

NTDM_C: In view of the NTDM solution methodology leading to (19), the following iterative solutions are derived.

$$p_0^C(x, t) = 1,$$

$$p_1^C(x, t) = \frac{6it^\mu}{\Gamma(\mu + 1)},$$

$$p_2^C(x, t) = -\frac{36t^{2\mu}}{\Gamma(2\mu + 1)},$$

⋮

Inserting $p_0^C(x, t)$, $p_1^C(x, t)$, \dots into (19), we arrive at the series solution

$$p^C(x, t) \approx 1 + \frac{6it^\mu}{\Gamma(\mu + 1)} - \frac{36t^{2\mu}}{\Gamma(2\mu + 1)} + \dots \quad (23)$$

NTDM_{CF}: In view of the NTDM solution methodology leading to (19), the iterative solutions are as follows:

$$p_0^{CF}(x, t) = 1,$$

$$p_1^{CF}(x, t) = 6i(\mu(t - 1) + 1),$$

$$p_2^{CF}(x, t) = -18(\mu^2(t^2 - 4t + 2) + 4\mu(t - 1) + 2),$$

⋮

Inserting these iterates into (19) gives the series solution

$$p^{CF}(x, t) \approx 1 + 6i(\mu(t-1) + 1) - 18(\mu^2(t^2 - 4t + 2) + 4\mu(t-1) + 2) + \dots \quad (24)$$

NTDM_{ABC}: In view of the NTDM solution methodology leading to (19), the following iterations are derived.

$$p_0^{ABC}(x, t) = 1,$$

$$p_1^{ABC}(x, t) = 6i \left(1 + \frac{\mu t^\mu}{\Gamma(\mu + 1)} - \mu \right),$$

$$p_2^{ABC}(x, t) = -36 \left((1 - \mu)^2 + \frac{\mu^2 t^{2\mu}}{\Gamma(2\mu + 1)} + \frac{2\mu(1 - \mu)t^\mu}{\Gamma(\mu + 1)} \right),$$

⋮

Inserting these iterates into (19) gives the series solution

$$p^{ABC}(x, t) \approx 1 + 6i \left(1 - \mu + \frac{\mu t^\mu}{\Gamma(\mu + 1)} \right) - 36 \left((1 - \mu)^2 + \frac{\mu^2 t^{2\mu}}{\Gamma(2\mu + 1)} + \frac{2\mu(1 - \mu)t^\mu}{\Gamma(\mu + 1)} \right) + \dots \quad (25)$$

Table 1 presents a comparison of numerical results for the real and imaginary parts of the NTDM_C, NTDM_{CF}, NTDM_{ABC} and exact solutions for Example 1 for $\mu = 1$ and $x = \pi - 1$. It is evident from the table that the approximate solution with respect to various derivatives is similar and has a good agreement with the exact solution. Figures 1 and 2 are 2D representations of the NTDM_C, NTDM_{CF} and NTDM_{ABC} and exact solutions, respectively, of the TFIOPM in Example 1 at $\mu = 1$ and $t = 1$. It is evident that as n increases, the approximate solution converges to the exact solution. The real and imaginary parts of successive order approximations of NTDM_C, NTDM_{CF} and NTDM_{ABC} and the exact solution of Example 1 are given in Figures 3 and 4, respectively. Figures 5 and 6 present the 3D simulations of the NTDM_C, NTDM_{CF}, and NTDM_{ABC} methods applied to Example 1 alongside the precise solution, for fractional order $\mu = 0.9$. Figures 7 and 8 present the approximate solutions for Example 2 for various n values. From the figures and table, we can conclude that the presented technique solutions with respect to various derivatives are similar, and when the μ approaches 1, it converges to the exact solution.

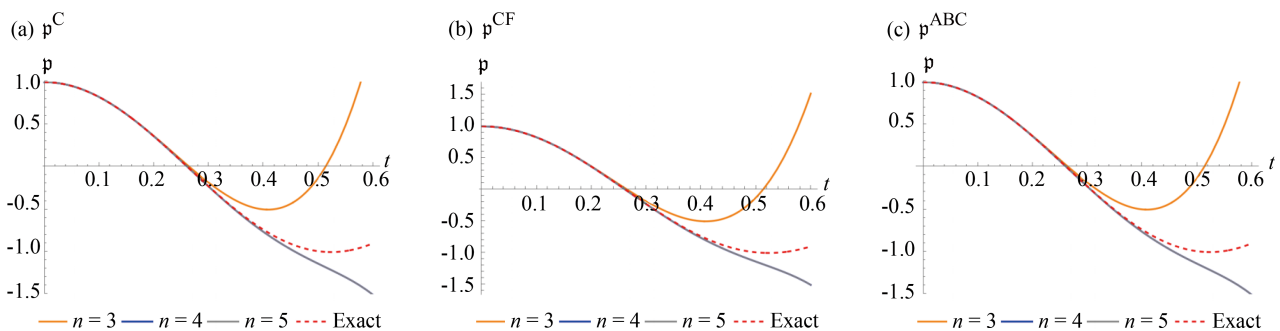


Figure 7. The real part of the estimated solution for Example 2

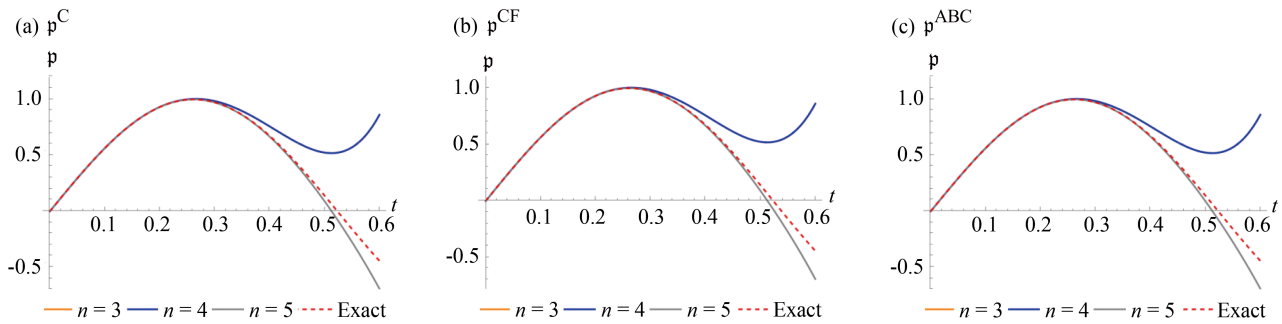


Figure 8. Imaginary component of the approximate solution for Example 2

6. Conclusions

This work investigates TFIOPM by natural transformation using Caputo, CF, and ABC derivatives. The numerical findings were compared with the existing literature. We have demonstrated that the current approach produces findings that align with the exact solution. Based on the table and figures, we can infer that the solutions provided by the technique are comparable for different derivatives and converge to the exact solution as the fractional order gets closer to 1. Though conceptually straightforward, the NTDM proves efficient in handling nonlinear fractional differential equations and offers a practical approach for numerous applications in option pricing models. The NTDM can be employed to study various NLFDEs that arise in biological and physical models.

Conflict of interest

The authors declare no competing financial interest.

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