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Research Article

Multiple Elliptic, Hyperbolic and Trigonometric Stochastic Solutions-for the Stochastic Coupled Schrödinger-KdV Equations in Dusty Plasma

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Abstract: In this paper, we consider the stochastic coupled Schrödinger-KdV equations forced by multiplicative Brownian motion in the Itô sense. By using a mapping method, we can obtain abundant elliptic, trigonometric, and hyperbolic stochastic solutions. The obtained results show that this method is an effective mathematical tool for finding analytical solutions to our equation. In the absence of noise, we get some previously solutions of the coupled Schrödinger-KdV equations. Because the coupled Schrö dinger-KdV equations have significant applications in dusty plasma, including Langmuir, electromagnetic waves, and dust-acoustic waves, these derived solutions may be used to analyze a wide range of essential physical phenomena. Moreover, the impacts of the noise term on the analytical solution of the stochastic coupled Schrödinger-KdV equations were demonstrated. We conclude that the multiplicative Brownian motion stabilizes the solutions to the stochastic coupled Schrödinger-KdV equations.

***Keywords*:** analytical solutions, stochastic process, brownian motion, mapping method, noise strength

MSC: 35A20, 60H15, 60H10, 35Q51

1. Introduction

Nonlinear Evolution Equations (NLEEs) are commonly employed to explain complicated phenomena in a variety of engineering and scientific areas, including fluid dynamics, acoustics, solid state physics, optical fibers, hydrodynamics and plasma physics. Consequently, it is crucial to solve these NLEEs. Recently, numerous powerful approaches for discovering the analytical solutions to NLEEs have been introduced, such as the sine-Gordon expansion technique [1], the F-expansion method [2], the generalizing Riccati equation mapping method [3], the Jacobi elliptic function expansion [4, 5], the $(G^{\backprime}/G)$-expansion [6], the sine-cosine method [7], the exp-function method [8], the $exp(-\phi(\varsigma))$-expansion method [9], the new extended auxiliary equation method [10, 11], the modified generalized Kudryashov's method [12], the modified extended tanh-function method [13, 14], the modified extended mapping method [15], the modified extended direct algebraic method [16], and etc.

One of these model is the Coupled Schrödinger-KdV (CSKdV) equations. The CSKdV model is a significant mathematical model in the field of mathematics and physics. It combines two important equations, the Schrödinger equation and the Korteweg-de Vries (KdV) equation, to describe the behavior of waves in certain physical systems. This

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equation was first introduced by Hiroshi Wadati in the 1970, and since then, it has been widely studied and used in dusty plasma, including Langmuir, electromagnetic waves, and dust-acoustic wave.

In many real-world situations, systems are subject to random fluctuations that can significantly impact their behavior. By introducing stochastic terms into the Schrödinger equation and the KdV equation, researchers can study how these random fluctuations affect the dynamics of quantum particles and wave propagation.

Another important aspect of the stochastic coupled Schrödinger-KdV equation is its ability to describe the interaction between quantum particles and waves. In many physical systems, such as plasmas or superfluids, quantum particles interact with waves in complex ways. The stochastic coupled equation allows researchers to study how these interactions evolve over time, shedding light on the underlying mechanisms that govern these systems.

In this paper we consider the following Stochastic Coupled Schrödinger-KdV (SCSKdV) system [17, 18]:

$$\begin{aligned} &i\Phi_t = \Phi_{xx} - \Phi\Psi + i\sigma\Phi\mathscr{B}_t, \\ &\Psi_t + 6\Psi\Psi_x + \Psi_{xxx} = \left(|\Phi|^2\right)_x + \sigma\Psi\mathscr{B}_t, \end{aligned} \tag{1}$$

where $\Phi(x,\, t)$ indicates the complex function while $\Psi(x,\, t)$ indicates the real-valued function and they present the short wave and long wave profiles, respectively; σ is the amplitude of noise; $\mathscr{B}$ is a Brownian motion and $\mathscr{B}_t = \frac{d\mathscr{B}}{dt}$. Here, we investigate the case where noise is a spatial constant. It is critical at this point to clarify the definition of a Brownian motion. The stochastic process $\mathscr{B}(t)$ is called Brownian motion if it satisfies the following conditions: (I) $\mathscr{B}(0) = 0$, (II) $\mathscr{B}(t)$ has continuous trajectories, (III) $\{\mathscr{B}(t)\}_{t=0}^{\infty}$ has stationary, independent increments, (IV) $\mathscr{B}(t)$ has a normal distribution [19, 20].

Because of the significance of the CSKdV equations in hydrodynamics, mathematical physics, and soliton theory, several researchers have obtained its analytical solutions using a variety of methods, including the $(G^{\backprime}/G)$-expansion method [21], the adomian's decomposition method [17], the complex hyperbolic-function method [22], the F-expansion method [18], the Petrov-Galerkin Method [23], the $\exp(-\Phi(\xi))$-expansion method [24], the Kudryashov method [25], the finite element method [26] and the mapping method [27].

The novelty of this paper is to get stochastic optical solutions to the SCSKdV in Eq. (1). The mapping method is used to find various types of solutions. To the best of our knowledge, the analysis presented in the article is not yet available in the literature. Additionally, we exhibit several two- and three-dimensional graphs to examine the impact of Brownian motion on the obtained solutions.

The paper's outline is as follows: Section 2 provides the wave equation of SCSKdV in Eq. (1), and Section 3 explains the mapping method. In Section 4, we get the solutions of the SCSKdV in Eq. (1). In Section 5, we investigate the impact of the stochastic term on the solutions of the SCSKdV in Eq. (1). Finally, the obtained results of this paper are presented.

2. Traveling wave equation for SCSKdV equations

To have the wave equation for SCSKdV in Eq. (1), we use the following transformation

$$\Phi(x,\, t) = \varphi(\eta)e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t} \text{ and } \Psi(x,\, t) = \psi(\eta)e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \;\; \eta = x + 2kt, \tag{2}$$

where φ and ψ are real deterministic functions; δ and k are non-zero real constants. We note that:

$$\Phi_t = [2k\varphi' + i\delta\varphi + \sigma\varphi\mathscr{B}_t - \frac{1}{2}\sigma^2\varphi + \frac{1}{2}\sigma^2\varphi]e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}$$

$$= [2k\varphi' + i\delta\varphi + \sigma\varphi\mathscr{B}_t]e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{3}$$

$$\Phi_x = (\varphi' + ik\varphi)e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t},\ \Phi_{xx} = (\varphi'' + 2ik\varphi' - k^2\varphi)e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t},$$

where $\frac{1}{2}\sigma^2\varphi$ is the Itô correction term. Similarly,

$$\Psi_t = [2k\psi' + \sigma\psi\mathscr{B}_t]e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t},\ \Psi_x = \psi' e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t},\ \Psi_{xxx} = \psi''' e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{4}$$

Substituting Eqs. (3) and (4) into Eqs. (1), we obtain the following system:

$$\begin{aligned} &\varphi'' + \left(\delta - k^2\right)\varphi + \varphi\psi e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t} = 0, \\ &2k\psi' + \psi''' + [6\psi\psi' - \left(\varphi^2\right)']e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t} = 0. \end{aligned} \tag{5}$$

Considering the expectation on both sides of Eq. (5) to arrive at:

$$\begin{aligned} &\varphi'' + \left(\delta - k^2\right)\varphi + \varphi\psi e^{-\frac{1}{2}\sigma^2 t}\mathbb{E}(e^{\sigma\mathscr{B}}) = 0, \\ &2k\psi' + \psi''' + [6\psi\psi' - \left(\varphi^2\right)']e^{-\frac{1}{2}\sigma^2 t}\mathbb{E}(e^{\sigma\mathscr{B}}) = 0. \end{aligned} \tag{6}$$

Since $\mathscr{B}(t)$ is a Gaussian process, then $\mathbb{E}(e^{\sigma\mathscr{B}(t)}) = e^{\frac{1}{2}\sigma^2 t}$ for any real number σ [28]. Hence, Eq. (6) becomes:

$$\begin{aligned} &\varphi'' + \left(\delta - k^2\right)\varphi + \varphi\psi = 0, \\ &2k\psi' + \psi''' + 6\psi\psi' - \left(\varphi^2\right)' = 0. \end{aligned} \tag{7}$$

Integrating second equation in Eq. (7) once, we get:

$$\begin{aligned} &\varphi'' + \left(\delta - k^2\right)\varphi + \varphi\psi = 0, \\ &\psi'' + 2k\psi + 3\psi^2 - \varphi^2 = 0. \end{aligned} \tag{8}$$

3. The clarification of the mapping method

Let us describe the mapping method that we utilize here (for more details, see [29–31]). Assuming the solution of Eq. (8) has the form:

$$\varphi(\eta) = \sum_{i=0}^{M_1} \ell_i \mathscr{M}^i(\eta),$$
$$\psi(\eta) = \sum_{i=0}^{M_2} \varpi_i \mathscr{M}^i(\eta), \tag{9}$$

where ℓ_i and ϖ_i are unknown constants to be determined and $\mathscr{M}$ solves the following auxiliary equation:

$$(\mathscr{M}^{\backprime})^2 = \hbar_1 \mathscr{M}^4 + \hbar_2 \mathscr{M}^2 + \hbar_3, \quad \hbar_1 \neq 0, \tag{10}$$

where $\hbar_1$, $\hbar_2$, and $\hbar_3$ are real numbers. Eq. (10) has several solutions for $\hbar_1$, $\hbar_2$ and $\hbar_3$ as follows:

Table 1. The solutions of Eq. (10)

case	$\hbar_1$	$\hbar_2$	$\hbar_3$	$\mathscr{M}(\eta)$
15	$\frac{1-\check{n}^2}{4}$	$\frac{(1+\check{n}^2)}{2}$	$\frac{(1-\check{n}^2)}{4}$	$nc(\eta) \pm sc(\eta)$
16	$\frac{\check{n}^2}{4}$	$\frac{(\check{n}^2-2)}{2}$	$\frac{\check{n}^2}{4}$	$\sqrt{1-\check{n}^2}(sd(\eta) \pm cd(\eta))$
17	$\frac{\check{n}^2-1}{4}$	$\frac{(1+\check{n}^2)}{2}$	$\frac{(\check{n}^2-1)}{4}$	$\check{n}sd(\eta) \pm nd(\eta)$
18	$\frac{\check{n}^2}{4}$	$\frac{(\check{n}^2-2)}{2}$	$\frac{1}{4}$	$\frac{sn(\eta)}{1 \pm dn(\eta)}$
19	$\frac{-1}{4}$	$\frac{(1+\check{n}^2)}{2}$	$\frac{(1-\check{n}^2)^2}{4}$	$\check{n}cn(\eta) \pm dn(\eta)$
20	$\frac{(1-\check{n}^2)^2}{4}$	$\frac{\check{n}^2+1}{2}$	$\frac{1}{4}$	$ds(\eta) \pm cs(\eta)$
21	$\frac{\check{n}^4(1-\check{n}^2)}{2(2-\check{n}^2)}$	$\frac{2(1-\check{n}^2)}{(\check{n}^2-2)}$	$\frac{(1-\check{n}^2)}{2(2-\check{n}^2)}$	$dc(\eta) \pm \sqrt{1-\check{n}^2}nc(\eta)$
22	1	$2-4\check{n}^2$	1	$\frac{sn(\eta)dn(\eta)}{cn(\eta)}$
23	$\check{n}^4$	2	1	$\frac{sn(\eta)cn(\eta)}{dn(\eta)}$
24	1	$\check{n}^2+2$	$(1-2\check{n}^2+\check{n}^4)$	$\frac{dn(\eta)cn(\eta)}{sn(\eta)}$
25	$\frac{-4}{\check{n}}$	$6\check{n}-\check{n}^2-1$	$(\check{n}^2-2\check{n}^3+\check{n}^4)$	$\frac{\check{n}dn(\eta)cn(\eta)}{\check{n}sn^2(\eta)+1}$
26	$\frac{-1}{4}$	$\frac{(1-2\check{n}^2)}{2}$	$\frac{1}{4}$	$\frac{sn(\eta)}{1 \pm cn(\eta)}$
27	$\frac{(1-\check{n}^2)}{4}$	$\frac{(1+\check{n}^2)}{2}$	$\frac{(1-\check{n}^2)}{4}$	$\frac{cn(\eta)}{1 \pm sn(\eta)}$

Where $ds(\eta) = ds(\eta, \check{n})$, $sn(\eta) = sn(\eta, \check{n})$, $sc(\eta) = sc(\eta, \check{n})$, $dn(\eta) = dn(\eta, \check{n})$, $cn(\eta) = cn(\eta, `\check{n})$, are the Jacobi Elliptic Functions (JEFs) for $0 < \check{n} \leq 1$.

When $\check{n} \to 0$, JEFs produce the trigonometric functions as follows:

$$sn(\eta) \to \sin(\eta),\ cs(\eta) \to \cot(\eta),$$

$$cn(\eta) \to \cos(\eta),\ ds \to \csc(\eta),$$

$$dn(\eta) \to 1,\ sc(\eta) \to \tan(\eta),$$

$$ns(\eta) \to \csc(\eta),\ ds(\eta) \to \csc(\eta).$$

When $\check{n} = 1$, JEFs produce the hyperbolic functions as follows:

$$dn(\eta) \to \operatorname{sech}(\eta),\ cn(\eta) \to \operatorname{sech}(\eta),$$

$$sn(\eta) \to \tanh(\eta),\ ns(\eta) \to \coth(\eta),$$

$$cs(\eta) \to \operatorname{csch}(\eta),\ ds \to \operatorname{csch}(\eta).$$

4. Optical solutions of the SCSKdV equations

First, let us equate φ'' (highest power order derivative term) with $\varphi\psi$ (highest power nonlinear term) and ψ'' (highest order derivative term) with φ^2 (highest nonlinear term) in Eq. (8) to calculate the parameters M_1 and M_2 as

$$M_1 + 2 = M_1 + M_2 \quad \text{and} \quad M_2 + 2 = 2M_1. \tag{11}$$

Hence

$$M_2 = 2 \quad \text{then} \quad M_2 = 2. \tag{12}$$

With $M_1 = 2$ and $M_2 = 2$, Eq. (9) takes the form

$$\begin{aligned} \varphi(\eta) &= \ell_0 + \ell_1 \mathscr{M}(\eta) + \ell_2 \mathscr{M}^2(\eta), \\ \psi(\eta) &= \varpi_0 + \varpi_1 \mathscr{M}(\eta) + \varpi_2 \mathscr{M}^2(\eta). \end{aligned} \tag{13}$$

Putting Eq. (13) into Eq. (8), we get:

$$
\begin{aligned}
&[\ell_2\varpi_2+6\hbar_1\ell_2]\mathscr{M}^4+[2\hbar_1\ell_1+\ell_1\varpi_2+\ell_2\varpi_1]\mathscr{M}^3\\
&\qquad+[4\hbar_2\ell_2+\left(\delta-k^2\right)\ell_2+\ell_0\varpi_2+\ell_2\varpi_0]\mathscr{M}^2\\
&\qquad+[\hbar_2\ell_1+\left(\delta-k^2\right)\ell_1+\ell_1\varpi_0+\ell_0\varpi_1]\mathscr{M}\\
&\qquad+[2\hbar_3\ell_2+\left(\delta-k^2\right)\ell_0+\ell_0\varpi_0]=0,
\end{aligned}
\tag{14}
$$

and

$$
\begin{aligned}
&(6\hbar_1\varpi_2-\ell_2^2+3\varpi_2^2)\mathscr{M}^4+(2\hbar_1\varpi_1+6\varpi_1\varpi_2-2\ell_1\ell_2)\mathscr{M}^3\\
&+(4\hbar_2\varpi_2+2k\varpi_2+6\varpi_0\varpi_2+2\ell_0\ell_2+3\varpi_1^2-\ell_1^2)\mathscr{M}^2\\
&+(\hbar_2\varpi_1+2k\varpi_1+6\varpi_0\varpi_1-2\ell_0\ell_1)\mathscr{M}\\
&+(2\hbar_3\varpi_2+2k\varpi_0+3\varpi_0^2-\ell_0^2)=0.
\end{aligned}
\tag{15}
$$

Equating each coefficient of $\mathscr{M}^j$ to zero for $j=4,\ 3,\ 2,\ 1,\ 0$, we have:

$$\ell_2\varpi_2+6\hbar_1\ell_2=0, \tag{16}$$

$$2\hbar_1\ell_1+\ell_1\varpi_2+\ell_2\varpi_1=0, \tag{17}$$

$$4\hbar_2+\left(\delta-k^2\right)\ell_2+\ell_0\varpi_2+\ell_2\varpi_0=0, \tag{18}$$

$$\hbar_2\ell_1+\left(\delta-k^2\right)\ell_1+\ell_1\varpi_0+\ell_0\varpi_1=0, \tag{19}$$

$$2\hbar_3\ell_2+\left(\delta-k^2\right)\ell_0+\ell_0\varpi_0=0, \tag{20}$$

and

$$6\hbar_1\varpi_2-\ell_2^2+3\varpi_2^2=0, \tag{21}$$

$$2\hbar_1\varpi_1+6\varpi_1\varpi_2-2\ell_1\ell_2=0, \tag{22}$$

$$4\hbar_2+2k\varpi_2+6\varpi_0\varpi_2-2\ell_0\ell_2+3\varpi_1^2-\ell_1^2=0, \tag{23}$$

$$\hbar_2\varpi_1+2k\varpi_1+6\varpi_0\varpi_1-2\ell_0\ell_1=0, \tag{24}$$

$$2\hbar_3\varpi_2+2k\varpi_0+3\varpi_0^2-\ell_0^2=0. \tag{25}$$

By solving the above equations, we have the following two cases:
Case I:

$$\begin{aligned}&\ell_0=\ell_2=0,\ \ell_1=\pm\,2\sqrt{\hbar_1\left(\hbar_2-k-3k^2+3\delta\right)},\\ &\varpi_0=k^2-\delta-\hbar_2,\ \varpi_1=0,\ \varpi_2=-2\hbar_1.\end{aligned} \tag{26}$$

Case II:

$$\begin{aligned}&\ell_0=\pm\,2\sqrt{2}\left(10\hbar_2-\sqrt{\hbar_2^2-3\hbar_1\hbar_3}\right),\ \ell_1=0,\ \ell_2=\pm6\sqrt{2}\hbar_1\\ &\varpi_0=-(2\hbar_2+\frac{1}{3}k+\frac{4}{3}\sqrt{\hbar_2^2-3\hbar_1\hbar_3}),\ \varpi_1=0,\ \varpi_2=-6\hbar_1,\\ &\delta=-k^2-\frac{1}{3}k\pm\frac{10}{3}\sqrt{\hbar_2^2-3\hbar_1\hbar_3}.\end{aligned} \tag{27}$$

Case I: Substituting Eq. (26) into Eq. (13) and using Eq. (2), we get the solutions of CSSKdV Eq. (1) as follows:

$$\Phi(x,\ t)=\pm\,2\sqrt{\hbar_1\left(\hbar_2-k-3k^2+3\delta\right)}\mathscr{M}\left(\eta\right)e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}, \tag{28}$$

$$\Psi(x,\ t)=\left\{k^2-\delta-\hbar_2-2\hbar_1\mathscr{M}^2\left(\eta\right)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}. \tag{29}$$

There are many cases for the solutions of Eq. (26) depending on $\hbar_1$, $\hbar_2$ and $\hbar_3$ as stated in Table 1 as follows:
Case I-1: If $\hbar_1=\check{n}^2,\ \hbar_2=-\left(1+\check{n}^2\right)$ and $\hbar_3=1$, then $\mathscr{M}\left(\eta\right)=sn(\eta)$, and Eq. (28) takes the form:

$$\Phi(x,\ t)=\left\{\pm\,2\check{n}\sqrt{-1-\check{n}^2-k-3k^2+3\delta}sn(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}, \tag{30}$$

$$\Psi(x,\ t)=\left\{k^2-\delta+1+\check{n}^2-2\check{n}^2sn^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}. \tag{31}$$

If $\check{n}=1$, then Eqs. (30) and (31) become:

$$\Phi(x,\ t)=\left\{\pm\,2\sqrt{-2-k-3k^2+3\delta}\tanh(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}, \tag{32}$$

$$\Psi(x,\ t)=\left\{k^2-\delta+2\mathrm{sech}^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}. \tag{33}$$

Case I-2: If $\hbar_1 = \check{n}^2,\ \hbar_2 = -\left(1+\check{n}^2\right)$ and $\hbar_3 = 1$, then $\mathscr{M}(\eta) = cd(\eta)$, and Eq. (28) takes the form:

$$\Phi(x,\ t) = \left\{\pm\, 2\check{n}\sqrt{-1-\check{n}^2-k-3k^2+3\delta}\, cd(x+2kt)\right\} e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{34}$$

$$\Psi(x,\ t) = \left\{k^2-\delta+1+\check{n}^2-2\check{n}^2 cd^2(x+2kt)\right\} e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{35}$$

Case I-3: If $\hbar_1 = -\check{n}^2,\ \hbar_2 = 2\check{n}^2-1$ and $\hbar_3 = 1-\check{n}^2$, then $\mathscr{M}(\eta) = cn(\eta)$, and Eq. (28) takes the form :

$$\Phi(x,\ t) = \left\{\pm\, 2\check{n}\sqrt{-2\check{n}^2+1+k+3k^2-3\delta}\, cn(x+2kt)\right\} e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{36}$$

$$\Psi(x,\ t) = \left\{k^2-\delta+1-2\check{n}^2+2\check{n}^2 cn^2(x+2kt)\right\} e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{37}$$

If $\check{n} = 1$, then Eqs. (36) and (37) become:

$$\Phi(x,\ t) = \left\{\pm\, 2\sqrt{-1+k+3k^2-3\delta}\, \mathrm{sech}(x+2kt)\right\} e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{38}$$

$$\Psi(x,\ t) = \left\{k^2-\delta-1+2\mathrm{sech}^2(\eta)\right\} e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{39}$$

Case I-4: If $\hbar_1 = -1,\ \hbar_2 = 2-\check{n}^2-1$ and $\hbar_3 = \check{n}^2-1$, then $\mathscr{M}(\eta) = dn(\eta)$, and Eq. (28) takes the form:

$$\Phi(x,\ t) = \left\{\pm\, 2\sqrt{-2+\check{n}^2+1+k+3k^2-3\delta}\, dn(x+2kt)\right\} e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{40}$$

$$\Psi(x,\ t) = \left\{k^2-\delta-2+\check{n}^2+2dn^2(x+2kt)\right\} e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{41}$$

If $\check{n} = 1$, then Eqs. (40) and (41) become: Eqs. (38) and (39).
Case I-5: If $\hbar_1 = 1,\ \hbar_2 = -\left(1+\check{n}^2\right)$ and $\hbar_3 = \check{n}^2$, then $\mathscr{M}(\eta) = ns(\eta)$, and Eq. (28) takes the form :

$$\Phi(x,\ t) = \left\{\pm\, 2\sqrt{-1-\check{n}^2-k-3k^2+3\delta}\, ns(x+2kt)\right\} e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{42}$$

$$\Psi(x,\ t) = \left\{k^2-\delta+1+\check{n}^2-2ns^2(x+2kt)\right\} e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{43}$$

If $\check{n} = 1$, then Eqs. (42) and (43) become:

$$\Phi(x,\ t) = \left\{\pm\, 2\sqrt{-2-k-3k^2+3\delta}\, \coth(x+2kt)\right\} e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{44}$$

$$\Psi(x,\ t)=\left\{k^2-\delta-2\operatorname{csch}^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{45}$$

While if $\check{n}\to 0$, then Eqs. (42) and (43) become:

$$\Phi(x,\ t)=\left\{\pm\, 2\sqrt{-1-k-3k^2+3\delta}\,\csc(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{46}$$

$$\Psi(x,\ t)=\left\{k^2-\delta+1-2\csc^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{47}$$

Case I-6: If $\hbar_1=1,\ \hbar_2=-\left(1+\check{n}^2\right)$ and $\hbar_3=\check{n}^2$, then $\mathscr{M}\left(\eta\right)=dc(\eta)$, and Eq. (28) takes the form :

$$\Phi(x,\ t)=\left\{\pm\, 2\sqrt{-1-\check{n}^2-k-3k^2+3\delta}\,dc(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{48}$$

$$\Psi(x,\ t)=\left\{k^2-\delta+1+\check{n}^2-2dc^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{49}$$

If $\check{n}\to 0$, then Eqs. (48) and (49) become:

$$\Phi(x,\ t)=\left\{\pm\, 2\sqrt{-1-k-3k^2+3\delta}\,\sec(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{50}$$

$$\Psi(x,\ t)=\left\{k^2-\delta+1-2\sec^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{51}$$

Case I-7: If $\hbar_1=1-\check{n}^2,\ \hbar_2=2\check{n}^2-1$ and $\hbar_3=-\check{n}^2$, then $\mathscr{M}\left(\eta\right)=nc(\eta)$, and Eq. (28) takes the form:

$$\Phi(x,\ t)=\left\{\pm\, 2\sqrt{(1-\check{n}^2)(2\check{n}^2-1-k-3k^2+3\delta)}\,nc(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{52}$$

$$\Psi(x,\ t)=\left\{k^2-\delta-2\check{n}^2+1-2(1-\check{n}^2)nc^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{53}$$

If $\check{n}\to 0$, then Eqs. (52) and (53) become Eqs. (50) and (51), respectively.
Case I-8: If $\hbar_1=\check{n}^2-1,\ \hbar_2=2-\check{n}^2$ and $\hbar_3=-1$, then $\mathscr{M}\left(\eta\right)=nd(\eta)$, and Eq. (28) takes the form:

$$\Phi(x,\ t)=\left\{\pm\, 2\sqrt{(\check{n}^2-1)(2-\check{n}^2-k-3k^2+3\delta)}\,nd(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{54}$$

$$\Psi(x,\ t)=\left\{k^2-\delta-2+\check{n}^2-2(\check{n}^2-1)nd^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{55}$$

Case I-9: If $\hbar_1=1-\check{n}^2,\ \hbar_2=2-\check{n}^2$ and $\hbar_3=1$, then $\mathscr{M}\left(\eta\right)=sc(\eta)$, and Eq. (28) takes the form:

$$\Phi(x,\ t)=\left\{\pm 2\sqrt{(1-\check{n}^2)(2-\check{n}^2-k-3k^2+3\delta)}sc(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{56}$$

$$\Psi(x,\ t)=\left\{k^2-\delta-2+\check{n}^2-2(1-\check{n}^2)sc^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{57}$$

If $\check{n}\to 0$, then Eqs. (56) and (57) become:

$$\Phi(x,\ t)=\left\{\pm 2\sqrt{(2-k-3k^2+3\delta)}\tan(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{58}$$

$$\Psi(x,\ t)=\left\{k^2-\delta-2\sec^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{59}$$

Case I-10: If $\hbar_1=-\check{n}^2(1-\check{n}^2),\ \hbar_2=2\check{n}^2-1$ and $\hbar_3=1$, then $\mathscr{M}(\eta)=sd(\eta)$, and Eq. (28) takes the form:

$$\Phi(x,\ t)=\left\{\pm 2\sqrt{(\check{n}^2-1)(2\check{n}^2-1-k-3k^2+3\delta)}sd(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{60}$$

$$\Psi(x,\ t)=\left\{k^2-\delta-2\check{n}^2+1-2\check{n}^2(1-\check{n}^2)sd^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{61}$$

Case I-11: If $\hbar_1=1,\ \hbar_2=2-\check{n}^2$ and $\hbar_3=1-\check{n}^2$, then $\mathscr{M}(\eta)=cs(\eta)$, and Eq. (28) takes the form:

$$\Phi(x,\ t)=\left\{\pm 2\sqrt{2-\check{n}^2-k-3k^2+3\delta}\,cs(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{62}$$

$$\Psi(x,\ t)=\left\{k^2-\delta-2+\check{n}^2-2cs^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{63}$$

If $\check{n}=1$, then Eqs. (62) and (63) become:

$$\Phi(x,\ t)=\left\{\pm 2\sqrt{1-k-3k^2+3\delta}\,\mathrm{csch}(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{64}$$

$$\Psi(x,\ t)=\left\{k^2-\delta-1-2\mathrm{csch}^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{65}$$

While, if $\check{n}\to 0$, then Eqs. (62) and (63) become:

$$\Phi(x,\ t)=\left\{\pm 2\sqrt{2-k-3k^2+3\delta}\cot(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{66}$$

$$\Psi(x,\ t=\left\{k^2-\delta-2-2\csc^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{67}$$

Case I-12: If $\hbar_1 = 1,\ \hbar_2 = 2\check{n}^2 - 1$ and $\hbar_3 = -\check{n}^2(1-\check{n}^2)$, then $\mathscr{M}(\eta) = ds(\eta)$, and Eq. (28) takes the form:

$$\Phi(x,\ t) = \left\{\pm 2\sqrt{(2\check{n}^2 - 1 - k - 3k^2 + 3\delta} ds(x+2kt)\right\} e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{68}$$

$$\Psi(x,\ t) = \left\{k^2 - \delta - 2\check{n}^2 + 1 - 2ds^2(x+2kt)\right\} e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{69}$$

Case I-13: If $\hbar_1 = \dfrac{1}{4},\ \hbar_2 = \dfrac{1-2\check{n}^2}{2}$ and $\hbar_3 = \dfrac{1}{4}$, then $\mathscr{M}(\eta) = ns(\eta) \pm cs(\eta)$, and Eq. (28) takes the form:

$$\Phi(x,\ t) = \left\{\pm \frac{1}{2}\sqrt{1-2\check{n}^2 - 2k - 6k^2 + 6\delta}\,(ns(x+2kt) \pm cs(x+2kt))\right\} e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{70}$$

$$\Psi(x,\ t) = \frac{1}{2}\left\{2k^2 - 2\delta - 1 + 2\check{n}^2 - (ns(x+2kt) \pm cs(x+2kt))^2\right\} e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{71}$$

Case I-14: If $\hbar_1 = \dfrac{1}{4},\ \hbar_2 = \dfrac{\check{n}^2-2}{2}$ and $\hbar_3 = \dfrac{\check{n}^2}{4}$, then $\mathscr{M}(\eta) = ns(\eta) \pm ds(\eta)$, and Eq. (28) takes the form:

$$\Phi(x,\ t) = \left\{\pm \frac{1}{2}\sqrt{\check{n}^2 - 2 - 2k - 6k^2 + 6\delta)}\,(ns(x+2kt) \pm ds(x+2kt))\right\} e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{72}$$

$$\Psi(x,\ t) = \frac{1}{2}\left\{2k^2 - 2\delta - \check{n}^2 + 2 - (ns(x+2kt) \pm ds(x+2kt))^2\right\} e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{73}$$

If $\check{n} = 1$, then Eqs. (70) and (71) [or Eqs. (72) and (73)] become:

$$\Phi(x,\ t) = \left\{\pm \frac{1}{2}\sqrt{-1 - 2k - 6k^2 + 6\delta}\,(\coth(x+2kt) \pm \operatorname{csch}(x+2kt))\right\} e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{74}$$

$$\Psi(x,\ t) = \frac{1}{2}\left\{2k^2 - 2\delta 1 + 1 - (\coth(x+2kt) \pm \operatorname{csch}(x+2kt))^2\right\} e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{75}$$

While, if $\check{n} \to 0$, then Eqs. (70) and (71) [or Eqs. (72) and (73)] become:

$$\Phi(x,\ t) = \left\{\pm \frac{1}{2}\sqrt{1 - 2k - 6k^2 + 6\delta}\,(\csc(x+2kt) \pm \cot(x+2kt))\right\} e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{76}$$

$$\Psi(x,\ t) = \frac{1}{2}\left\{2k^2 - 2\delta - 1 - (\csc(x+2kt) \pm \cot(x+2kt))^2\right\} e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{77}$$

Case I-15: If $\hbar_1 = \dfrac{1-\check{n}^2}{4},\ \hbar_2 = \dfrac{1+\check{n}^2}{2}$ and $\hbar_3 = \dfrac{1-\check{n}^2}{4}$, then $\mathscr{M}(\eta) = nc(\eta) \pm sc(\eta)$, and Eq. (28) takes the form:

$$\Phi(x,\,t)=\left\{\pm\,\frac{1}{2}\sqrt{(1-\check{n}^2)\,(1+\check{n}^2-2k-6k^2+6\delta)}(nc(x+2kt)\pm sc(x+2kt))\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{78}$$

$$\Psi(x,\,t)=\frac{1}{2}\left\{2k^2-2\delta-1-\check{n}^2-(1-\check{n}^2)(nc(x+2kt)\pm sc(x+2kt))^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{79}$$

If $\check{n}\to 0$, then Eqs. (78) and (79) become:

$$\Phi(x,\,t)=\left\{\pm\,\frac{1}{2}\sqrt{(1-2k-6k^2+6\delta)}(\sec(x+2kt)\pm\tan(x+2kt))\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{80}$$

$$\Psi(x,\,t)=\frac{1}{2}\left\{2k^2-2\delta-1-(\sec(x+2kt)\pm\tan(x+2kt))^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{81}$$

Case I-16: If $\hbar_1=\frac{\check{n}^2}{4}$, $\hbar_2=\frac{\check{n}^2-2}{2}$ and $\hbar_3=\frac{\check{n}^2}{4}$, then $\mathscr{M}(\eta)=\sqrt{1-\check{n}^2}\,(sd(\eta)\pm cd(\eta))$, and Eq. (28) takes the form:

$$\Phi(x,\,t)=\left\{\pm\,\check{n}\sqrt{\frac{\check{n}^2-2-2k-6k^2+6\delta}{2}}\sqrt{1-\check{n}^2}\,(sd(x+2kt)\pm cd(x+2kt))\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{82}$$

$$\Psi(x,\,t)=\frac{1}{2}\left\{2k^2-2\delta-\check{n}^2+2-\check{n}^2\left(\sqrt{1-\check{n}^2}sd(x+2kt)\pm cd(x+2kt)\right)^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{83}$$

Case I-17: If $\hbar_1=\frac{\check{n}^2-1}{4}$, $\hbar_2=\frac{\check{n}^2+1}{2}$ and $\hbar_3=\frac{\check{n}^2-1}{4}$, then $\mathscr{M}(\eta)=\check{n}sd(\eta)\pm nd(\eta)$, and Eq. (28) takes the form:

$$\Phi(x,\,t)=\left\{\pm\,\sqrt{\frac{(\check{n}^2-1)\,(\check{n}^2+1-2k-6k^2+6\delta)}{2}}\,(\check{n}sd(x+2kt)\pm nd(x+2kt))\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{84}$$

$$\Psi(x,\,t)=\frac{1}{2}\left\{2k^2-2\delta-\check{n}^2-1-\left(\check{n}^2-1\right)(\check{n}sd(x+2kt)\pm nd(x+2kt))^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{85}$$

Case I-18: If $\hbar_1=\frac{\check{n}^2}{4}$, $\hbar_2=\frac{\check{n}^2-2}{2}$ and $\hbar_3=\frac{1}{4}$, then $\mathscr{M}(\eta)=\frac{sn(\eta)}{1\pm dn}$, and Eq. (28) takes the form:

$$\Phi(x,\,t)=\left\{\pm\,\check{n}\sqrt{\frac{(\check{n}^2-2-2k-6k^2+6\delta)}{2}}\left(\frac{sn(x+2kt)}{1\pm dn(x+2kt)}\right)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{86}$$

$$\Psi(x,\,t)=\frac{1}{2}\left\{2k^2-2\delta-\check{n}^2+2-\check{n}^2\left(\frac{sn(x+2kt)}{1\pm dn(x+2kt)}\right)^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{87}$$

If $\check{n} = 1$, then Eqs. (86) and (87) become:

$$\Phi(x,\ t) = \left\{ \pm \sqrt{\frac{(-1-2k-6k^2+6\delta)}{2}} \left(\frac{\tanh(x+2kt)}{1 \pm \operatorname{sech}(x+2kt)} \right) \right\} e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{88}$$

$$\Psi(x,\ t) = \frac{1}{2} \left\{ 2k^2 - 2\delta + 1 - \left(\frac{\tanh(x+2kt)}{1 \pm \operatorname{sech}(x+2kt)} \right)^2 \right\} e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{89}$$

Case I-19: If $\hbar_1 = \dfrac{-1}{4}$, $\hbar_2 = \dfrac{\check{n}^2+1}{2}$ and $\hbar_3 = \dfrac{\left(1-\check{n}^2\right)^2}{4}$, then $\mathscr{M}(\eta) = \check{n}cn(\eta) \pm dn(\eta)$, and Eq. (28) takes the form:

$$\Phi(x,\ t) = \left\{ \pm \sqrt{\frac{-\check{n}^2+1+2k+6k^2-6\delta}{2}} \left(\check{n}cn(x+2kt) \pm dn(x+2kt) \right) \right\} e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{90}$$

$$\Psi(x,\ t) = \frac{1}{2} \left\{ 2k^2 - 2\delta - \check{n}^2 - 1 + \left(\check{n}cn(x+2kt) \pm dn(x+2kt) \right)^2 \right\} e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{91}$$

Case I-20: If $\hbar_1 = \dfrac{\left(1-\check{n}^2\right)^2}{4}$, $\hbar_2 = \dfrac{\check{n}^2+1}{2}$ and $\hbar_3 = \dfrac{1}{4}$, then $\mathscr{M}(\eta) = \left(ds(\eta) \pm cs(\eta) \right)$, and Eq. (28) takes the form:

$$\Phi(x,\ t) = \left\{ \pm \left(1-\check{n}^2\right) \sqrt{\frac{\check{n}^2+1-2k-6k^2+6\delta}{2}} \left(ds(x+2kt) \pm cs(x+2kt) \right) \right\} e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{92}$$

$$\Psi(x,\ t) = \frac{1}{2} \left\{ 2k^2 - 2\delta - \check{n}^2 - 1 - \left(1-\check{n}^2\right)^2 \left(ds(x+2kt) \pm cs(x+2kt) \right)^2 \right\} e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{93}$$

If $\check{n} \to 0$, then Eqs. (92) and (93) become:

$$\Phi(x,\ t) = \left\{ \pm \sqrt{\frac{1-2k-6k^2+6\delta}{2}} \left(\sec(x+2kt) \pm \tan(x+2kt) \right) \right\} e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{94}$$

$$\Psi(x,\ t) = \frac{1}{2} \left\{ 2k^2 - 2\delta - 1 - \left(\sec(x+2kt) \pm \tan(x+2kt) \right)^2 \right\} e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{95}$$

Case I-21: If $\hbar_1 = \dfrac{\check{n}^4(1-\check{n}^2)}{2(2-\check{n}^2)}$, $\hbar_2 = \dfrac{2(1-\check{n}^2)}{\check{n}^2-2}$ and $\hbar_3 = \dfrac{1-\check{n}^2}{2(2-\check{n}^2)}$, then $\mathscr{M}(\eta) = dc(\eta) \pm \sqrt{1-\check{n}^2}nc(\eta)$, and Eq. (28) takes the form:

$$\Phi(x,\ t)=\left\{\frac{\sqrt{2}\check{n}^2}{2-\check{n}^2}\sqrt{(\check{n}^2-1)\left[2(1-\check{n}^2)+(\check{n}^2-2)(-k-3k^2+3\delta)\right]}\right. \tag{96}$$

$$\left.\left(dc(x+2kt)\pm\sqrt{1-\check{n}}nc(x+2kt)\right)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t},$$

$$\Psi(x,\ t)=\frac{1}{\check{n}^2-2}\left\{\left(\check{n}^2-2\right)\left(k^2-\delta\right)+\left(1-\check{n}^2\right)\right. \tag{97}$$

$$\left.\times\left[-2+\check{n}^4\left(dc(x+2kt)\pm\sqrt{1-\check{n}}nc(x+2kt)\right)^2\right]\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}.$$

Case I-22: If $\hbar_1=1,\ \hbar_2=2-4\check{n}^2$ and $\hbar_3=1$, then $\mathscr{M}(\eta)=\dfrac{sn(\eta)dn(\eta)}{cn(\eta)}$, and Eq. (28) takes the form:

$$\Phi(x,\ t)=\left\{\pm 2\sqrt{2-4\check{n}^2-k-3k^2+3\delta}\frac{sn(x+2kt)dn(x+2kt)}{cn(x+2kt)}\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{98}$$

$$\Psi(x,\ t)=\left\{k^2-\delta-2+4\check{n}^2-\frac{2sn^2(x+2kt)dn^2(x+2kt)}{cn^2(x+2kt)}\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{99}$$

Case I- 23: If $\hbar_2=\check{n}^4,\ \hbar_1=2$ and $\hbar_3=1$, then $\mathscr{M}(\eta)=\dfrac{sn(\eta)cn(\eta)}{dn(\eta)}$, and Eq. (28) takes the form:

$$\Phi(x,\ t)=\left\{\pm 2\check{n}\sqrt{2-k-3k^2+3\delta}\frac{sn(x+2kt)cn(x+2kt)}{dn(x+2kt)}\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{100}$$

$$\Psi(x,\ t)=\left\{k^2-\delta-2-\frac{2\check{n}^2sn^2(x+2kt)cn^2(x+2kt)}{dn^2(x+2kt)}\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{101}$$

If $\check{n}=1$, then Eqs. (100) and (101) become:

$$\Phi(x,\ t)=\left\{\pm 2\sqrt{2-k-3k^2+3\delta}\tanh(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{102}$$

$$\Psi(x,\ t)=\left\{k^2-\delta-2-2(\tanh(x+2kt))^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{103}$$

Case I-24: If $\hbar_1=1,\ \hbar_2=\check{n}^2+2$, and $\hbar_3=1-2\check{n}^2+\check{n}^4$, then $\mathscr{M}(\eta)=\dfrac{dn(\eta)cn(\eta)}{sn(\eta)}$, and Eq. (28) takes the form:

$$\Phi(x,\ t)=\left\{\pm 2\sqrt{\check{n}^2+2-k-3k^2+3\delta}\frac{dn(x+2kt)cn(x+2kt)}{sn(x+2kt)}\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{104}$$

$$\Psi(x,\ t)=\left\{k^2-\delta-\check{n}^2-2-\frac{2dn^2(x+2kt)cn^2(x+2kt)}{sn^2(x+2kt)}\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{105}$$

If $\check{n}=1$, then Eqs. (104) and (105) become:

$$\Phi(x,\ t)=\left\{\pm 2\sqrt{3-k-3k^2+3\delta}\operatorname{sech}(x+2kt)\operatorname{csch}(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{106}$$

$$\Psi(x,\ t)=\left\{k^2-\delta-3-2(\operatorname{sech}(x+2kt)\operatorname{csch}(x+2kt))^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{107}$$

Case I-25: If $\hbar_1=-\frac{4}{\check{n}}$, $\hbar_2=6\check{n}-\check{n}^2-1$ and $\hbar_3=-2\check{n}^3+\check{n}^4+\check{n}^2$, then $\mathscr{M}(\eta)=\frac{\check{n}cn(\eta)dn(\eta)}{\check{n}sn^2(\eta)+1}$, and Eq. (28) takes the form:

$$\Phi(x,\ t)=\left\{\pm 4\sqrt{\check{n}^2-6\check{n}+1+k+3k^2-3\delta}\,\frac{\sqrt{\check{n}}cn(x+2kt)dn(x+2kt)}{\check{n}sn^2(x+2kt)+1}\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{108}$$

$$\Psi(x,\ t)=\left\{k^2-\delta+\check{n}^2-6\check{n}+1+\frac{8\check{n}cn^2(x+2kt)dn^2(x+2kt)}{(\check{n}sn^2(x+2kt)+1)^2}\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{109}$$

If $\check{n}=1$, then Eqs. (108) and (109) become:

$$\Phi(x,\ t)=\left\{\pm 4\sqrt{-4+k+3k^2-3\delta}\,\frac{\operatorname{sech}^2(x+2kt)}{1+\tanh^2(x+2kt)}\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{110}$$

$$\Psi(x,\ t)=\left\{k^2-\delta+\check{n}^2-6\check{n}+1+\frac{8\operatorname{sech}^4(x+2kt)}{(1+\tanh^2(x+2kt))^2}\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{111}$$

Case I-26: If $\hbar_1=\frac{1}{4}$, $\hbar_2=\frac{1-2\check{n}^2}{2}$ and $\hbar_3=\frac{1}{4}$, then $\mathscr{M}(\eta)=\frac{sn(\eta)}{1\pm cn(\eta)}$, and Eq. (28) takes the form:

$$\Phi(x,\ t)=\left\{\pm\sqrt{\frac{1-2\check{n}^2-2k-6k^2+6\delta}{2}}\,\frac{sn(x+2kt)}{1\pm cn(x+2kt)}\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{112}$$

$$\Psi(x,\ t)=\frac{1}{2}\left\{2k^2-2\delta-1+2\check{n}^2-\frac{sn^2(x+2kt)}{(1\pm cn(x+2kt))^2}\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{113}$$

If $\check{n}\to 0$, then Eqs. (108) and (109) become:

$$\Phi(x,\ t)=\left\{\pm\sqrt{\frac{1-2k-6k^2+6\delta}{2}}(\csc(x+2kt)\pm\cot(x+2kt))\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{114}$$

$$\Psi(x,\ t)=\frac{1}{2}\left\{2k^2-2\delta-1-(\csc(x+2kt)\pm\cot(x+2kt))^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{115}$$

Case I-27: If $\hbar_1=\dfrac{1-\check{n}^2}{4}$, $\hbar_2=\dfrac{1+\check{n}^2}{2}$ and $\hbar_3=\dfrac{1-\check{n}^2}{4}$, then $\mathscr{M}(\eta)=\dfrac{cn(\eta)}{1\pm sn(\eta)}$, and Eq. (28) takes the form:

$$\Phi(x,\ t)=\left\{\pm\sqrt{\frac{(1-\check{n}^2)(1+\check{n}^2-2k-6k^2+6\delta)}{2}}\frac{cn(x+2kt)}{1\pm sn(x+2kt)}\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{116}$$

$$\Psi(x,\ t)=\frac{1}{2}\left\{2k^2-2\delta-1-\check{n}^2-\frac{cn^2(x+2kt)}{(1\pm sn(x+2kt))^2}\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{117}$$

If $\check{n}\to 0$, then Eqs. (108) and (109) become:

$$\Phi(x,\ t)=\left\{\pm\sqrt{\frac{(1-2k-6k^2+6\delta)}{2}}(\sec(x+2kt)\pm\tan(x+2kt))\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{118}$$

$$\Psi(x,\ t)=\frac{1}{2}\left\{2k^2-2\delta-1-(\sec(x+2kt)\pm\tan(x+2kt))^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{119}$$

Remark 1 If we put $\sigma=0$ in Eqs. (30) to (119), then we have the same solutions that stated in [18].
Case II: Substituting Eq. (27) and using Eq. (13) Eq. (2), we get the solutions of CSSKdV Eq. (1) as follows:

$$\Phi(x,\ t)=\left[2\sqrt{2}\left(10\hbar_2-\sqrt{\hbar_2^2-3\hbar_1\hbar_3}\right)+6\sqrt{2}\hbar_1\mathscr{M}^2(\eta)\right]e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{120}$$

$$\Psi(x,\ t)=\left[-2\hbar_2-\frac{1}{3}k-\frac{4}{3}\sqrt{\hbar_2^2-3\hbar_1\hbar_3}-6\hbar_1\mathscr{M}^2(\eta)\right]e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{121}$$

There are many cases for the solutions of Eq. (26) as stated in Table 1 as follows:
Case II-1: If $\hbar_1=\check{n}^2$, $\hbar_2=-\left(1+\check{n}^2\right)$ and $\hbar_3=1$, then $\mathscr{M}(\eta)=sn(\eta)$, and Eq. (120) takes the form

$$\Phi(x,\ t)=\left\{2\sqrt{2}\left(-10\left(1+\check{n}^2\right)-\sqrt{\check{n}^4-\check{n}^2+1}\right)+6\sqrt{2}\check{n}^2 sn^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{122}$$

$$\Psi(x,\ t)=\left\{2\left(1+\check{n}^2\right)-\frac{1}{3}k-\frac{4}{3}\sqrt{\check{n}^4-\check{n}^2+1}-6\check{n}^2 sn^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{123}$$

If $\check{n}=1$, then Eqs. (122) and (123) become:

$$\Phi(x,\ t)=\left\{-42\sqrt{2}+6\sqrt{2}\tanh^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{124}$$

$$\Psi(x,\ t)=\left\{-\frac{8}{3}-\frac{1}{3}k-6\tanh^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}. \tag{125}$$

Case II- 2: If $\hbar_1=\check{n}^2,\ \hbar_2=-\left(1+\check{n}^2\right)$ and $\hbar_3=1$, then $\mathscr{M}(\eta)=cd(\eta)$, and Eq. (120) takes the form:

$$\Phi(x,\ t)=\left\{2\sqrt{2}\left(-10\left(1+\check{n}^2\right)-\sqrt{\check{n}^4-\check{n}^2+1}\right)+6\sqrt{2}\check{n}^2cd^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}, \tag{126}$$

$$\Psi(x,\ t)=\left\{2\left(1+\check{n}^2\right)-\frac{1}{3}k-\frac{4}{3}\sqrt{\check{n}^4-\check{n}^2+1}-6\check{n}^2cd^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}. \tag{127}$$

Case II-3: If $\hbar_1=-\check{n}^2,\ \hbar_2=2\check{n}^2-1$ and $\hbar_3=1-\check{n}^2$, then $\mathscr{M}(\eta)=cn(\eta)$, and Eq. (120) takes the form:

$$\Phi(x,\ t)=\left\{2\sqrt{2}\left(10(2\check{n}^2-1)-\sqrt{\check{n}^4-\check{n}^2+1}\right)-6\sqrt{2}\check{n}^2cn^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}, \tag{128}$$

$$\Psi(x,\ t)=\left\{-2(2\check{n}^2-1)-\frac{1}{3}k-\frac{4}{3}\sqrt{\check{n}^4-\check{n}^2+1}+6\check{n}^2cn^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}. \tag{129}$$

If $\check{n}=1$, then Eqs. (128) and (129) become:

$$\Phi(x,\ t)=\left\{18\sqrt{2}-6\sqrt{2}\operatorname{sech}^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}, \tag{130}$$

$$\Psi(x,\ t)=\left\{-\frac{1}{3}k-\frac{10}{3}+6\operatorname{sech}^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}. \tag{131}$$

Case II-4: If $\hbar_1=-1,\ \hbar_2=2-\check{n}^2$ and $\hbar_3=\check{n}^2-1$, then $\mathscr{M}(\eta)=dn(\eta)$, and Eq. (120) takes the form:

$$\Phi(x,\ t)=\left\{2\sqrt{2}\left(10(2-\check{n}^2)-\sqrt{\check{n}^4-\check{n}^2+1}\right)-6\sqrt{2}dn^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}, \tag{132}$$

$$\Psi(x,\ t)=\left\{-2(2-\check{n}^2)-\frac{1}{3}k-\frac{4}{3}\sqrt{\check{n}^4-\check{n}^2+1}+6dn^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}. \tag{133}$$

Case II-5: If $\hbar_1=1,\ \hbar_2=-\left(1+\check{n}^2\right)$ and $\hbar_3=\check{n}^2$, then $\mathscr{M}(\eta)=ns(\eta)$, and Eq. (120) takes the form:

$$\Phi(x,\ t)=\left\{2\sqrt{2}\left(-10\left(1+\check{n}^2\right)-\sqrt{\check{n}^4-\check{n}^2+1}\right)+6\sqrt{2}ns^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}, \tag{134}$$

$$\Psi(x,\ t)=\left\{-2\left(1+\check{n}^2\right)-\frac{1}{3}k-\frac{4}{3}\sqrt{\check{n}^4-\check{n}^2+1}-6ns^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}. \tag{135}$$

If $\check{n}=1$, then Eqs. (134) and (135) become:

$$\Phi(x,\ t)=\left\{-42\sqrt{2}+6\sqrt{2}\coth^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{136}$$

$$\Psi(x,\ t)=\left\{-\frac{1}{3}k-\frac{16}{3}-6\coth^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{137}$$

While if $\check{n}\to 0$, then Eqs. (134) and (135) become:

$$\Phi(x,\ t)=\left\{-22\sqrt{2}+6\sqrt{2}\csc^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{138}$$

$$\Psi(x,\ t)=\left\{-\frac{1}{3}k-\frac{10}{3}-6\csc^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{139}$$

Case II-6: If $\hbar_1=1,\ \hbar_2=-\left(1+\check{n}^2\right)$ and $\hbar_3=\check{n}^2$, then $\mathscr{M}(\eta)=dc(\eta)$, and Eq. (120) takes the form:

$$\Phi(x,\ t)=\left\{2\sqrt{2}\left(-10\left(1+\check{n}^2\right)-\sqrt{\check{n}^4-\check{n}^2+1}\right)+6\sqrt{2}dc^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{140}$$

$$\Psi(x,\ t)=\left\{-2\left(1+\check{n}^2\right)-\frac{1}{3}k-\frac{4}{3}\sqrt{\check{n}^4-\check{n}^2+1}-6dc^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{141}$$

If $\check{n}\to 0$, then Eqs. (140) and (141) become:

$$\Phi(x,\ t)=\left\{18\sqrt{2}+6\sqrt{2}\sec^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{142}$$

$$\Psi(x,\ t)=\left\{-\frac{1}{3}k-\frac{10}{3}-6\sec^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{143}$$

Case II-7: If $\hbar_1=1-\check{n}^2,\ \hbar_2=2\check{n}^2-1$ and $\hbar_3=-\check{n}^2$, then $\mathscr{M}(\eta)=nc(\eta)$, and Eq. (120) takes the form:

$$\Phi(x,\ t)=\left\{2\sqrt{2}\left(10(2\check{n}^2-1)-\sqrt{\check{n}^4-\check{n}^2+1}\right)+6\sqrt{2}(1-\check{n}^2)nc^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{144}$$

$$\Psi(x,\ t)=\left\{-2(2\check{n}^2-1)-\frac{1}{3}k-\frac{4}{3}\sqrt{\check{n}^4-\check{n}^2+1}-6(1-\check{n}^2)nc^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{145}$$

Case II-8: If $\hbar_1=\check{n}^2-1,\ \hbar_2=2-\check{n}^2$ and $\hbar_3=-1$, then $\mathscr{M}(\eta)=nd(\eta)$, and Eq. (120) takes the form:

$$\Phi(x,\ t)=\left\{2\sqrt{2}\left(10(2-\check{n}^2)-\sqrt{\check{n}^4-\check{n}^2+1}\right)+6\sqrt{2}(\check{n}^2-1)nd^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{146}$$

$$\Psi(x,\ t)=\left\{-2(2-\check{n}^2)-\frac{1}{3}k-\frac{4}{3}\sqrt{\check{n}^4-\check{n}^2+1}-6(\check{n}^2-1)nd^2(x+2kt)\right\}. \tag{147}$$

Case II-9: If $\hbar_1=1-\check{n}^2,\ \hbar_2=2-\check{n}^2$ and $\hbar_3=1$, then $\mathscr{M}(\eta)=sc(\eta)$, and Eq. (120) takes the form:

$$\Phi(x,\ t)=\left\{2\sqrt{2}\left(10(2-\check{n}^2)-\sqrt{\check{n}^4-\check{n}^2+1}\right)+6\sqrt{2}(1-\check{n}^2)sc^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{148}$$

$$\Psi(x,\ t)=\left\{-2(2-\check{n}^2)-\frac{1}{3}k-\frac{4}{3}\sqrt{\check{n}^4-\check{n}^2+1}-6(1-\check{n}^2)sc^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{149}$$

If $\check{n}\to 0$, then Eqs. (148) and (149) become:

$$\Phi(x,\ t)=\left\{38\sqrt{2}+6\sqrt{2}\tan^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{150}$$

$$\Psi(x,\ t)=\left\{-\frac{1}{3}k-\frac{16}{3}-6\tan^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{151}$$

Case II-10: If $\hbar_1=-\check{n}^2(1-\check{n}^2),\ \hbar_2=2\check{n}^2-1$ and $\hbar_3=1$, then $\mathscr{M}(\eta)=sd(\eta)$, and Eq. (120) takes the form :

$$\Phi(x,\ t)=\left\{2\sqrt{2}\left(10(2\check{n}^2-1)-\sqrt{\check{n}^4-\check{n}^2+1}\right)-6\sqrt{2}\check{n}^2(1-\check{n}^2)sd^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{152}$$

$$\Psi(x,\ t)=\left\{-2(2\check{n}^2-1)-\frac{1}{3}k-\frac{4}{3}\sqrt{\check{n}^4-\check{n}^2+1}+6\check{n}^2(1-\check{n}^2)sd^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{153}$$

Case II-11: If $\hbar_1=1,\ \hbar_2=2-\check{n}^2$ and $\hbar_3=1-\check{n}^2$, then $\mathscr{M}(\eta)=cs(\eta)$, and Eq. (120) takes the form :

$$\Phi(x,\ t)=\left\{2\sqrt{2}\left(10(2-\check{n}^2)-\sqrt{\check{n}^4-\check{n}^2+1}\right)+6\sqrt{2}cs^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{154}$$

$$\Psi(x,\ t)=\left\{-2(2-\check{n}^2)-\frac{1}{3}k-\frac{4}{3}\sqrt{\check{n}^4-\check{n}^2+1}-6cs^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{155}$$

If $\check{n}=1$, then Eqs. (154) and (155) become:

$$\Phi(x,\ t)=\left\{18\sqrt{2}+6\sqrt{2}\operatorname{csch}^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{156}$$

$$\Psi(x,\ t)=\left\{-\frac{1}{3}k-\frac{10}{3}-6\operatorname{csch}^2(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{157}$$

While, if $\check{n}\to 0$, then Eqs. (154) and (155) become:

$$\Phi(x,\ t)=\left\{38\sqrt{2}+6\sqrt{2}\cot^{2}(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^{2}t}, \tag{158}$$

$$\Psi(x,\ t)=\left\{-\frac{1}{3}k-\frac{16}{3}-6\cot^{2}(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^{2}t}. \tag{159}$$

Case II-12: If $\hbar_1=1,\ \hbar_2=2\check{n}^2-1$ and $\hbar_3=-\check{n}^2(1-\check{n}^2)$, then $\mathscr{M}(\eta)=ds(\eta)$, and Eq. (120) takes the form:

$$\Phi(x,\ t)=\left\{2\sqrt{2}\left(10(2\check{n}^2-1)-\sqrt{\check{n}^4-\check{n}^2+1}\right)+6\sqrt{2}ds^{2}(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^{2}t}, \tag{160}$$

$$\Psi(x,\ t)=\left\{-2(2\check{n}^2-1)-\frac{1}{3}k-\frac{4}{3}\sqrt{\check{n}^4-\check{n}^2+1}-6ds^{2}(x+2kt)\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^{2}t}. \tag{161}$$

Case II-13: If $\hbar_1=\frac{1}{4},\ \hbar_2=\frac{1-2\check{n}^2}{2}$ and $\hbar_3=\frac{1}{4}$, then $\mathscr{M}(\eta)=ns(\eta)\pm cs(\eta)$, and Eq. (120) takes the form:

$$\Phi(x,\ t)=\left\{\sqrt{2}\left(10(1-2\check{n}^2)-\frac{1}{2}\sqrt{16\check{n}^4-16\check{n}^2+1}\right)+\frac{3\sqrt{2}}{2}\left(ns(x+2kt)\pm cs(x+2kt)\right)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^{2}t}, \tag{162}$$

$$\Psi(x,\ t)=\left\{-(1-2\check{n}^2)-\frac{1}{3}k-\frac{2}{3}\sqrt{16\check{n}^4-16\check{n}^2+1}-\frac{3}{2}(ns(x+2kt)\pm cs(x+2kt))^{2}\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^{2}t}. \tag{163}$$

Case II-14: If $\hbar_1=\frac{1}{4},\ \hbar_2=\frac{\check{n}^2-2}{2}$ and $\hbar_3=\frac{\check{n}^2}{4}$, then $\mathscr{M}(\eta)=ns(\eta)\pm ds(\eta)$, and Eq. (120) takes the form :

$$\Phi(x,\ t)=\left\{2\sqrt{2}\left(10(\check{n}^2-2)-\frac{1}{2}\sqrt{4\check{n}^4-19\check{n}^2+16}\right)+\frac{3\sqrt{2}}{2}\left(ns(x+2kt)\pm ds(x+2kt)\right)^{2}\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^{2}t}, \tag{164}$$

$$\Psi(x,\ t)=\left\{-(\check{n}^2-2)-\frac{1}{3}k-\frac{2}{3}\sqrt{4\check{n}^4-19\check{n}^2+16}-\frac{3}{2}\left(ns(x+2kt)\pm ds(x+2kt)\right)^{2}\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^{2}t}. \tag{165}$$

If $\check{n}=1$, then Eqs. (164) and (165) [or Eqs. (162) and (163)] become:

$$\Phi(x,\ t)=\left\{-21\sqrt{2}+\frac{3\sqrt{2}}{2}\left(\coth(x+2kt)\pm\operatorname{csch}(x+2kt)\right)^{2}\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^{2}t}, \tag{166}$$

$$\Psi(x,\ t)=\left\{\frac{1}{3}-\frac{1}{3}k-\frac{3}{2}\left(\coth(x+2kt)\pm\operatorname{csch}(x+2kt)\right)^{2}\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^{2}t}. \tag{167}$$

While, if $\check{n}\to 0$, then Eqs. (164) and (165) [or Eqs. (162) and (163)] become:

$$\Phi(x,\ t)=\left\{-44\sqrt{2}+\frac{3\sqrt{2}}{2}\left(\csc(x+2kt)\pm\cot(x+2kt)\right)^2\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{168}$$

$$\Psi(x,\ t)=\left\{\frac{-2}{3}-\frac{1}{3}k-\frac{3}{2}\left(\csc(x+2kt)\pm\cot(x+2kt)\right)^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{169}$$

Case II-15: If $\hbar_1=\dfrac{1-\check{n}^2}{4}$, $\hbar_2=\dfrac{1+\check{n}^2}{2}$ and $\hbar_3=\dfrac{1-\check{n}^2}{4}$, then $\mathscr{M}(\eta)=nc(\eta)\pm sc(\eta)$, and Eq. (120) takes the form:

$$\Phi(x,\ t)=\left\{\sqrt{2}\left(10(1+\check{n}^2)-\frac{1}{2}\sqrt{\check{n}^4+14\check{n}^2+1}\right)+\frac{3\sqrt{2}(1-\check{n}^2)}{2}(nc(x+2kt)\pm sc(x+2kt))^2\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{170}$$

$$\Psi(x,\ t)=\left\{-(1+\check{n}^2)-\frac{1}{3}k-\frac{2}{3}\sqrt{\check{n}^4+14\check{n}^2+1}-\frac{3(1-\check{n}^2)}{2}(nc(x+2kt)\pm sc(x+2kt))^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{171}$$

If $\check{n}\to 0$, then Eqs. (170) and (171) become:

$$\Phi(x,\ t)=\left\{\frac{19\sqrt{2}}{2}+\frac{3\sqrt{2}}{2}(\sec(x+2kt)\pm\tan(x+2kt))^2\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{172}$$

$$\Psi(x,\ t)=\left\{-\frac{1}{3}k-\frac{5}{3}-\frac{3}{2}(\sec(x+2kt)\pm\tan(x+2kt))^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{173}$$

Case II-16: If $\hbar_1=\dfrac{\check{n}^2}{4}$, $\hbar_2=\dfrac{\check{n}^2-2}{2}$ and $\hbar_3=\dfrac{\check{n}^2}{4}$, then $\mathscr{M}(\eta)=\sqrt{1-\check{n}^2}\,(sd(\eta)\pm cd(\eta))$, and Eq. (120) takes the form:

$$\Phi(x,\ t)=\left\{\sqrt{2}\left(10(\check{n}^2-2)-\frac{1}{2}\sqrt{\check{n}^4-16\check{n}^2+16}\right)+\frac{3\sqrt{2}\check{n}^2}{2}\left(\sqrt{1-\check{n}^2}sd(x+2kt)\pm cd(x+2kt)\right)^2\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{174}$$

$$\Psi(x,\ t)=\left\{-(\check{n}^2-2)-\frac{1}{3}k-\frac{2}{3}\sqrt{\check{n}^4-16\check{n}^2+16}-\frac{3\check{n}^2}{2}\left(\sqrt{1-\check{n}^2}sd(x+2kt)\pm cd(x+2kt)\right)^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{175}$$

Case II-17: If $\hbar_1=\dfrac{\check{n}^2-1}{4}$, $\hbar_2=\dfrac{\check{n}^2+1}{2}$ and $\hbar_3=\dfrac{\check{n}^2-1}{4}$, then $\mathscr{M}(\eta)=\check{n}sd(\eta)\pm nd(\eta)$, and Eq. (120) takes the form:

$$\Phi(x,\ t)=\left\{\sqrt{2}\left(10(1+\check{n}^2)-\frac{1}{2}\sqrt{\check{n}^4+14\check{n}^2+1}\right)+\frac{3\sqrt{2}(\check{n}^2-1)}{2}\left(\check{n}sd(x+2kt)\pm nd(x+2kt)\right)^2\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t},\tag{176}$$

$$\Psi(x,\ t)=\frac{1}{2}\left\{-(1+\check{n}^2)-\frac{1}{3}k-\frac{2}{3}\sqrt{\check{n}^4+14\check{n}^2+1}-\frac{3(\check{n}^2-1)}{2}\left(\check{n}sd(x+2kt)\pm nd(x+2kt)\right)^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}.\tag{177}$$

Case II-18: If $\hbar_1=\frac{\check{n}^2}{4}$, $\hbar_2=\frac{\check{n}^2-2}{2}$ and $\hbar_3=\frac{1}{4}$, then $\mathscr{M}(\eta)=\frac{sn(\eta)}{1\pm dn}$, and Eq. (120) takes the form:

$$\Phi(x,\ t)=\left\{\sqrt{2}\left(10(\check{n}^2-2)-\frac{1}{2}\sqrt{4\check{n}^4-19\check{n}^2+16}\right)+\frac{3\sqrt{2}\check{n}^2}{2}\left(\frac{sn(\eta)}{1\pm dn(\eta)}\right)^2\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t},\tag{178}$$

$$\Psi(x,\ t)=\left\{-(\check{n}^2-2)-\frac{1}{3}k-\frac{2}{3}\sqrt{4\check{n}^4-19\check{n}^2+16}-\frac{3\check{n}^2}{2}\left(\frac{sn(\eta)}{1\pm dn(\eta)}\right)^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}.\tag{179}$$

If $\check{n}=1$, then Eqs. (178) and (179) become:

$$\Phi(x,\ t)=\left\{-21\sqrt{2}+\frac{3\sqrt{2}}{2}\left(\frac{\tanh(\eta)}{1\pm\operatorname{sech}(\eta)}\right)^2\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t},\tag{180}$$

$$\Psi(x,\ t)=\left\{\frac{1}{3}-\frac{1}{3}k-\frac{3}{2}\left(\frac{\tanh(\eta)}{1\pm\operatorname{sech}(\eta)}\right)^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}.\tag{181}$$

Case II-19: If $\hbar_1=\frac{-1}{4}$, $\hbar_2=\frac{\check{n}^2+1}{2}$ and $\hbar_3=\frac{\left(1-\check{n}^2\right)^2}{4}$, then $\mathscr{M}(\eta)=\check{n}cn(\eta)\pm dn(\eta)$, and Eq. (120) takes the form:

$$\Phi(x,\ t)=\left\{\sqrt{2}\left(10(\check{n}^2+1)-\frac{1}{2}\sqrt{7\check{n}^4+2\check{n}^2+7}\right)-\frac{3\sqrt{2}}{4}\left(\check{n}cn(x+2kt)\pm dn(x+2kt)\right)^2\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t},\tag{182}$$

$$\Psi(x,\ t)=\left\{-(\check{n}^2+1)-\frac{1}{3}k-\frac{2}{3}\sqrt{7\check{n}^4+2\check{n}^2+7}-\frac{3}{4}\left(\check{n}cn(x+2kt)\pm dn(x+2kt)\right)^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}.\tag{183}$$

Case II-20: If $\hbar_1=\frac{\left(1-\check{n}^2\right)^2}{4}$, $\hbar_2=\frac{\check{n}^2+1}{2}$ and $\hbar_3=\frac{1}{4}$, then $\mathscr{M}(\eta)=(ds(\eta)\pm cs(\eta))$, and Eq. (120) takes the form:

$$\Phi(x,\ t)=\left\{\sqrt{2}\left(10(\check{n}^2+1)-\frac{1}{2}\sqrt{\check{n}^4+14\check{n}^2+1}\right)+3\sqrt{2}\left(1-\check{n}^2\right)^2\left(ds(x+2kt)\pm cs(x+2kt)\right)^2\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{184}$$

$$\Psi(x,\ t)=\left\{-(\check{n}^2+1)-\frac{1}{3}k-\frac{2}{3}\sqrt{\check{n}^4+14\check{n}^2+1}-3\left(1-\check{n}^2\right)^2\left(ds(x+2kt)\pm cs(x+2kt)\right)^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{185}$$

If $\check{n}\to 0$, then Eqs. (184) and (185) become:

$$\Phi(x,\ t)=\left\{\frac{19\sqrt{2}}{2}+\frac{6\sqrt{2}}{2}\left(\sec(x+2kt)\pm\tan(x+2kt)\right)^2\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{186}$$

$$\Psi(x,\ t)=\left\{-\frac{1}{3}k-\frac{5}{3}-\frac{6}{2}\left(\sec(x+2kt)\pm\tan(x+2kt)\right)^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{187}$$

Case II-21: If $\hbar_1=1,\ \hbar_2=2-4\check{n}^2$ and $\hbar_3=1$, then $\mathscr{M}(\eta)=\dfrac{sn(\eta)dn(\eta)}{cn(\eta)}$, and Eq. (120) takes the form:

$$\Phi(x,\ t)=\left\{2\sqrt{2}\left(10(2-4\check{n}^2)-\sqrt{16\check{n}^4-16\check{n}^2+1}\right)+6\sqrt{2}\frac{sn^2(x+2kt)dn^2(x+2kt)}{cn^2(x+2kt)}\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{188}$$

$$\Psi(x,\ t)=\left\{-2(2-4\check{n}^2)-\frac{1}{3}k+\frac{4}{3}\sqrt{16\check{n}^4-16\check{n}^2+1}-\frac{6sn^2(x+2kt)dn^2(x+2kt)}{cn^2(x+2kt)}\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{189}$$

Case II- 22: If $\hbar_1=\check{n}^4,\ \hbar_2=2$ and $\hbar_3=1$, then $\mathscr{M}(\eta)=\dfrac{sn(\eta)cn(\eta)}{dn(\eta)}$, and Eq. (120) takes the form:

$$\Phi(x,\ t)=\left\{2\sqrt{2}\left(20-\sqrt{4-3\check{n}^4}\right)+6\sqrt{2}\check{n}^4\frac{sn^2(x+2kt)cn^2(x+2kt)}{dn^2(x+2kt)}\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{190}$$

$$\Psi(x,\ t)=\left\{-4-\frac{1}{3}k-\frac{4}{3}\sqrt{4-3\check{n}^4}-6\check{n}^4\frac{sn^2(x+2kt)cn^2(x+2kt)}{dn^2(x+2kt)}\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{191}$$

If $\check{n}=1$, then Eqs. (190) and (191) become:

$$\Phi(x,\ t)=\left\{38\sqrt{2}+6\sqrt{2}\tanh^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{192}$$

$$\Psi(x,\ t)=\left\{-\frac{1}{3}k-\frac{16}{3}-6(\tanh(x+2kt))^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{193}$$

Case II-23: If $\hbar_1 = 1,\ \hbar_2 = \check{n}^2+2$, and $\hbar_3 = 1-2\check{n}^2+\check{n}^4$, then $\mathscr{M}(\eta) = \dfrac{dn(\eta)cn(\eta)}{sn(\eta)}$, and Eq. (120) takes the form:

$$\Phi(x,\ t) = \left\{2\sqrt{2}\left(10(\check{n}^2+2)-\sqrt{-2\check{n}^4+10\check{n}^2+1}\right)+6\sqrt{2}\frac{dn^2(x+2kt)cn^2(x+2kt)}{sn^2(x+2kt)}\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{194}$$

$$\Psi(x,\ t) = \left\{-2(\check{n}^2+2)-\frac{1}{3}k-\frac{4}{3}\sqrt{-2\check{n}^4+10\check{n}^2+1}-\frac{6dn^2(x+2kt)cn^2(x+2kt)}{sn^2(x+2kt)}\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{195}$$

If $\check{n}=1$, then Eqs. (194) and (195) become:

$$\Phi(x,\ t) = \left\{54\sqrt{2}+6\sqrt{2}\text{sech}^2(x+2kt)\text{csch}^2(x+2kt)\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{196}$$

$$\Psi(x,\ t) = \left\{-10-\frac{1}{3}k-6(\text{sech}(x+2kt)\text{csch}(x+2kt))^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{197}$$

Case II-24: If $\hbar_1 = -\dfrac{4}{\check{n}}$, $\hbar_2 = 6\check{n}-\check{n}^2-1$ and $\hbar_3 = -2\check{n}^3+\check{n}^4+\check{n}^2$, then $\mathscr{M}(\eta) = \dfrac{\check{n}cn(\eta)dn(\eta)}{\check{n}sn^2(\eta)+1}$, and Eq. (120) takes the form:

$$\Phi(x,\ t) = \left\{2\sqrt{2}\left(10(6\check{n}-\check{n}^2-1)-\sqrt{\check{n}^4+14\check{n}^2+1}\right)-24\sqrt{2}\frac{cn^2(x+2kt)dn^2(x+2kt)}{(\check{n}sn^2(x+2kt)+1)^2}\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{198}$$

$$\Psi(x,\ t) = \left\{-12\check{n}+2\check{n}^2+2-\frac{1}{3}k-\frac{4}{3}\sqrt{\check{n}^4+14\check{n}^2+1}+\frac{24cn^2(x+2kt)dn^2(x+2kt)}{(\check{n}sn^2(x+2kt)+1)^2}\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{199}$$

If $\check{n}=1$, then Eqs. (198) and (199) become:

$$\Phi(x,\ t) = \left\{72\sqrt{2}-\frac{24\sqrt{2}\text{sech}^4(x+2kt)}{(1+\tanh^2(x+2kt))^2}\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{200}$$

$$\Psi(x,\ t) = \left\{-\frac{1}{3}k-\frac{40}{3}+\frac{24\text{sech}^4(x+2kt)}{(1+\tanh^2(x+2kt))^2}\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}. \tag{201}$$

Case II-25: If $\hbar_1 = \dfrac{1}{4}$, $\hbar_2 = \dfrac{1-2\check{n}^2}{2}$ and $\hbar_3 = \dfrac{1}{4}$, then $\mathscr{M}(\eta) = \dfrac{sn(\eta)}{1\pm cn(\eta)}$, and Eq. (120) takes the form:

$$\Phi(x,\ t) = \left\{2\sqrt{2}\left(10(1-2\check{n}^2)-\frac{1}{2}\sqrt{16\check{n}^4-16\check{n}^2+1}\right)+\frac{3\sqrt{2}}{2}\frac{sn^2(x+2kt)}{[1\pm cn(x+2kt)]^2}\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2 t}, \tag{202}$$

$$\Psi(x,\ t)=\left\{-(1-2\check{n}^2)-\frac{1}{3}k-\frac{2}{3}\sqrt{16\check{n}^4-16\check{n}^2+1}-\frac{3}{2}\frac{sn^2(x+2kt)}{(1\pm cn(x+2kt))^2}\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}. \tag{203}$$

If $\check{n}\to 0$, then Eqs. (202) and (203) become:

$$\Phi(x,\ t)=\left\{19\sqrt{2}+\frac{3\sqrt{2}}{2}(\csc(x+2kt)\pm\cot(x+2kt))^2\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}, \tag{204}$$

$$\Psi(x,\ t)=\left\{-\frac{1}{3}k-\frac{5}{3}-\frac{3}{2}(\csc(x+2kt)\pm\cot(x+2kt))^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}. \tag{205}$$

Case II-26: If $\hbar_1=\dfrac{1-\check{n}^2}{4}$, $\hbar_2=\dfrac{1+\check{n}^2}{2}$ and $\hbar_3=\dfrac{1-\check{n}^2}{4}$, then $\mathscr{M}(\eta)=\dfrac{cn(\eta)}{1\pm sn(\eta)}$, and Eq. (120) takes the form:

$$\Phi(x,\ t)=\left\{\sqrt{2}\left(10(1+\check{n}^2)-\frac{1}{2}\sqrt{\check{n}^4+14\check{n}^2+1}\right)+\frac{3\sqrt{2}(1-\check{n}^2)}{2}\frac{(cn^2(x+2kt)}{(1\pm sn(x+2kt))^2}\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}, \tag{206}$$

$$\Psi(x,\ t)=\left\{-(1+\check{n}^2)-\frac{1}{3}k-\frac{2}{3}\sqrt{\check{n}^4+14\check{n}^2+1}-\frac{3(1-\check{n}^2)}{2}\frac{(cn^2(x+2kt)}{(1\pm sn(x+2kt))^2}\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}. \tag{207}$$

If $\check{n}\to 0$, then Eqs. (206) and (207) become:

$$\Phi(x,\ t)=\left\{\frac{19\sqrt{2}}{2}+\frac{3\sqrt{2}}{2}(\sec(x+2kt)\pm\tan(x+2kt))^2\right\}e^{i(kx+\delta t)+\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}, \tag{208}$$

$$\Psi(x,\ t)=\frac{1}{2}\left\{-\frac{5}{3}-\frac{1}{3}k-\frac{3}{2}(\sec(x+2kt)\pm\tan(x+2kt))^2\right\}e^{\sigma\mathscr{B}-\frac{1}{2}\sigma^2t}. \tag{209}$$

Remark 2 If we put $\sigma=0$ in Eqs. (122), (123), (124), (125), (128), (129), (130), (131), (132), and (133), then we have the same solutions that stated in [27].

5. Discussion and effect of noise

Discussion: The presence of multiplicative noise in the Schrödinger-KdV equation can lead to the emergence of complex and chaotic behavior. Nonlinear equations like the Schrödinger-KdV equation are known to exhibit a wide range of dynamical phenomena, including soliton solutions, turbulence, and pattern formation. When multiplicative noise is introduced into the equation, it can interact with the nonlinear terms and lead to novel and unpredictable dynamics. Understanding and analyzing these chaotic behaviors can provide valuable insights into the underlying mechanisms that govern the system, and may ultimately lead to new breakthroughs in various fields of physics and engineering.

Effect of noise: Now, let us show the effect of noise on the analytical solution of the SCSKdV in Eq. (1). To illustrate the behavior of some of the discovered solutions, many figures are provided, including Eqs. (30), (31), (32), (33), (88) and (99) as follows:

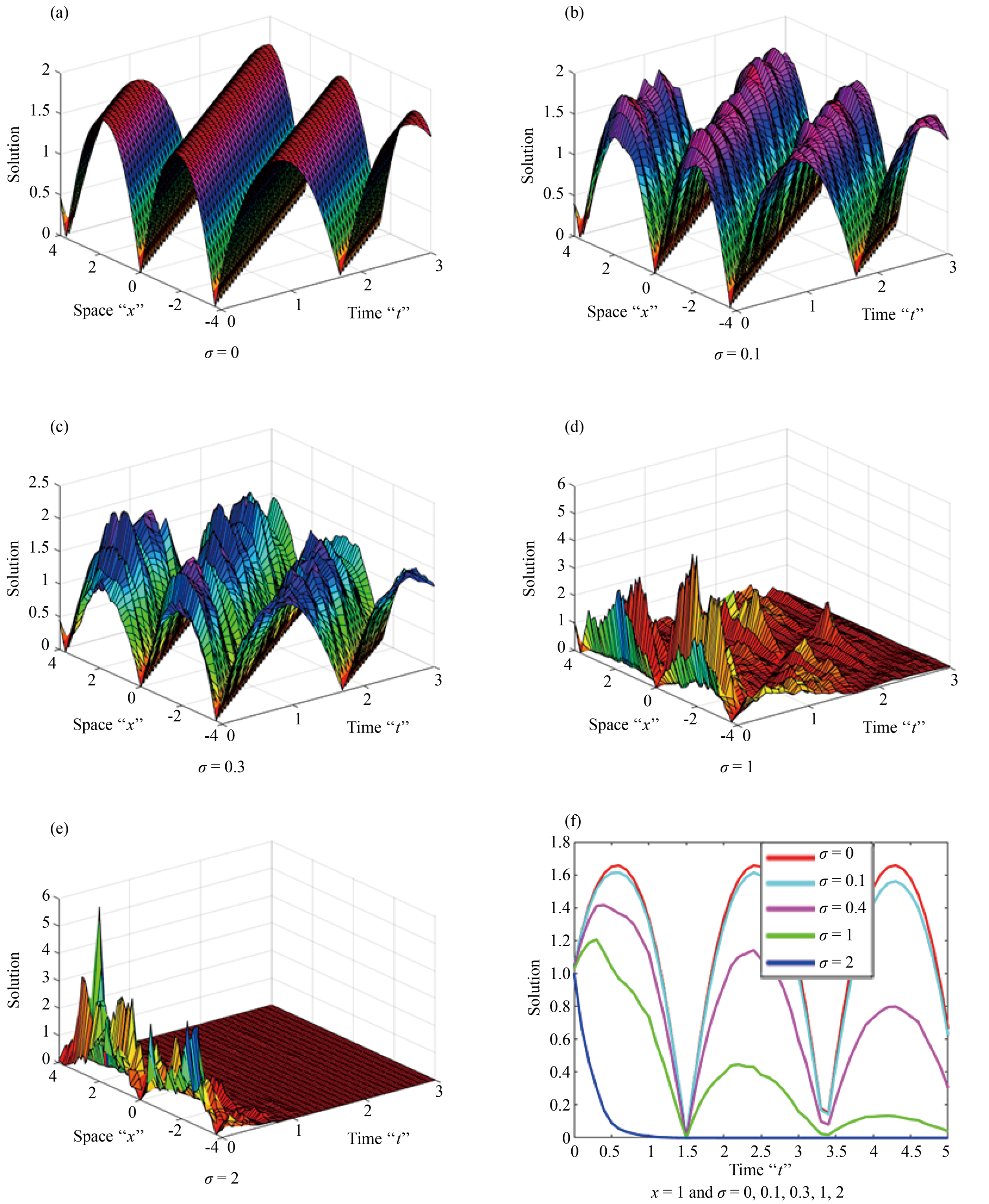

Figure 1. (a-c) display 3D-shape of the periodic soliton solution $\Phi(x, t)$ in Eq (30) with $\check{n} = 0.5$, $k = -1$, $\delta = 2$, $x \in [0, 4]$, $t \in [0, 3]$ and $\sigma = 0, 0.1, 0.3, 1, 2$ (d) shows 2D-shape of Eq. (30) with different σ

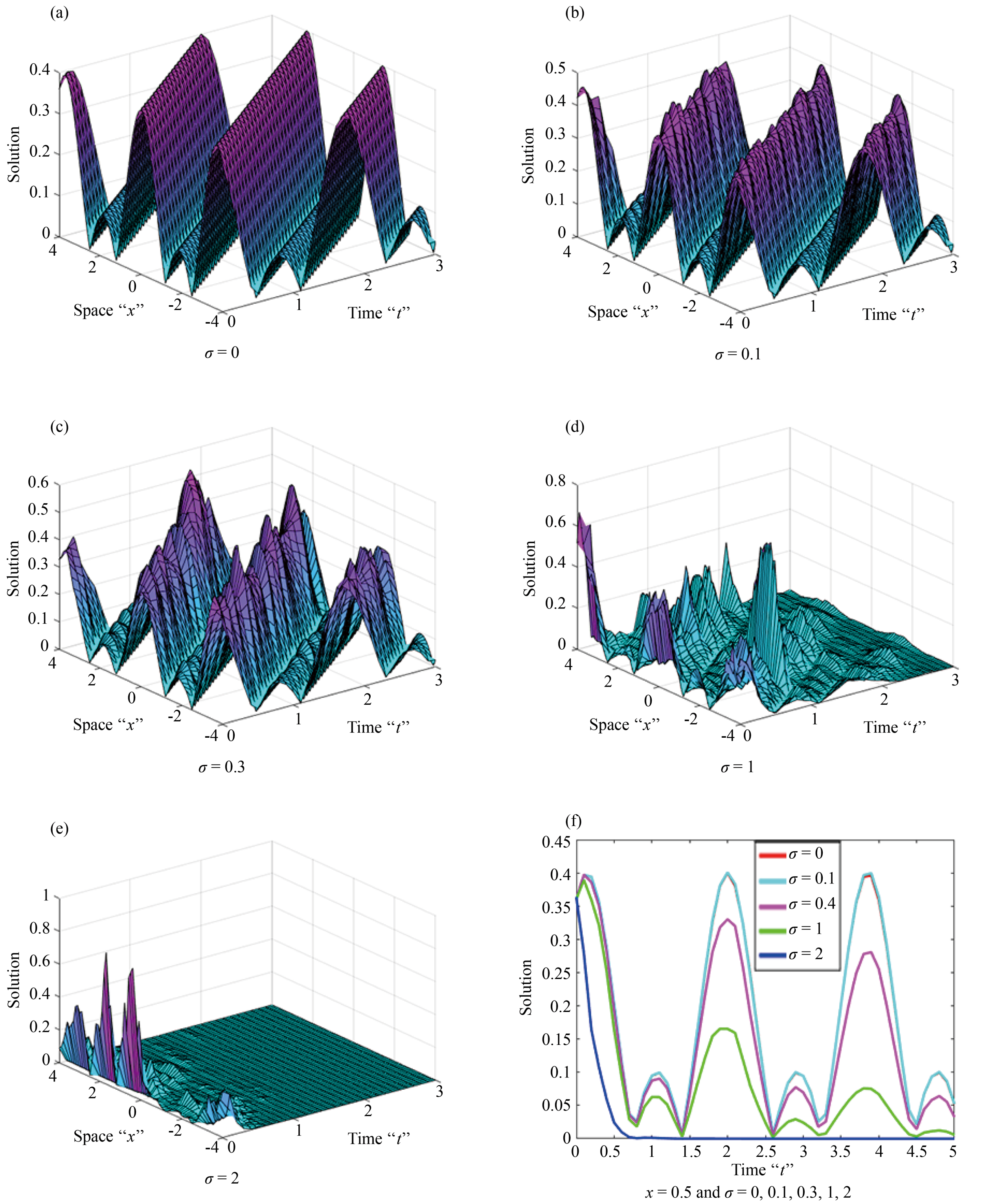

Figure 2. (a-e) display 3D-shape of the periodic soliton solution $\Psi(x,\ t)$ in Eq. (31) with $\check{n} = 0.5$, $k = -1$, $\delta = 2$, $x \in [0,\ 4]$, $t \in [0,\ 3]$ and $\sigma = 0,\ 0.1,\ 0.3,\ 1,\ 2$ (f) shows 2D-shape of Eq. (31) with different σ

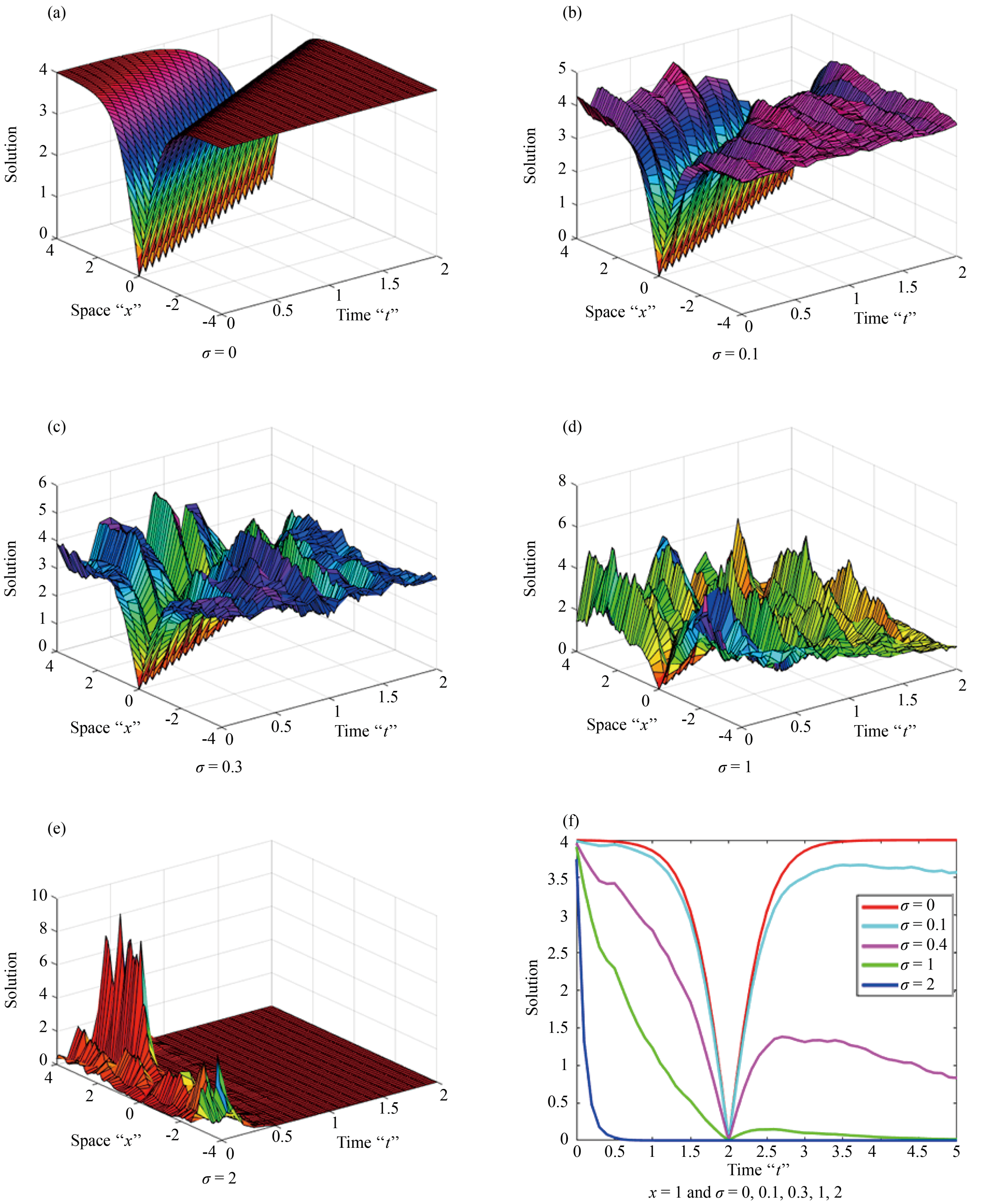

Figure 3. (a-e) display 3D-shape of the dark soliton solution $\Phi(x, t)$ in Eq. (32) with $k = -1$, $\delta = 2$, $x \in [0, 4]$, $t \in [0, 3]$ and $\sigma = 0, 0.1, 0.3, 1, 2$ (f) shows 2D-shape of Eq. (32) with different σ

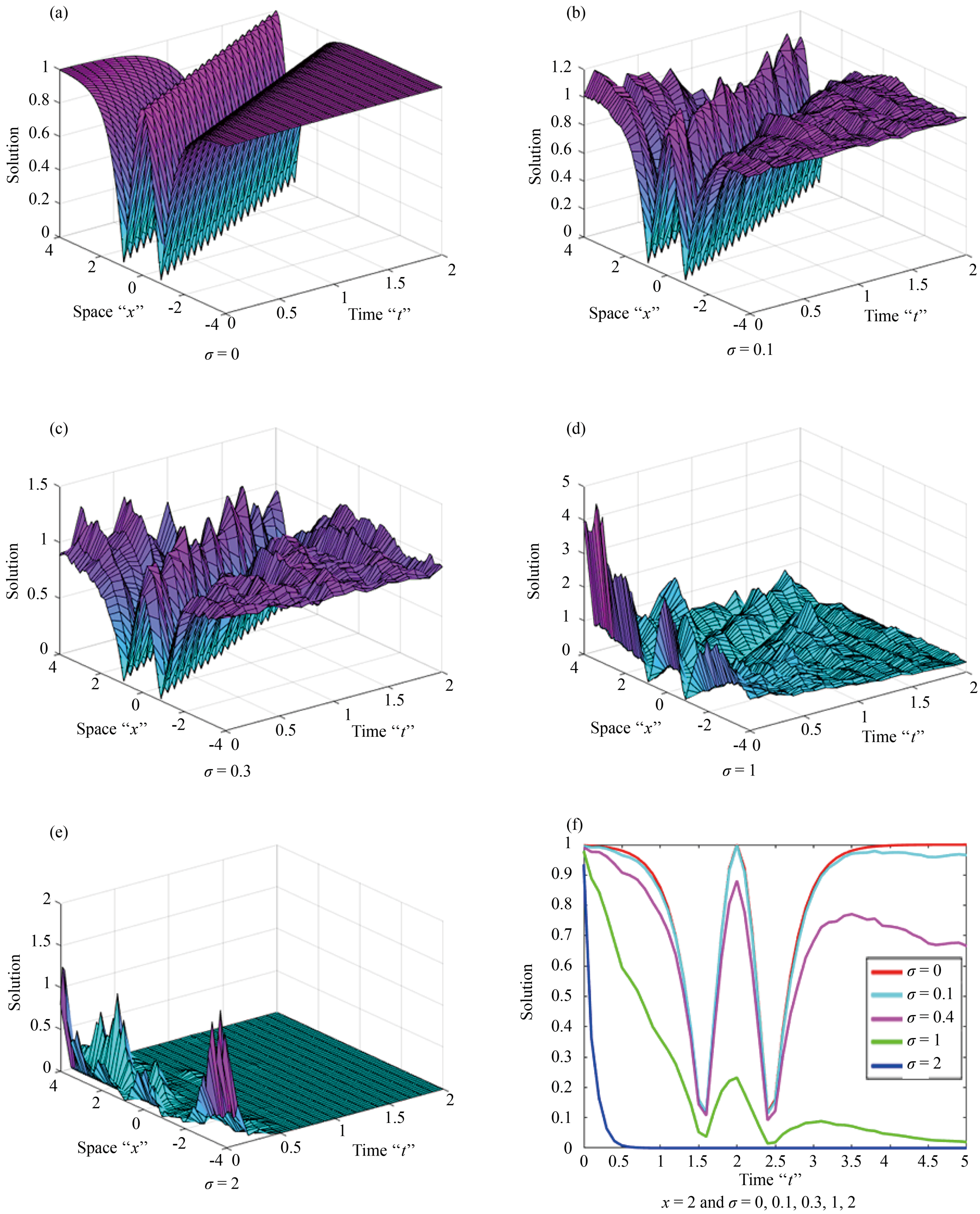

Figure 4. (a-c) display 3D-shape of solution $\Psi(x, t)$ in Eq. (33) with $k = 2$ and $\sigma = 0, 0.1, 0.3, 1, 2$ (d) shows 2D-shape of Eq. (33) with different σ

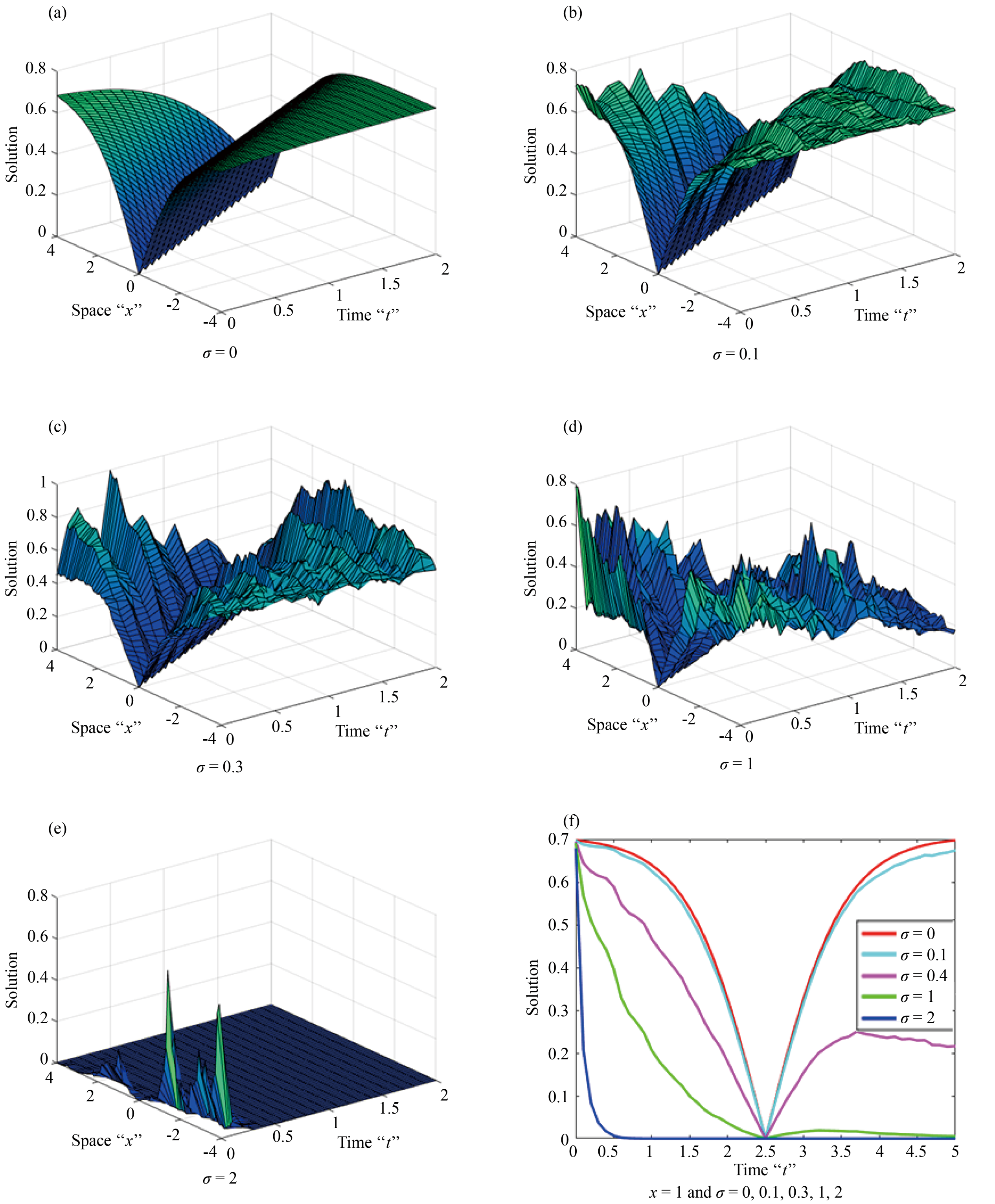

Figure 5. (a-c) display 3D-shape of the dark soliton solution $\Phi(x, t)$ in Eq. (88) with $k = 2$ and $\sigma = 0, 0.1, 0.3, 1, 2$ (d) shows 2D-shape of Eq. (88) with different σ

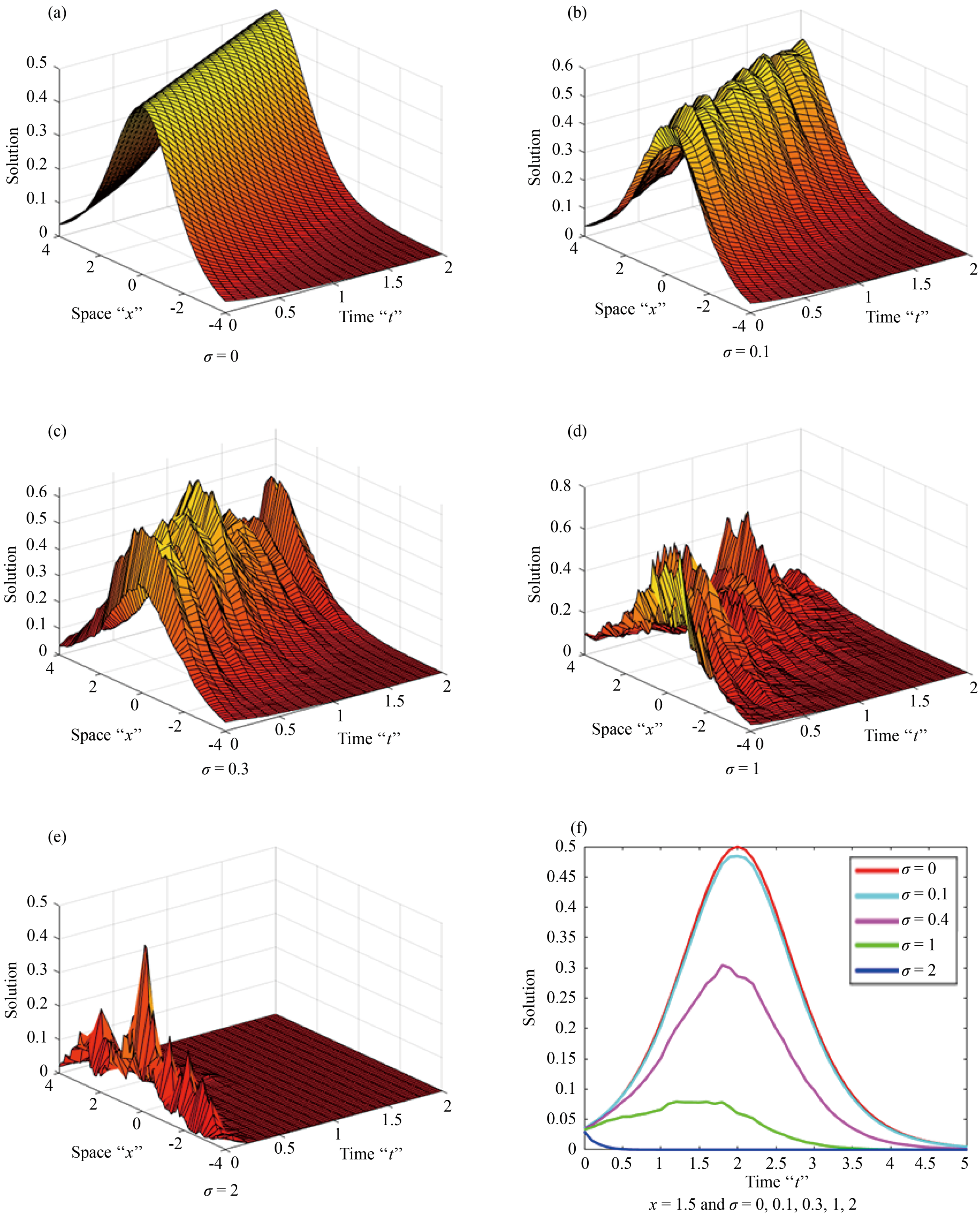

Figure 6. (a-c) display 3D-shape of the bright soliton solution $\Psi(x, t)$ in Eq. (89) with $k = 2$ and $\sigma = 0, 0.1, 0.3, 1, 2$ (d) shows 2D-shape of Eq. (89) with different σ

Now, Figures 1-6 show that there are several additional kinds of solutions, including periodic soliton solutions, kink soliton solutions, bright soliton solutions, dark soliton solutions and so on, when the multiplicative noise is disregarded (i.e. when $\sigma = 0$). The surface becomes flatter after brief transit patterns when noise is added and its amplitude is raised by $\sigma = 0.1,\ 0.3,\ 1,\ 2$. This indicates that SCSKdV equations solutions are influenced by multiplicative Brownian motion and are kept stable around zero.

6. Conclusions

In this paper, the stochastic coupled Schrödinger-KdV equations in Eq. (1) forced by multiplicative Brownian motion in the Itô sense was considered. By utilizing a mapping method, we were able to obtain new trigonometric, rational, hyperbolic, and elliptic stochastic solutions. The results demonstrate that this method is an effective mathematical tool for finding exact solutions to our equation. Also, it is a potential method for solving other nonlinear evolution equations. In the absence of noise, we obtained some previously solutions of CSKdV such as the solutions stated in [18, 27]. These obtained solutions can be applied to the analysis of a wide variety of crucial physical phenomena because of the coupled Schrödinger-KdV equations have important applications in dusty plasma, including Langmuir, electromagnetic waves, and dust-acoustic wave. Furthermore, the noise term impacts on the analytical solution of the SCSKdV equations in Eq. (1) were shown using the MATLAB software. We displayed that multiplicative Brownian motion stabilized the solutions at zero. In the future work, we can obtain the exact solutions for coupled Schrödinger-KdV equations with additive noise.

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Data availability statement

The data used to support the findings of this study are included within the article.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Baskonus HM, Bulut H, Sulaiman TA. New complex hyperbolic structures to the Lonngren-wave equation by using sine-gordon expansion method. *Applied Mathematics and Nonlinear Science*. 2019; 4: 129-138. Available from: https://doi.org/10.2478/AMNS.2019.1.00013.

[2] Lu B. The first integral method for some time fractional differential equations. *Applied Mathematics and Nonlinear Science*. 2012; 395: 684-693. Available from: https://doi.org/10.1016/j.jmaa.2012.05.066.

[3] Al-Askar FM, Mohammed WW. Abundant optical solutions for the Sasa-Satsuma equation with M-truncated derivative. *Frontiers in Physics*. 2023; 1216451: 740. Available from: https://doi.org/10.3389/fphy.2023.1216451.

[4] Yan ZL. Abunbant families of Jacobi elliptic function solutions of the dimensional integrable Davey-Stewartson-type equation via a new method. *Chaos Solitons Fractals*. 2003; 18: 299-309. Available from: https://doi.org/10.1016/S0960-0779(02)00653-7.

[5] Al-Askar FM, Cesarano C, Mohammed WW. Effects of the wiener process and beta derivative on the exact solutions of the kadomtsev-petviashvili equation. *Axioms*. 2023; 12: 748. Available from: https://doi.org/10.3390/axioms12080748.

[6] Wang ML, Li XZ, Zhang JL. The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Physics Letters A*. 2008; 372: 417-423. Available from: https://doi.org/10.1016/j.physleta.2007.07.051.

[7] Wazwaz AM. The sine-cosine method for obtaining solutions with compact and noncompact structures. *Applied Mathematics a Computation*. 2004; 159: 559-576. Available from: https://doi.org/10.1016/j.amc.2003.08.136.

[8] He JH, Wu XH. Exp-function method for nonlinear wave equations. *Chaos Solitons Fractals*. 2006; 30: 700-708. Available from: https://doi.org/10.1177/14613484241275313.

[9] Khan K, Akbar MA. The $exp(-\phi(\varsigma))$-expansion method for finding travelling wave solutions of Vakhnenko-Parkes equation. *International Journal of Dynamical Systems and Differential Equations*. 2014; 5: 72-83. Available from: https://doi.org/10.1504/IJDSDE.2014.067119.

[10] Albayrak P, Ozisik M, Bayram M, Secer A, Das SE, Biswas A, et al. Pure-cubic optical solitons and stability analysis with kerr law nonlinearity. *Contemporary Mathematics*. 2025; 4: 530-48. Available from: https://ojs.wiserpub.com/index.php/CM/article/view/3308.

[11] Ozisik M, Secer A, Bayram M. Dispersive optical solitons of Biswas-Arshed equation with a couple of novel approaches. *Optik*. 2022; 265: 169547. Available from: https://doi.org/10.1016/j.ijleo.2022.169547.

[12] Ozisik M, Secer A, Bayram M, Yusuf A, Sulaiman T. Soliton solutions of the (2 + 1)-dimensional Kadomtsev-Petviashvili equation via two different integration schemes. *International Journal of Modern Physics B*. 2023; 37: 2350212. Available from: https://doi.org/10.1142/S0217979223502120.

[13] Hamza AE, Alshammari M, Atta D, Mohammed WW. Fractional-stochastic shallow water equations and its analytical solutions. *Results in Physics*. 2023; 53: 106953. Available from: https://doi.org/10.1016/j.rinp.2023.106953.

[14] Samir I, Ahmed KK, Ahmed HM, Emadifar H, Rabie WB. Extraction of newly soliton wave structure of generalized stochastic NLSE with standard Brownian motion, quintuple power law of nonlinearity and nonlinear chromatic dispersion. *Physics Open*. 2024; 21: 100232. Available from: https://doi.org/10.1016/j.physo.2024.100232.

[15] Ahmed KK, Ahmed H, Badra NM, Mirzazadeh M, Rabie W, Eslami M. Diverse exact solutions to Davey-Stewartson model using modified extended mapping method. *Nonlinear Analysis: Modelling and Control*. 2024; 29: 983-1002. Available from: https://doi.org/10.15388/namc.2024.29.36103.

[16] Ahmed KK, Ahmed HM, Rabie WB, Shehab MF. Effect of noise on wave solitons for (3 + 1)-dimensional nonlinear Schrödinger equation in optical fiber. *Indian Journal of Physics*. 2024; 98: 4863-4882. Available from: https://doi.org/10.1007/s12648-024-03222-3.

[17] Kaya D, El-Sayed SM. On the solution of the coupled Schrödinger-KdV equation by the decomposition method. *Physics Letters A*. 2003; 313: 82-88. Available from: https://doi.org/10.1016/S0375-9601(03)00723-0.

[18] Filiz A, Ekici M, Sonmezoglu A. F-expansion method and new exact solutions of the Schrödinge-KdV equation. *Scientific World Journal*. 2014; 2014: 534063. Available from: https://doi.org/10.1155/2014/534063.

[19] Arnous AH, Elsherbeny AM, Secer A, Ozisik M, Bayram M, Shah NA, et al. Optical solitons for the dispersive concatenation model with spatio-temporal dispersion having multiplicative white noise. *Results in Physics*. 2024; 56: 107299. Available from: https://doi.org/10.1016/j.rinp.2023.1072994.

[20] Zayed EME, Arnous AH, Secer A, Ozisik M, Bayram M, Shah NA, et al. High dispersion and cubic-quintic-septic-nonic nonlinearity effects on optical solitons in the complex Ginzburg-Landau equation of eighth-order with multiplicative white noise in the Itô sense. *Results in Physics*. 2024; 58: 107439. Available from: https://doi.org/10.1016/j.rinp.2024.107439.

[21] Rezazadeh H, Davodi AG, Gholami D. Combined formal periodic wave-like and soliton-like solutions of the conformable Schrödinger-KdV equation using the (G'/G)-expansion technique. *Results in Physics*. 2023; 47: 106352. Available from: https://doi.org/10.1016/j.rinp.2023.106352.

[22] Bai CL, Zhao H. Complex hyperbolic-function method and its applications to nonlinear equations. *Physics Letters A*. 2006; 355: 32-38. Available from: https://doi.org/10.1016/j.physleta.2006.01.094.

[23] Ismail MS, Mosally FM, Alamoudi KM. Petrov-Galerkin method for the Coupled Schrödinger-KdV Equation. *Physics Letters A*. 2014; 2014: 705204. Available from: https://doi.org/10.1155/2014/705204.

[24] Rahhman MM, Aktar A, Roy KC. Analytical solutions of nonlinear Coupled Schrödinger-KdV Equation via Advance exponential expansion. *American Journal of Mathematical and Computer Modelling*. 2018; 3: 46-51. Available from: https://doi.org/10.11648/j.ajmcm.20180303.11.

[25] Lee J, Sakthivel R. Exact travelling wave solutions for some important nonlinear physical models. *Pramana Journal of Physics*. 2013; 80: 757-769. Available from: https://doi.org/10.1007/s12043-013-0520-9.

[26] Bai D, Zhang L. The finite element method for the coupled Schrödinger-KdV equations. *Pramana Journal of Physics*. 2009; 373: 2237-2244. Available from: https://doi.org/10.1016/j.physleta.2009.04.043.

[27] Peng YZ. Exact periodic wave solutions to the coupled Schrödinger-KdV equation and DS equations. *Pramana Journal of Physics*. 2004; 62: 33-941. Available from: https://doi.org/10.1007/BF02706141.

[28] Calin O. *An Informal Introduction to Stochastic Calculus with Applications*. 2nd ed. USA: World Scientific; 2021.

[29] Peng Y. Exact solutions for some nonlinear partial differential equations. *Physics Letters A*. 2003; 314: 401-408. Available from: https://doi.org/10.1016/S0375-9601(03)00909-5.

[30] Zhang D. Doubly periodic solutions of the modified Kawahara equation. *Chaos, Solitons and Fractals*. 2005; 25: 1155-1160. Available from: https://doi.org/10.1016/j.chaos.2004.11.084.

[31] Liu Q, Zhu J. Exact Jacobian elliptic function solutions and hyperbolic function solutions for Sawada-Kotere equation with variable coefficient. *Chaos, Solitons and Fractals*. 2006; 352: 233-238. Available from: https://doi.org/10.1016/j.physleta.2005.12.007.