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Research Article

Fractional View Analysis of Algebra Application for Nonlinear Physical Model

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Abstract: This paper examines the application of algebraic principles to solve nonlinear physical models described by fractional-order differential equations. Utilizing the Aboodh Transform Iteration Method and the Aboodh Residual Power Series Method within the Caputo operator framework, the study explores innovative solutions for these models. By bridging algebraic techniques with advanced mathematical methods, it underscores the power of these approaches in unraveling the complexities of systems governed by fractional calculus. This research not only deepens our understanding of nonlinear phenomena but also opens new avenues for the practical application of algebra in solving real-world challenges, showcasing the boundless potential of mathematical innovation.

***Keywords*:** nonlinear physical model, fractional order differential equation, aboodh residual power series method, aboodh transform iteration method

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1. Introduction

There has been significant improvement during the past few years in the creation of efficient and rapid methods that can be used to solve fractional differential equations. This progress is mostly due to the fact that fractional derivatives are more powerful to describe many complex phenomena, particularly for those with memory effects, mechanical performances of materials, anomalous diffusion processes and groundwater flow problems, and systems from control theory. Dynamic models of natural systems are frequently improved by models with fractional orders of differentiation. Although concentrated-type fractional derivatives are the most commonly used here and going forward, well-known fractional derivatives like the Riemann-Liouville and Caputo have their own set of problems, especially when using singular kernels, and may present some challenges when it comes to physical modelling [1–3].

A new fractional operator with a nonsingular exponential kernel was presented by Caputo and Fabrizio in 2015 [4]. This also allows for a better representation of heterogeneous distributions and evolving systems, unrepresentable by conventional models based on a single kernel. Many works have been examined using the Caputo-Fabrizio (CF) operator in the context of groundwater flow [5], the Oscillation phenomenon [6], fractional-order models for the transmission of a computer virus [7], groundwater pollution problems [8], and evolution equations [9].

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Motivated by non-singular and non-local kernels, Atangana and Baleanu introduced a fractional derivative of the Mittag-Leffler type [10]. This originated from the motivation to correct depictions of non local systems and material heterogeneities across multiple scales, where traditional local or singular-kernel wrongs [11, 12]. The obtained Atangana-Baleanu (AB) derivative has been subsequently utilized in many fields. For instance, Gomez Aguilar et al. utilized the AB operator for investigating other solutions of electromagnetic wave propagation in dielectric medium [13]. In another application, Alkahtani [14] studied dynamic analysis of Chua's circuit by applying the recently defined AB derivative which revealed further chaotic cases who also used the AB derivatives for his article. Also, Owolabi applied AB derivative in the modeling and simulation of ecological system by means of two-step Adams-Bashforth method [15].

In the last few years, a great deal of research on fractional-order differential equations has been done and has attracted a lot of interest, because these equations can adequately describe the memory-dependent and hereditary phenomena presented in many physical, biological, and engineering systems. Such equations offer a more realistic physics in the description of complex dynamics of evolution in phenomena like anomalous diffusion, nonlinear wave propagation, phase-transition, and viscoelastic effects [11, 16, 17]. For solving such models, many analytical and semi-analytical methods have been developed and modified; among them are the Homotopy Analysis Method (HAM) [18], Adomian Decomposition Method (ADM) [19], Variational Iteration Method (VIM) [20], q-homotopy methods [21], in order to address the some advantages of the other methods under certain circumstances. It is pointed out that in more recent times, transform methods, such as the Laplace-Adomian [22] and the Elzaki transforms [23] and methods based on conformable and Caputo-Fabrizio derivatives [4], have played a significant role.

Take into consideration the Cahn-Allen Equation (CAE) of the following form:

$$\frac{\partial\lambda(\rho,\,\zeta)}{\partial\zeta}-\frac{\partial^2\lambda(\rho,\,\zeta)}{\partial\rho^2}+\lambda^3(\rho,\,\zeta)-\lambda(\rho,\,\zeta)=0, \tag{1}$$

Initial guess

$$\lambda(\rho,\,0)=\lambda_0(\rho). \tag{2}$$

The Equation (1) link to numerous physical problems, including phase separation, crystal growth, and image processing, has attracted a lot of research recently. The novel fractional version of the nonlinear CAE is given as follows:

$$D_{\zeta}^{p}\lambda(\rho,\,\zeta)-\frac{\partial^2\lambda(\rho,\,\zeta)}{\partial\rho^2}+\lambda^3(\rho,\,\zeta)-\lambda(\rho,\,\zeta)=0,\ \text{where } 0<p\leq 1 \tag{3}$$

Initial guess

$$\lambda(\rho,\,0)=\lambda_0(\rho). \tag{4}$$

Here, the fractional-order Caputo's derivative is represented by D_{ζ}^{p}. The order of the fractional derivative is determined by the parameter p. When $p=1$ the fractional-order nonlinear CAE simplifies to the conventional nonlinear CAE. The CAE [24] describes the order-disorder transition and phase separation in iron alloys and finds various scientific applications including mathematical biology, plasma physics, and quantum mechanics. Many numerical techniques have been investigated for the CAE problem: the Haar wavelet method [25], the homotopy analysis transform method [26], the homotopy analysis method [27], the residual power series method [28], and the first integral method [29]. With numerical simulations run using the Crank Nicholson method, Algahtani [30] contrasts the Caputo-Fabrizio and Atangana-Baleanu fractional operators for the CAE.

Combining KdV and modified KdV equations produced the Gardner equation [31], which characterises internal solitary waves in shallow water. The Gardner equation is used in physics, particularly plasma physics, fluid dynamics and quantum field theory [32, 33]. Furthermore reported in the solid state and plasma are several wave behaviours [34]. The Gardner equation of time fractional order is given as follows:

$$D_{\zeta}^{p}\lambda(\rho,\ \zeta)+\frac{\partial^{3}\lambda(\rho,\ \zeta)}{\partial\rho^{3}}-6k^{2}\lambda^{2}(\rho,\ \zeta)\frac{\partial\lambda(\rho,\ \zeta)}{\partial\rho}+6\lambda(\rho,\ \zeta)\frac{\partial\lambda(\rho,\ \zeta)}{\partial\rho}=0,\ \text{where } 0<p\leq 1 \tag{5}$$

Initial guess

$$\lambda(\rho,\ 0)=\lambda_{0}(\rho), \tag{6}$$

In this equation, the parameter k modulates the strength and form of the nonlinear interaction terms, playing a key role in the adjustment between quadratic and cubic nonlinearity. Physically, it is intimately associated with the amplitude, width and stability of soliton-like wave structures in dispersive media and in this case study, we have taken $k = 0.1$. The initial conditions were selected on physical grounds and also in accordance with the classical solutions:

For the Cahn-Allen equation, the initial condition $\lambda(\rho,\ 0)=\frac{1}{e^{\frac{\rho}{\sqrt{2}}}+1}$ gives an sigmoidal profile signifying a phase boundary, and this is a common feature of phase separation in alloys and polymer blends.

For the Gardner equation, $\lambda(\rho,\ 0)=\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{\rho}{2}\right)$ serves as a solitary wave profile, which arise in the context of non-linear plasma physics, and shallow water waves.

The semi-analytical method known as Residual Power Series Method (RPSM) which use the Taylor series with the residual error function was developed by Arqub [35]. This method was used for many fuzzy and other types of differential equations [36–41].

The Aboodh Residual Power Series Method (ARPSM) is an approach that combines the Aboodh transformation with the residual power series method to solve differential equations of fractional order [42, 43]. By utilizing the Aboodh transform along with the iterative procedure of residual corrections, ARPSM series solutions are accurate and quickly converging without resorting to linearization, discretization, or perturbation. This method is especially attractive in dealing with the nonlinear and Fractional Partial Differential Equations (FPDEs) where classical techniques fail due to memory phenomena, and the non-local operators that are present in the fractional calculus. Many examples, such as the CAE and the Gardner equation, have been used to implement the ARPSM for various FPDEs, which illustrate that the proposed approach could not only preserve the physical feature of the problem, but also enhance the computational efficiency. The method also shows flexibility to deal with the initial and boundary value problems under the Caputo fractional derivative paradigm and as such, model complex physical phenomena.

Among the most important mathematical achievements for fractional partial differential equations is the Aboodh Transform Iterative Technique (ATIM). Computational complexity and convergence problems may develop when attempting to solve partial differential equations with fractional derivatives with traditional approaches. Our new method keeps a steady processing economy while continually improving approximations to overcome these constraints, hence increasing the accuracy with time. This result enhances our capacity to identify and understand complex systems under control by fractional partial differential equations, therefore facilitating the solution of difficult problems in the domains of applied mathematics, engineering, and physics [44–46]. The Aboodh transform is very useful in cases where:

• The power terms or fractional derivatives are involved in the system,
• The domain is semi-infinite or starts form zero,
• The initial values are given in an explicit analytical form.

As opposed to the Laplace transform, which can yield complex inverse operations in the presence of nonlinearity, the Aboodh transform results in direct inversion formulas while without loss of time-domain interpretability, and accordingly is more manageable for nonlinear fractional Partial Differential Equations (PDEs). It also obeys the avoidance of the convolution integrals which simplifies the iterative structure of the ATIM. Nonetheless, we understand that it is possible that in some multidimensional or oscillatory problem, Fourier or Laplace transform could be better.

In this paper we will investigate the the Cahn-Allen Equation (CAE) and Gardner equation physical models using two different method known as Aboodh Residual Power Series Method (ARPSM) and Aboodh Transform Iterative Method (ATIM). These two methods given the approximate solution close to the exact solution showing that these two method are efficient and reliable for the solution of the fractional partial differential equations. Tables and graphs are given in this paper which shows that these two method provide the best possible result than other analytical and numerical methods.

2. Foundational ideas

Definition 1 [47] Consider the function $\lambda(\rho,\ \zeta)$ foe which the Aboodh Transform (AT) is describe as,

$$A[\lambda(\rho,\ \zeta)] = \Psi(\rho,\ \xi) = \frac{1}{\xi}\int_0^\infty \lambda(\rho,\ \zeta)e^{-\zeta\xi}d\zeta,\ r_1 \le \xi \le r_2. \tag{7}$$

The Inverse Aboodh Transform (AIT) is describe as,

$$A^{-1}[\Psi(\rho,\ \xi)] = \lambda(\rho,\ \zeta) = \frac{1}{2\pi i}\int_{u-i\infty}^{u+i\infty} \Psi(\rho,\ \zeta)\xi e^{\zeta\xi}d\zeta \tag{8}$$

Where $\rho = (\rho_1,\ \rho_2,\ \cdots,\ \rho_p) \in \mathbb{R}$ and $p \in \mathbb{N}$.

Lemma 1 [48, 49] Consider $\lambda_1(\rho,\ \zeta)$ and $\lambda_2(\rho,\ \zeta)$ are continuous functions on $[0,\ \infty]$ and having exponential order. Let $A[\lambda_1(\rho,\ \zeta)] = \Psi_1(\rho,\ \zeta)$, $A[\lambda_2(\rho,\ \zeta)] = \Psi_2(\rho,\ \zeta)$ and $\chi_1,\ \chi_2$ are constants. Some of the characteristics of AT is given below,

1. $A[\chi_1\lambda_1(\rho,\ \zeta)+\chi_2\lambda_2(\rho,\ \zeta)] = \chi_1\Psi_1(\rho,\ \xi)+\chi_2\Psi_2(\rho,\ \zeta)$,
2. $A^{-1}[\chi_1\Psi_1(\rho,\ \zeta)+\chi_2\Psi_2(\rho,\ \zeta)] = \chi_1\lambda_1(\rho,\ \xi)+\chi_2\lambda_2(\rho,\ \zeta)$,
3. $A[J_\zeta^p\lambda(\rho,\ \zeta)] = \dfrac{\Psi(\rho,\ \xi)}{\xi^p}$,
4. $A[D_\zeta^p\lambda(\rho,\ \zeta)] = \xi^p\Psi(\rho,\ \xi) - \sum_{K=0}^{r-1}\dfrac{\lambda^K(\rho,\ 0)}{\xi^{K-p+2}},\ r-1 < p \le r,\ \ r \in \mathbb{N}$.

Definition 2 [50] For the function $\lambda(\rho,\ \zeta)$ the Caputo's fractional-order derivative is given as,

$$D_\zeta^p\lambda(\rho,\ \zeta) = J_\zeta^{m-p}\lambda^{(m)}(\rho,\ \zeta),\ \ r \ge 0,\ \ m-1 < p \le m, \tag{9}$$

where $\rho = (\rho_1,\ \rho_2,\ \cdots,\ \rho_p) \in \mathbb{R}^p$ and $m,\ p \in R$, J_ζ^{m-p} is the R-L integral of $\lambda(\rho,\ \zeta)$.

Definition 3 [51] The power series form is stated as:

$$\sum_{r=0}^{\infty}\vartheta_r(\rho)(\zeta-\zeta_0)^{rp} = \vartheta_0(\zeta-\zeta_0)^0 + \vartheta_1(\zeta-\zeta_0)^p + \vartheta_2(\zeta-\zeta_0)^{2p} + \cdots, \tag{10}$$

where $\rho = (\rho_1,\ \rho_2,\ \cdots,\ \rho_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$.

Lemma 2 The AT for the function $\lambda(\rho,\ \zeta)$ is given as:

$$A\left[D_{\zeta}^{rp}\lambda(\rho,\ \zeta)\right]=\xi^{rp}\Psi(\rho,\ \xi)-\sum_{j=0}^{r-1}\xi^{p(r-j)-2}D_{\zeta}^{jp}\lambda(\rho,\ 0),\ 0<p\leq 1, \tag{11}$$

where $\rho=(\rho_1,\ \rho_2,\ \cdots,\ \rho_p)\in\mathbb{R}^p$ and $p\in\mathbb{N}$ and $D_{\zeta}^{rp}=D_{\zeta}^{p}.\ D_{\zeta}^{p}.\ \cdots.\ D_{\zeta}^{p}(r-\text{times})$.

Proof. Using the principle of mathematical induction we will prove Eq. (11). Put $r=1$ in Eq. (11).

$$A\left[D_{\zeta}^{2p}\lambda(\rho,\ \zeta)\right]=\xi^{2p}\Psi(\rho,\ \xi)-\xi^{2p-2}\lambda(\rho,\ 0)-\xi^{p-2}D_{\zeta}^{p}\lambda(\rho,\ 0). \tag{12}$$

With the help of Lemma 1, it is proved that Eq. (11) is true for $r=1$. Put $r=2$ in Eq. (11), we obtain

$$A[D_{r}^{2p}\lambda(\rho,\ \zeta)]=\xi^{2p}\Psi(\rho,\ \xi)-\xi^{2p-2}\lambda(\rho,\ 0)-\xi^{p-2}D_{\zeta}^{p}\lambda(\rho,\ 0). \tag{13}$$

From the Left-Hand Side (LHS) of Eq. (13), we obtain

$$LHS=A\left[D_{\zeta}^{2p}\lambda(\rho,\ \zeta)\right]. \tag{14}$$

Alternatively, Eq. (14) can be written as,

$$LHS=A\left[D_{\zeta}^{p}D_{\zeta}^{p}\lambda(\rho,\ \zeta)\right]. \tag{15}$$

Let

$$z(\rho,\ \zeta)=D_{\zeta}^{p}\lambda(\rho,\ \zeta). \tag{16}$$

Substitute Eq. (16) in Eq. (15),

$$LHS=A\left[D_{\zeta}^{p}z(\rho,\ \zeta)\right]. \tag{17}$$

By implementing the derivative of Caputo's, Eq. (17) becomes

$$LHS=A[J^{1-p}z^{'}(\rho,\ \zeta)]. \tag{18}$$

By utilizing the integral of Riemann-Liouvulle (RL), Eq. (18) becomes

$$LHS=\frac{A[z^{'}(\rho,\ \zeta)]}{\xi^{1-p}}. \tag{19}$$

The AT derivative property is applied on Eq. (19) to make the following modification,

$$LHS = \xi^{p} Z(\rho,\ \xi) - \frac{z(\rho,\ 0)}{\xi^{2-p}}, \tag{20}$$

From Eq. (16), we obtain:

$$Z(\rho,\ \xi) = \xi^{p} \Psi(\rho,\ \xi) - \frac{\lambda(\rho,\ 0)}{\xi^{2-p}}, \tag{21}$$

where $A[z(\rho,\ \zeta)] = Z(\rho,\ \xi)$. Alternatively, Eq. (20) is written as,

$$LHS = \xi^{2p} \Psi(\rho,\ \xi) - \frac{\lambda(\rho,\ 0)}{\xi^{2-2p}} - \frac{D_{\zeta}^{p} \lambda(\rho,\ 0)}{\xi^{2-p}}, \tag{22}$$

Let us assumed that for $r = K$, Eq. (11) holds. Now, in Eq. (11) substitute $r = K$,

$$A\left[D_{\zeta}^{Kp} \lambda(\rho,\ \zeta)\right] = \xi^{Kp} \Psi(\rho,\ \xi) - \sum_{j=0}^{K-1} \xi^{p(K-j)-2} D_{\zeta}^{jp} D_{\zeta}^{jp} \lambda(\rho,\ 0),\ \ 0 < p \leq 1. \tag{23}$$

Now, substitute $r = K + 1$ in Eq. (11), we obtain

$$A\left[D_{\zeta}^{(K+1)p} \lambda(\rho,\ \zeta)\right] = \xi^{(K+1)p} \Psi(\rho,\ \xi) - \sum_{j=0}^{K} \xi^{p((K+1)-j)-2} D_{\zeta}^{jp} \lambda(\rho,\ 0). \tag{24}$$

Eq. (24) gives us the subsequent result

$$LHS = A\left[D_{\zeta}^{Kp} (D_{\zeta}^{Kp})\right]. \tag{25}$$

Let

$$D_{\zeta}^{Kp} = g(\rho,\ \zeta). \tag{26}$$

From Eq. (25), we obtain

$$LHS = A\left[D_{\zeta}^{p} g(\rho,\ \zeta)\right]. \tag{27}$$

The Caputo derivative and the integral of RL gives us the following result,

$$LHS = \xi^{p} A\left[D_{\zeta}^{Kp}\lambda(\rho,\ \zeta)\right] - \frac{g(\rho,\ 0)}{\xi^{2-p}}. \tag{28}$$

Using (23), we obtain Eq. (28)

$$LHS = \xi^{rp}\Psi(\rho,\ \xi) - \sum_{j=0}^{r-1} \xi^{p(r-j)-2} D_{\zeta}^{jp}\lambda(\rho,\ 0), \tag{29}$$

Eq. (29) is alternatively written as

$$LHS = A[D_{\zeta}^{rp}\lambda(\rho,\ 0)]. \tag{30}$$

Hence, Eq. (11) is true for $r = K+1$. Thus, for any positive integers, Eq. (11) is holds using the mathematical induction. □

Lemma 3 The AT for the function $\lambda(\rho,\ \zeta)$ in the Multiple Fractional Taylor's Series (MFTS) form is given below

$$\Psi(\rho,\ \xi) = \sum_{r=0}^{\infty} \frac{\vartheta_r(\rho)}{\xi^{rp+2}},\ \xi > 0, \tag{31}$$

where, $\rho = (s_1,\ \rho_2,\ \cdots,\ \rho_p) \in \mathbb{R}^p,\ p \in \mathbb{N}$.

Proof. Consider the Taylor series

$$\lambda(\rho,\ \zeta) = \vartheta_0(\rho) + \vartheta_1(\rho)\frac{\zeta^p}{\Gamma[p+1]} + + \vartheta_2(\rho)\frac{\zeta^{2p}}{\Gamma[2p+1]} + \cdots. \tag{32}$$

Subject AT to Eq. (32)

$$A[\lambda(\rho,\ \zeta)] = A[\vartheta_0(\rho)] + A\left[\vartheta_1(\rho)\frac{\zeta^p}{\Gamma[p+1]}\right] + A\left[\vartheta_1(\rho)\frac{\zeta^{2p}}{\Gamma[2p+1]}\right] + \cdots \tag{33}$$

With the application of AT's characteristic, we obtain

$$A[\lambda(\rho,\ \zeta)] = \vartheta_0(\rho)\frac{1}{\xi^2} + \vartheta_1(\rho)\frac{\Gamma[p+1]}{\Gamma[p+1]}\frac{1}{\xi^{p+2}} + \vartheta_2(\rho)\frac{\Gamma[2p+1]}{\Gamma[2p+1]}\frac{1}{\xi^{2p+2}}\cdots \tag{34}$$

Hence, we obtain a novel Taylor series in AT's framework. □

Lemma 4 The MFPS form of the function $A[\lambda(\rho,\ \zeta)] = \Psi(\rho,\ \xi)$ using novel Taylor's series (31) is given as,

$$\vartheta_0(\rho) = \lim_{\xi\to\infty} \xi^2\Psi(\rho,\ \xi) = \lambda(\rho,\ 0). \tag{35}$$

Proof. Consider the Taylor series

$$\vartheta_0(\rho) = \xi^2\Psi(\rho,\ \xi) - \frac{\vartheta_1(\rho)}{\xi^p} - \frac{\vartheta_2(\rho)}{\xi^{2p}} - \cdots \tag{36}$$

Calculating limit in Eq. (35) yield Eq. (36). □

Theorem 1 The MFPS form of the function $A[\lambda(\rho,\ \zeta)] = \Psi(\rho,\ \xi)$ is given as

$$\Psi(\rho,\ \xi) = \sum_{0}^{\infty} \frac{\vartheta_r(\rho)}{\xi^{rp+2}},\ \ \xi > 0, \tag{37}$$

where $\rho = (\rho_1,\ \rho_2,\ \cdots,\ \rho_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$. Then we have

$$\vartheta_r(\rho) = D_r^{rp}\lambda(\rho,\ 0), \tag{38}$$

where, $D_\zeta^{rp} = D_\zeta^p.\ D_\zeta^p.\ \cdots.\ D_\zeta^p(r - \text{times})$.

Proof. Consider the Taylor series

$$\vartheta_1(\rho) = \xi^{p+2}\Psi(\rho,\ \xi) - \xi^p\vartheta_0(\rho) - \frac{\vartheta_2(\rho)}{\xi^p} - \frac{\vartheta_3(\rho)}{\xi^{2p}} - \cdots \tag{39}$$

Taking limit of Eq. (39),

$$\vartheta_1(\rho) = \lim_{\xi\to\infty}(\xi^{p+2}\Psi(\rho,\ \xi) - \xi^p\vartheta_0(\rho)) - \lim_{\xi\to\infty}\frac{\vartheta_2(\rho)}{\xi^p} - \lim_{\xi\to\infty}\frac{\vartheta_3(\rho)}{\xi^{2p}} - \cdots \tag{40}$$

Simplifying limit, we obtain

$$\vartheta_1(\rho) = \lim_{\xi\to\infty}(\xi^{p+2}\Psi(\rho,\ \xi) - \xi^p\vartheta_0(\rho)). \tag{41}$$

Implement Lemma 2 on Eq. (41),

$$\vartheta_1(\rho) = \lim_{\xi\to\infty}(\xi^2 A\left[D_\zeta^p\lambda(\rho,\ \zeta)\right](\xi)). \tag{42}$$

Implement Lemma 3 on Eq. (42),

$$\vartheta_1(\rho) = D_\zeta^p\lambda(\rho,\ 0). \tag{43}$$

Using Taylor series and taking limit $\xi \to \infty$ again, we obatain

$$\vartheta_2(\rho) = \xi^{2p+2}\Psi(\rho,\ \xi) - \xi^{2p}\vartheta_0(\rho) - \xi^{p}\vartheta_1(\rho) - \frac{\vartheta_3(\rho)}{\xi^{p}} - \cdots \tag{44}$$

Using Lemma 3, we obtain

$$\vartheta_2(\rho) = \lim_{\xi\to\infty} \xi^2(\xi^{2p}\Psi(\rho,\ \xi) - \xi^{2p-2}\vartheta_0(\rho) - \xi^{p-2}\vartheta_1(\rho)). \tag{45}$$

By implement Lemmas 2 and 4 on Eq. (45), we obtain

$$\vartheta_2(\rho) = D_{\zeta}^{2p}\lambda(\rho,\ 0). \tag{46}$$

Using the same procedure, we obtain

$$\vartheta_3(\rho) = \lim_{\xi\to\infty} \xi^2(A\left[D_{\zeta}^{2p}\lambda(\rho,\ p)\right](\xi)). \tag{47}$$

Using Lemma 4, we obtain

$$\vartheta_3(\rho) = D_{\zeta}^{3p}\lambda(\rho,\ 0). \tag{48}$$

Generally

$$\vartheta_r(\rho) = D_{\zeta}^{rp}\lambda(\rho,\ 0). \tag{49}$$

Hence, proved. □

Hence, the Taylor's series is convergent.

Theorem 2 The MFTS expression for the function $A[\lambda(\rho,\ \zeta)] = \Psi(\rho,\ \wp)$ converge, when $\left|\wp^{a}A\left[D_{\zeta}^{(K+1)p}\lambda(\rho,\ \zeta)\right]\right| \leq T$, for all $0 < \wp \leq s$ and $0 < p \leq 1$, if the residual $R_K(\rho,\ \wp)$ satisfies the inequality,

$$|R_K(\rho,\ \wp)| \leq \frac{T}{\wp^{(K-1)p+2}},\ \ 0 < \wp \leq s. \tag{50}$$

Proof. Assume $A\left[D_{\zeta}^{rp}\lambda(\rho,\ \zeta)\right](\wp)$ is defined on $0 < \wp \leq s$ for $r = 0,\ 1,\ 2,\ \cdots,\ K+1$. Proceed with the assumption that $\left|\wp^{2}A\left[D_{\zeta^{K+1}}\lambda(\rho,\ \zeta)\right]\right| \leq T,$ on $0 < \wp \leq s$. Using the new Taylor's series, establish the following relationship:

$$R_K(\rho,\ \wp) = \Psi(\rho,\ \wp) - \sum_{r=0}^{K}\frac{\vartheta_r(\rho)}{\wp^{rp+2}}. \tag{51}$$

By using Theorem 1, Eq. (51) becomes

$$R_K(\rho,\ \wp) = \Psi(\rho,\ \wp) - \sum_{r=0}^{K} \frac{D_\zeta^{rp}\lambda(\rho,\ 0)}{\wp^{rp+2}}. \tag{52}$$

Multiply Eq. (52) by $\wp^{(K+1)a+2}$,

$$\wp^{(K+1)p+2}R_K(\rho,\ \wp) = \wp^2(\wp^{(K+1)p}\Psi(\rho,\ \wp) - \sum_{r=0}^{K} \wp^{(K+1-r)p-2}D_\zeta^{rp}\lambda(\rho,\ 0)). \tag{53}$$

By implementing Lemma 2 on Eq. (53), we obtain

$$\wp^{(K+1)p+2}R_K(\rho,\ \wp) = \wp^2 A[D_\zeta^{(K+1)p}\lambda(\rho,\ \zeta)]. \tag{54}$$

Taking absolute of Eq. (54),

$$|\wp^{(K+1)p+2}R_K(\rho,\ \wp)| = |\wp^2 A[D_\zeta^{(K+1)p}\lambda(\rho,\ \zeta)]|. \tag{55}$$

Applying the stated criteria in Eq. (55),

$$\frac{-T}{\wp^{(K+1)p+2}} \leq R_K(\rho,\ \wp) \leq \frac{T}{\wp^{(K+1)p+2}}. \tag{56}$$

Eq. (56) can be alternatively written as

$$|R_K(\rho,\ \wp)| \leq \frac{T}{\wp^{(K+1)p+2}}. \tag{57}$$

Hence, it is proved that the series converge under this condition. □

3. Proposed methodologies

3.1 *The solution of PDE via ARPSM technique*

In this section we explain the ARPSM methodology, and its application for solving the fractional-order PDEs.
Step 1: Consider the nonlinear and homogeneous PDE of time fractional-order,

$$D_\zeta^{pq}\lambda(\rho,\ \zeta) + \varphi(\rho)N(\lambda) - \delta(\rho,\ \lambda) = 0, \tag{58}$$

Step 2: Both sides of Eq. (58) are subjected to the AT

$$A[D_{\zeta}^{pq}\lambda(\rho,\ \zeta)+\varphi(\rho)N(\lambda)-\delta(\rho,\ \lambda)]=0, \tag{59}$$

Utilizing Lemma 2, Eq. (59) becomes

$$\Psi(\rho,\ s)=\sum_{j=0}^{q-1}\frac{D_{\zeta}^{j}\lambda(\rho,\ 0)}{s^{pq+2}}-\frac{\varphi(\rho)Y(s)}{s^{pq}}+\frac{F(\rho,\ s)}{s^{pq}}, \tag{60}$$

where, $A[\delta(\rho,\ \lambda)]=F(\rho,\ s)$, $A[N(\lambda)]=Y(s)$.

Step 3: Eq. (60) has the following series form solution,

$$\Psi(\rho,\ s)=\sum_{r=0}^{\infty}\frac{\vartheta_r(\rho)}{s^{rp+2}},\ \ s>0, \tag{61}$$

Step 4: Here are the steps to proceed:

$$\vartheta_0(\rho)=\lim_{s\to\infty}s^2\Psi(\rho,\ s)=\lambda(\rho,\ 0), \tag{62}$$

Utilizing Theorem 2, we obtain

$$\begin{aligned}
&\vartheta_1(\rho)=D_{\zeta}^{p}\lambda(\rho,\ 0),\\
&\vartheta_2(\rho)=D_{\zeta}^{2p}\lambda(\rho,\ 0),\\
&\vdots\\
&\vartheta_w(\rho)=D_{\zeta}^{zp}\lambda(\rho,\ 0),
\end{aligned} \tag{63}$$

Step 5: The following is the formula for finding the K^{th} truncated series $\Psi(\rho,\ s)$,

$$\Psi_K(\rho,\ s)=\sum_{r=0}^{K}\frac{\vartheta_r(\rho)}{s^{rp+2}},\ \ s>0, \tag{64}$$

$$\Psi_K(\rho,\ s)=\frac{\vartheta_0(\rho)}{s^2}+\frac{\vartheta_1(\rho)}{s^{p+2}}+\cdots+\frac{\vartheta_z(\rho)}{s^{zp+2}}+\sum_{r=z+1}^{K}\frac{\vartheta_r(\rho)}{s^{rp+2}}, \tag{65}$$

Step 6: Note that the K^{th}-truncated Aboodh residual function and the Aboodh Residual Function (ARF) from 60 and must be considered separately to obtain:

$$ARes(\rho,\ s) = \Psi(\rho,\ s) - \sum_{j=0}^{q-1} \frac{D_{\zeta}^{j}\lambda(\rho,\ 0)}{s^{pj+2}} + \frac{\varphi(\rho)Y(s)}{s^{pj}} - \frac{F(\rho,\ s)}{s^{pj}}, \tag{66}$$

and

$$ARes_K(\rho,\ s) = \Psi_K(\rho,\ s) - \sum_{j=0}^{q-1} \frac{D_{\zeta}^{j}\lambda(\rho,\ 0)}{s^{pj+2}} + \frac{\varphi(\rho)Y(s)}{s^{pj}} - \frac{F(\rho,\ s)}{s^{pj}}. \tag{67}$$

Step 7: Substitute $\Psi_K(\rho,\ s)$ in Eq. (67).

$$ARes_K(\rho,\ s) = \left(\frac{\vartheta_0(\rho)}{s^2} + \frac{\vartheta_1(\rho)}{s^{p+2}} + \cdots + \frac{\vartheta_z(\rho)}{s^{zp+2}} + \sum_{r=z+1}^{K} \frac{\vartheta_r(\rho)}{s^{rp+2}}\right) - \sum_{j=0}^{q-1} \frac{D_{\zeta}^{j}\lambda(\rho,\ 0)}{s^{pj+2}} + \frac{\varphi(\rho)Y(s)}{s^{pj}} - \frac{F(\rho,\ s)}{s^{pj}}. \tag{68}$$

Step 8: Multiply Eq. (68), by s^{Kp+2}.

$$\begin{aligned} s^{Kp+2}ARes_K(\rho,\ s) = s^{Kp+2}\Bigg(&\frac{\vartheta_0(\rho)}{s^2} + \frac{\vartheta_1(\rho)}{s^{p+2}} + \cdots + \frac{\vartheta_z(\rho)}{s^{zp+2}} + \sum_{r=z+1}^{K} \frac{\vartheta_r(\rho)}{s^{rp+2}} - \sum_{j=0}^{q-1} \frac{D_{\zeta}^{j}\lambda(\rho,\ 0)}{s^{pj+2}} \\ &+ \frac{\varphi(\rho)Y(s)}{s^{pj}} - \frac{F(\rho,\ s)}{s^{pj}}\Bigg). \end{aligned} \tag{69}$$

Step 9: Taking $\lim_{s\to\infty}$ of Eq. (69), we obtain

$$\begin{aligned} \lim_{s\to\infty} s^{Kp+2}ARes_K(\rho,\ s) = \lim_{s\to\infty} s^{Kp+2}\Bigg(&\frac{\vartheta_0(\rho)}{s^2} + \frac{\vartheta_1(\rho)}{s^{p+2}} + \cdots + \frac{\vartheta_z(\rho)}{s^{zp+2}} + \sum_{r=z+1}^{K} \frac{\vartheta_r(\rho)}{s^{rp+2}} \\ &- \sum_{j=0}^{q-1} \frac{D_{\zeta}^{j}\lambda(\rho,\ 0)}{s^{pj+2}} + \frac{\varphi(\rho)Y(s)}{s^{pj}} - \frac{F(\rho,\ s)}{s^{pj}}\Bigg). \end{aligned} \tag{70}$$

Step 10: From the equation below, we can find $\vartheta_K(\rho)$

$$\lim_{s\to\infty}(s^{Kp+2}ARes_K(\rho,\ s)) = 0, \tag{71}$$

where $K = z+1,\ z+2,\ \cdots$.

Step 11: For the K^{th} approximate solution of Eq. (60), put $\vartheta_K(\rho)$ in place of K-truncated series of $\Psi(\rho,\ s)$.

Step 12: Subject AIT to $\Psi_K(\rho,\ s)$, we obtain the solution $\lambda_K(\rho,\ \zeta)$.

3.2 *Problem 1*

Consider the fractional order CAE equation

$$D_{\zeta}^{p}\lambda(\rho,\ \zeta)-\frac{\partial^{2}\lambda(\rho,\ \zeta)}{\partial\rho^{2}}+\lambda^{3}(\rho,\ \zeta)-\lambda(\rho,\ \zeta)=0,\ \text{where } 0<p\leq 1 \tag{72}$$

Initial guess

$$\lambda(\rho,\ 0)=\frac{1}{e^{\frac{\rho}{\sqrt{2}}}+1}. \tag{73}$$

The AT is subjected to Eq. (72) and use the initial guess Eq. (73), we obtain

$$\lambda(\rho,\ s)-\frac{\frac{1}{e^{\frac{\rho}{\sqrt{2}}}+1}}{s^{2}}-\frac{1}{s^{p}}\left[\frac{\partial^{2}\lambda(\rho,\ \zeta)}{\partial\rho^{2}}\right]+\frac{1}{s^{p}}A_{\zeta}\left[A_{\zeta}^{-1}\lambda^{3}(\rho,\ \zeta)\right]-\frac{1}{s^{p}}\left[\lambda(\rho,\ \zeta)\right]=0, \tag{74}$$

Thus, the k^{th}-truncated term series is written as,

$$\lambda(\rho,\ s)=\frac{\frac{1}{e^{\frac{\rho}{\sqrt{2}}}+1}}{s^{2}}+\sum_{r=1}^{k}\frac{f_{r}(\rho,\ s)}{s^{rp+1}},\ r=1,\ 2,\ 3,\ 4\cdots \tag{75}$$

Aboodh Residual Function (ARF) is written as,

$$A_{\zeta}Res(\rho,\ s)=\lambda(\rho,\ s)-\frac{\frac{1}{e^{\frac{\rho}{\sqrt{2}}}+1}}{s^{2}}-\frac{1}{s^{p}}\left[\frac{\partial^{2}\lambda(\rho,\ \zeta)}{\partial\rho^{2}}\right]+\frac{1}{s^{p}}A_{\zeta}\left[A_{\zeta}^{-1}\lambda^{3}(\rho,\ \zeta)\right]-\frac{1}{s^{p}}\left[\lambda(\rho,\ \zeta)\right]=0, \tag{76}$$

and the k^{th}-ARF as:

$$A_{\zeta}Res_{k}(\rho,\ s)=\lambda_{k}(\rho,\ s)-\frac{\frac{1}{e^{\frac{\rho}{\sqrt{2}}}+1}}{s^{2}}-\frac{1}{s^{p}}\left[\frac{\partial^{2}\lambda_{k}(\rho,\ \zeta)}{\partial\rho^{2}}\right]+\frac{1}{s^{p}}A_{\zeta}\left[A_{\zeta}^{-1}\lambda_{k}^{3}(\rho,\ \zeta)\right]-\frac{1}{s^{p}}\left[\lambda_{k}(\rho,\ \zeta)\right]=0, \tag{77}$$

In order to determine $f_r(\rho,\ s)$, for $r=1,\ 2,\ 3,\ \ldots$, we multiply the whole equation with s^{rp+1} and taking $\lim_{s\to\infty}$. In the r^{th}-truncated series Eq. (75) put r^{th}-Aboodh residual function Eq. (77). $A_{\zeta}Res_{\lambda,\ r}(\rho,\ s))=0$, for $r=1,\ 2,\ 3,\ \cdots$. The first few terms using this procedure is given below,

$$f_{1}(\rho,\ s)=\frac{3}{8}\text{sech}^{2}\left(\frac{\rho}{2\sqrt{2}}\right), \tag{78}$$

$$f_2(\rho,\ s) = \frac{9}{2}\sinh^4\left(\frac{\rho}{2\sqrt{2}}\right)\operatorname{csch}^3\left(\frac{\rho}{\sqrt{2}}\right), \tag{79}$$

and so on.

Put $f_r(\rho,\ s)$ values in Eq. (75), we obtain

$$\lambda(\rho,\ s) = \frac{1}{s^2\left(e^{\frac{\rho}{\sqrt{2}}}+1\right)} + \frac{3\operatorname{sech}^2\left(\frac{\rho}{2\sqrt{2}}\right)}{8s^{p+1}} + \frac{9\sinh^4\left(\frac{\rho}{2\sqrt{2}}\right)\operatorname{csch}^3\left(\frac{\rho}{\sqrt{2}}\right)}{2s^{2p+1}} + \cdots \tag{80}$$

With the application of AIT, we obtain the required solution.

$$\lambda(\rho,\ \eta,\ \zeta) = \frac{1}{e^{\frac{\rho}{\sqrt{2}}}+1} + \frac{3\zeta^p\operatorname{sech}^2\left(\frac{\rho}{2\sqrt{2}}\right)}{8\Gamma(p+1)} + \frac{9\zeta^{2p}\sinh^4\left(\frac{\rho}{2\sqrt{2}}\right)\operatorname{csch}^3\left(\frac{\rho}{\sqrt{2}}\right)}{2\Gamma(2p+1)} + \cdots \tag{81}$$

The exact solution for $p = 1$ is given below,

$$\lambda(\rho,\ \zeta) = \frac{1}{e^{\frac{\rho}{\sqrt{2}}-\frac{3\zeta}{2}}+1}. \tag{82}$$

3.3 *Problem 2*

Consider the fractional-order Gardner equation

$$D_\zeta^p\lambda(\rho,\ \zeta) + \frac{\partial^3\lambda(\rho,\ \zeta)}{\partial\rho^3} - 6k^2\lambda^2(\rho,\ \zeta)\frac{\partial\lambda(\rho,\ \zeta)}{\partial\rho} + 6\lambda(\rho,\ \zeta)\frac{\partial\lambda(\rho,\ \zeta)}{\partial\rho} = 0,\ \text{where } 0 < p \leq 1 \tag{83}$$

Initial guess

$$\lambda(\rho,\ 0) = \frac{1}{2}\tanh\left(\frac{\rho}{2}\right) + \frac{1}{2}, \tag{84}$$

The AT is subjected to Eq. (83) and use the initial guess Eq. (84), we obtain

$$\begin{aligned}
&\lambda(\rho,\ s) - \frac{\frac{1}{2}\tanh\left(\frac{\rho}{2}\right)+\frac{1}{2}}{s^2} + \frac{1}{s^p}\left[\frac{\partial^3\lambda(\rho,\ \zeta)}{\partial\rho^3}\right] - \frac{6}{s^p}A_\zeta\left[k^2A_\zeta^{-1}\lambda^2(\rho,\ \zeta)\times\frac{\partial A_\zeta^{-1}\lambda(\rho,\ \zeta)}{\partial\rho}\right] \\
&+\frac{6}{s^p}A_\zeta\left[A_\zeta^{-1}\lambda(\rho,\ \zeta)\times\frac{\partial A_\zeta^{-1}\lambda(\rho,\ \zeta)}{\partial\rho}\right] = 0,
\end{aligned} \tag{85}$$

Thus, the term series that are k^{th}-truncated is written as,

$$\lambda(\rho,\ s)=\frac{\frac{1}{2}\tanh\left(\frac{\rho}{2}\right)+\frac{1}{2}}{s^2}+\sum_{r=1}^{k}\frac{f_r(\rho,\ s)}{s^{rp+1}},\quad r=1,\ 2,\ 3,\ 4\cdots. \tag{86}$$

Aboodh Residual Function (ARF) is written as,

$$\begin{aligned}A_\zeta Res(\rho,\ s)=\lambda(\rho,\ s)-\frac{\frac{1}{2}\tanh\left(\frac{\rho}{2}\right)+\frac{1}{2}}{s^2}+\frac{1}{s^p}\left[\frac{\partial^3\lambda(\rho,\ \zeta)}{\partial\rho^3}\right]-\frac{6}{s^p}A_\zeta\left[k^2A_\zeta^{-1}\lambda^2(\rho,\ \zeta)\times\frac{\partial A_\zeta^{-1}\lambda(\rho,\ \zeta)}{\partial\rho}\right]\\+\frac{6}{s^p}A_\zeta\left[A_\zeta^{-1}\lambda(\rho,\ \zeta)\times\frac{\partial A_\zeta^{-1}\lambda(\rho,\ \zeta)}{\partial\rho}\right]=0,\end{aligned} \tag{87}$$

and the k^{th}-ARFs as:

$$\begin{aligned}A_\zeta Res_k(\rho,\ \eta,\ s)=\lambda_k(\rho,\ s)-\frac{\frac{1}{2}\tanh\left(\frac{\rho}{2}\right)+\frac{1}{2}}{s^2}+\frac{1}{s^p}\left[\frac{\partial^3\lambda_k(\rho,\ \zeta)}{\partial\rho^3}\right]-\frac{6}{s^p}A_\zeta\left[k^2A_\zeta^{-1}\lambda_k^2(\rho,\ \zeta)\times\frac{\partial A_\zeta^{-1}\lambda_k(\rho,\ \zeta)}{\partial\rho}\right]\\+\frac{6}{s^p}A_\zeta\left[A_\zeta^{-1}\lambda_k(\rho,\ \zeta)\times\frac{\partial A_\zeta^{-1}\lambda_k(\rho,\ \zeta)}{\partial\rho}\right]=0,\end{aligned} \tag{88}$$

In order to determine $f_r(\rho,\ s)$, for $r=1,\ 2,\ 3,\ ...$, we multiply the whole equation with s^{rp+1} and taking $\lim_{s\to\infty}$. In the r^{th}-truncated series Eq. (86) put r^{th}-Aboodh residual function Eq. (88). $A_\zeta Res_{\lambda,\ r}(\rho,\ s))=0$, for $r=1,\ 2,\ 3,\ \cdots$. The first few terms using this procedure is given below,

$$f_1(\rho,\ s)=-\frac{1}{8}\text{sech}^4\left(\frac{\rho}{2}\right)\left(-3\left(k^2-1\right)\sinh(\rho)+\left(4-3k^2\right)\cosh(\rho)+1\right), \tag{89}$$

$$\begin{aligned}f_2(\rho,\ s)=-\Big(e^{-\frac{3\rho}{2}}\text{sech}^5\left(\frac{\rho}{2}\right)\Big(e^\rho\Big(2\Big(90k^2-91\Big)+e^\rho\Big(-228k^2+e^\rho\Big(-180k^4+\Big(7-6k^2\Big)^2e^\rho\\+444k^2-261\Big)+230\Big)\Big)+\sinh(\rho)-\cosh(\rho)-3\Big)\Big)/\Big(64(\cosh(\rho)+1)\Big),\end{aligned} \tag{90}$$

and so on.

Put $f_r(\rho,\ s)$ values in Eq. (86), we obtain

$$\lambda(\rho,\ s)=\frac{\frac{1}{2}\tanh\left(\frac{\rho}{2}\right)+\frac{1}{2}}{s}-\frac{\operatorname{sech}^4\left(\frac{\rho}{2}\right)\left(-3\left(k^2-1\right)\sinh(\rho)+\left(4-3k^2\right)\cosh(\rho)+1\right)}{8s^{p+1}}$$

$$-\left(e^{-\frac{3\rho}{2}}\operatorname{sech}^5\left(\frac{\rho}{2}\right)\left(e^{\rho}\left(2\left(90k^2-91\right)+e^{\rho}\left(-228k^2+e^{\rho}\left(-180k^4+\left(7-6k^2\right)^2e^{\rho}\right.\right.\right.\right.\right. \tag{91}$$

$$\left.\left.\left.\left.\left.+444k^2-261\right)+230\right)\right)+\sinh(\rho)-\cosh(\rho)-3\right)\right)/\left(64s^{2p+1}(\cosh(\rho)+1)\right)+\cdots.$$

With the application of AIT, we obtain the required solution.

$$\lambda(\rho,\ \zeta)=\frac{1}{2}\tanh\left(\frac{\rho}{2}\right)+\frac{1}{2}-\frac{\operatorname{sech}^4\left(\frac{\rho}{2}\right)\left(-3\left(k^2-1\right)\sinh(\rho)+\left(4-3k^2\right)\cosh(\rho)+1\right)}{8\Gamma(p+1)}$$

$$-\left(e^{-\frac{3\rho}{2}}\operatorname{sech}^5\left(\frac{\rho}{2}\right)\left(e^{\rho}\left(2\left(90k^2-91\right)+e^{\rho}\left(-228k^2+e^{\rho}\left(-180k^4+\left(7-6k^2\right)^2e^{\rho}\right.\right.\right.\right.\right. \tag{92}$$

$$\left.\left.\left.\left.\left.+444k^2-261\right)+230\right)\right)+\sinh(\rho)-\cosh(\rho)-3\right)\right)/\left(64\Gamma(2p+1)(\cosh(\rho)+1)\right)+\cdots.$$

The exact solution for $p=1$ is given below,

$$\lambda(\rho,\ \zeta)=\frac{1}{2}\tanh\left(\frac{\rho-\zeta}{2}\right)+\frac{1}{2}. \tag{93}$$

3.4 *The solution of PDE via ATIM*

Consider the nonlinear PDE of fractional-order,

$$D_{\zeta}^{p}\lambda(\rho,\ \zeta)=\Theta\Big(\lambda(\rho,\ \zeta),\ D_{\rho}^{\zeta}\lambda(\rho,\ \zeta),\ D_{\rho}^{2\zeta}\lambda(\rho,\ \zeta),\ D_{\rho}^{3\zeta}\lambda(\rho,\ \zeta)\Big),\ \ 0<p,\ \zeta\leq 1, \tag{94}$$

Initial guess

$$\lambda^{(k)}(\rho,\ 0)=h_k,\ \ k=0,\ 1,\ 2,\ \cdots,\ m-1, \tag{95}$$

The function $\lambda(\rho,\ \zeta)$ is unknown, while $\Theta\Big(\lambda(\rho,\ \zeta),\ D_{\rho}^{\zeta}\lambda(\rho,\ \zeta),\ D_{\rho}^{2\zeta}\lambda(\rho,\ \zeta),\ D_{\rho}^{3\zeta}\lambda(\rho,\ \zeta)\Big)$ may be a nonlinear operator or linear of $\lambda(\rho,\ \zeta)$, $D_{\rho}^{\zeta}\lambda(\rho,\ \zeta)$, $D_{\rho}^{2\zeta}\lambda(\rho,\ \zeta)$ and $D_{\rho}^{3\zeta}\lambda(\rho,\ \zeta)$. The following equation may be obtained by subjecting AT to Eq. (94).

$$A[\lambda(\rho,\ \zeta)] = \frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\lambda^{(k)}(\rho,\ 0)}{s^{2-p+k}} + A\left[\Theta\Big(\lambda(\rho,\ \zeta),\ D_\rho^{\zeta}\lambda(\rho,\ \zeta),\ D_\rho^{2\zeta}\lambda(\rho,\ \zeta),\ D_\rho^{3\zeta}\lambda(\rho,\ \zeta)\Big)\right]\right), \tag{96}$$

Subject AIT to Eq. (96)

$$\lambda(\rho,\ \zeta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\lambda^{(k)}(\rho,\ 0)}{s^{2-p+k}} + A\left[\Theta\Big(\lambda(\rho,\ \zeta),\ D_\rho^{\zeta}\lambda(\rho,\ \zeta),\ D_\rho^{2\zeta}\lambda(\rho,\ \zeta),\ D_\rho^{3\zeta}\lambda(\rho,\ \zeta)\Big)\right]\right)\right]. \tag{97}$$

Using ATIM, we obtain this series form solution,

$$\lambda(\rho,\ \zeta) = \sum_{i=0}^{\infty}\lambda_i. \tag{98}$$

The operators $\Theta\Big(\lambda,\ D_\rho^{\zeta}\lambda,\ D_\rho^{2\zeta}\lambda,\ D_\rho^{3\zeta}\lambda\Big)$ decompose in the subsequent manner,

$$\begin{aligned}\Theta\Big(\lambda,\ D_\rho^{\zeta}\lambda,\ D_\rho^{2\zeta}\lambda,\ D_\rho^{3\zeta}\lambda\Big) &= \Theta\Big(\lambda_0,\ D_\rho^{\zeta}\lambda_0,\ D_\rho^{2\zeta}\lambda_0,\ D_\rho^{3\zeta}\lambda_0\Big) + \sum_{i=0}^{\infty}\left(\Theta\left(\sum_{k=0}^{i}\Big(\lambda_k,\ D_\rho^{\zeta}\lambda_k,\ D_\rho^{2\zeta}\lambda_k,\ D_\rho^{3\zeta}\lambda_k\Big)\right)\right. \\ &\left. -\Theta\left(\sum_{k=1}^{i-1}\Big(\lambda_k,\ D_\rho^{\zeta}\lambda_k,\ D_\rho^{2\zeta}\lambda_k,\ D_\rho^{3\zeta}\lambda_k\Big)\right)\right).\end{aligned} \tag{99}$$

Insert Eq. (99) and Eq. (98) in Eq. (97),

$$\begin{aligned}\sum_{i=0}^{\infty}\lambda_i(\rho,\ \zeta) &= A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\lambda^{(k)}(\rho,\ 0)}{s^{2-p+k}} + A\left[\Theta\left(\lambda_0,\ D_\rho^{\zeta}\lambda_0,\ D_\rho^{2\zeta}\lambda_0,\ D_\rho^{3\zeta}\lambda_0\right)\right]\right)\right] \\ &+ A^{-1}\left[\frac{1}{s^p}\left(A\left[\sum_{i=0}^{\infty}\left(\Theta\sum_{k=0}^{i}(\lambda_k,\ D_\rho^{\zeta}\lambda_k,\ D_\rho^{2\zeta}\lambda_k,\ D_\rho^{3\zeta}\lambda_k)\right)\right]\right)\right] \\ &- A^{-1}\left[\frac{1}{s^p}\left(A\left[\left(\Theta\sum_{k=1}^{i-1}(\lambda_k,\ D_\rho^{\zeta}\lambda_k,\ D_\rho^{2\zeta}\lambda_k,\ D_\rho^{3\zeta}\lambda_k)\right)\right]\right)\right]\end{aligned} \tag{100}$$

$$\lambda_0(\rho,\ \zeta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\lambda^{(k)}(\rho,\ 0)}{s^{2-p+k}}\right)\right],$$

$$\lambda_1(\rho,\ \zeta) = A^{-1}\left[\frac{1}{s^p}\left(A\left[\Theta\left(\lambda_0,\ D_\rho^\zeta\lambda_0,\ D_\rho^{2\zeta}\lambda_0,\ D_\rho^{3\zeta}\lambda_0\right)\right]\right)\right],$$

$$\vdots \tag{101}$$

$$\lambda_{m+1}(\rho,\ \zeta) = A^{-1}\left[\frac{1}{s^p}\left(A\left[\sum_{i=0}^{\infty}\left(\Theta\sum_{k=0}^{i}(\lambda_k,\ D_\rho^\zeta\lambda_k,\ D_\rho^{2\zeta}\lambda_k,\ D_\rho^{3\zeta}\lambda_k)\right)\right]\right)\right]$$

$$-A^{-1}\left[\frac{1}{s^p}\left(A\left[\left(\Theta\sum_{k=1}^{i-1}\left(\lambda_k,\ D_\rho^\zeta\lambda_k,\ D_\rho^{2\zeta}\lambda_k,\ D_\rho^{3\zeta}\lambda_k\right)\right)\right]\right)\right],\ m = 1,\ 2,\ \cdots.$$

The m^{th}-term solution of Eq. (94) is given as,

$$\lambda(\rho,\ \zeta) = \sum_{i=0}^{m-1}\lambda_i. \tag{102}$$

3.4.1 *Problem with ATIM*

Examine the fractional order CAE equation

$$D_\zeta^p\lambda(\rho,\ \zeta) = \frac{\partial^2\lambda(\rho,\ \zeta)}{\partial\rho^2} - \lambda^3(\rho,\ \zeta) + \lambda(\rho,\ \zeta),\ \text{where } 0 < p \le 1 \tag{103}$$

Initial guess

$$\lambda(\rho,\ 0) = \frac{1}{e^{\frac{\rho}{\sqrt{2}}} + 1}. \tag{104}$$

and exact solution

$$\lambda(\rho,\ \zeta) = \frac{1}{e^{\frac{\rho}{\sqrt{2}} - \frac{3\zeta}{2}} + 1}. \tag{105}$$

We get the following outcome by applying the AT on Eq. (103),

$$A\left[D_\zeta^p\lambda(\rho,\ \zeta)\right]=\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\lambda^{(k)}(\rho,\ 0)}{s^{2-p+k}}+A\left[\frac{\partial^2\lambda(\rho,\ \zeta)}{\partial\rho^2}-\lambda^3(\rho,\ \zeta)+\lambda(\rho,\ \zeta)\right]\right), \tag{106}$$

We get the following outcome by using the AIT on either side of (106):

$$\lambda(\rho,\ \zeta)=A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\lambda^{(k)}(\rho,\ 0)}{s^{2-p+k}}+A\left[\frac{\partial^2\lambda(\rho,\ \zeta)}{\partial\rho^2}-\lambda^3(\rho,\ \zeta)+\lambda(\rho,\ \zeta)\right]\right)\right]. \tag{107}$$

The following is the equation that results from applying the AT iteratively:

$$\begin{aligned}\lambda_0(\rho,\ \zeta)&=A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\lambda^{(k)}(\rho,\ 0)}{s^{2-p+k}}\right)\right]\\ &=A^{-1}\left[\frac{\lambda(\rho,\ \eta,\ 0)}{s^2}\right]\\ &=\frac{1}{e^{\frac{\rho}{\sqrt{2}}}+1},\end{aligned} \tag{108}$$

By implementing the RL integral on Eq. (103), we obtain

$$\lambda(\rho,\ \zeta)=\frac{1}{e^{\frac{\rho}{\sqrt{2}}}+1}-A\left[\frac{\partial^2\lambda(\rho,\ \zeta)}{\partial\rho^2}-\lambda^3(\rho,\ \zeta)+\lambda(\rho,\ \zeta)\right]. \tag{109}$$

By using ATIM, we get the subsequent terms of the solution,

$$\lambda_0(\rho,\ \zeta)=\frac{1}{e^{\frac{\rho}{\sqrt{2}}}+1},$$

$$\lambda_1(\rho,\ \zeta)=\frac{3\zeta^p\operatorname{sech}^2\left(\frac{\rho}{2\sqrt{2}}\right)}{8\Gamma(p+1)},$$

$$\lambda_2(\rho,\,\zeta)=\frac{3\zeta^{2p}}{2048}\left(\frac{9\zeta^{p}}{\Gamma(p+1)^3}\left(-\frac{3\zeta^{p}\Gamma(3p)\operatorname{sech}^6\left(\frac{\rho}{2\sqrt{2}}\right)}{\Gamma(4p)}-\frac{512e^{\sqrt{2}\rho}\Gamma(p+1)\Gamma(2p+1)}{\left(e^{\frac{\rho}{\sqrt{2}}}+1\right)^5\Gamma(3p+1)}\right)\right.$$
$$\left.+\frac{384\tanh\left(\frac{\rho}{2\sqrt{2}}\right)\operatorname{sech}^2\left(\frac{\rho}{2\sqrt{2}}\right)}{\Gamma(2p+1)}\right). \tag{110}$$

The final ATIM solution is provided as follows:

$$\lambda(\rho,\,\zeta)=\lambda_0(\rho,\,\zeta)+\lambda_1(\rho,\,\zeta)+\lambda_2(\rho,\,\zeta)+\cdots. \tag{111}$$

$$\lambda(\rho,\,\zeta)=\frac{1}{e^{\frac{\rho}{\sqrt{2}}}+1}+\frac{3\zeta^{p}\operatorname{sech}^2\left(\frac{\rho}{2\sqrt{2}}\right)}{8\Gamma(p+1)}+\frac{3\zeta^{2p}}{2048}\left(\frac{9\zeta^{p}}{\Gamma(p+1)^3}\left(-\frac{3\zeta^{p}\Gamma(3p)\operatorname{sech}^6\left(\frac{\rho}{2\sqrt{2}}\right)}{\Gamma(4p)}\right.\right.$$
$$\left.\left.-\frac{512e^{\sqrt{2}\rho}\Gamma(p+1)\Gamma(2p+1)}{\left(e^{\frac{\rho}{\sqrt{2}}}+1\right)^5\Gamma(3p+1)}\right)+\frac{384\tanh\left(\frac{\rho}{2\sqrt{2}}\right)\operatorname{sech}^2\left(\frac{\rho}{2\sqrt{2}}\right)}{\Gamma(2p+1)}\right)+\cdots. \tag{112}$$

3.4.2 *Problem with ATIM*

Examine the fractional-order Gardner equation that follows:

$$D_{\zeta}^{p}\lambda(\rho,\,\zeta)=-\frac{\partial^3\lambda(\rho,\,\zeta)}{\partial\rho^3}+6k^2\lambda^2(\rho,\,\zeta)\frac{\partial\lambda(\rho,\,\zeta)}{\partial\rho}-6\lambda(\rho,\,\zeta)\frac{\partial\lambda(\rho,\,\zeta)}{\partial\rho},\ \text{where } 0<p\leq 1 \tag{113}$$

Having IC's:

$$\lambda(\rho,\,0)=\frac{1}{2}\tanh\left(\frac{\rho}{2}\right)+\frac{1}{2}, \tag{114}$$

and exact solution

$$\lambda(\rho,\,\zeta)=\frac{1}{2}\tanh\left(\frac{\rho-\zeta}{2}\right)+\frac{1}{2}. \tag{115}$$

We get the following outcome by using the AT on either side of Eq. (113):

$$A\left[D_{\zeta}^{p}\lambda(\rho,\ \zeta)\right]=\frac{1}{s^{p}}\left(\sum_{k=0}^{m-1}\frac{\lambda^{(k)}(\rho,\ 0)}{s^{2-p+k}}+A\left[-\frac{\partial^{3}\lambda(\rho,\ \zeta)}{\partial\rho^{3}}+6k^{2}\lambda^{2}(\rho,\ \zeta)\frac{\partial\lambda(\rho,\ \zeta)}{\partial\rho}-6\lambda(\rho,\ \zeta)\frac{\partial\lambda(\rho,\ \zeta)}{\partial\rho}\right]\right),\tag{116}$$

We get the following outcome by using the AIT on either side of Eq. (116):

$$\lambda(\rho,\ \zeta)=A^{-1}\left[\frac{1}{s^{p}}\left(\sum_{k=0}^{m-1}\frac{\lambda^{(k)}(\rho,\ 0)}{s^{2-p+k}}+A\left[-\frac{\partial^{3}\lambda(\rho,\ \zeta)}{\partial\rho^{3}}+6k^{2}\lambda^{2}(\rho,\ \zeta)\frac{\partial\lambda(\rho,\ \zeta)}{\partial\rho}-6\lambda(\rho,\ \zeta)\frac{\partial\lambda(\rho,\ \zeta)}{\partial\rho}\right]\right)\right].\tag{117}$$

The iterative procedure of the AT is used to derive this equation:

$$\begin{aligned}\lambda_{0}(\rho,\ \zeta)&=A^{-1}\left[\frac{1}{s^{p}}\left(\sum_{k=0}^{m-1}\frac{\lambda^{(k)}(\rho,\ 0)}{s^{2-p+k}}\right)\right]\\&=A^{-1}\left[\frac{\lambda(\rho,\ 0)}{s^{2}}\right]\\&=\frac{1}{2}\tanh\left(\frac{\rho}{2}\right)+\frac{1}{2},\end{aligned}\tag{118}$$

By implementing the RL integral on Eq. (83), we obtain

$$\lambda(\rho,\ \zeta)=\frac{1}{2}\tanh\left(\frac{\rho}{2}\right)+\frac{1}{2}-A\left[-\frac{\partial^{3}\lambda(\rho,\ \zeta)}{\partial\rho^{3}}+6k^{2}\lambda^{2}(\rho,\ \zeta)\frac{\partial\lambda(\rho,\ \zeta)}{\partial\rho}-6\lambda(\rho,\ \zeta)\frac{\partial\lambda(\rho,\ \zeta)}{\partial\rho}\right].\tag{119}$$

By using ATIM, we get the subsequent terms of the solution,

$$\lambda_{0}(\rho,\ \zeta)=\frac{1}{2}\tanh\left(\frac{\rho}{2}\right)+\frac{1}{2},$$

$$\lambda_{1}(\rho,\ \zeta)=\frac{e^{\rho}\left(e^{\rho}\left(e^{\rho}\left(6k^{2}-7\right)-2\right)-1\right)\zeta^{p}}{\left(e^{\rho}+1\right)^{4}\Gamma(p+1)},$$

$$\begin{aligned}\lambda_{2}(\rho,\ \zeta)=&\Big(e^{\rho}\zeta^{2p}\Big(-\Big(6e^{\rho}\left(e^{\rho}+1\right)^{3}\left(e^{\rho}\left(e^{\rho}\left(6k^{2}-7\right)-2\right)-1\right)\Big(e^{\rho}\Big(3k^{2}+e^{\rho}\Big(22k^{2}+e^{\rho}\Big(-42k^{4}\\&+53k^{2}+e^{\rho}\left(12k^{4}-20k^{2}+7\right)-10\Big)-18\Big)-2\Big)-1\Big)\zeta^{p}\Gamma(2p+1)\Big)/\left(\Gamma(p+1)^{2}\Gamma(3p+1)\right)\\&-\Big(\left(e^{\rho}+1\right)^{6}\Big(e^{\rho}\Big(e^{\rho}\Big(2\left(90k^{2}-91\right)+e^{\rho}\Big(-228k^{2}+e^{\rho}\Big(-180k^{4}+e^{\rho}\left(7-6k^{2}\right)^{2}+444k^{2}\end{aligned}$$

$$-261\Big)230\Big)\Big)-3\Big)-1\Big)\Big)/\Big(\Gamma(2p+1)\Big)-\Big(6e^{2\rho}\Big(-6e^{2\rho}k^3+e^{\rho}\Big(7e^{\rho}+2\Big)k+k\Big)^2\Big(e^{\rho}+e^{3\rho}\Big(6k^2-7\Big)$$

$$+e^{2\rho}\Big(17-18k^2\Big)+1\Big)\zeta^{2p}\Gamma(3p+1)\Big)/\Big(\Gamma(p+1)^3\Gamma(4p+1)\Big)\Big)\Big)/\Big(\Big(e^{\rho}+1\Big)^{13}\Big), \tag{120}$$

The final solution of the ATIM procedure are as follows:

$$\lambda(\rho,\ \zeta)=\lambda_0(\rho,\ \zeta)+\lambda_1(\rho,\ \zeta)+\lambda_2(\rho,\ \zeta)+\cdots \tag{121}$$

$$\lambda(\rho,\ \zeta)=\frac{1}{2}\tanh\left(\frac{\rho}{2}\right)+\frac{1}{2}+\frac{e^{\rho}\left(e^{\rho}\left(e^{\rho}\left(6k^2-7\right)-2\right)-1\right)\zeta^{p}}{\left(e^{\rho}+1\right)^4\Gamma(p+1)}$$

$$+\Big(e^{\rho}\zeta^{2p}\Big(-\Big(6e^{\rho}\Big(e^{\rho}+1\Big)^3\Big(e^{\rho}\Big(e^{\rho}\Big(6k^2-7\Big)-2\Big)-1\Big)\Big(e^{\rho}\Big(3k^2+e^{\rho}\Big(22k^2+e^{\rho}\Big(-42k^4$$

$$+53k^2+e^{\rho}\Big(12k^4-20k^2+7\Big)-10\Big)-18\Big)-2\Big)-1\Big)\zeta^{p}\Gamma(2p+1)\Big)/\Big(\Gamma(p+1)^2\Gamma(3p+1)\Big)$$

$$-\Big(\Big(e^{\rho}+1\Big)^6\Big(e^{\rho}\Big(e^{\rho}\Big(2\Big(90k^2-91\Big)+e^{\rho}\Big(-228k^2+e^{\rho}\Big(-180k^4+e^{\rho}\Big(7-6k^2\Big)^2+444k^2$$

$$-261\Big)230\Big)\Big)-3\Big)-1\Big)\Big)/\Big(\Gamma(2p+1)\Big)-\Big(6e^{2\rho}\Big(-6e^{2\rho}k^3+e^{\rho}\Big(7e^{\rho}+2\Big)k+k\Big)^2\Big(e^{\rho}+e^{3\rho}\Big(6k^2-7\Big)$$

$$+e^{2\rho}\Big(17-18k^2\Big)+1\Big)\zeta^{2p}\Gamma(3p+1)\Big)/\Big(\Gamma(p+1)^3\Gamma(4p+1)\Big)\Big)\Big)/\Big(\Big(e^{\rho}+1\Big)^{13}\Big)+\cdots. \tag{122}$$

4. Physical significance of the models and discussion

The fractional CAE and the fractional Gardner equation are of basic importance for describing non-linear dynamical systems with memory dependent properties. The CAE is a generaization of standard phase-field models of spinodal decomposition and phase separation in binary alloys and other materials. The nonlinearity is due to the double-well potential of the Ginzburg-Landau energy and allows to introduce bistable phases, while the second-order spatial derivative takes into account diffusion. The provision of the term D_{ζ}^{p} introduces nonlocality in time to the equation so that the equation can model complex hereditary nature that arises in many cases such as viscoelastic materials, porous media and anomalous diffusion processes.

Similarly, the fractional Gardner equation combines properties of the Korteweg-de Vries (KdV) and the modified KdV equations, which qualifies it to model the nonlinear wave propagation in plasma, surface water waves, and optical fibers. The fractional operator improves its capability to describe systems with long-term memory, for a more realistic modelling of dissipative or subdiffusive environments. Such generalizations are particularly important for describing physical aspects of materials and wave problems that cannot be captured by the classical integer-order models.

Table 1. The effect of p values on the solution $\lambda(\rho, \zeta)$ obtain through ARPSM of problem 1

ζ	ρ	$\lambda(\rho, \zeta)_{\text{ARPSM}}$	$\lambda(\rho, \zeta)_{\text{ARPSM}}$	$\lambda(\rho, \zeta)_{\text{ARPSM}}$		$\text{Error}_{\text{ARPSM}}$
		$p = 0.64$	$p = 0.84$	$p = 1.00$	Exact	$p = 1.00$
0.1	-5.0	0.980065	0.977279	0.975530	0.975530	3.619844×10^{-7}
	-2.5	0.893790	0.880219	0.871888	0.871889	8.529974×10^{-7}
	0.0	0.593651	0.557183	0.537430	0.537430	1.578437×10^{-7}
	2.5	0.203229	0.177297	0.165514	0.165513	9.907927×10^{-7}
	5.0	0.042000	0.035518	0.032750	0.032750	3.712018×10^{-7}
0.05	-5.0	0.977477	0.974980	0.973674	0.973674	2.277507×10^{-8}
	-2.5	0.881292	0.869354	0.863276	0.863276	5.540595×10^{-8}
	0.0	0.560831	0.532068	0.518741	0.518741	4.941035×10^{-9}
	2.5	0.180302	0.162694	0.155412	0.155412	5.971528×10^{-8}
	5.0	0.036297	0.032116	0.030456	0.030456	2.306360×10^{-8}
0.03	-5.0	0.976033	0.973881	0.972894	0.972894	2.959327×10^{-9}
	-2.5	0.874386	0.864245	0.859696	0.859696	7.290677×10^{-9}
	0.0	0.544046	0.520901	0.511248	0.511248	3.843547×10^{-10}
	2.5	0.169870	0.156607	0.151515	0.151515	7.625824×10^{-9}
	5.0	0.033806	0.030729	0.029582	0.029582	2.981770×10^{-9}
0.01	-5.0	0.973959	0.972580	0.972092	0.972092	3.662836×10^{-11}
	-2.5	0.864629	0.858264	0.856038	0.856038	9.137757×10^{-11}
	0.0	0.521877	0.508311	0.503750	0.503750	1.581956×10^{-12}
	2.5	0.157215	0.150019	0.147699	0.147699	9.275694×10^{-11}
	5.0	0.030873	0.029249	0.028733	0.028733	3.672090×10^{-11}

Table 2. The effect of p values on the solution $\lambda(\rho, \zeta)$ obtain through ARPSM of problem 2

ζ	ρ	$\lambda(\rho, \zeta)_{\text{ARPSM}}$	$\lambda(\rho, \zeta)_{\text{ARPSM}}$	$\lambda(\rho, \zeta)_{\text{ARPSM}}$		$\text{Error}_{\text{ARPSM}}$
		$p = 0.64$	$p = 0.84$	$p = 1.00$	Exact	$p = 1.00$
0.1	-5.0	0.005296	0.005763	0.006060	0.006059	1.039104×10^{-6}
	-2.5	0.060693	0.065925	0.069145	0.069138	6.727202×10^{-6}
	0.0	0.436269	0.461664	0.475000	0.475021	2.081250×10^{-5}
	2.5	0.903564	0.912575	0.916834	0.916827	6.805638×10^{-6}
	5.0	0.991314	0.992198	0.992610	0.992608	1.089414×10^{-6}
0.05	-5.0	0.005728	0.006151	0.006368	0.006368	1.314186×10^{-7}
	-2.5	0.065504	0.070107	0.072427	0.072426	8.436272×10^{-7}
	0.0	0.459103	0.478584	0.487500	0.487503	2.603515×10^{-6}
	2.5	0.911559	0.917881	0.920562	0.920561	8.485406×10^{-7}
	5.0	0.992097	0.992710	0.992967	0.992966	1.345624×10^{-7}
0.03	-5.0	0.005972	0.006333	0.006496	0.006496	2.852031×10^{-8}
	-2.5	0.068167	0.072056	0.073781	0.073781	1.824485×10^{-7}
	0.0	0.470508	0.486056	0.492500	0.492501	5.624493×10^{-7}
	2.5	0.915292	0.920124	0.922012	0.922012	1.830856×10^{-7}
	5.0	0.992459	0.992924	0.993105	0.993105	2.892774×10^{-8}
0.01	-5.0	0.006320	0.006547	0.006626	0.006626	1.061302×10^{-9}
	-2.5	0.071906	0.074321	0.075160	0.075160	6.765458×10^{-9}
	0.0	0.485400	0.494459	0.497500	0.497500	2.083312×10^{-9}
	2.5	0.919906	0.922571	0.923438	0.923438	6.773325×10^{-9}
	5.0	0.992903	0.993158	0.993240	0.993240	1.066332×10^{-9}

The ARPSM and the ATIM were used to solve the models. The numerical findings confirm an excellent-reconciliation between the approximate and accurate expressions, in addition to very good absolute deviations even for very small values of the fractional order p. As observed from the Table 1, the ARPSM solution $\lambda(\rho, \zeta)$ for fractional CAE demonstrates the excellent accuracy for different ζ and ρ. The error at $p = 1$ is always quite low, showing that the method is efficient and accurate. The solution profiles become smoother and more diffusive, typical for fractional dynamics, when decreasing p is depicted in Figures 1-4 for different fractional orders. Similarly, in Table 2, we demonstrate the practice of ARPSM towards the Gardner equation. The approach is able to effectively follow the evolution of the wave profile over fractional orders preserving an error in the approximation to the exact solution. In Figure 5 we show the numerical approximations for different p, and in Figures 6 and 7 we compare them with the exact profile, showing this simple method is robust enough to represent the ARPSM under many different dynamics.

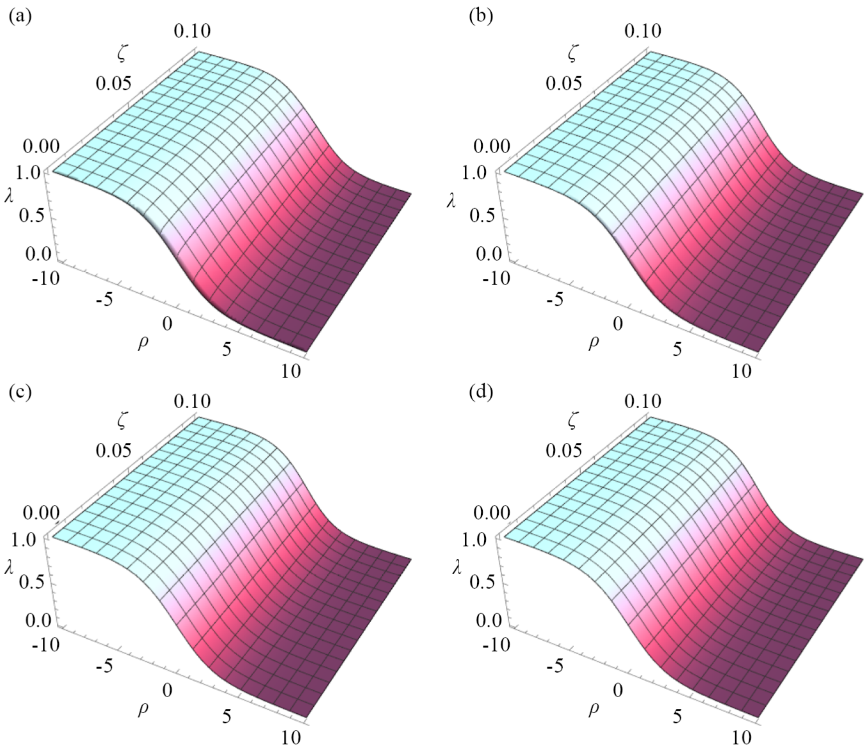

Figure 1. The comparison of the approximate solution $\lambda(\rho,\ \zeta)$ obtain through ARPSM of problem 1 at $\zeta = 0.1$ for different fractional order p. (a) shows the behaviour of fraction-order $p = 0.44$, (b) shows the behaviour of fraction-order $p = 0.64$, (c) shows the behaviour of fraction-order $p = 0.84$ and (d) shows the behaviour of fraction-order $p = 1.00$

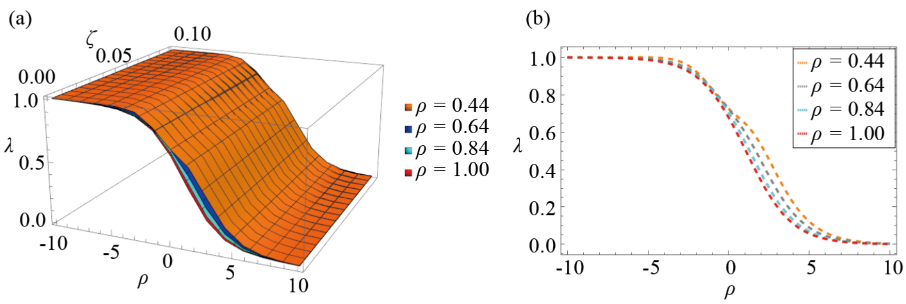

Figure 2. The effect of the values of p on the solution $\lambda(\rho,\ \zeta)$ obtain through ARPSM of problem 1 at $\zeta = 0.1$

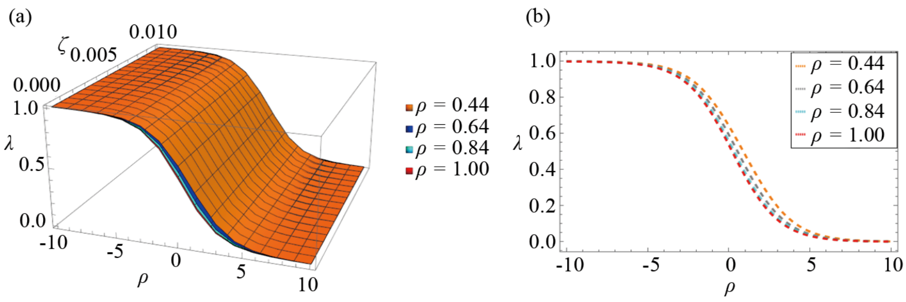

Figure 3. The effect of the values of p on the solution $\lambda(\rho,\ \zeta)$ obtain through ARPSM of problem 1 at $\zeta = 0.05$

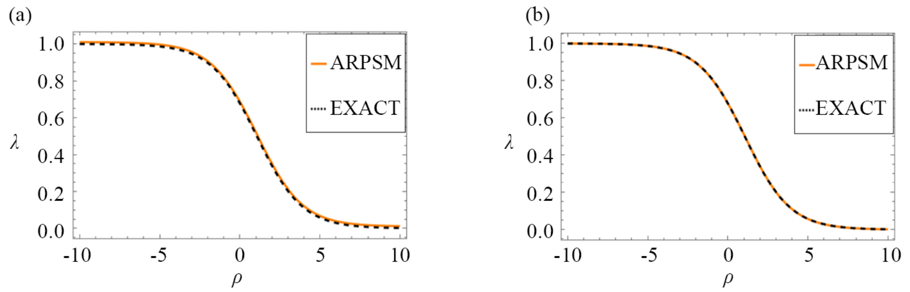

Figure 4. In figure, (a) shows the Exact solution comparison with the approximate solution obtain through ARPSM at $\zeta = 0.1$, (b) shows the Exact solution comparison with the approximate solution obtain through ARPSM at $\zeta = 0.05$

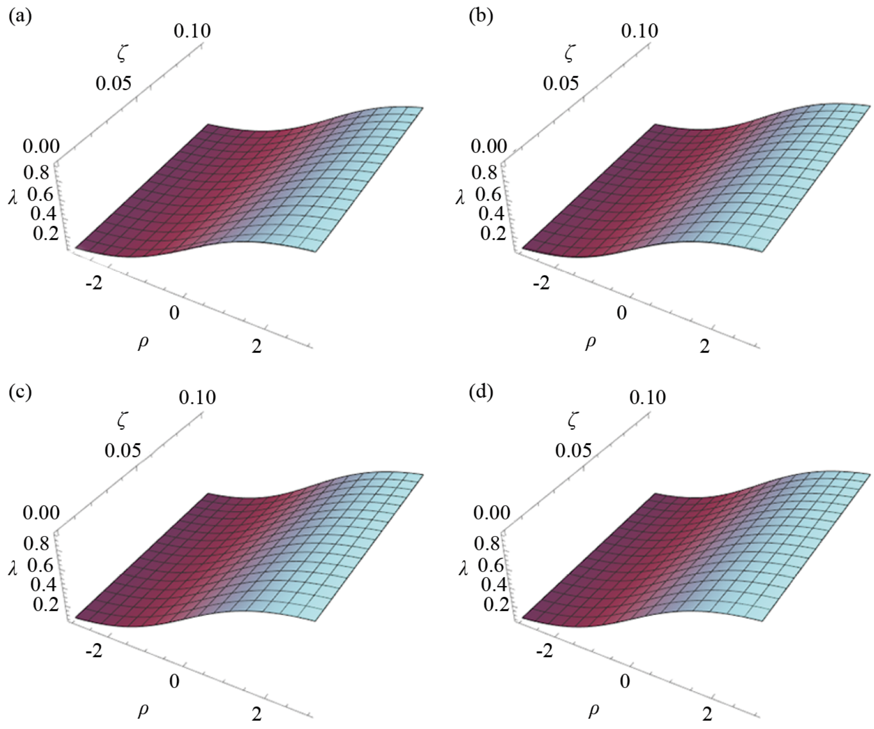

Figure 5. The comparison of the approximate solution $\lambda(\rho,\ \zeta)$ obtain through ARPSM of problem 2 at $\zeta = 0.1$ for different fractional order p. (a) shows the behaviour of fraction-order $p = 0.44$, (b) shows the behaviour of fraction-order $p = 0.64$, (c) shows the behaviour of fraction-order $p = 0.84$ and (d) shows the behaviour of fraction-order $p = 1.00$

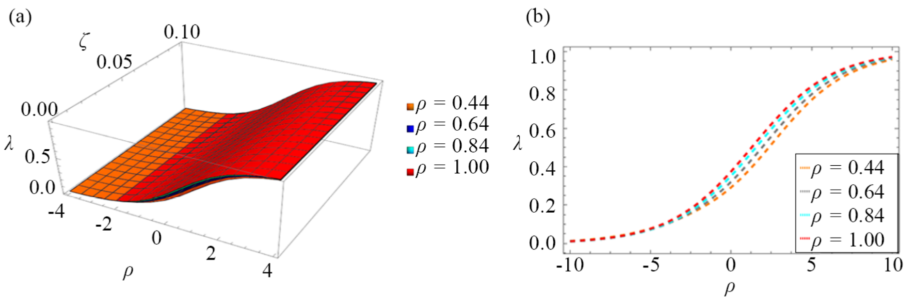

Figure 6. The effect of the values of p on the solution $\lambda(\rho, \zeta)$ obtain through ARPSM of problem 2 at $\zeta = 0.1$

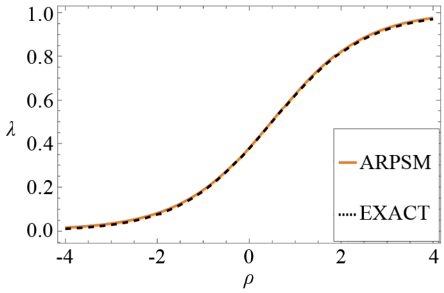

Figure 7. ARPSM and Exact solution comparison for $\zeta = 0.1$

In comparison the ATIM results are given in Tables 3 and 4. ATIM is still a desirable iterative scheme, despite the fact that it is considerably less accurate than ARPSM in terms of the absolute error. The error level of $p = 1$ remains remarkably low, often falling within the range of 10^{-6} to 10^{-8}. This level of accuracy is more than sufficient for practical uses.

Table 3. The effect of p values on the solution $\lambda(\rho, \zeta)$ obtain through ATIM of problem 1

ζ	ρ	$\lambda(\rho, \zeta)_{\text{ATIM}}$	$\lambda(\rho, \zeta)_{\text{ATIM}}$	$\lambda(\rho, \zeta)_{\text{ATIM}}$		$\text{Error}_{\text{ARPSM}}$
		$p = 0.64$	$p = 0.84$	$p = 1.00$	Exact	$p = 1.00$
0.1	-5.0	0.979499	0.977201	0.975516	0.975530	1.42175×10^{-5}
	-2.5	0.891934	0.879965	0.871841	0.871889	4.85353×10^{-5}
	0.0	0.593546	0.557173	0.537428	0.537430	1.47620×10^{-6}
	2.5	0.202332	0.177174	0.165491	0.165513	2.19646×10^{-5}
	5.0	0.041496	0.035449	0.032737	0.032750	1.33436×10^{-5}
0.05	-5.0	0.977327	0.974966	0.973673	0.973674	1.79955×10^{-6}
	-2.5	0.880802	0.869310	0.863270	0.863276	6.00550×10^{-6}
	0.0	0.560814	0.532067	0.518741	0.518741	8.73384×10^{-8}
	2.5	0.180066	0.162673	0.155410	0.155412	2.79951×10^{-6}
	5.0	0.036164	0.032104	0.030454	0.030456	1.64450×10^{-6}
0.03	-5.0	0.975977	0.973878	0.972894	0.972894	3.90654×10^{-7}
	-2.5	0.874203	0.864233	0.859695	0.859696	1.29163×10^{-6}
	0.0	0.544041	0.520900	0.511248	0.511248	1.10630×10^{-8}
	2.5	0.169782	0.156601	0.151514	0.151515	6.09087×10^{-7}
	5.0	0.033756	0.030726	0.029582	0.029582	3.53204×10^{-7}
0.01	-5.0	0.973952	0.972580	0.972092	0.972092	1.45412×10^{-8}
	-2.5	0.864607	0.858264	0.856038	0.856038	4.76269×10^{-8}
	0.0	0.521876	0.508311	0.503750	0.503750	1.33417×10^{-10}
	2.5	0.157204	0.150018	0.147699	0.147699	2.27158×10^{-8}
	5.0	0.030867	0.029249	0.028733	0.028733	1.30075×10^{-8}

Table 4. The effect of p values on the solution $\lambda(\rho,\ \zeta)$ obtain through ATIM of problem 2

ζ	ρ	$\lambda(\rho,\ \zeta)_{\text{ATIM}}$	$\lambda(\rho,\ \zeta)_{\text{ATIM}}$	$\lambda(\rho,\ \zeta)_{\text{ATIM}}$		$\text{Error}_{\text{ARPSM}}$
		$p = 0.64$	$p = 0.84$	$p = 1.00$	Exact	$p = 1.00$
0.1	-5.0	0.005294	0.005763	0.006060	0.006059	9.536058×10^{-7}
	-2.5	0.060513	0.065895	0.069138	0.069138	2.995709×10^{-7}
	0.0	0.437134	0.461807	0.475031	0.475021	1.043750×10^{-5}
	2.5	0.903390	0.912546	0.916828	0.916827	4.656842×10^{-7}
	5.0	0.991311	0.992197	0.992609	0.992608	1.004002×10^{-6}
0.05	-5.0	0.005727	0.006151	0.006368	0.006368	1.207340×10^{-7}
	-2.5	0.065457	0.070102	0.072426	0.072426	4.291322×10^{-8}
	0.0	0.459331	0.478609	0.487504	0.487503	1.302734×10^{-6}
	2.5	0.911513	0.917876	0.920562	0.920561	5.330642×10^{-8}
	5.0	0.992096	0.992709	0.992967	0.992966	1.238832×10^{-7}
0.03	-5.0	0.005972	0.006333	0.006496	0.006496	2.621267×10^{-8}
	-2.5	0.068150	0.072054	0.073781	0.073781	9.731047×10^{-9}
	0.0	0.470593	0.486063	0.492501	0.492501	2.813006×10^{-7}
	2.5	0.915275	0.920122	0.922012	0.922012	1.107831×10^{-8}
	5.0	0.992459	0.992924	0.993105	0.993105	2.662080×10^{-8}
0.01	-5.0	0.006320	0.006547	0.006626	0.006626	9.758428×10^{-10}
	-2.5	0.071904	0.074321	0.075160	0.075160	3.772810×10^{-10}
	0.0	0.485410	0.494459	0.497500	0.497500	1.041687×10^{-8}
	2.5	0.919904	0.922571	0.923438	0.923438	3.939158×10^{-10}
	5.0	0.992903	0.993158	0.993240	0.993240	9.808813×10^{-10}

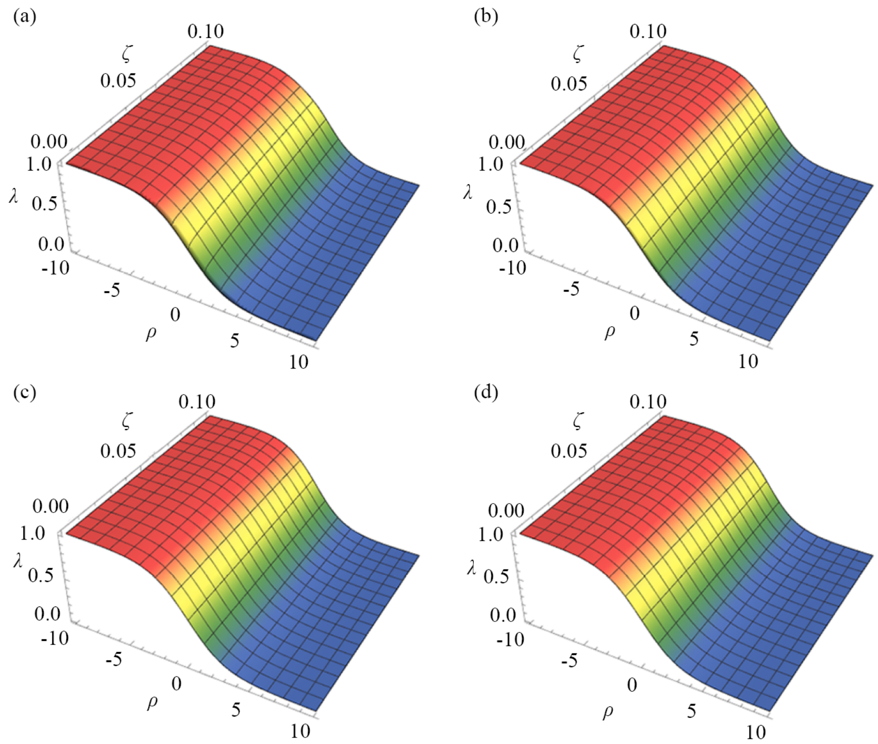

Figure 8. The comparison of the approximate solution $\lambda(\rho,\ \zeta)$ obtain through ATIM of problem 1 at $\zeta = 0.1$ for different fractional order p. (a) shows the behaviour of fraction-order $p = 0.44$, (b) shows the behaviour of fraction-order $p = 0.64$, (c) shows the behaviour of fraction-order $p = 0.84$ and (d) shows the behaviour of fraction-order $p = 1.00$

Figures 8-11 for the fractional CAE and Figures 12-14 for the Gardner equation graphically validate the ATIM approach with the exact solutions. How $\lambda(\rho, \zeta)$ behaves for different p is a reflection of the capacity of the fractional derivative to capture delayed diffusion and long-memory jumps, e.g., for $\zeta = 0.1$ and $\zeta = 0.05$.

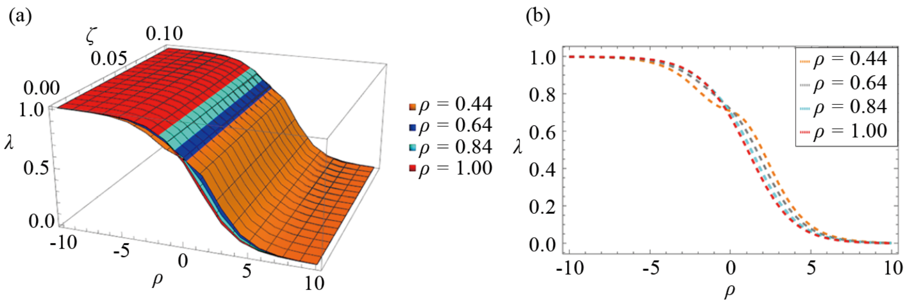

Figure 9. The effect of the values of p on the solution $\lambda(\rho, \zeta)$ obtain through ATIM of problem 1 at $\zeta = 0.1$

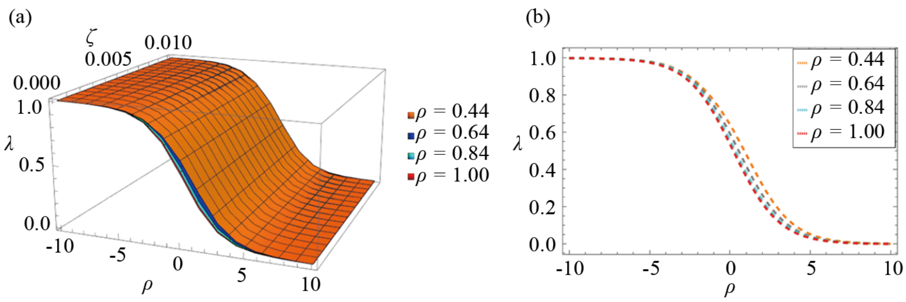

Figure 10. The effect of the values of p on the solution $\lambda(\rho, \zeta)$ obtain through ATIM of problem 1 at $\zeta = 0.05$

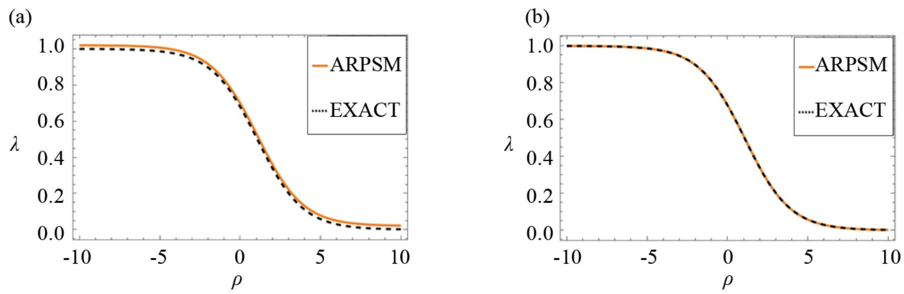

Figure 11. In figure, (a) shows the Exact solution comparison with the approximate solution obtain through ATIM at $\zeta = 0.1$, (b) shows the Exact solution comparison with the approximate solution obtain through ATIM at $\zeta = 0.05$

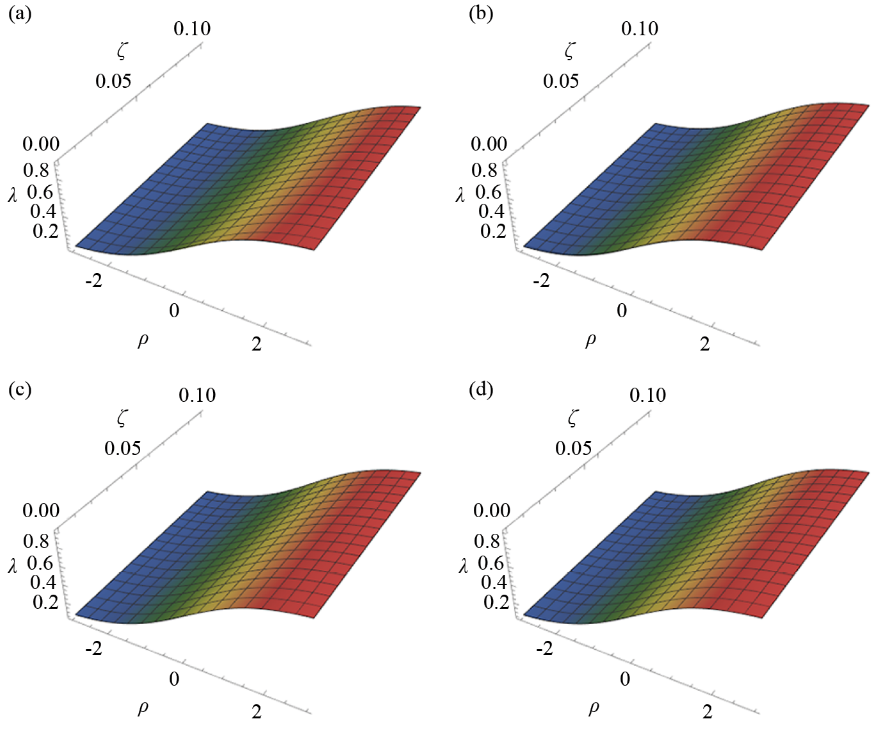

Figure 12. The comparison of the approximate solution $\lambda(\rho, \zeta)$ obtain through ATIM of problem 2 at $\zeta = 0.1$ for different fractional order p. (a) shows the behaviour of fraction-order $p = 0.44$, (b) shows the behaviour of fraction-order $p = 0.64$, (c) shows the behaviour of fraction-order $p = 0.84$ and (d) shows the behaviour of fraction-order $p = 1.00$

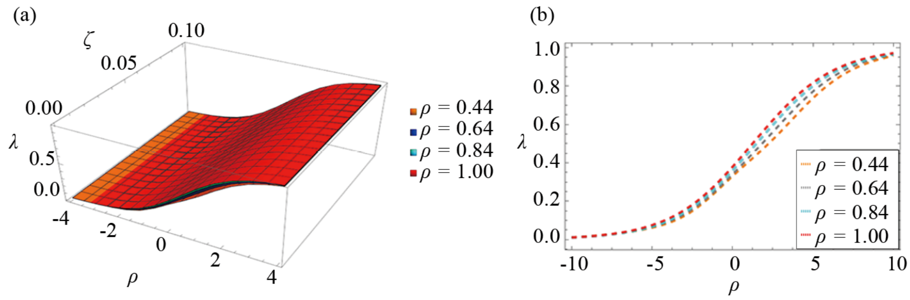

Figure 13. The effect of the values of p on the solution $\lambda(\rho, \zeta)$ obtain through ATIM of problem 2 at $\zeta = 0.1$

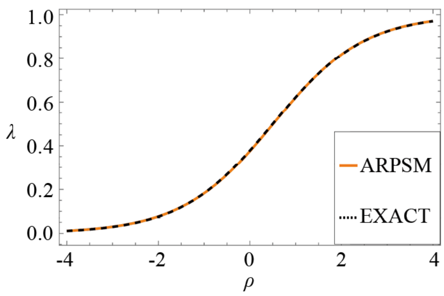

Figure 14. ATIM and Exact solution comparison for $\zeta = 0.1$

Moreover, the relative results in Tables 5 and 6 demonstrate the efficacy of ARPSM. For any ρ, ζ we have both ARPSM and ATIM giving less absolute errors, and this is also the case on the best results achieved at lower ζ which entails ARPSM dominance in term of tackling stiff fractional PDEs. The results of both methods illustrate that, for $0 < p < 1$, fractional models not only give more insight to the physics but also make effective prediction.

Table 5. The example 1 solution comparison for $p = 1.00$ and absolute error of the proposed methods

ζ	ρ	Exact	$ATIM_{p=1}$	$ARPSM_{p=1}$	$Error_{ATIM}$	$Error_{ARPSM}$
0.1	-5.0	0.975530	0.975516	0.975530	1.42175×10^{-5}	3.61984×10^{-7}
	-2.5	0.871889	0.871841	0.871888	4.85353×10^{-5}	8.52997×10^{-7}
	0.0	0.537430	0.537428	0.537430	1.47620×10^{-6}	1.57843×10^{-7}
	2.5	0.165513	0.165491	0.165514	2.19646×10^{-5}	9.90792×10^{-7}
	5.0	0.032750	0.032737	0.032750	1.33436×10^{-5}	3.71201×10^{-7}
0.05	-5.0	0.973674	0.973673	0.973674	1.79955×10^{-6}	2.27750×10^{-8}
	-2.5	0.863276	0.863270	0.863276	6.00550×10^{-6}	5.54059×10^{-8}
	0.0	0.518741	0.518741	0.518741	8.73384×10^{-6}	4.94103×10^{-8}
	2.5	0.155412	0.155410	0.155412	2.79951×10^{-6}	5.97152×10^{-8}
	5.0	0.030456	0.030454	0.030456	1.64450×10^{-6}	2.30636×10^{-8}
0.03	-5.0	0.972894	0.972894	0.972894	3.90654×10^{-7}	2.95932×10^{-9}
	-2.5	0.859696	0.859695	0.859696	1.29163×10^{-6}	7.29067×10^{-9}
	0.0	0.511248	0.511248	0.511248	1.10630×10^{-8}	3.84354×10^{-10}
	2.5	0.151515	0.151514	0.151515	6.09087×10^{-7}	7.62582×10^{-9}
	5.0	0.029582	0.029582	0.029582	3.53204×10^{-7}	2.98177×10^{-9}
0.01	-5.0	0.972092	0.972092	0.972092	1.45412×10^{-8}	3.66284×10^{-11}
	-2.5	0.856038	0.856038	0.856038	4.76269×10^{-8}	9.13776×10^{-11}
	0.0	0.503750	0.503750	0.503750	1.33417×10^{-10}	1.58194×10^{-12}
	2.5	0.147699	0.147699	0.147699	2.27158×10^{-8}	9.27569×10^{-11}
	5.0	0.028733	0.028733	0.028733	1.30075×10^{-8}	3.67209×10^{-11}

Table 6. The example 2 solution comparison for $p = 1.00$ and absolute error of the proposed methods

ζ	ρ	Exact	$ATIM_{p=1}$	$ARPSM_{p=1}$	$Error_{ATIM}$	$Error_{ARPSM}$
0.1	-5.0	0.006059	0.006060	0.006060	9.536058×10^{-7}	1.039104×10^{-6}
	-2.5	0.069138	0.069138	0.069145	2.995709×10^{-7}	6.727202×10^{-6}
	0.0	0.475021	0.475031	0.475000	1.043750×10^{-5}	2.081250×10^{-5}
	2.5	0.916827	0.916828	0.916834	4.656842×10^{-7}	6.805638×10^{-6}
	5.0	0.992608	0.992609	0.992610	1.004002×10^{-6}	1.089414×10^{-6}
0.05	-5.0	0.006368	0.006368	0.006368	1.207340×10^{-7}	1.314186×10^{-7}
	-2.5	0.072426	0.072426	0.072427	4.291322×10^{-8}	8.436272×10^{-7}
	0.0	0.487503	0.487504	0.487500	1.302734×10^{-6}	2.603515×10^{-6}
	2.5	0.920561	0.920562	0.920562	5.330642×10^{-8}	8.485406×10^{-7}
	5.0	0.992966	0.992967	0.992967	1.238832×10^{-7}	1.345624×10^{-7}
0.03	-5.0	0.006496	0.006496	0.006496	2.621267×10^{-8}	2.852031×10^{-8}
	-2.5	0.073781	0.073781	0.073781	9.731047×10^{-9}	1.824485×10^{-7}
	0.0	0.492501	0.492501	0.492500	2.813006×10^{-7}	5.624493×10^{-7}
	2.5	0.922012	0.922012	0.922012	1.107831×10^{-8}	1.830856×10^{-7}
	5.0	0.993105	0.993105	0.993105	2.662080×10^{-8}	2.892774×10^{-8}
0.01	-5.0	0.006626	0.006626	0.006626	9.758428×10^{-10}	1.061300×10^{-9}
	-2.5	0.075160	0.075160	0.075160	3.772810×10^{-10}	6.765458×10^{-9}
	0.0	0.497500	0.497500	0.497500	1.041687×10^{-9}	2.083312×10^{-8}
	2.5	0.923438	0.923438	0.923438	3.939158×10^{-10}	6.773325×10^{-9}
	5.0	0.993240	0.993240	0.993240	9.808814×10^{-10}	1.066332×10^{-9}

The inclusion of fractional derivatives in the CAE and Gardner equations provides a wider description capacity of physical problems with memory and anomalous behaviors in real systems. The ARPSM and ATIM methods presented powerful promises in solving fractional PDEs, where ARPSM attains slightly better in both accuracy and convergence than ATIM. These results are an important step towards a more widespread use of fractional models in materials science, wave mechanics and simulations of complex systems.

5. Conclusion

This paper provides a comparative analysis of the ARPSM and ATIM for the solution of the fractional-order nonlinear PDEs, the CAE and Gardner equations with the aid of the Caputo derivative. They are very accurate and computationally efficient, accounting for nonlocal as well as memory effects associated to fractional systems. Our findings demonstrate the effect of fractional order p on phase separation kinetics and nonlinear wave dispersion, which is important in understanding phenomena in the framework of physical processes, like material response, behavior of solitons and time evolution in materials with hereditary effects. For the initial conditions used, we have chosen physically motivated parameters which have been tuned to mimic real profile for such systems.

Both ARPSM and ATIM yield solutions that compare well with exact solution, ARPSM slightly outperforms. The small constraining absolute errors of the proposed methods verify the reliability and convergence of the methods. The Aboodh transform is not frequently employed as classical integral transforms. However, its algebraic formulators simplify its iterations by utilising the minimum rule iteration intention. These results indicate the general character of the techniques used, directly applicable to other models (such as those in plasma physics, optical fibers, and fluid dynamics). The work also fills gaps in the literature by the use of physically based initial conditions, the non-singular kernel dynamics and the error-controlled semi-analytical solutions. The techniques developed may be generalized to multi-dimensional problems, or performing experimental data fitting, in future work.

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Data availability

No datasets were generated or analysed during the current study.

Conflict of interest

The authors declare no competing interests.

References

[1] Al-Sawalha MM, Khan A, Ababneh OY, Botmart T. Fractional view analysis of Kersten-Krasil'shchik coupled KdV-mKdV systems with non-singular kernel derivatives. *AIMS Mathematics*. 2022; 7(10): 18334-18359.

[2] Elsayed EM, Shah R, Nonlaopon K. The analysis of the fractional-order Navier-Stokes equations by a novel approach. *Journal of Function Spaces*. 2022; 2022: 8979447.

[3] Naeem M, Rezazadeh H, Khammash AA, Shah R, Zaland S. Analysis of the fuzzy fractional-order solitary wave solutions for the KdV equation in the sense of Caputo-Fabrizio derivative. *Journal of Mathematics*. 2022; 2022(1): 3688916.

[4] Caputo M, Fabrizio M. A new definition of fractional derivative without singular kernel. *Progress in Fractional Differentiation and Applications*. 2015; 1(2): 73-85.

[5] Atangana A, Baleanu D. Caputo-Fabrizio derivative applied to groundwater flow within confined aquifer. *Journal of Engineering Mechanics*. 2017; 143(5): D4016005.

[6] Singh J, Kumar D, Nieto JJ. Analysis of an El Nino-Southern Oscillation model with a new fractional derivative. *Chaos, Solitons and Fractals*. 2017; 99: 109-115.

[7] Singh J, Kumar D, Hammouch Z, Atangana A. A fractional epidemiological model for computer viruses pertaining to a new fractional derivative. *Applied Mathematics and Computation*. 2018; 316: 504-515.

[8] Atangana A, Alqahtani RT. Numerical approximation of the space-time Caputo-Fabrizio fractional derivative and application to groundwater pollution equation. *Advances in Difference Equations*. 2016; 2016: 1-13.

[9] Atangana A. Non validity of index law in fractional calculus: A fractional differential operator with Markovian and non-Markovian properties. *Physica A: Statistical Mechanics and Its Applications*. 2018; 505: 688-706.

[10] Atangana A, Baleanu D. New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. *Thermal Science*. 2016; 20(2): 763-769.

[11] Podlubny I. *Fractional Differential Equations*. San Diego: Academic Press; 1999.

[12] Kilbas AA, Srivastava HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations*. Amsterdam: Elsevier; 2006.

[13] Gomez-Aguilar JF, Escobar-Jimenez RF, Lopez-Lopez MG. Atangana-Baleanu fractional derivative applied to electromagnetic waves in dielectric media. *Journal of Electromagnetic Waves and Applications*. 2016; 30(15): 1937-1952.

[14] Alkahtani BST. Chua's circuit model with Atangana-Baleanu derivative with fractional order. *Chaos, Solitons and Fractals*. 2016; 89: 547-551.

[15] Owolabi KM. Modelling and simulation of a dynamical system with the Atangana-Baleanu fractional derivative. *European Physical Journal Plus*. 2018; 133(1): 15.

[16] Hilfer R. *Applications of Fractional Calculus in Physics*. Singapore: World Scientific; 2000.

[17] Sabatier J, Agrawal OP, Machado JT. *Advances in Fractional Calculus*. Dordrecht, The Netherlands: Springer; 2007.

[18] Liao S. *Beyond Perturbation: Introduction to the Homotopy Analysis Method*. Boca Raton, FL: Chapman and Hall/CRC; 2003.

[19] Adomian G. *Solving Frontier Problems of Physics: The Decomposition Method*. Dordrecht: Springer Science and Business Media; 2013.

[20] He JH. Variational iteration method a kind of non-linear analytical technique: Some examples. *International Journal of Non-Linear Mechanics*. 1999; 34(4): 699-708.

[21] Maitama S, Zhao W. New homotopy analysis transform method for solving multidimensional fractional diffusion equations. *Arab Journal of Basic and Applied Sciences*. 2020; 27(1): 27-44.

[22] Wazwaz AM. The combined Laplace transform-Adomian decomposition method for handling nonlinear Volterra integro-differential equations. *Applied Mathematics and Computation*. 2010; 216(4): 1304-1309.

[23] Maitama S, Abdullahi I. A new analytical method for solving linear and nonlinear fractional partial differential equations. *Progress in Fractional Differentiation and Applications*. 2016; 2(4): 247-256.

[24] Allen SM, Cahn JW. A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta Metallurgica*. 1979; 27(6): 1085-1095.

[25] Hariharan G, Kannan K. Haar wavelet method for solving Cahn-Allen equation. *Applied Mathematical Sciences*. 2009; 3(51): 2523-2533.

[26] Kumar D, Singh J, Baleanu D. Analytic study of Allen-Cahn equation of fractional order. *Bulletin of Mathematical Analysis and Applications*. 2016; 1: 31-40.

[27] Esen A, Yagmurlu NM, Tasbozan O. Approximate analytical solution to time-fractional damped Burger and Cahn-Allen equations. *Applied Mathematics and Information Sciences*. 2013; 7(5): 1951-1956.

[28] Tariq H, Akram G. New traveling wave exact and approximate solutions for the nonlinear Cahn-Allen equation: Evolution of a nonconserved quantity. *Nonlinear Dynamics*. 2017; 88: 581-594.

[29] Tascan F, Bekir A. Travelling wave solutions of the Cahn-Allen equation by using first integral method. *Applied Mathematics and Computation*. 2009; 207(1): 279-282.

[30] Algahtani OJJ. Comparing the Atangana-Baleanu and Caputo-Fabrizio derivative with fractional order: Allen Cahn model. *Chaos, Solitons and Fractals*. 2016; 89: 552-559.

[31] Gardner GHF, Gardner LW, Gregory A. Formation velocity and density: The diagnostic basics for stratigraphic traps. *Geophysics*. 1974; 39(6): 770-780.

[32] Fu Z, Liu S, Liu S. New kinds of solutions to Gardner equation. *Chaos, Solitons and Fractals*. 2004; 20(2): 301-309.

[33] Xu GQ, Li ZB, Liu YP. Exact solutions to a large class of nonlinear evolution equations. *Chinese Journal of Physics*. 2003; 41(3): 232-241.

[34] Kuo CK. New solitary solutions of the Gardner equation and Whitham-Broer-Kaup equations by the modified simplest equation method. *Optik*. 2017; 147: 128-135.

[35] Arqub OA. Series solution of fuzzy differential equations under strongly generalized differentiability. *Journal of Advanced Research in Applied Mathematics*. 2013; 5(1): 31-52.

[36] Abu Arqub O, Abo-Hammour Z, Al-Badarneh R, Momani S. A reliable analytical method for solving higher-order initial value problems. *Discrete Dynamics in Nature and Society*. 2013; 2013: 673829.

[37] Arqub OA, El-Ajou A, Zhour ZA, Momani S. Multiple solutions of nonlinear boundary value problems of fractional order: A new analytic iterative technique. *Entropy*. 2014; 16(1): 471-493.

[38] El-Ajou A, Arqub OA, Momani S. Approximate analytical solution of the nonlinear fractional KdV-Burgers equation: A new iterative algorithm. *Journal of Computational Physics*. 2015; 293: 81-95.

[39] Xu F, Gao Y, Yang X, Zhang H. Construction of fractional power series solutions to fractional Boussinesq equations using residual power series method. *Mathematical Problems in Engineering*. 2016; 2016: 9675089.

[40] Zhang J, Wei Z, Li L, Zhou C. Least-squares residual power series method for the time-fractional differential equations. *Complexity*. 2019; 2019: 6764043.

[41] Zhang MF, Liu YQ, Zhou XS. Efficient homotopy perturbation method for fractional non-linear equations using Sumudu transform. *Thermal Science*. 2015; 19(4): 1167-1171.

[42] Liaqat MI, Etemad S, Rezapour S, Park C. A novel analytical Aboodh residual power series method for solving linear and nonlinear time-fractional partial differential equations with variable coefficients. *AIMS Mathematics*. 2022; 7(9): 16917-16948.

[43] Liaqat MI, Akgul A, Abu-Zinadah H. Analytical investigation of some time-fractional Black-Scholes models by the Aboodh residual power series method. *Mathematics*. 2023; 11(2): 276.

[44] Ojo GO, Mahmudov NI. Aboodh transform iterative method for spatial diffusion of a biological population with fractional-order. *Mathematics*. 2011; 9(2): 155.

[45] Awuya MA, Ojo GO, Mahmudov NI. Solution of space-time fractional differential equations using Aboodh transform iterative method. *Journal of Mathematics*. 2022; 2022: 6454342.

[46] Awuya MA, Subasi D. Aboodh transform iterative method for solving fractional partial differential equation with Mittag-Leffler kernel. *Symmetry*. 2021; 13(11): 2055.

[47] Aboodh KS. The new integral transform 'Aboodh transform'. *Global Journal of Pure and Applied Mathematics*. 2013; 9(1): 35-43.

[48] Aggarwal S, Chauhan R. A comparative study of Mohand and Aboodh transforms. *International Journal of Research in Advent Technology*. 2019; 7(1): 520-529.

[49] Benattia ME, Belghaba K. Application of the Aboodh transform for solving fractional delay differential equations. *Universal Journal of Mathematics and Applications*. 2020; 3(3): 93-101.

[50] Delgado BB, Macias-Diaz JE. On the general solutions of some non-homogeneous Div-curl systems with Riemann-Liouville and Caputo fractional derivatives. *Fractal and Fractional*. 2021; 5(3): 117.

[51] Alshammari S, Al-Smadi M, Hashim I, Alias MA. Residual power series technique for simulating fractional Bagley-Torvik problems emerging in applied physics. *Applied Sciences*. 2019; 9(23): 5029.