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Block-Transitive 3-(v, 5, λ) Designs with Sporadic or Alternating Socle

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Abstract: This paper presents a systematic classification of 3-designs with block size 5. Let $\mathscr{D} = (\mathscr{P}, \mathscr{B})$ be a nontrivial 3-$(v, 5, \lambda)$ design, and let $G \leq \mathrm{Aut}(\mathscr{D})$ act transitively on blocks. We classify all such pairs $(\mathscr{D}, G)$ where the socle of G is either: one of the 26 sporadic simple groups, or an alternating group A_n $(n \geq 5)$.

***Keywords*:** 3-design, block-transitive, automorphism, sporadic group, alternating group

MSC: 20B25, 05B25

1. Introduction

A 3-(v, k, λ) *design* $\mathscr{D}$ is an incidence structure $(\mathscr{P}, \mathscr{B})$ satisfying:
(i) $\mathscr{P}$ is a finite set with $|\mathscr{P}| = v$ (*points*);
(ii) $\mathscr{B}$ is a set of k-element subsets of $\mathscr{P}$ (*blocks*);
(iii) For any three distinct elements in $\mathscr{P}$, there are exactly λ blocks that contain all three of them.

As for a 3-design $\mathscr{D}$ like this, we define $b = |\mathscr{B}|$ (the size of $\mathscr{B}$), and $r = |\mathscr{P}(\alpha)|$ where $\mathscr{P}(\alpha)$ is the set of blocks containing any given point $\alpha \in \mathscr{P}$. The parameters v, k, λ, b, r of $\mathscr{D}$ are positive integers. Additionally, $\mathscr{D}$ is called *complete* if $\mathscr{B}$ comprises all possible k-element subsets of $\mathscr{P}$, i.e., $\mathscr{B} = \binom{\mathscr{P}}{k} = \{B \subseteq \mathscr{P} \mid |B| = k\}$. All designs $\mathscr{D}$ considered in this paper are *nontrivial*, that is, $3 < k < v - 1$.

Let σ be a permutation of the set $\mathscr{P}$ (i.e., $\sigma \in \mathrm{Sym}(\mathscr{P})$). If $B^\sigma \in \mathscr{B}$ for any $B \in \mathscr{B}$ then σ is defined as an *automorphism* of $\mathscr{D}$. The *full automorphism group* $\mathrm{Aut}(\mathscr{D})$ of a design $\mathscr{D}$ is the subgroup of $\mathrm{Sym}(\mathscr{P})$ consisting of all such automorphisms. For a group $G \leq \mathrm{Aut}(\mathscr{D})$, a design $\mathscr{D}$ is called *point-transitive* (*block-transitive*) when its automorphism group G is transitive on the points (blocks). We say $\mathscr{D}$ is *flag-transitive* if G acts transitively on flag set $\mathscr{F}$, where $\mathscr{F} = \{(\alpha, B) \mid \alpha \in \mathscr{P}, B \in \mathscr{B} \text{ and } \alpha \in B\}$.

For any subset $B \subseteq \mathscr{P}$, G_B denotes the *setwise stabilizer* subgroup of B in G. In particular, if $B = \{\alpha\}$, then the stabilizer G_B coincides with the *point stabilizer* of α, denoted by G_α. The *rank* of G (denoted $\mathrm{rank}(G)$) equals the number of G_α-orbits on $\mathscr{P}$. The size of each orbit is called a *subdegree*, labeled as d [1, 2].

Transitivity plays a fundamental role in characterizing the homogeneity properties of incidence structures $\mathscr{D} = (\mathscr{P}, \mathscr{B})$. For instance, flag-transitivity has constituted a major research focus in design theory since the 1960s. The systematic investigation of flag-transitive 2-designs dates back to 1961, and Higman and McLaughlin [3] showed that

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for any flag-transitive Steiner 2-design $\mathscr{D}$, $G \le \mathrm{Aut}(\mathscr{D})$ acts primitively on $\mathscr{P}$. Applying the O'Nan-Scott theorem-which provides a complete classification of finite primitive permutation groups, Buekenhout [4] demonstrated that any such automorphism group G ought to be almost simple type or affine type, thereby laying a foundation for subsequent classification. In 2001, Huber [5] completely classified all Steiner 3-designs $\mathscr{D}$ with block size $k \le 7$ whose automorphism group G acts transitively on flags. By applying the classification theorem of 2-transitive permutation groups, Huber [6] completely classified all flag-transitive Steiner 3-designs in 2005. It can be seen that research on flag-transitive 3-designs has progressed rapidly in recent decades. In contrast, block-transitive 3-designs have been less extensively studied compared to their flag-transitive counterparts.

In their seminal 2001 work [7], Mann and Tuan established a fundamental limitation for 3-designs: any block-transitive, point-imprimitive design must satisfy the strict inequality $v \le \frac{k(k-1)}{2} + 1$. This foundational result was later extended by Tuan in 2003 [8], who demonstrated that for certain specific values of k, the number of 3-designs (G is point-imprimitive and block-transitive) adhering to this upper bound is finite. After that, in 2022, Zhan et al. [9] proved that if $\mathscr{D}$ is a 3-$(v,\ k,\ \lambda)$ design satisfying $k \le 6$ and $G \le \mathrm{Aut}(\mathscr{D})$ acts as both a point-primitive and block-transitive group the design $\mathscr{D}$, then G belongs to either the almost simple or the affine type. Subsequently, Lan et al. [10] gave some results about block-transitive Steiner 3-designs where $\mathrm{Soc}(G) \cong A_n$ (alternating group). In the same year, Zhan and Pang [11] subsequently classified block-transitive 3-$(v,\ 4,\ \lambda)$ design with alternating and sporadic socle. Recently, Gong [12] classified the block-transitive and point-primitive automorphism groups of 3-$(v,\ k,\ 2)$ designs, showing that they must be of affine or almost simple type. This result establishes a foundation for further investigation.

According to current results [9], a block-transitive automorphism group G of a 3-$(v,\ 5,\ \lambda)$ design $\mathscr{D}$ necessarily acts point-primitively on $\mathscr{D}$. Also, it must be of either almost simple or affine type. For this conclusion, in our paper, we would like to make a classification about the pairs $(\mathscr{D},\ G)$, where $\mathscr{D}$ is a 3-$(v,\ 5,\ \lambda)$ design and G is a block-transitive automorphism group with sporadic or alternating socle.

Now, we show our results in the following Theorem 1 comprehensively.

Theorem 1 Let $\mathscr{D} = (\mathscr{P},\ \mathscr{B})$ be a nontrivial 3-$(v,\ 5,\ \lambda)$ design. Suppose that $G \le \mathrm{Aut}(\mathscr{D})$ acts transitively on $\mathscr{B}$ and $\mathrm{Soc}(G)$ is either a sporadic simple group or an alternating group, then one of the following occurs:

(A) Sporadic socle:

(i) If $G \cong M_{11}$ then $\mathscr{D}$ is the unique 3-$(11,\ 5,\ \lambda)$ design with $\lambda \in \{4,\ 24\}$ or the unique 3-$(12,\ 5,\ \lambda)$ design with $\lambda \in \{6,\ 30\}$;

(ii) If $G \cong M_{12}$ then $\mathscr{D}$ is the unique 3-$(12,\ 5,\ 36)^*$ design;

(iii) If $G \cong M_{22}$ then $\mathscr{D}$ is the unique 3-$(22,\ 5,\ \lambda)$ design with $\lambda \in \{3,\ 24,\ 120\}$;

(iv) If $G \cong M_{22} : 2$ then $\mathscr{D}$ is the unique 3-$(22,\ 5,\ \lambda)$ design with $\lambda \in \{3,\ 48,\ 120\}$;

(v) If $G \cong M_{23}$ then $\mathscr{D}$ is the unique 3-$(23,\ 5,\ \lambda)$ design with $\lambda \in \{30,\ 160\}$;

(vi) If $G \cong M_{24}$ then $\mathscr{D}$ is the unique 3-$(24,\ 5,\ 210)^*$ design.

(B) Alternating socle:

(i) If $G \cong A_n$ or S_n $(n \ge 7)$ then $\mathscr{D}$ is the unique complete 3-$\left(n,\ 5,\ \frac{(n-3)(n-4)}{2}\right)^*$ design;

(ii) If $G \cong A_6$ then $\mathscr{D}$ is the unique 3-$(10,\ 5,\ 3)$ design;

(iii) If $G \cong S_6$ then $\mathscr{D}$ is the unique 3-$(10,\ 5,\ 6)$ design;

(iv) If $G \cong M_{10}$ then $\mathscr{D}$ is the unique 3-$(10,\ 5,\ \lambda)$ design with $\lambda \in \{3,\ 15\}$;

(v) If $G \cong PGL_2(9)$ or $P\Gamma L_2(9)$ then $\mathscr{D}$ is the unique 3-$(10,\ 5,\ \lambda)$ design with $\lambda \in \{6,\ 15\}$.

Remark 1

(1) All 3-$(v,\ 5,\ \lambda)^*$ designs listed in this paper mean that $\mathscr{D}$ are complete designs;

(2) The designs with the same parameters mentioned above are all isomorphic.

Next, some useful and important results will be collected in Section 2. In Section 3, we will concentrate mainly on automorphism groups G is a sporadic group. In Section 4, then we will investigate the case that G is an alternating group A_n or a symmetric group S_n.

2. Preliminaries

We begin by stating a fundamental hypothesis that will underpin our subsequent analysis.

Hypothesis $\mathcal{H}$: Let $\mathscr{D}$ be a nontrivial 3-$(v,\ 5,\ \lambda)$ design, and let $G \leq \mathrm{Aut}(\mathscr{D})$ act transitively on $\mathscr{B}$.

As preparatory material, we present fundamental results from design and group theories as prerequisite knowledge, while also supplying formal proofs for two specific theoretical cases.

Lemma 1 [13, Corollary 1.4] For a 3-$(v,\ k,\ \lambda)$ design $\mathscr{D}$, all parameters satisfy:

(i) $vr = bk$;

(ii) $\lambda \binom{v}{3} = \binom{k}{3} b$;

(iii) $\lambda \binom{v-1}{2} = \binom{k-1}{2} r$;

(iv) $r \geq k$ and $b \geq v$.

Lemma 2 [9, Lemma 2.2] For a 3-$(v,\ k,\ \lambda)$ design $\mathscr{D}$ admits a block-transitive automorphism group G:

(i) $r \mid k \cdot |G_\alpha|$;

(ii) For nontrivial subdegree d of G, $(v-1)(v-2)$ is a divisor of $k(k-1)(k-2)\binom{d}{2}$.

Lemma 3 Let B be a 5-subset of v point set $\mathscr{P}$. Assume that G is a 3-homogeneous group on $\mathscr{P}$, then the pair $(\mathscr{P},\ B^G)$ forms a block-transitive 3-$(v,\ 5,\ \lambda)$ design with $\lambda = \dfrac{60|B^G|}{v(v-1)(v-2)}$. Moreover, if G is 5-homogeneous, then $(\mathscr{P},\ B^G)$ is the 3-$\left(v,\ 5,\ \dfrac{(v-3)(v-4)}{2}\right)$ design.

Proof. The first assertion can be obtained from [14, Theorem1.9]. For the 5-homogeneous case, $\mathscr{D}$ is complete with $b = \binom{v}{5}$ as $k = 5$. Lemmas 1 and 2 yield:

$$\lambda = \frac{b\binom{k}{3}}{\binom{v}{3}} = \binom{v}{5} \cdot \frac{\binom{k}{3}}{\binom{v}{3}} = \frac{(v-3)(v-4)}{2}. \tag{1}$$

□

Lemma 4 Suppose that G acts block-transitively on a 3-$(v,\ k,\ \lambda)$ design $\mathscr{D}$. If $k = 5$ and G is an almost simple group, then $\mathrm{rank}(G) \leq 6$.

Proof. Set $\mathrm{rank}(G) = n+1$, since the socle of G is a non-abelian simple group, there exists a subdegree d of G satisfy $1 < d \leq \dfrac{v-1}{n}$. Then $30d(d-1) \geq (v-1)(v-2)$ by Lemma 2 (ii). Hence, we have $30d(d-1) \geq nd(nd-1)$. Because of $d > 1$,

$$n \geq (n^2 - 30)d \geq 2(n^2 - 30). \tag{2}$$

Eventually, we obtain the upper bound $n \leq 5$ and the rank of G is at most 6. □

3. Sporadic groups case

Here assume that $\mathscr{D}$ is a 3-$(v,\ 5,\ \lambda)$ design admitting a block-transitive automorphism group G. Let $\mathrm{Soc}(G)$ be a sporadic simple group. Relying on Lemma 4, we are able to gain that the rank of G is at most 6. With the help of this result, we find the 95 candidates for G are as in Table 1 from [15, Theorem 1.1] at first. Secondly, we eliminate unsuitable cases through two straightforward steps. Finally, we will construct designs for the remaining cases by using Lemma 3.

Table 1. Primitive sporadic groups with rank at most 6

Case	G	v	rank	TD	Case	G	v	rank	TD
1	M_{11}	11	2	4	49	$J_2:2$	100	3	
2		12	2	3	50		280	4	
3		55	3		51		315	5	
4		66	4		52		525	6	
5	M_{12}	12	2	5	53	Suz	1,782	3	
6		66	3		54		22,880	5	
7		144	5		55		32,760	6	
8		220	5		56	$Suz:2$	1,782	3	
9	$M_{12}:2$	144	4		57		22,880	5	
10	M_{22}	22	2	3	58		32,760	6	
11		77	3		59	HS	100	3	
12		176	3		60		176	2	2
13		231	4		61		1,100	5	
14		330	5		62	$HS:2$	100	3	
15		616	5		63		1,100	5	
16		672	6		64	McL	275	3	
17	$M_{22}:2$	22	2	3	65		2,025	4	
18		77	3		66		7,128	5	
19		231	4		67		15,400	5	
20		330	5		68		22,275	6	
21		616	5		69	$McL:2$	275	3	
22		672	6		70		7,128	5	
23	M_{23}	23	2	4	71		15,400	5	
24		253	3		72		22,275	6	
25		506	4		73	He	2,058	5	
26		1,288	4		74	$He:2$	2,058	4	
27	M_{24}	24	2	5	75		8,330	6	
28		276	3		76	$O'N$	122,760	5	
29		759	4		77	Ru	4,060	3	
30		1,288	3		78	Fi_{22}	3,510	3	
31		1,771	4		79		14,080	3	
32		2,024	5		80		61,776	4	
33		3,795	5		81	$Fi_{22}:2$	3,510	3	
34	Co_1	98,280	4		82		61,776	4	
35		1,545,600	5		83	Fi_{23}	31,671	3	
36		8,292,375	6		84		137,632	3	
37	Co_2	2,300	3		85	Fi'_{24}	306,936	3	
38		46,575	5		86	$Fi'_{24}:2$	306,936	3	
39		47,104	6		87	Ly	8,835,156	5	
40		56,925	5		88		9,606,125	5	
41	Co_3	276	2	2	89	${}^2F_4(2)'$	1,600	4	
42		11,178	5		90		1,755	5	
43		128,800	6		91		2,925	6	
44	J_1	266	5		92	${}^2F_4(2)$	1,755	5	
45	J_2	100	3		93		2,304	6	
46		280	4		94		2,925	5	
47		315	6		95	B	13,571,955,000	5	
48		525	6						

Remark The abbreviation TD appearing in Table 1 represents the transitivity degree for each automorphism group under consideration.

Proposition 1 Under HYPOTHESIS $\mathcal{H}$, if G has a sporadic socle then $\mathrm{rank}(G) = 2$.

Proof. It follows from Lemma 2 (ii) that $(v-1)(v-2)$ is one divisor of $30d(d-1)$. Simple calculations show this: 86 potential cases that the rank unequal to 2 are systematically eliminated. Considering the remaining 9 cases, we can clearly konw that G must be 2-transitive. □

Corollary 1 Under HYPOTHESIS $\mathcal{H}$, if G has a sporadic socle. Then G acts as a 3-transitive group on $\mathscr{P}$.

Proof. By arithmetic constraints imposed by block-transitivity and nontriviality of $\mathscr{D}$, the tuple $(b,\ v,\ r,\ \lambda)$ must fulfill the conditions as follows:

(i) $v \geq 7$ and $b \mid |G|$ (see Lemma 2(i));

(ii) $b = \dfrac{vr}{5}$ is an integer and $b \geq v$ (Lemma 1(i)(iv));

(iii) $r \mid 5|G_\alpha|$;

(iv) $\lambda = \dfrac{12r}{(v-1)(v-2)} \in \mathbb{Z}$ (refer to Lemma 1(iii)).

Therefore, all possible parameter sets for a given group G can be obtained using the GAP system [16]. However, performing the aforementioned calculation, we found that Co_3 and HS fail to yield appropriate parameters to satisfy the conditions (i)-(iv). In this process, by Atlas [17], we spotted that the remaining 7 cases are at least 3-transitive. □

We analyze the 7 cases satisfying the two-stage criteria. For each admissible group G, the following proposition holds:

Proposition 2 If $G \cong M_{12}$ or M_{24}, then $\mathscr{D}$ is a complete 3-$(12,\ 5,\ 36)$ design or a complete 3-$(24,\ 5,\ 210)$ design, respectively.

Proof. It is clear that G is 5-transitive, therefore, this conclusion can be gained immediately from Lemma 3. □

Next, we examine the remaining 5 cases where the groups G are not 5-transitive.

Proposition 3 If the transitivity degree of G less than 5, then the design $\mathscr{D}$ must be one of the designs listed in Table 2.

Proof. Here we only consider the case where $G \cong M_{11}$, as the others can be proved similarly. The Mathieu group M_{11} acting on $\mathscr{P} = \{1,\ 2,\ \ldots,\ 11\}$ has primitive permutation representations with generators:

$$g_1 = (1,\ 10)(2,\ 8)(3,\ 11)(5,\ 7);\ g_2 = (1,\ 4,\ 7,\ 6)(2,\ 11,\ 10,\ 9). \tag{3}$$

Through Magma computations [18], we determine that the group action of G on the collection of all 5-subsets of $\mathscr{P}$ produces exactly two orbits, which we designate as O_1 (of size 66) and O_2 (of size 396). Then by Lemma 3 and the 3-transitivity of G, $\mathscr{D}_i = (\mathscr{P},\ O_i)$ is a block-transitive 3-$(11,\ 5,\ \lambda)$ design, where $i = 1,\ 2$. Moreover, making use of Lemma 1, we would have $\lambda \in \{4,\ 24\}$. □

Since the discussion of the remaining 4 cases is completely identical, we present only the results in Table 2 for brevity.

Table 2. All 3-$(v,\ 5,\ \lambda)$ designs with spodic socle

Number	G	b	$\mathscr{D}$	Number	G	b	$\mathscr{D}$
1	M_{11}	66	3-(11, 5, 4)	7		18,480	3-(22, 5, 120)
2		396	3-(11, 5, 24)	8	$M_{22}:2$	462	3-(22, 5, 3)
3		132	3-(12, 5, 6)	9		7,392	3-(22, 5, 48)
4		660	3-(12, 5, 30)	10		18,480	3-(22, 5, 120)
5	M_{22}	462	3-(22, 5, 3)	11	M_{23}	5,313	3-(23, 5, 30)
6		3,696	3-(22, 5, 24)	12		28,336	3-(23, 5, 160)

Theorem 1 (**A**) is proved by applying Propositions 1-3 together. □

4. Alternating groups case

In the subsequent step, we start to prove Theorem 1 (**B**) by exploring alternating socle structures. In the first place, assume that $(\mathscr{D}, G)$ satisfy Hypothesis $\mathcal{H}$ and $\mathrm{Soc}(G)$ is an alternating group A_n with $n \geq 5$. Then by Lemma 4, we need to consider the groups with rank at most 6 which are listed in Table 3 (also see [15, Theorem 1.2]). In the following, we deal with the groups of given point in Cases 1-19 one by one. At last, we will provide a detailed discussion of four special groups.

Table 3. Primitive alternating and symmetric groups with rank at most 6

Case	G	$\mathrm{rank}(G)$	Subdegrees	Case	G	$\mathrm{rank}(G)$	Subdegrees
1	A_5	2	1, 5	13	$P\Gamma L_2(9)$	5	1, 4, 8, 16, 16
2	S_5	2	1, 5	14	A_7	2	1, 14
3	A_6	2	1, 5	15	A_8	2	1, 14
4	S_6	2	1, 5	16	S_8	5	1, 14, 21, 28, 56
5	M_{10}	2	1, 9	17	A_9	3	1, 56, 63
6	M_{10}	4	1, 5, 10, 20	18	A_{11}	5	1, 110, 330, 495, 1,584
7	M_{10}	5	1, 4, 8, 16, 16	19	A_{12}	4	1, 440, 495, 1,584
8	$PGL_2(9)$	2	1, 9	20	A_n	$m+1$	
9	$PGL_2(9)$	5	1, 5, 10, 10, 10	21	S_n	$m+1$	
10	$PGL_2(9)$	6	1, 4, 8, 8, 8, 16	22	A_{2s}	$\left\lceil \frac{s+1}{2} \right\rceil$	
11	$P\Gamma L_2(9)$	2	1, 9	23	S_{2s}	$\left\lceil \frac{s+1}{2} \right\rceil$	
12	$P\Gamma L_2(9)$	4	1, 5, 10, 20				

Proposition 4 Under Hypothesis $\mathcal{H}$, if $\mathrm{Soc}(G) \cong A_n$ then $\mathrm{rank}(G) = 2$.

Proof. It is apparent that $(v-1)(v-2)$ is not a divisor of $30d(d-1)$, we can exclude 10 cases by a uncomplicated calculation. Continuously, we noticed the remaining 9 cases are all equal to 2, it implies that G must be 2-transitive. □

Proposition 5 Cases 1, 2, 3, 4, 14, 15 of Table 3 are impossible.

Proof. As we all know, the nontriviality conditions in our designs which suggest the $v > 6$. What is more, for $v = 15$ and $G \cong A_7$ or A_8, we detect that there are no parameter sets meet the conditions listed in Lemmas 1 and 2. □

Proposition 6 If $G \cong M_{10}$, then $\mathscr{D}$ is the 3-$(10,\ 5,\ \lambda)$ design with $\lambda \in \{3,\ 15\}$.

Proof. For $G \cong M_{10}$, its action partitions the collection of all 5-element subsets of a 10-point set into two orbits of lengths 30 and 180. There exist two block-transitive 3-designs admitting G as a 3-transitive automorphism group. With the aid of Lemma 1, they are the 3-$(10,\ 5,\ 3)$ design and the 3-$(10,\ 5,\ 15)$ design respectively. □

Proposition 7 Suppose that $G \cong PGL_2(9)$ or $P\Gamma L_2 9$, then $\mathscr{D}$ is the 3-$(10,\ 5,\ \lambda)$ design with $\lambda \in \{6,\ 15\}$.

Proof. The 3-transitive action of G on $\mathscr{P}$ induces a partition of the 5-subsets of $\mathscr{P}$ into two orbits. Then we can get that the sizes of Orbit O_1 and Orbit O_2 are 72 and 180 respectively. Moreover, they are adhere to Lemma 1, so we can acquire two designs: a 3-$(10,\ 5,\ 6)$ design and a 3-$(10,\ 5,\ 15)$ design. □

We now examine groups whose socle is the alternating group A_n of degree $\binom{n}{m}$, establishing the following structural result:

Proposition 8 Let G be isomorphic to either the alternating group A_n or the symmetric group S_n. If $v = \binom{n}{m}$, then $m = 1$ and $\mathscr{D}$ is the 3-$\left(v,\ 5,\ \frac{(v-3)(v-4)}{2}\right)$ design with $n \geq 7$.

Proof. Note that the rank of G is $m+1$, and all subdegrees of G are (see [19, Proposition 15]):

$$n_0 = 1,\ n_{i+1} = \binom{n-m}{m-i} \cdot \binom{m}{i},\ i = 0,\ 1,\ \ldots,\ m-1. \tag{4}$$

Applying Lemma 4, we immediately obtain $m \le 5$.

Case 1: A_n or S_n acts $n-2$-transitively on $\mathscr{P}$ when $m = 1$. Lemma 3 yields that $\mathscr{D}$ is the complete 3-design with degree n, and owing to nontrivial feature, $n \ge 7$.

Case 2: If $2 \le m \le 5$ and $v = \binom{n}{m}$. Through elementary computations, we observe that the subdegrees of G contradict Lemma 2 (ii). This contradiction necessarily implies the non-existence of any block-transitive 3-design design with block size 5. ☐

Proposition 9 Let G be isomorphic to either the alternating group A_{2s} or the symmetric group S_{2s}. If $v = \frac{(2s)!}{2s!^2}$, then $s = 3$ and $\mathscr{D}$ is isomorphic to the 3-$(10,\ 5,\ 3)$ or the 3-$(10,\ 5,\ 6)$ design.

Proof. Here G has rank $\left\lceil \frac{s+1}{2} \right\rceil$. Using conditions Lemma 4 and $2s \ge 5$, we can compute $3 \le s \le 11$.

Table 4. Subdegree of A_{2s} or S_{2s} with $4 \le s \le 11$

s	Subdegrees	s	Subdegrees
$s = 4$	1, 16, 18	$s = 8$	1, 64, 784, 2,450, 3,136
$s = 5$	1, 25, 100	$s = 9$	1, 81, 1,296, 7,056, 15,876
$s = 6$	1, 36, 200, 225	$s = 10$	1, 100, 2,025, 14,400, 31,752, 44,100
$s = 7$	1, 49, 441, 1,225	$s = 11$	1, 121, 3,025, 27,225, 108,900, 213,444

If $s = 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11$, then $G \cong A_{2s}$ or S_{2s} acts on $\frac{(2s)!}{2s!^2}$ points. Checking all subdegrees d listed in Table 4, we find that $(v-1)(v-2)$ is not a divisor of $30d(d-1)$. This contradicts Lemma 2 (ii). ☐

For the remaining case $s = 3$. Here $\mathscr{P}$=$\{1,\ 2,\ \ldots,\ 10\}$ and $G \cong A_6$ with two generators:

$$g_1 = (1,\ 2,\ 3)(4,\ 5,\ 6)(7,\ 8,\ 9);\ g_2 = (2,\ 4,\ 3,\ 7)(5,\ 6,\ 9,\ 8);\ g_3 = (1,\ 10)(4,\ 7)(5,\ 6)(8,\ 9). \tag{5}$$

By using Magma, there exists four G-orbits on all 5-element subsets of $\mathscr{P}$, namely O_i where $i \in \{1,\ 2,\ 3,\ 4\}$ and $|O_1| = |O_2| = 36$ and $|O_3| = |O_4| = 90$. Note that G is a 2-transitive but not 3-transitive group on $\mathscr{P}$. Also by using commands `Design<3, 10|O>` and `IsIsomorphic(D1, D2)`, we found that both $\mathscr{D}_1 = (\mathscr{P},\ O_1)$ and $\mathscr{D}_2 = (\mathscr{P},\ O_2)$ are block-transitive 3-$(10,\ 5,\ 3)$ designs, and $\mathscr{D}_1 \cong \mathscr{D}_2$.

For $G \cong S_6$, the same construction yields the unique (up to isomorphism) 3-$(10,\ 5,\ 6)$ design.

The proof of Theorem 1 (**B**) is completed in Propositions 4-9.

Lastly, during the process of the aboving proof, we have discovered a crucial result, which will have significant influence for futher rerserch.

Corollary 2 Under Hypothesis $\mathcal{H}$, if $\mathrm{Soc}(G)$ is either an alternating group or a sporadic simple group, then G is a 2-transitive group on $\mathscr{P}$.

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Conflict of interest

The authors affirm there are no financial or non-financial competing interests associated with this research.

References

[1] Colbourn C, Dinitz JH. *Handbook of Combinatorial Designs*. 2nd ed. UK: Chapman & Hall/CRC; 2007.

[2] Wielandt H. *Finite Permutation Groups*. New York: Academic press; 1964.

[3] Higman D, McLaughlin J. Geometric ABA-groups. *Illinois Journal of Mathematics*. 1961; 5(3): 382-397.

[4] Buekenhout F, Delandtsheer A, Doyen J. Finite linear spaces with flag-transitive groups. *Journal of Combinatorial Theory, Series A*. 1988; 49(2): 268-293. Available from: https://doi.org/10.1016/0097-3165(88)90056-8.

[5] Huber M. *Klassifikationen Fahnentransitiver Steiner Designs*. Germany: University of Tübingen; 2001.

[6] Huber M. The classification of flag-transitive steiner 3-designs. In: *Flag-transitive Steiner Designs*. Switzerland: Birkhäuser Basel; 2009. Available from: https://doi.org/10.1007/978-3-0346-0002-6_7.

[7] Mann A, Tuan ND. Block-transitive point-imprimitive t-designs. *Geometriae Dedicata*. 2001; 88(1): 81-90. Available from: https://doi.org/10.1023/A:1013184931427.

[8] Tuan ND. A note on block-transitive point-imprimitive designs. *European Journal of Combinatorics*. 2003; 24(1): 113-119. Available from: https://doi.org/10.1016/S0195-6698(02)00118-X.

[9] Zhan X, Pang X, Wang Y. Block-transitive 3-designs with block size at most 6. *Graphs and Combinatorics*. 2022; 38(5): 1-14. Available from: https://doi.org/10.1007/s00373-022-02544-5.

[10] Lan T, Liu W, Yin FG. Block-transitive 3-$(v, k, 1)$ designs associated with alternating groups. *Designs, Codes and Cryptography*. 2023; 91(8): 2791-2807. Available from: https://doi.org/10.1007/s10623-023-01215-7.

[11] Pang X, Zhan X. Block-transitive 3-$(v, 4, \lambda)$ designs with sporadic or alternating socle. *Designs, Codes and Cryptography*. 2023; 91(12): 3825-3835. Available from: https://doi.org/10.1007/s10623-023-01275-9.

[12] Gong L, Liu W, Dai S. The reduction of block-transitive 3-$(v, k, 2)$ designs. *Applied Mathematics and Computation*. 2025; 500: 129435. Available from: https://doi.org/10.1016/j.amc.2025.129435.

[13] Hughes DR, Piper FC. *Design Theory*. Cambridge: Cambridge University Press; 1985.

[14] Cusack CA, Magliveras SS. Semiregular large sets. *Designs, Codes and Cryptography*. 1999; 18(1): 81-87. Available from: https://doi.org/10.1023/A:1008381018661.

[15] Muzychuk M, Spiga P. Finite primitive groups of small rank: Symmetric and sporadic groups. *Journal of Algebraic Combinatorics*. 2020; 52(2): 103-136. Available from: https://doi.org/10.1007/s10801-019-00896-5.

[16] The GAP Group. *GAP-groups, algorithms, and programming, version 4.8.6*. Available from: http://www.gap-system.org [Accessed 9th September 2023].

[17] Conway J, Curtis R, Norton S, Parker R, Wilson R. *Atlas of Finite Groups*. Eynsham: Oxford University Press; 1985.

[18] Bosma W, Cannon J, Playoust C. The Magma algebra system I: The user language. *Journal of Symbolic Computation*. 1997; 24(3-4): 235-265. Available from: https://doi.org/10.1006/jsco.1996.0125.

[19] Zhou S, Wang Y. Flag-transitive non-symmetric 2-designs with $(r, \lambda) = 1$ and alternating socle. *The Electronic Journal of Combinatorics*. 2015; 22(2): P2.6. Available from: https://doi.org/10.37236/4664.