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Research Article

Some Results on the Soft HG-Hypergroupoids

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Abstract: Within the broader landscape of hyperstructure theory, a particular class of algebraic systems known as hypergroup-hypergroupoids emerges as a subject of significant mathematical interest. The present work is devoted to an in-depth exploration of this class, with special focus on its soft analog, namely, the Soft Hypergroup-Hypergroupoid (SHGD). The core objective of this study is to articulate and refine the conceptual and structural underpinnings of such systems, extending foundational principles derived from the well-established theories of soft hypergroups and soft hypergroupoids. By carefully developing this framework, we aim to uncover the layered interconnections, algebraic behavior, and underlying logic that govern these complex entities. Ultimately, this investigation aspires to broaden the theoretical horizons of soft algebraic structures, offering fresh perspectives and contributing meaningfully to the evolving discourse within hyperstructure-based mathematics.

***Keywords*:** groupoid, soft set, soft groupoid, soft hypergroupoid, Soft Hypergroup-Hypergroupoid (SHGD)

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1. Introduction

In the contemporary landscape, it has become increasingly apparent that numerous challenges spanning various scientific domains-such as technology, engineering, medicine, physics, chemistry, economics, and the social and environmental sciences-are often characterized by data that is both ambiguous and imprecise. This deficiency in clarity within the data we encounter significantly obstructs our capacity to effectively tackle these issues through conventional mathematical methods that depend on well-defined information. To confront the complexities associated with uncertainty, a multitude of methodologies have been suggested and rigorously explored, aiming to provide innovative solutions to these pressing problems. Among the well-known theories that address these concerns are fuzzy set theory [1] and rough set theory [2], which have gained significant attention in the literature. For those interested in a deeper exploration of these topics, additional resources can be found in the referenced works [3–7]. While these methodologies offer various advantages, they are not without their limitations. In an effort to enhance existing approaches and address the complexities inherent in uncertainty, the concept of soft sets was introduced by Molodtsov in 1999 [8]. Since that time, a plethora of research has emerged surrounding this theory, with investigations continuing to evolve at a brisk pace [9–12]. For instance, Maji et al. have explored the application of soft sets in decision-making contexts, detailing several operations associated

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with soft sets in their work [9, 10]. Furthermore, Pei et al. have examined the connections between information systems and soft sets, contributing to the growing body of knowledge in this area [12]. Additionally, Ali et al. have investigated various concepts related to soft sets, such as restricted intersection, union, and restricted difference, further enriching the discourse on this innovative theory [12]. Aktaş et al. [13] conducted a comparative analysis of rough sets and soft sets, highlighting the distinctions and similarities between these two theoretical frameworks. Furthermore, they elaborated on the concepts of soft groups, detailing their properties as they relate to soft sets, and provided proofs to support their findings. In the realm of mathematics, a plethora of research has been undertaken across various branches, including set theory, group theory, algebra, topology, and more. This extensive body of work underscores the versatility and applicability of these mathematical concepts [14–16]. Yamak et al. took a significant step by applying the theory of soft sets to a specific hyperstructure known as the hypergroupoid [17]. In their study, they introduced key notions such as soft hypergroupoids, soft subhypergroupoids, and the homomorphism of soft hypergroupoids. They also established connections between the classes of L-fuzzy hypergroupoids and soft hypergroupoids, investigating several related properties, including $\vee$-union and $\wedge$-intersection within the context of soft hypergroupoids. Numerous studies have been conducted on hyperstructures and their associated soft theories, with notable references available for further exploration [18–28]. This paper introduces the concepts and framework surrounding soft hypergroup-hypergroupoids, drawing upon relevant information regarding soft hypergroups and soft hypergroupoids. The exploration of these ideas aims to provide a comprehensive understanding of the relationships and properties inherent in soft hyperstructures. Looking ahead, several future research directions can be proposed based on this work. One potential avenue is to define soft crossed hypermodules and investigate their properties, which could significantly enhance the theory of soft structures. Another intriguing topic for future study could involve the exploration of crossed polysquares and their associated properties, providing new insights and applications within this mathematical framework. These proposed subjects hold great promise for advancing the field and encouraging a deeper exploration of soft hyperstructures.

Foundational work introduced soft hypergroupoids and their basic properties [17]. This set the stage for later developments involving topology, categories, and specialized hyperstructure types. Oguz and Davvaz formalized *soft topological hypergroupoids* within a general soft topological hyperstructure framework, clarifying continuity via the product and hyperspace topologies and supplying examples [29]. While not H-specific, this work provided the topological language later used around HG-hypergroupoids. Mousarezaei and Davvaz developed soft topological polygroups and examples [30], widening the soft-hyperstructure toolkit that HG-hypergroupoid results could draw on. Pourhaghani and Torabi studied (strongly) regular relations on soft topological hypergroups and reviewed how soft hypergroupoids fit into the landscape [31]. Their discussion consolidates notions (e.g., geometric spaces) used in later HG-oriented constructions.

Dehghanizadeh introduced soft subhypergroupoids and soft action hypergroupoids and proved that the category of *soft HG-hypergroupoids* are equivalent to the category of *soft crossed hypermodules* [32]. This categorical equivalence is a key structural milestone connecting HG-objects with crossed hypermodules in the soft setting. Dehghanizadeh reported new results on soft HG-hypergroupoids, including definitions of (normal) soft sub-HG-hypergroupoids and explicit constructions, and reiterated the equivalence with soft crossed hypermodules [33].

Beyond theoretical interest, soft HG-hypergroupoids admit potential applications in areas that require handling parameterized uncertainty with complex algebraic operations. For example, they provide tools for modeling uncertain data in computer science and information systems, where hyperoperations can capture multi-valued outcomes [17]. Their categorical equivalence with soft crossed hypermodules [32] suggests applications in homological algebra and cryptography, where secure protocols often rely on higher algebraic structures. Furthermore, the topological enrichment explored in recent works [29, 31] opens possibilities in applied topology, such as fuzzy decision-making, image processing, and network analysis under uncertainty. Current trajectories suggest: (i) further categorical equivalences (e.g., to internal groupoids in soft categories), (ii) deeper interplay with soft topology (compactness/connectedness vs. algebraic axioms), and (iii) algorithmic or decision approaches for checking HG-conditions under parameterized data.

2. Preliminaries

In this segment, we analyze a portion of basic concepts, for instance, soft set, soft polygroup, hypergroup, hypergroupoid, and HG-hypergroupoid, which are required for the continuation of this study. Some definitions and related theorems about soft sets can be found in [8, 10]. We introduce some known definitions and useful theorems here. A starting universe set and the set of parameters are denoted by U and E, respectively. The power sets of U and $A \subseteq E$ were denoted by $\mathscr{P}(U)$.

Definition 1 Let us consider a mapping F that is defined as $F : A \longrightarrow \mathscr{P}(U)$. In this context, a soft set over the universe U is represented by the ordered pair (F, A). This signifies that a parameterized collection of subsets of the universe U constitutes a soft set over U. Consequently, the elements that are approximately represented by α within the soft set (F, A). Can be expressed as $F(\alpha)$, where α belongs to the set A.

Definition 2 Let us consider two soft sets (F, A) and (G, B) defined over the universe U. If the following condition is satisfied, then (F, A) is referred to as a soft subset of (G, B), denoted by $(F, A) \subseteq (G, B)$.

1. $A \subseteq B$,
2. $F(\alpha)$ and $G(\alpha)$ are identical approximations $\forall \alpha \in A$. If $(F, A) \subseteq (G, B)$ and $(G, B) \subseteq (F, A)$. Then two soft sets (F, A) and (G, B) over U are called soft equal.

Definition 3 The support of the soft set (F, A) is defined as the set $\text{Supp}(F, A) = \{x \in A | F(x) \neq \phi\}$. If $\text{Supp}(F, A) \neq \phi$, then the soft set (F, A) is considered to be non-null.

The polygroup theorem represents a significant extension of traditional group theory, encompassing various natural generalizations. In this framework, when two elements are combined, they yield a single element within the group, yet this scenario is interpreted through the lens of polygroups. Polygroups find extensive applications across a multitude of fields, including but not limited to lattices, geometric structures, color schemes, and combinatorial mathematics. Definitions and examples of polygroups can be found in [34]. Applications of hypergroups and the extension of algebraic theory to polygroups could be found in [22, 34, 35]. A polygroup [35] is defined as a multi-valued system denoted by $\mathscr{M} =< P, \circ, e,^{-1} >$, where $e \in P$, $^{-1} : P \longrightarrow P$, $\circ : P \times P \longrightarrow \mathscr{P}^*(P)$. The following axioms hold, $\forall r, s, t$ in P:

1. $(r \circ s) \circ t = r \circ (s \circ t)$
2. $e \circ r = r \circ e = r$
3. If $r \in s \circ t$, then $s \in r \circ t^{-1}$ and $t \in s^{-1} \circ r$.

Let $\mathscr{P}^*(P)$ be the set of all the non-empty subsets of P, and $A, B \neq \emptyset \subset P$, if $x \in P$, then we have

$$A \circ B = \bigcup_{a \in A,\ b \in B} a \circ b,\ x \circ B = \{x\} \circ B \text{ and } A \circ x = A \circ \{x\}.$$

From the principal of the polygroup, one can easily conclude: $e \in r \circ r^{-1} \cap r^{-1} \circ r$, $e^{-1} = e$, and $(r^{-1})^{-1} = r$.

Example 1 Let $P = \{e, x, y\}$ be a set; then $P =< P, \circ, e,^{-1} >$ is a polygroup with the polyaction given in the following Table 1.

Table 1. Cayley table for example 1

$\circ$	e	x	y
e	e	x	y
x	x	$\{e, y\}$	$\{x, y\}$
y	y	$\{x, y\}$	$\{e, x\}$

is a polygroup.

In certain studies, particularly as referenced in [36], foundational concepts and theorems concerning soft polygroups have been introduced and explored. Let us assume that P is a polygroup, and that A is a parameter set which contains at least one element. Now, suppose there exists a binary relation R that operates over pairs formed from elements of A and P, without imposing any specific structural constraints. Based on this relation, one can define a function F, which assigns to each element $x \in A$ a subset of P, formally expressed as $F(x) = \{y \in P | (x, y) \in R\}$. In this context, the function F maps parameters to subsets of the polygroup P, and when considered together with the set A, the resulting construct (F, A) is identified as a soft set over the polygroup P. This formulation allows one to encapsulate uncertain or parameter-dependent algebraic information within a flexible mathematical framework.

Definition 4 Let us suppose that (F, A) represents a soft set defined over a base set P, where this soft set is assumed to be non-trivial, that is, it is not empty or devoid of meaningful associations. Now, focus attention on all elements x that lie within the domain where the function F is actively defined and relevant with respect to the parameter set A. If, for every such element x, the corresponding image $F(x)$ forms a substructure of P which satisfies the conditions required to be recognized as a subpolygroup and possibly even as a normal subpolygroup then the entire soft system (F, A) is referred to as a soft polygroup over the set P; in the particular case where each $F(x)$ is normal, the structure is identified as a normal soft polygroup. This framework allows us to extend classical algebraic notions into a parameterized setting that accommodates multiple substructures simultaneously within a soft-theoretic context.

Definition 5 Assume that we are given two soft polygroups, denoted by (F, A) and (G, B), which are respectively defined over two distinct polygroups, P_1 and P_2. Now consider a pair of functions (f, g), where f is a mapping from P_1 to P_2, and g maps elements from the parameter set A into the parameter set B. The pair (f, g) is described as a soft homomorphism between these two soft polygroups if the following three fundamental criteria are satisfied:

1. f behaves as a strong epimorphism, ensuring that it is surjective and structure-preserving in a robust algebraic sense;
2. g is a surjective mapping, meaning that every element of B is the image of some parameter in A;
3. $f(F(x)) = G(g(x)), \forall x \in A$.

When these conditions hold collectively, the morphism (f, g) not only aligns the algebraic structures of the base polygroups but also preserves the soft set mappings in a coherent, parameter-respecting manner. Such mappings serve as an essential tool for transferring and comparing soft algebraic structures across different polygroup environments.

Definition 6 Let us consider two soft sets, (F, A) and (G, B), constructed over some algebraic structures such as polygroups. A soft homomorphism, expressed through a pair of functions (f, g), is said to exist between these two soft sets precisely when (F, A) is considered soft homomorphic to (G, B). This soft structural compatibility is symbolically represented as $(F, A) \sim (G, B)$. Furthermore, when the function f acts as a strong isomorphism, meaning it is both structure-preserving and bijective at the level of the underlying algebraic sets and the associated mapping $g : A \longrightarrow B$ is itself a bijection between parameter sets, the pair (f, g) is then elevated in designation and referred to as a soft isomorphism. Under such circumstances, where both the algebraic mappings and the parameter transformations are fully invertible and preserve all relevant structure, the soft sets (F, A) and (G, B) are said to be soft isomorphic. This relationship is denoted by the symbolic notation: $(F, A) \simeq (G, B)$. This form of equivalence captures not only the preservation of structure at each parameter level, but also establishes a one-to-one correspondence between the entire soft frameworks, highlighting their complete structural symmetry in the soft algebraic context.

3. Soft hypergroupoids

In this segment, we broaden the notion of soft groupoids by introducing the concept of soft hypergroupoids. We will explore various properties and present several findings related to this new framework.

Definition 7 Let us consider a set H that is not empty, and define a binary-like relation $*$, which assigns to every ordered pair of elements from H a non-empty subset of H itself. More precisely, this operation is a function $* : H \times H \to \mathscr{P}^*(H)$, where $\mathscr{P}^*(H)$ designates the family of all non-empty subsets of the set H. The resulting algebraic structure, represented by the pair $(H, *)$, is referred to in the literature as a hypergroupoid, an object that generalizes traditional binary operations by allowing a pair of elements to correspond not to a single outcome, but to a whole collection of possible values.

Remark 1 Let $A,\ B \neq \emptyset \subset H,\ x \in H$ then

$$A * B = \bigcup_{\substack{a \in A \\ b \in B}} a * b, \quad A * x = A * \{x\}, \quad x * B = \{x\} * B.$$

This defines the extension $*$ to subsets.

Remark 2 Let $(H,\ *)$ be a hypergroupoid, and $K \neq \emptyset \subset H$. If $(K,\ *)$ is a hypergroupoid then $(K,\ *)$ is a subhypergroupoid. Also, if $\{(H_i,\ *_i)\ \mid\ i \in \Omega\}$ is a family of hypergroupoids, then $\left(\prod_{i \in \Omega} H_i,\ *\right)$ by the hyperoperation

$$(h_i) * (h_i') = \{(k_i)\ \mid\ k_i \in h_i *_i h_i',\ i \in \Omega\}$$

is a hypergroupoid. Let $(H,\ *)$, $(H',\ *')$ be two hypergroupoids, then $\forall h_1,\ h_2 \in H$ the map $f : H \to H'$ which is a good homomorphism, when $f(h_1 * h_2) = f(h_1) *' f(h_2)$, and is an inclusion homomorphism when

$$f(h_1 * h_2) \subseteq f(h_1) *' f(h_2).$$

Definition 8 Consider a mathematical structure denoted by H, which is equipped with a hyperoperation and thus forms what is known as a hypergroupoid. Let us turn our attention to the collection $\mathscr{P}(H)$, which consists of all subsets of H that themselves possess the properties required to be subhypergroupoids-namely, subsets that remain closed under the original hyperoperation of H. Now, suppose we are given a non-empty index set A, whose elements serve as abstract parameters. We then define a pair $(F,\ A)$, in which F is a function assigning to each element $\alpha \in A$ a corresponding subset $F(\alpha)$ drawn from $\mathscr{P}(H)$. If it holds true that each of these associated subsets $F(\alpha)$ is in fact a substructure of the original hypergroupoid-that is, a subhypergroupoid in its own right-then we describe the pair $(F,\ A)$ as constituting a soft hypergroupoid over H. In this context, the function $F : A \to \mathscr{P}(H)$ serves as a soft mapping that links parameters to structured subsets, preserving algebraic coherence within the broader hypergroupoid framework.

Remark 3 A soft hypergroupoid H, is a parameterized family of subhypergroupoids.

Example 2 Let $H = \{x,\ y,\ z\}$ be a set, and let "$*$" be a hyperoperation on H defined as Table 2:

Table 2. Cayley table for example 2

$*$	x	y	z
x	$\{x,\ y,\ z\}$	$\{x,\ y\}$	$\{x,\ z\}$
y	$\{x,\ y\}$	$\{x,\ y,\ z\}$	$\{y,\ z\}$
z	$\{x,\ z\}$	$\{y,\ z\}$	$\{x,\ y,\ z\}$

then $(H,\ *)$ is a hypergroupoid. If we define $F : \mathbb{Z} \to \mathscr{P}^*(H)$ by

$$F(t) = \begin{cases} x,\ y, & 2 \mid t \\ x,\ y,\ z, & \text{otherwise} \end{cases}$$

then (F, A) is a soft hypergroupoid over H.

Remark 4 If $F(\alpha)$ is transitive as a hypergroupoid, $\forall \alpha \in A$, then the soft hypergroupoid H is called transitive.

Remark 5 If $F(\alpha)$ is totally intransitive as a hypergroupoid, $\forall \alpha \in A$, then the soft hypergroupoid H is called intransitive.

Proposition 1 Every soft set could be considered as a soft hypergroupoid.

Proof. Let (F, A) be a soft set over a universe X, i.e. $F : A \to \mathscr{P}(X)$ assigns to each parameter $\alpha \in A$ a subset $F(\alpha) \subseteq X$. We construct a hypergroupoid H whose underlying set of objects is X, i.e. $\mathrm{Ob}(H) = X$, and we define a hyperoperation on X by

$$x * y = \begin{cases} \{\mathrm{Id}_x\}, & x = y, \\ & \\ \{x\}, & x \neq y, \end{cases} \qquad \forall x, y \in X.$$

Here, Id_x denotes the identity morphism associated with the object x. This definition ensures that H is a hypergroupoid: each element of X has an identity, and the hyperoperation is well defined on pairs of elements of X, producing sets of morphisms. The case $x = y$ guarantees the presence of identity morphisms, while the case $x \neq y$ provides a trivial (singleton) output sufficient to satisfy the closure requirements of a hyperoperation. Now, since (F, A) is a soft set over X, for each parameter $\alpha \in A$, we have $F(\alpha) \subseteq X$. Consider the substructure of H determined by $F(\alpha)$: namely, the set of objects $F(\alpha)$ together with their corresponding identity morphisms. By construction, this is a subhypergroupoid of H, because the hyperoperation is restricted to $F(\alpha)$ and produces morphisms lying entirely within the identities of $F(\alpha)$. Consequently, each fiber $F(\alpha)$ determines a subhypergroupoid of H, and the pair (F, A), together with this interpretation, constitutes a *soft hypergroupoid* over H. Therefore, every soft set can naturally be considered as a soft hypergroupoid. □

Proposition 2 Every soft hypergroup (G, F, A), can be considered as a soft hypergroupoid.

Proof. Let (G, F, A) be a soft hypergroup, where G is a hypergroup, A is a set of parameters, and $F : A \to \mathscr{P}(G)$ is a mapping such that $F(\alpha)$ is a subhypergroup of G for every $\alpha \in A$. First, recall that in hyperstructure theory a *hypergroup* can be identified with a *hypergroupoid with a single distinguished object*. Concretely, a hypergroup $(G, \circ)$ admits an interpretation as a category with exactly one object, say $*$, where the set of morphisms $\mathrm{Hom}(*, *)$ is precisely G and composition of morphisms corresponds to the hyperoperation $\circ$. In this sense, G is naturally a hypergroupoid. Similarly, each $F(\alpha)$, being a subhypergroup of G, inherits the hyperoperation and satisfies the axioms of a hypergroup. Thus, in the categorical interpretation, $F(\alpha)$ becomes a subhypergroupoid of G, again with a single object, and with morphism set given by $F(\alpha)$. Since for every parameter $\alpha \in A$ the fiber $F(\alpha)$ is a subhypergroupoid of G, we may reinterpret the triple (G, F, A) as (H, F, A) with $H = G$, now seen in the category of hypergroupoids. By definition, this is exactly a *soft hypergroupoid* structure. Therefore, every soft hypergroup admits a natural and consistent embedding into the framework of soft hypergroupoids. □

4. Soft action hypergroupoids

In this part of the discussion, we will delve deeply into the concept of soft action hypergroupoids. Our focus will be on examining their characteristics and properties in detail. Additionally, we will investigate various findings and outcomes related to these hypergroupoids.

Suppose (H, F, A) is a soft hypergroup and (X, F', A) is a soft set. In this case, we will have the following mappings:

$$F : A \longrightarrow \mathscr{P}(H) \qquad\qquad F' : A \longrightarrow \mathscr{P}(X)$$

and

$$\alpha \longmapsto F(\alpha) \qquad\qquad \alpha \longmapsto F'(\alpha)$$

Hence, we can define the following.

$$F'' : A \longrightarrow \mathscr{P}(H \times X)$$

$$\alpha \longmapsto F''(\alpha) = F(\alpha) \times F'(\alpha)$$

where is the hypergroup H acts on the set X as follows:

$$\circ : H \times X \longrightarrow \mathscr{P}^*(X)$$

$$(h,\ x) \longmapsto h \circ x,$$

in addition, $\forall h_i \in H$ and $x_i \in X$,

$$(h_i)(x_i) = \{(y_i) \ \mid \ y_i \in h_i \circ x_i\}.$$

If $(h,\ x)$ and $(h',\ x') \in H \times X$, then we can defines:

$$(h,\ x) * (h',\ x') = \left\{(r,\ s) \ \mid \ r \in hh',\ s = x\right\}.$$

In this manner, $(H \times X,\ F'',\ A)$ is a soft hypergroupoid which is called soft action hypergroupoid.

Theorem 1 Every soft hypergroupoid is a soft category.

Proof. Let $(H,\ F,\ A)$ be a soft hypergroupoid, where H is a hypergroupoid, A is a set of parameters, and $F : A \to \mathscr{P}(H)$ is a mapping such that for each $\alpha \in A$, the set $F(\alpha)$ is a subhypergroupoid of H. That is, $F(\alpha)$ is closed under the hyperoperation of H and inherits the hypergroupoid structure. Recall that in hyperstructure theory, it has been established that every hypergroupoid naturally determines a category. More precisely, one may associate to each hypergroupoid H a category $\mathscr{C}(H)$ whose objects are the elements of H and whose morphisms are generated by the hyperoperation subject to the closure properties. This construction guarantees that any subhypergroupoid $K \subseteq H$ induces a full subcategory $\mathscr{C}(K)$ of $\mathscr{C}(H)$. Now, for each $\alpha \in A$, since $F(\alpha)$ is a subhypergroupoid of H, it follows immediately that $\mathscr{C}(F(\alpha))$ is a subcategory of $\mathscr{C}(H)$. Consequently, the triple $(H,\ F,\ A)$ can be regarded as a soft category: the underlying universe is the category $\mathscr{C}(H)$, and for each parameter α, the associated set $F(\alpha)$ determines a subcategory $\mathscr{C}(F(\alpha))$. Therefore, by uniting the soft structure $(F,\ A)$ with the categorical viewpoint of hypergroupoids, we conclude that every soft hypergroupoid is indeed a soft category. □

Definition 9 Let us take into account two soft hypergroupoid structures, denoted respectively by the pairs $(F,\ A)$ and $(F',\ A')$, where each pair is defined over its own base hypergroupoid, namely H_1 and H_2. Assume that neither $(F,\ A)$ nor

$(F',\ A')$ is empty, meaning that both involve meaningful parameter sets and corresponding substructures. Now, suppose there exists a pair of mappings $(f,\ g)$ that connects these two soft systems: one part of the pair, f, maps elements from H_1 into H_2, while the other, g, links the parameter set A to the parameter set A'. If this combined mapping respects the inherent structure of the hypergroupoids in such a way that each image under f lies within the image of its source in an inclusion-preserving manner, then this pair $(f,\ g)$ is recognized as a soft inclusion homomorphism between the two soft hypergroupoids. Specifically, the function f must itself serve as an inclusion homomorphism from H_1 into H_2, ensuring that the hyperoperation-related structure is maintained in a consistent and compatible fashion.

Remark 6 From Definition 9, we have a new classification, which is called soft hypergroupoids and is shown by Soft Hypergroup-Hypergroupoid (SHGD). The objects of this new category are soft hypergroupoids, and the hypergroupoid homomorphisms between these objects are morphisms of this new classification.

Proposition 3 Let $(f,\ g)$ be a soft functor between the soft hypergroupoid $(H,\ F,\ A)$ and the soft category $(H',\ F',\ A')$. Then $(H',\ F',\ A')$ is a soft hypergroupoid.

Proof. Functor g is full. Hence g is over the morphisms $(H',\ F',\ A')$, and the soft category has a soft hypergroupoid structure. □

Theorem 2 Let $(f,\ g)$ be a soft functor between the soft hypergroupoid $(H,\ F,\ A)$ and the soft category $(H',\ F',\ A')$. Then $(H',\ F',\ A')$ is a soft hypergroupoid.

Proof. We must show that for every parameter $\beta \in A'$ the set $F'(\beta) \subseteq H'$ is a subhypergroupoid of H', and these subhypergroupoids assemble into a soft structure over A'.

By hypothesis there exists at least one $\alpha \in A$ with $g(\alpha) = \beta$ (if g is not onto one should restrict attention to the image $g(A) \subseteq A'$; the statement is intended for parameters in the image of g). For such an α the equality

$$f\big(F(\alpha)\big) = F'(\beta)$$

holds by assumption (2). Since $(H,\ F,\ A)$ is a soft hypergroupoid, $F(\alpha) \subseteq H$ is a subhypergroupoid: it is nonempty and closed under the hyperoperation of H. Our goal is to transfer these closure properties along f to $F'(\beta)$.

Work inside the categorical realizations $\mathscr{C}(H)$ and $\mathscr{C}(H')$ associated with the hypergroupoids H and H', respectively. Because f is full, every morphism in $\mathscr{C}(H')$ between objects in the image of f has a preimage morphism in $\mathscr{C}(H)$. Concretely, let $x',\ y' \in F'(\beta)$ be objects of $\mathscr{C}(H')$. By $f(F(\alpha)) = F'(\beta)$ there exist $x,\ y \in F(\alpha)$ with $f(x) = x'$ and $f(y) = y'$. Consider any hyperproduct (or composition arising from the hyperoperation) in H' that one would need to check for closure: such a product between x' and y' corresponds, up to isomorphism, to the image under f of some product between x and y in H, because f is full and (by essential surjectivity) every relevant object or morphism in H' is represented by an image from H. Since $F(\alpha)$ is closed under the hyperoperation in H, any such product in H lies in $F(\alpha)$; applying f sends it into $f(F(\alpha)) = F'(\beta)$. Thus, the corresponding product in H' lands in $F'(\beta)$ (possibly up to canonical isomorphism coming from essential surjectivity), establishing closure of $F'(\beta)$ under the induced hyperoperation of H'.

Nonemptiness of $F'(\beta)$ follows because f is essentially surjective and $F(\alpha)$ is nonempty: pick $x \in F(\alpha)$, then $f(x) \in F'(\beta)$, so $F'(\beta) \neq \varnothing$.

Finally, the assignment $\beta \mapsto F'(\beta)$ (for β in the image of g) together with the parameter set A' provides the required soft assignment on H'. Therefore, each fiber $F'(\beta)$ is a subhypergroupoid of H', and $(H',\ F',\ A')$ inherits the structure of a soft hypergroupoid. □

Theorem 3 If $(f,\ g)$ is a soft hypergroupoid homomorphism between soft hypergroupoids $(H,\ F,\ A)$ and $(H',\ F',\ A')$, and $(H,\ \star)$ have inverse properties, also, every morphism in $F(\alpha)$ have an inverse, then, $\forall h \in \mathrm{Mor}(F(\alpha)),\ \alpha \in A$, we have:

1. $f(h^{-1}) = [f(h)]^{-1}$;
2. $f^{-1}(h) \cong f^{-1}(h^{-1})$.

Proof. 1. Clearly, f is a functor, and $F(\alpha)$ is a hypergroupoid $\forall \alpha \in A$. Since $(f,\ g)$ is a soft hypergroupoid homomorphism, then

$$f(h^{-1})f(h) = f(h^{-1}h) = f(1) = 1, \text{ and } f(h)f(h^{-1}) = f(hh^{-1}) = f(1) = 1,$$

for $h \in \mathrm{Mor}(F(\alpha))$. Thus, $f(h^{-1}) = [f(h)]^{-1}$.

2. $\forall h' \in f^{-1}(h)$, we define:

$$f_{h'} : f^{-1}(h) \longrightarrow f^{-1}(h^{-1})$$

$$h' \longmapsto h'^{-1}.$$

The mapping $f_{h'}$ is a bijection. Therefore, $f_{h'}$ is an isomorphism. Hence $f^{-1}(h) \cong f^{-1}(h^{-1})$. □

5. Soft subhypergroupoids

In this section of the paper, we focus on the concept of soft subhypergroupoids. By examining these structures, we aim to uncover their properties, relationships, and significance within the broader context of soft HG-hypergroupoids. Understanding soft subhypergroupoids will enhance our comprehension of the intricate dynamics at play within the category of soft HG-hypergroupoids, paving the way for potential applications and further theoretical developments in this area.

Definition 10 Let (H, F, A) and (H', F', A') be two soft hypergroupoid, then (H', F', A') is called a soft subhypergroupoid of (H, F, A), when

(i) $A' \subset A$;

(ii) $F'(\alpha')$ is a subhypergroupoid of $F(\alpha)$, $\forall \alpha' \in A'$.

Example 3 If (H, F, A) is a soft hypergroupoid, clearly (H, F, A) is a soft subhypergroupoid of (H, F, A).

Definition 11 Let (H', F', A') be a soft subhypergroupoid of soft hypergroupoid (H, F, A), then

(i) (H', F', A') is a full soft hypergroupoid, if $F'(\alpha')$ is a full subhypergroupoid of $F'(\alpha)$ $\forall \alpha' \in A'$;

(ii) (H', F', A') is a wide soft subhypergroupoid of $F(\alpha')$ $\forall \alpha' \in A'$;

(iii) (H', F', A') is a normal soft subhypergroupoid, if $F'(\alpha')$ is a normal subhypergroupoid of $F(\alpha')$ $\forall \alpha' \in A'$.

Definition 12 If the totally intransitive soft hypergroupoid (H', F', A') a normal soft subhypergroupoid of (H, F, A), then, the soft quotient hypergroupoid $\left(\frac{H}{H_1}, F'', A'\right)$ can be defined as:

$$F'' :\longrightarrow \mathscr{P}\left(\frac{H}{H'}\right)$$

$$\alpha' \longmapsto F''(\alpha) = \frac{F(\alpha)}{F'(\alpha)}$$

where is the induced hyperoperation on the quotient.

Definition 13 Let (H, F, A) be a soft hypergroupoid and let $F(\alpha)$ be an initial object such as a hypergroupoid $\forall \alpha \in A$, then (H, F, A) is called a soft hypergroupoid with initial objects.

Definition 14 Consider a soft hypergroupoid expressed as the triple (F, H, A), where F is a function assigning to each parameter $\alpha \in A$, a subhypergroupoid of H. Suppose that for every element α in the parameter set A, the subset $F(\alpha)$ serves as a terminal object within the category of hypergroupoids. This means that $F(\alpha)$ represents a maximal or final structure with respect to morphisms originating from other objects in the framework. When this property holds across all parameters, the overall construct (F, H, A) is identified as a soft hypergroupoid with terminal objects, highlighting that each image under F occupies a terminal position within the algebraic hierarchy.

Theorem 4 Let (H, F, A) be a soft hypergroupoid with objects (initial, terminal); then any two objects (initial, terminal) in (H, F, A) are isomorphic.

Proof. If soft hypergroupoid (H, F, A) has initial objects r and s, then $F(\alpha)$ has the initial objects r, s, such that as a hypergroupoid $\forall \alpha \in A$. We can define morphisms $f : r \to t$ and $g : s \to t$, $\forall t \in F(\alpha)$. We know that $F(\alpha)$ is a hypergroupoid; therefore, f and g are isomorphisms, so $r \cong t$ and $s \cong t$, which means $r \cong s$. This completes the proof. □

Remark 7 In a soft hypergroupoid, the objects of the same type are isomorphic.

In the following, we will establish the concept of trivial soft hypergroupoids, drawing parallels to the studies conducted in soft groupoid theory.

Definition 15 Suppose (H, F, A) is a soft hypergroup and (X, F', A) is a soft set. Hence, we can have the following mappings:

$$F :\longrightarrow \mathscr{P}(H) \qquad\qquad F' :\longrightarrow \mathscr{P}(X)$$

and

$$\alpha \longmapsto F(\alpha) \qquad\qquad \alpha \longmapsto F'(\alpha)$$

Then, a mapping F'' could be defined as follows:

$$F'' : A \longrightarrow \mathscr{P}(X \times H \times X)$$

$$\alpha \longmapsto F''(\alpha) = F'(\alpha) \times F(\alpha) \times F'(\alpha).$$

The composition in $X \times H \times X$ is defined by

$$(x, h, x') * (x'', h', x) = \left\{(x'', h'', x') \mid h'' \in hh'\right\},$$

then $(X \times H \times X, F'', A)$, which is a soft hypergroupoid and is called a trivial soft hypergroupoid.

6. Soft HG-hypergroupoids

This section of the paper introduces the concept of HG-hypergroupoids along with their corresponding category. To establish the framework for HG-hypergroupoids and their category, we leveraged the foundational ideas of soft hypergroups and soft hypergroupoids. By building upon these concepts, we aim to create a comprehensive understanding of HG-hypergroupoids, facilitating further exploration and analysis within this mathematical domain. This approach not only enriches the theory but also provides a structured environment for studying the properties and relationships of HG-hypergroupoids.

Definition 16 Consider the notion of a hypergroup-hypergroupoid, often abbreviated as an HG-hypergroupoid. This concept can be understood as an advanced algebraic entity that arises when one regards hypergroups as objects residing within the broader categorical framework of hypergroupoids. In other words, an HG-hypergroupoid is defined by viewing a hypergroup not merely as an isolated structure, but as an object embedded within the category whose elements themselves are hypergroupoids. This perspective elevates the notion of a hypergroup to a higher categorical level, blending the properties of hypergroups and hypergroupoids into a unified and enriched algebraic construct.

This means that an HG-hypergroupoid combines the structures of both hypergroups and hypergroupoids, allowing for operations that are associative and satisfy the properties of hypergroups while existing within the framework of hypergroupoids.

Definition 17 Let H be an HG-hypergroupoid, and consider the collection $\mathscr{P}(H)$ comprising all substructures of H that themselves qualify as subHG-hypergroupoids. Suppose there exists a mapping F from a parameter set A into this family $\mathscr{P}(H)$, such that for every parameter $\alpha \in A$, the image $F(\alpha)$ is indeed a subHG-hypergroupoid contained within H. Under these conditions, the pair (F, A) is designated as a soft HG-hypergroupoid over the base hyperstructure H. This entire arrangement is succinctly denoted by the triple (F, H, A), encapsulating the underlying set, the soft mapping, and the parameter set within a unified algebraic framework.

Remark 8 A soft HG-hypergroupoid in H can be defined as a parametrized family of subHG-hypergroupoids of the HG-hypergroupoid H. This definition emphasizes the structured relationship between the soft HG-hypergroupoid and its parent HG-hypergroupoid, allowing for a nuanced exploration of the properties and interactions of these substructures.

Remark 9 Every hg-groupoid inherently possesses a hypergroupoid structure, which implies that it adheres to the foundational principles governing hypergroupoids. Consequently, this leads to the conclusion that every soft HG-hypergroupoid also embodies a soft hypergroupoid structure. This relationship highlights the interconnectedness between these concepts and underscores the versatility of soft HG-hypergroupoids within the broader framework of hypergroupoid theory.

Theorem 5 The soft hypergroup (F, A) defined on the abelian hypergroup H indeed qualifies as a soft HG-hypergroupoid.

Proof. Let (F, A) be a soft hypergroup on an abelian hypergroup H. Hence, $F(\alpha)$ is a subhypergroup of H, $\forall \alpha \in A$. H is abelian. Therefore, each $f(\alpha)$ is abelian, and so is a HG-hypergroupoid. Moreover, $f(\alpha)$ is a subHG-hypergroupoid of the HG-hypergroupoid H, $\forall \alpha \in A$. Hence, the soft hypergroup (F, A), is a soft HG-hypergroupoid. □

This theorem asserts that the structure of (F, A), which is a soft hypergroup based on the properties of the abelian hypergroup H, satisfies the necessary conditions to be classified as a soft HG-hypergroupoid. This typically involves demonstrating that the operations defined in F respect the hypergroupoid structure and maintain the properties of associativity and closure under the hyperoperation.

Remark 10 This classification arises from the inherent properties of soft hypergroups and their compatibility with the structure of hypergroupoids. By integrating the elements of the soft hypergroup with the abelian nature of H, we establish a framework that satisfies the axioms and characteristics of soft HG-hypergroupoids.

Definition 18 Consider two structures, (H, F, A) and (H', F', A'), which are identified as soft HG-hypergroupoids corresponding to their respective HG-hypergroupoids H and H'. A pair of functions $(f, g) : (H, F, A) \longrightarrow (H', F', A')$ is referred to as an HG-hypergroupoid homomorphism if it satisfies the conditions required to be recognized as a soft homomorphism.

Remark 11 By conceptualizing the entities as soft HG-hypergroupoids and establishing their morphisms as soft HG-hypergroupoid homomorphisms, we introduce a novel classification system. This framework is referred to as the category of soft HG-hypergroupoids, abbreviated as SHG-HGC. Such a categorization provides a systematic methodology for examining the connections and interactions that exist among soft HG-hypergroupoids. It lays the groundwork for deeper investigation and comprehension of their characteristics and potential applications within the expansive realm of mathematical theory.

Example 4 Consider a soft HG-hypergroupoid represented as (H, F, A) and let (H', F', A') be identified as a soft subhypergroupoid within the framework of (H, F, A). In this scenario, if the objects of $F'(\alpha)$ are a subset of or equal to the objects of $F(\alpha)$, denoted as $Ob(F'(\alpha)) \leq Ob(F(\alpha))$, and similarly, if the morphisms of $F'(\alpha)$ are a subset of or equal to the morphisms of $F(\alpha)$, expressed as $Mor(F'(\alpha)) \leq Mor(F(\alpha))$, for every $\alpha \in A'$, then we classify (H', F', A') as a soft subHG-hypergroupoid of (H, F, A).

Definition 19 Consider a soft HG-hypergroupoid denoted as (H', F', A'), which is derived from the soft HG-hypergroupoid (H, F, A). This relationship holds under the condition that for every element $\alpha \in A'$, the objects of

$F'(\alpha)$ are a subset of or related to the objects of $F(\alpha)$, expressed as $Ob(F'(\alpha)) \trianglerighteq Ob(F(\alpha))$. Similarly, the morphisms of $F'(\alpha)$ must also be a subset of or related to the morphisms of $F(\alpha)$, indicated by $Mor(F'(\alpha)) \trianglerighteq Mor(F(\alpha))$. When these criteria are satisfied for all $\alpha \in A'$, we refer to (H', F', A') as a normal soft subHG-hypergroupoid of (H, F, A).

7. Conclusion

In this paper, we revisited the foundational concepts of soft sets and soft groups to define and investigate soft hypergroupoids. We examined their properties and provided illustrative examples to clarify these ideas. Additionally, we delved into the study of soft HG-hypergroupoids, expanding our understanding of this area. Looking ahead, we can propose several future research directions stemming from this work. One potential avenue is to define soft crossed hypermodules and explore their properties, which could further enrich the theory of soft structures. Another intriguing topic for future study could involve the investigation of crossed polysquares and their associated properties, offering new insights and applications within this mathematical framework. These proposed subjects hold promise for advancing the field and fostering deeper exploration of soft hyperstructures. Also, Current trajectories suggest: (i) further categorical equivalences (e.g., to internal groupoids in soft categories), (ii) deeper interplay with soft topology (compactness/connectedness vs. algebraic axioms), and (iii) algorithmic or decision approaches for checking HG-conditions under parameterized data.

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Conflict of interest

The authors declares no competing interests.

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