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Research Article

Differential Subordination Properties for Non-Bazilevič Functions Connected with a q-Calculus Operator

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Abstract: This paper demonstrates applications of q-calculus operators in geometric function theory with the use of differential subordination properties on non-Bazilevič functions with respect to k-symmetric in this study. The method of the differential subordination theory is applied in developing subordination results involving the new class of univalent functions introduced and studied in the work. Several geometric properties of the class of functions are established in the study. The results generalize many known results in literature. Such previously proved results are presented as corollaries. Different examples are also provided based on the results.

Keywords: convex function, starlike function, komatu integral operator, differential subordination, q-Ruscheweyh derivative operator

MSC: 30C80, 30C45

1. Introduction and preliminaries

The advantages of adding q-calculus in research pertaining to geometric function theory, as emphasized by Srivastava's recent review article [1], have inspired and stimulated new research that links these widely used tools to univalent functions theory. The same study [1] also emphasizes the usefulness of q-analogous operators in geometric function theory. With the introduction of the concepts of q-integral and q-derivative, Jackson [2, 3] provided the initial mathematical applications of q-calculus. When Ismail et al. [4] investigated a class of q-stalike functions, a relationship connecting q-calculus and univalent functions theory was revealed. Srivastava's 1989 book chapter [5] supplied a general framework for studies incorporating features of quantum calculus in geometric function theory, and as a result, research following this line of research began to develop.

With the introduction of q-analog operators, q-calculus found numerous applications on univalent functions. Kanas and Răducanu [6] used convolution to define the Ruscheweyh's differential operator q-analog. Furthermore, Aldweby and Darus [7] and Mahmood and Sokół [8] conducted additional research on the use of this differential operator. Recent studies continue to apply this operator for obtaining interesting outcome as seen in [9–11]. Sălăgean's differential operator q-analog was defined in [12], following the same pattern as for the Ruscheweyh's differential operator q-analog and inspired numerous applications [13–16]. Examples of several q-analogue operators investigated recently using differential subordination theory applied for certain classes of analytic functions are given in [17–19] and other q-analogue operators are introduced in the recent works [20, 21], where both differential subordination theory and its dual, differential superordination theory, are employed.

Motivated by of the numerous q-operators providing means for obtaining noteworthy outcome in geometric function theory, the investigation presented by this paper deals with a linear q-differ-integral operator defined by convolution as a generalization of certain operators previously studied. Following the prolific line of research in geometric function theory related to new classes of functions being defined and studied by means of differential subordination, using this new operator, an investigation focuses on a previously unexplored class of analytic functions by applying the tools offered by differential subordination theory.

For introducing the new findings, we convey the general context in which the research is carried out.

The class of analytic functions $\mathscr{H}(c,\ n)$ is considered, with functions having the following form:

$$f(\mathrm{z}) = c + \sum_{j=n}^{\infty} c_j \mathrm{z}^j, \quad (\mathrm{z} \in \mathbb{D}),$$

$\mathbb{D} = \{\mathrm{z} : \mathrm{z} \in \mathbb{C} \text{ and } |\mathrm{z}| < 1\}$.

Also, let $\mathscr{A}$ denote the subclass of $\mathscr{H}(0,\ 1)$ consisting of functions normalized by $f(0) = 0$ and $f'(0) = 1$. Considering that $f \in \mathscr{A}$, it can be defined by the following series expansion

$$f(\mathrm{z}) = \mathrm{z} + \sum_{j=2}^{\infty} c_j \mathrm{z}^j, \quad (\mathrm{z} \in \mathbb{D}). \tag{1}$$

As usual, $\mathscr{S}$ denotes the class of all univalent functions $f \in \mathscr{A}$.

Notable subclasses of $\mathscr{S}$ comprise starlike functions,

$$\mathscr{S}^* = \left\{ f \in \mathscr{S} :\ \mathrm{Re}\frac{zf'(z)}{f(z)} > 0,\ z \in \mathbb{D} \right\},$$

and convex functions

$$K = \left\{ f \in \mathscr{S} :\ \mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1 \right) > 0,\ z \in \mathbb{D} \right\}.$$

Definition 1 [22] If $f,\ h \in \mathscr{A}$, h is said to be superordinate to f or (equivalently) f is said to be subordinate to h, written as $f(\mathrm{z}) \prec h(\mathrm{z})\,(\mathrm{z} \in \mathbb{D})$, if there exists a Schwarz function ω analytic in $\mathbb{D}$, with $\omega(0) = 0$ and $|\omega(\mathrm{z})| < 1$, such that $f(\mathrm{z}) = h(\omega(\mathrm{z}))$. In particular, if $h \in \mathscr{S}$, the following equivalence holds [23, 24]:

$$f(\mathrm{z}) \prec h(\mathrm{z}) \quad (\mathrm{z} \in \mathbb{D}) \quad \Longleftrightarrow \quad f(0) = h(0),\ f(\mathbb{D}) \subset h(\mathbb{D}).$$

Definition 2 The convolution of $f \in \mathscr{A}$, defined by (1) and $h \in \mathscr{A}$, defined by

$$h(\mathrm{z}) = \mathrm{z} + \sum_{j=2}^{\infty} b_j \mathrm{z}^j, \quad (\mathrm{z} \in \mathbb{D}),$$

is given by

$$(f * h)(\mathrm{z}) = \mathrm{z} + \sum_{j=2}^{\infty} c_j b_j \mathrm{z}^j = (h * f)(\mathrm{z}).$$

The subclass of functions $\psi(\mathrm{z}) \in K$, satisfying $\psi(0) = 1$ and $\mathrm{Re}(\psi(\mathrm{z})) > 0$ is denoted by P.
For $f \in \mathscr{A}$ and $\varepsilon_\kappa = e^{\frac{2\pi i}{\kappa}}$, let

$$f_\kappa(\mathrm{z}) = \frac{1}{\kappa} \sum_{j=0}^{\kappa-1} \varepsilon_\kappa^{-j} f(\varepsilon_\kappa^j \mathrm{z}), \quad (\kappa \in \mathbb{N} = \{1,\ 2,\ 3,\ ...\}).$$

The function f satisfying the inequality:

$$\mathrm{Re}\left(\frac{\mathrm{z} f'(\mathrm{z})}{f_\kappa(\mathrm{z})}\right) > 0, \quad (\mathrm{z} \in \mathbb{D}), \tag{2}$$

is identified as starlike with respect to κ-symmetric points. We denote the subclass of $\mathscr{A}$ containing all such functions by $\mathscr{S}_s^{(\kappa)}$. Sakaguchi [25] introduced the class $\mathscr{S}_s^{(2)}$. Also, several subclasses of $\mathscr{S}_s^{(\kappa)}$ are generated by substituting (2) with

$$\mathrm{Re}\left(\frac{\mathrm{z} f'(\mathrm{z})}{f_\kappa(\mathrm{z})}\right) \prec \psi(\mathrm{z}), \quad (\mathrm{z} \in \mathbb{D}),$$

where $\mathrm{Re}(\psi(\mathrm{z})) > 0$, $\psi(0) = 1$ and $\psi(\mathrm{z})$ is convex function.

We now go over the fundamental terms and specifics of concepts pertaining to q-calculus that will be utilized throughout this study.

Definition 3 [2, 3] The q-derivative operator $\check{D}_q$ of $f(\mathrm{z})$ defined by

$$(\check{D}_q f)(\mathrm{z}) = \begin{cases} f'(0) = 1 & (\mathrm{z} = 0), \\ \dfrac{f(\mathrm{z}) - f(q\mathrm{z})}{(1-q)\mathrm{z}} & (q \in (0,\ 1), \mathrm{z} \neq 0). \end{cases} \tag{3}$$

The definition above indicates that:

$$\lim_{q \to 1^-} (\check{D}_q f)(\mathrm{z}) = f'(\mathrm{z}),$$

in a specified subset of $\mathbb{C}$, for a differentiable function $f \in \mathscr{A}$ of the form (1). Also, it follows that:

$$(\check{D}_q f)(z) = 1 + \sum_{j=2}^{\infty} [j]_q c_j z^{j-1}, \quad (z \in \mathbb{D}), \tag{4}$$

where the q-number $[j]_q$ is provided by:

$$[j]_q = \begin{cases} 0 & (j = 0), \\ \sum_{\iota=0}^{j-1} q^{\iota} = 1 + q + q^2 + q^3 + \ldots + q^{j-1} & (j \in \mathbb{N}). \end{cases} \tag{5}$$

Definition 4 [26] For $q \in (0,\ 1)$, the q-factorial $[\ell,\ q]!$ defined by

$$[\ell,\ q]! = \begin{cases} \prod_{\iota=1}^{\ell} [\iota]_q & (\ell \in \mathbb{N}), \\ 1 & (\ell = 0). \end{cases}$$

It was established that

$$\lim_{q \to 1^-} [\ell,\ q]! = \ell!,$$

$$\lim_{q \to 1^-} [\ell]_q = \ell.$$

Definition 5 [26] Let $0 < q < 1$ and define the q-Pochhammer symbol $[\lambda,\ q]_\kappa\,(\kappa \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \lambda \in \mathbb{C})$ by

$$[\lambda,\ q]_\kappa = \frac{(q^\lambda;\ q)_\kappa}{(1-q)^\kappa} = \begin{cases} \prod_{\iota=0}^{\kappa-1} [\lambda + \iota]_q & (\kappa \in \mathbb{N}), \\ 1 & (\kappa = 0). \end{cases}$$

The following recurrence relation also defines the q-gamma function:

$$\Gamma_q(1) = 1 \text{ and } \Gamma_q(\lambda + 1) = [\lambda]_q \Gamma_q(\lambda).$$

The q-Ruscheweyh derivative operator $\mathscr{R}_q^\alpha$ was introduced in [6] as:

$$\mathscr{R}_q^\alpha f(z) = \mathscr{G}_q^{\alpha+1}(z) * f(z), \quad (\alpha > -1,\ z \in \mathbb{D}), \tag{6}$$

where

$$\mathscr{G}_q^{\alpha+1}(\mathrm{z}) = \mathrm{z} + \sum_{j=2}^{\infty} \frac{\Gamma_q(\alpha+j)}{[j-1,\ q]!\Gamma_q(\alpha+1)} \mathrm{z}^j = \mathrm{z} + \sum_{j=2}^{\infty} \frac{[\alpha+1,\ q]_{j-1}}{[j-1,\ q]!} \mathrm{z}^j. \tag{7}$$

Considering (6), the following forms are obtained:

$$\mathscr{R}_q^0 f(\mathrm{z}) = f(\mathrm{z}), \quad \mathscr{R}_q^1 f(\mathrm{z}) = \mathrm{z}(\check{D}_q f)(\mathrm{z}),$$

and

$$\mathscr{R}_q^n f(\mathrm{z}) = \frac{\mathrm{z}(\check{D}_q^n(\mathrm{z}^{n-1} f(\mathrm{z})))}{[n,\ q]!}, \quad (n \in \mathbb{N}).$$

Making use of (6) and (7), $\mathscr{R}_q^\alpha f(\mathrm{z})$ can be written as:

$$\mathscr{R}_q^\alpha f(\mathrm{z}) = \mathrm{z} + \sum_{j=2}^{\infty} \frac{[\alpha+1,\ q]_{j-1}}{[j-1,\ q]!} c_j \mathrm{z}^j, \quad (\mathrm{z} \in \mathbb{D},\ \alpha > -1). \tag{8}$$

Note that $\lim_{q\to 1^-} \mathscr{R}_q^\alpha f(\mathrm{z}) = \mathscr{R}^\alpha f(\mathrm{z})$ (see [27]).

For defining the new linear q-differ-integral operator used in this investigation, we recall previously studied operators that we need to apply.

Komatu [28] defined the operator $\mathscr{I}_\sigma^\tau$ as follows:

$$\mathscr{I}_\sigma^\tau f(\mathrm{z}) = \frac{\sigma^\tau}{\Gamma(\tau)} \int_0^1 \left(\log \frac{1}{t}\right)^{\tau-1} t^{\sigma-2} f(\mathrm{z}t) dt, \quad (\sigma > 0,\ \tau \geq 0).$$

It is obvious that

$$\mathscr{I}_\sigma^\tau f(\mathrm{z}) = \mathrm{z} + \sum_{j=2}^{\infty} \left(\frac{\sigma}{\sigma+j-1}\right)^\tau c_j \mathrm{z}^j, \quad (\sigma > 0,\ \tau \geq 0,\ \mathrm{z} \in \mathbb{D}). \tag{9}$$

It can be noticed that:

- For $\tau = 0$, we obtain $\mathscr{I}_\sigma^0 f(\mathrm{z}) = f(\mathrm{z})$;
- For $\sigma = 2$, we get the operator $\mathscr{I}^\tau f(\mathrm{z})$, (see [29]);
- For $\tau = 1$ and $\sigma = 2$ we have Libera operator [30];
- For $\tau = 1$ and $\sigma = a+1$, $a > 0$ we obtain Bernardi operator [31];
- For $\tau = k$ and $\sigma = 1$, $k \in \mathbb{N}$, we obtain $\mathscr{I}^k f(\mathrm{z})$, (see [32]);
- For $\tau = n$ and $\sigma = 2$, $n \in \mathbb{N}$, we get the operator $\mathscr{I}^n f(\mathrm{z})$, (see [33]).

For $\alpha > -1$, $\sigma > 0$ and $\tau \geq 0$, we define the operator $\Upsilon_{\sigma,\ q}^{\tau,\ \alpha} : \mathscr{A} \to \mathscr{A}$ by

$$\Upsilon_{\sigma,\, q}^{\tau,\, \alpha} f(z) = \mathscr{G}_q^{\alpha+1}(z) * \mathscr{I}_\sigma^\tau f(z)$$

$$= z + \sum_{j=2}^{\infty} \Psi_{q,\, \sigma}^{\tau}(\alpha;\, j) c_j z^j \qquad (0 < q < 1, z \in \mathbb{D}), \tag{10}$$

where, using (7) and (9),

$$\Psi_{q,\, \sigma}^{\tau}(\alpha;\, j) = \left(\frac{[\alpha+1,\, q]_{j-1}}{[j-1,\, q]!} \right) \left(\frac{\sigma}{\sigma+j-1} \right)^{\tau}. \tag{11}$$

The new linear q-differ-integral operator $\mathfrak{T}_{\sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu)$ used as tool for this investigation can now be introduced as follows:

Definition 6 The operator $\mathfrak{T}_{\sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu) : \mathscr{A} \to \mathscr{A}$ is given by:

$$\mathfrak{T}_{\sigma,\, q}^{0,\, \tau,\, \alpha}(\beta,\, \nu) f(\mathrm{z}) = f(\mathrm{z})$$

$$\mathfrak{T}_{\sigma,\, q}^{1,\, \tau,\, \alpha}(\beta,\, \nu) f(\mathrm{z}) = \left(1 - \frac{\beta}{\nu+1}\right) \Upsilon_{\sigma,\, q}^{\tau,\, \alpha} f(\mathrm{z}) + \frac{\beta}{\nu+1} \mathrm{z} \check{D}_q \left(\Upsilon_{\sigma,\, q}^{\tau,\, \alpha} f(\mathrm{z}) \right) = \mathfrak{T}_{\sigma,\, q}^{\tau,\, \alpha}(\beta,\, \nu) f(\mathrm{z})$$

$$= \mathrm{z} + \sum_{j=2}^{\infty} \left(\frac{\nu + 1 + \beta([j]_q - 1)}{\nu+1} \right) \Psi_{q,\, \sigma}^{\tau}(\alpha;\, j) c_j \mathrm{z}^j \tag{12}$$

$$\vdots$$

$$\mathfrak{T}_{\sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu) f(\mathrm{z}) = \mathfrak{T}_{\sigma,\, q}^{\tau,\, \alpha}(\beta,\, \nu) \left(\mathfrak{T}_{\sigma,\, q}^{m-1,\, \tau,\, \alpha}(\beta,\, \nu) f(\mathrm{z}) \right),$$

where $\Psi_{q,\, \sigma}^{\tau}(\alpha;\, j)$ given by (11), $\beta \geq 0$, $m \in \mathbb{N}_0$, $\alpha > -1$, $\sigma > 0$, $\nu > -1$, $\tau \geq 0$ and $0 < q < 1$.

From (12), it follows that

$$\mathfrak{T}_{\sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu) f(\mathrm{z}) = \mathrm{z} + \sum_{j=2}^{\infty} \Theta_{\sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu,\, j) c_j \mathrm{z}^j, \tag{13}$$

where

$$\Theta_{\sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu,\, j) = \left(\frac{\nu + 1 + \beta([j]_q - 1)}{\nu+1} \Psi_{q,\, \sigma}^{\tau}(\alpha;\, j) \right)^m.$$

By virtue of (10) and (13), $\mathfrak{T}_{\sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu) f(\mathrm{z})$ can be described as:

$$\mathfrak{T}_{\sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu)f(z) = \underbrace{\left[\left(\Upsilon_{\sigma,\, q}^{\tau,\, \alpha}f(z) * \mathfrak{P}_q(\beta,\, \nu)(z)\right) * \ldots * \left(\Upsilon_{\sigma,\, q}^{\tau,\, \alpha}f(z) * \mathfrak{P}_q(\beta,\, \nu)(z)\right)\right]}_{m-\text{times}} * f(z),$$

where

$$\mathfrak{P}_q(\beta,\, \nu)(z) = \frac{z - \left(1 - \dfrac{\beta}{1+\nu}\right)qz^2}{(1-qz)(1-z)}.$$

It can be noticed that the operator $\mathfrak{T}_{\sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu)$ expands upon a number of well-known, earlier studied operators. We highlight now a few significant special cases:

• For $m = 1$, and $\tau = \beta = \nu = 0$, we obtain q-Ruscheweyh derivative operator studied by Kanas and Răducanu [6] (see also [26]);

• For $m = 1$, $\tau = \beta = \nu = 0$ and $q \to 1^-$, we obtain Ruscheweyh derivative operator defined by Ruscheweyh [27];

In this research, we suppose that:

$$\mathfrak{K}_{\kappa,\, \sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu;\, z) = \frac{1}{\kappa}\sum_{j=0}^{\kappa-1} \varepsilon_\kappa^{-j}\left(\mathfrak{T}_{\sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu)f\left(\varepsilon_\kappa^j z\right)\right), \quad (f \in \mathscr{A}). \tag{14}$$

For $\kappa = 1$, we have $\mathfrak{K}_{1,\, \sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu;\, z) = \mathfrak{T}_{\sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu)f(z)$.

We now present the new class of analytic functions defined and investigated for this research using the generalized linear q-differ-integral operator introduced in Definition 6.

Definition 7

A function $f \in \mathfrak{N}_{\kappa,\, q,\, m}^{\tau,\, \nu,\, \beta}(\sigma,\, \alpha,\, \mu;\, \psi)$ if and only if

$$\left[(\mathfrak{T}_{\sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu)f(z))'\left(\frac{z}{\mathfrak{K}_{\kappa,\, \sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu;\, z)}\right)\right]^{\mu+1} \prec \psi(z), \quad (z \in \mathbb{D}),$$

where $\psi \in \mathrm{P}$, $\mu \in [0,\, 1]$ and $\mathfrak{K}_{\kappa,\, \sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu;\, z) \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$.

It is simple to confirm the relationships listed below:

(a) $\mathfrak{N}_{1,\, q,\, 0}^{\tau,\, \nu,\, \beta}\left(\sigma,\, \alpha,\, 0;\, \dfrac{1+z}{1-z}\right) = \mathscr{S}^*$;

(b) $\mathfrak{N}_{\kappa,\, q,\, 0}^{\tau,\, \nu,\, \beta}\left(\sigma,\, \alpha,\, 0;\, \dfrac{1+z}{1-z}\right) = \mathscr{S}_s^{(\kappa)}$;

(c) $\mathfrak{N}_{1,\, q,\, 0}^{\tau,\, \nu,\, \beta}\left(\sigma,\, \alpha,\, \mu;\, \dfrac{1+z}{1-z}\right) = \mathscr{N}(\mu)$, (see [34]).

The new outcome exposed by this article proposes differential subordination results involving functions from the new class $\mathfrak{N}_{\kappa,\, q}^{\tau,\, \nu,\, \beta}(\sigma,\, \alpha,\, \mu;\, \psi)$ introduced in Definition 7.

The following known results are employed for proving the new findings presented in the next section.

Lemma 1 [24, 35] For $\gamma \neq 0$, $\mathrm{Re}(\gamma) \geq 0$ and $\chi(z)$ be a convex in $\mathbb{D}$, with $\chi(0) = c$. If $g \in \mathscr{H}(c,\, n)$ and

$$g(z)+\frac{zg'(z)}{\gamma}\prec\chi(z), \tag{15}$$

then

$$g(z)\prec\varphi(z)=\frac{\gamma}{nz^{\gamma/n}}\int_0^z\chi(s)s^{(\gamma/n)-1}ds\prec\chi(z),$$

where $\varphi(z)$ is convex, $\varphi(z)\in\mathscr{H}(c,\ n)$, and is the best dominant of (15).

Lemma 2 [24, 36] For $\chi(z)$ be a starlike in $\mathbb{D}$, with $\chi(0)=0$. If $g\in\mathscr{H}(c,\ n)$ and

$$zg'(z)\prec\chi(z), \tag{16}$$

then

$$g(z)\prec\varphi(z)=c+\left(\frac{1}{n}\right)\int_0^z\chi(s)s^{-1}ds,$$

where $\varphi(z)\in\mathscr{H}(c,\ n)$, and $\varphi(z)$ is the best dominant of (16).

Lemma 3 [37, 38] For $a,\ b,\ c$ real numbers other than $0,\ -1,\ -2,\ \ldots$ and $c>b>0$, we have

$$\int_0^1 s^{b-1}(1-s)^{c-b-1}(1-sz)^{-a}ds=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a,\ b;\ c;\ z),\ z<|1| \tag{17}$$

$${}_2F_1(a,\ b;\ c;\ z)=(1-z)^{-a}_2F_1(a,\ c-b;\ c;\ z/(z-1)) \tag{18}$$

$${}_2F_1(1,\ 1;\ 2;\ z)=-(1/z)ln(1-z) \tag{19}$$

$${}_2F_1(1,\ 1;\ 2;\ \rho z/(\rho z+1))=\frac{(1+\rho z)ln(1+\rho z)}{\rho z}. \tag{20}$$

Lemma 4 [24] Let $\mathbf{r}(z)$ be univalent in $\mathbb{D}$ and $h(z)$ be analytic in a domain D containing $\mathbf{r}(\mathbb{D})$. If $z\mathbf{r}'(z)h(\mathbf{r}(z))$ is starlike in $\mathbb{D}$ and

$$z\mathbf{p}'(z)h(\mathbf{p}(z))\prec z\mathbf{r}'(z)h(\mathbf{r}(z)),$$

then $\mathbf{p}(z)\prec\mathbf{r}(z)$ and $\mathbf{r}(z)$ is the best dominant.

2. Results

Unless otherwise indicated, the rest of this work is based on the following assumptions: $\nu,\ \alpha > -1,\ \sigma > 0,\ \beta,\ \tau \geq 0,\ \psi(z) \in P,\ 0 \leq \mu \leq 1,\ 0 < q < 1,\ f \in \mathscr{A}$ and $\mathfrak{T}_{\sigma,\ q}^{m,\ \tau,\ \alpha}(\beta,\ \nu)f(z)$ and $\mathfrak{K}_{\kappa,\ \sigma,\ q}^{m,\ \tau,\ \alpha}(\beta,\ \nu;\ z) \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$.

Theorem 1 Let $\psi(z) \in P$. If

$$\left(\left(\mathfrak{T}_{\sigma,\ q}^{m,\ \tau,\ \alpha}(\beta,\ \nu)f(z)\right)' \left(\frac{z}{\mathfrak{K}_{\kappa,\ \sigma,\ q}^{m,\ \tau,\ \alpha}(\beta,\ \nu;\ z)} \right)^{\mu+1} \right)^2 \left(3 + 2\mu + \frac{2z\left(\mathfrak{T}_{\sigma,\ q}^{m,\ \tau,\ \alpha}(\beta,\ \nu)f(z)\right)''}{\left(\mathfrak{T}_{\sigma,\ q}^{m,\ \tau,\ \alpha}(\beta,\ \nu)f(z)\right)'} \right. \tag{21}$$
$$\left. -2(\mu+1)\frac{z\left(\mathfrak{K}_{\kappa,\ \sigma,\ q}^{m,\ \tau,\ \alpha}(\beta,\ \nu;\ z)\right)'}{\mathfrak{K}_{\kappa,\ \sigma,\ q}^{m,\ \tau,\ \alpha}(\beta,\ \nu;\ z)} \right) \prec \psi(z),$$

then

$$\left(\mathfrak{T}_{\sigma,\ q}^{m,\ \tau,\ \alpha}(\beta,\ \nu)f(z)\right)' \left(\frac{z}{\mathfrak{K}_{\kappa,\ \sigma,\ q}^{m,\ \tau,\ \alpha}(\beta,\ \nu;\ z)} \right)^{\mu+1} \prec \varphi(z) = \sqrt{\xi(z)}, \tag{22}$$

where

$$\xi(z) = \frac{1}{z}\int_0^z \psi(s)ds,$$

and the function $\varphi(z) \in K$ is the best dominant.

Proof. Take

$$\mathscr{T}(z) = \left(\mathfrak{T}_{\sigma,\ q}^{m,\ \tau,\ \alpha}(\beta,\ \nu)f(z)\right)' \left(\frac{z}{\mathfrak{K}_{\kappa,\ \sigma,\ q}^{m,\ \tau,\ \alpha}(\beta,\ \nu;\ z)} \right)^{\mu+1}.$$

With this definition, $\mathscr{T}(z) \neq 0$ in $\mathbb{D}$ and $\mathscr{T}(z) \in \mathscr{H}(1,\ 1)$. It's straightforward to validate that $\xi(z) \in K$ and $\xi(z) \in \mathscr{S}$ because $\psi(z) \in K$. Next, consider $\mathscr{D}(z) = \mathscr{T}^2(z)$, which gives that $\mathscr{D}(z) \neq 0$ in $\mathbb{D}$ and $\mathscr{D}(z) \in \mathscr{H}(1,\ 1)$. By differentiating in the logarithmic sense, we obtain

$$\frac{z\mathscr{D}'(z)}{\mathscr{D}(z)} = \frac{2z\left(\mathfrak{T}_{\sigma,\ q}^{m,\ \tau,\ \alpha}(\beta,\ \nu)f(z)\right)''}{\left(\mathfrak{T}_{\sigma,\ q}^{m,\ \tau,\ \alpha}(\beta,\ \nu)f(z)\right)'} + 2(\mu+1)\left(1 - \frac{z\left(\mathfrak{K}_{\kappa,\ \sigma,\ q}^{m,\ \tau,\ \alpha}(\beta,\ \nu;\ z)\right)'}{\mathfrak{K}_{\kappa,\ \sigma,\ q}^{m,\ \tau,\ \alpha}(\beta,\ \nu;\ z)} \right).$$

By considering (21), we deduce:

$$\mathscr{D}(z) + z\mathscr{D}'(z) \prec \psi(z). \tag{23}$$

By employing Lemma 1 considering $\gamma = 1$, the conclusion is that

$$\mathscr{D}(\mathrm{z}) \prec \xi(\mathrm{z}) \prec \psi(\mathrm{z}),$$

with $\xi(\mathrm{z})$ the best dominant of (23). Also, we deduce that $\mathrm{Re}(\xi(\mathrm{z})) > 0$, because $\mathrm{Re}(\psi(\mathrm{z})) > 0$ and $\xi(\mathrm{z}) \prec \psi(\mathrm{z})$. Since $\xi(\mathrm{z}) \in \mathscr{S}$, we have that $\varphi(\mathrm{z}) = \sqrt{\xi(\mathrm{z})} \in \mathscr{S}$ and

$$\mathscr{D}(\mathrm{z}) = \mathscr{T}^2(\mathrm{z}) \prec \xi(\mathrm{z}) = \varphi^2(\mathrm{z}),$$

concluding that $\mathscr{D}(\mathrm{z}) \prec \varphi(\mathrm{z})$. Knowing that $\xi(\mathrm{z})$ is the best dominant for (23), we conclude that $\varphi(z)$ is the best dominant for the subordination (22).

Corollary 1 Let $\psi(\mathrm{z}) = \dfrac{1+A\mathrm{z}}{1+B\mathrm{z}} \in \mathrm{P}\ (-1 \leq B < A < 1)$. If

$$\left(\left(\mathfrak{T}^{m,\ \tau,\ \alpha}_{\sigma,\ q}(\beta,\ \nu) f(\mathrm{z})\right)' \left(\frac{\mathrm{z}}{\mathfrak{K}^{m,\ \tau,\ \alpha}_{\kappa,\ \sigma,\ q}(\beta,\ \nu;\ z)} \right)^{\mu+1} \right)^2 \left(3 + 2\mu + \frac{2\mathrm{z}\left(\mathfrak{T}^{m,\ \tau,\ \alpha}_{\sigma,\ q}(\beta,\ \nu) f(\mathrm{z})\right)''}{\left(\mathfrak{T}^{m,\ \tau,\ \alpha}_{\sigma,\ q}(\beta,\ \nu) f(\mathrm{z})\right)'} \right.$$

$$\left. -2(\mu+1) \frac{\mathrm{z}\left(\mathfrak{K}^{m,\ \tau,\ \alpha}_{\kappa,\ \sigma,\ q}(\beta,\ \nu;\ \mathrm{z})\right)'}{\mathfrak{K}^{m,\ \tau,\ \alpha}_{\kappa,\ \sigma,\ q}(\beta,\ \nu;\ \mathrm{z})} \right) \prec \frac{1+A\mathrm{z}}{1+B\mathrm{z}},$$

then

$$\left(\mathfrak{T}^{m,\ \tau,\ \alpha}_{\sigma,\ q}(\beta,\ \nu) f(\mathrm{z})\right)' \left(\frac{\mathrm{z}}{\mathfrak{K}^{m,\ \tau,\ \alpha}_{\kappa,\ \sigma,\ q}(\beta,\ \nu;\ \mathrm{z})} \right)^{\mu+1} \prec \varphi(\mathrm{z}) = \sqrt{\theta(A,\ B;\ \mathrm{z})}, \tag{24}$$

where

$$\theta(A,\ B;\ z) = \begin{cases} 1 + \dfrac{Az}{2} & B = 0, \\ \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right) \dfrac{\ln(1+Bz)}{B\mathrm{z}} & B \neq 0, \end{cases}$$

with function $\varphi(\mathrm{z})$ the best dominant. Furthermore,

$$\mathrm{Re}\left(\left(\mathfrak{T}^{m,\ \tau,\ \alpha}_{\sigma,\ q}(\beta,\ \nu) f(\mathrm{z})\right)' \left(\frac{\mathrm{z}}{\mathfrak{K}^{m,\ \tau,\ \alpha}_{\kappa,\ \sigma,\ q}(\beta,\ \nu;\ \mathrm{z})} \right)^{\mu+1} \right) > \sqrt{\theta(A,\ B)}, \tag{25}$$

where

$$\theta(A,\ B)=\begin{cases}1-\dfrac{A}{2} & B=0,\\ \dfrac{A}{B}+\left(1-\dfrac{A}{B}\right)\dfrac{\ln(1-B)}{-B} & B\neq 0.\end{cases}$$

Proof. Take $\psi(z)=\dfrac{1+Az}{1+Bz}$ with $-1\leq B<A<1$. By applying Theorem 1 we get that $\mathrm{Re}(\xi(z))>0$ and $\xi(z)\in K$. Using Lemma 3, the function $\xi(z)$ can be written as follows:

$$\xi(z)=\frac{1}{z}\int_0^z\frac{1+As}{1+Bs}ds=\frac{1}{z}\int_0^1\frac{1+Azt}{1+Bzt}(zdt)$$

$$=\frac{A}{B}+\left(1-\frac{A}{B}\right)\int_0^1(1+Bzt)^{-1}dt$$

$$=\frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1,\ 1,\ 2;\ \frac{Bz}{Bz+1}\right)$$

$$=\frac{A}{B}+\left(1-\frac{A}{B}\right)\frac{\ln(1+Bz)}{Bz}.$$

Hence, (24) is proved. We also have

$$\min_{|z|\leq 1}\mathrm{Re}(\varphi(z))=\min_{|z|\leq 1}\mathrm{Re}(\sqrt{\xi(z)})=\sqrt{\xi(-1)}.$$

The outcome given by (25) is the best possible since $\xi(z)$ is the best dominant of (24), and this completes the proof. Taking $B\neq 0$ in Corollary 1, we obtain.

Corollary 2 If

$$\left(\left(\mathfrak{T}^{m,\ \tau,\ \alpha}_{\sigma,\ q}(\beta,\ \nu)f(z)\right)'\left(\frac{z}{\mathfrak{K}^{m,\ \tau,\ \alpha}_{\kappa,\ \sigma,\ q}(\beta,\ \nu;\ z)}\right)^{\mu+1}\right)^2\left(3+2\mu+\frac{2z\left(\mathfrak{T}^{m,\ \tau,\ \alpha}_{\sigma,\ q}(\beta,\ \nu)f(z)\right)''}{\left(\mathfrak{T}^{m,\ \tau,\ \alpha}_{\sigma,\ q}(\beta,\ \nu)f(z)\right)'}\right.$$

$$\left.-2(\mu+1)\frac{z\left(\mathfrak{K}^{m,\ \tau,\ \alpha}_{\kappa,\ \sigma,\ q}(\beta,\ \nu;\ z)\right)'}{\mathfrak{K}^{m,\ \tau,\ \alpha}_{\kappa,\ \sigma,\ q}(\beta,\ \nu;\ z)}\right)\prec\frac{1+\dfrac{B\ln(1-B)}{B+\ln(1-B)}z}{1+Bz},$$

then

$$\operatorname{Re}\left(\left(\mathfrak{T}_{\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\ \nu)f(\mathrm{z})\right)'\left(\frac{\mathrm{z}}{\mathfrak{K}_{\kappa,\,\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\ \nu;\ \mathrm{z})}\right)^{\mu+1}\right)>0.$$

Taking $B=-1$ in Corollary 2, we obtain.

Example 1 If

$$\operatorname{Re}\left(\left(\left(\mathfrak{T}_{\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\ \nu)f(\mathrm{z})\right)'\left(\frac{\mathrm{z}}{\mathfrak{K}_{\kappa,\,\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\ \nu;\ \mathrm{z})}\right)^{\mu+1}\right)^{2}\left(3+2\mu+\frac{2\mathrm{z}\left(\mathfrak{T}_{\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\ \nu)f(\mathrm{z})\right)''}{\left(\mathfrak{T}_{\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\ \nu)f(\mathrm{z})\right)'}\right.\right.$$

$$\left.\left.-2(\mu+1)\frac{\mathrm{z}\left(\mathfrak{K}_{\kappa,\,\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\ \nu;\ \mathrm{z})\right)'}{\mathfrak{K}_{\kappa,\,\sigma,\,\alpha,\,q}^{m,\,\tau,\,\alpha}(\beta,\ \nu;\ \mathrm{z})}\right)>\frac{2\ln 2-1}{2\ln 2-2}\cong-0.61,$$

then

$$\operatorname{Re}\left(\left(\mathfrak{T}_{\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\ \nu)f(\mathrm{z})\right)'\left(\frac{\mathrm{z}}{\mathfrak{K}_{\kappa,\,\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\ \nu;\ \mathrm{z})}\right)^{\mu+1}\right)>0.$$

Taking $B=-1$ and $A=1-2\rho\ (0\leq\rho<1)$ in Corollary 1, we have.

Corollary 3 Suppose that $\operatorname{Re}(\varpi(\mathrm{z}))>\rho\ (0\leq\rho<1)$. If

$$\varpi(\mathrm{z})=\left(\left(\mathfrak{T}_{\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\ \nu)f(\mathrm{z})\right)'\left(\frac{\mathrm{z}}{\mathfrak{K}_{\kappa,\,\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\ \nu;\ \mathrm{z})}\right)^{\mu+1}\right)^{2}\left(3+2\mu+\frac{2\mathrm{z}\left(\mathfrak{T}_{\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\ \nu)f(\mathrm{z})\right)''}{\left(\mathfrak{T}_{\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\ \nu)f(\mathrm{z})\right)'}\right.$$

$$\left.-2(\mu+1)\frac{\mathrm{z}\left(\mathfrak{K}_{\kappa,\,\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\ \nu;\ \mathrm{z})\right)'}{\mathfrak{K}_{\kappa,\,\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\ \nu;\ \mathrm{z})}\right),$$

then

$$\operatorname{Re}\left(\left(\mathfrak{T}_{\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\ \nu)f(\mathrm{z})\right)'\left(\frac{\mathrm{z}}{\mathfrak{K}_{\kappa,\,\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\ \nu;\ \mathrm{z})}\right)^{\mu+1}\right)>\theta(\rho),$$

where

$$\theta(\rho)=\sqrt{(2\rho-1)+2(1-\rho)\ln 2}.$$

Considering $m=0$ in Corollary 3, we obtain.

Corollary 4 [39] Suppose that $f \in \mathscr{A}$ with $f_\kappa(\mathrm{z})$ and $f'(\mathrm{z}) \neq 0$ for all $\mathrm{z} \in \mathbb{D} \setminus \{0\}$. If

$$\operatorname{Re}\left(\left(f'(\mathrm{z})\left(\frac{\mathrm{z}}{f_\kappa(\mathrm{z})}\right)^{\mu+1}\right)^2\left(3+2\mu+\frac{2\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}-2(\mu+1)\frac{\mathrm{z}f'_\kappa(\mathrm{z})}{f_\kappa(\mathrm{z})}\right)\right) > \rho,$$

then

$$\operatorname{Re}\left(f'(\mathrm{z})\left(\frac{\mathrm{z}}{f_\kappa(\mathrm{z})}\right)^{\mu+1}\right) > \theta(\rho),$$

where

$$\theta(\rho) = \sqrt{(2\rho-1)+2(1-\rho)\ln 2}.$$

Taking $\mu = 0$ in Corollary 4, we obtain.
Corollary 5[39] Let $f \in \mathscr{A}$ with $f_\kappa(\mathrm{z})$ and $f'(\mathrm{z}) \neq 0$ for all $\mathrm{z} \in \mathbb{D} \setminus \{0\}$. If

$$\operatorname{Re}\left(\left(\frac{\mathrm{z}f'(\mathrm{z})}{f_\kappa(\mathrm{z})}\right)^2\left(3+\frac{2\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}-2\frac{\mathrm{z}f'_\kappa(\mathrm{z})}{f_\kappa(\mathrm{z})}\right)\right) > \rho,$$

then

$$\operatorname{Re}\left(\frac{\mathrm{z}f'(\mathrm{z})}{f_\kappa(\mathrm{z})}\right) > \theta(\rho),$$

where

$$\theta(\rho) = \sqrt{(2\rho-1)+2(1-\rho)\ln 2}.$$

Taking $\rho = 0$ in Corollary 3, we have.
Example 2 Let $\operatorname{Re}(\varpi(\mathrm{z})) > 0$. If

$$\varpi(\mathrm{z}) = \left(\left(\mathfrak{T}_{\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\,\nu)f(\mathrm{z})\right)'\left(\frac{\mathrm{z}}{\mathfrak{K}_{\kappa,\,\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\,\nu;\,\mathrm{z})}\right)^{\mu+1}\right)^2\left(3+2\mu+\frac{2\mathrm{z}\left(\mathfrak{T}_{\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\,\nu)f(\mathrm{z})\right)''}{\left(\mathfrak{T}_{\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\,\nu)f(\mathrm{z})\right)'}\right.$$

$$\left.-2(\mu+1)\frac{\mathrm{z}\left(\mathfrak{K}_{\kappa,\,\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\,\nu;\,\mathrm{z})\right)'}{\mathfrak{K}_{\kappa,\,\sigma,\,q}^{m,\,\tau,\,\alpha}(\beta,\,\nu;\,\mathrm{z})}\right),$$

then

$$\mathrm{Re}\left(\left(\mathfrak{T}_{\sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu)f(\mathrm{z})\right)'\left(\frac{\mathrm{z}}{\mathfrak{K}_{\kappa,\, \sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu;\, \mathrm{z})}\right)^{\mu+1}\right) > \sqrt{2\ln 2-1} \cong 0.62.$$

Taking $m=0$ in Corollary 2, we obtain.

Corollary 6 Suppose that $f \in \mathscr{A}$ satisfying $f_\kappa(\mathrm{z})$ and $f'(\mathrm{z}) \neq 0$ for all $\mathrm{z} \in \mathbb{D}\setminus\{0\}$. If

$$\mathrm{Re}\left(\left(f'(\mathrm{z})\left(\frac{\mathrm{z}}{f_\kappa(\mathrm{z})}\right)^{\mu+1}\right)^2\left(3+2\mu+\frac{2\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}-2(\mu+1)\frac{\mathrm{z}f_\kappa'(\mathrm{z})}{f_\kappa(\mathrm{z})}\right)\right) > 0,$$

then

$$\mathrm{Re}\left(f'(\mathrm{z})\left(\frac{\mathrm{z}}{f_\kappa(\mathrm{z})}\right)^{\mu+1}\right) > \sqrt{2\ln 2-1} \cong 0.62.$$

By considering $\mu=0$ in Corollary 6, we obtain.

Example 3 Suppose that $f \in \mathscr{A}$ with $f_\kappa(\mathrm{z})$ and $f'(\mathrm{z}) \neq 0$ for all $\mathrm{z} \in \mathbb{D}\setminus\{0\}$. If

$$\mathrm{Re}\left(\left(\frac{\mathrm{z}f'(\mathrm{z})}{f_\kappa(\mathrm{z})}\right)^2\left(3+\frac{2\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}-2\frac{\mathrm{z}f_\kappa'(\mathrm{z})}{f_\kappa(\mathrm{z})}\right)\right) > 0,$$

then

$$\mathrm{Re}\left(\frac{\mathrm{z}f'(\mathrm{z})}{f_\kappa(\mathrm{z})}\right) > \sqrt{2\ln 2-1} \cong 0.62.$$

Theorem 2 Suppose that $\chi(\mathrm{z}) \in \mathscr{S}^*$ with $\chi(0)=0$. If

$$\frac{\mathrm{z}\left(\mathfrak{T}_{\sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu)f(\mathrm{z})\right)''}{\left(\mathfrak{T}_{\sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu)f(\mathrm{z})\right)'}+(\mu+1)\left(1-\frac{\mathrm{z}\left(\mathfrak{K}_{\kappa,\, \sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu;\, \mathrm{z})\right)'}{\mathfrak{K}_{\kappa,\, \sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu;\, \mathrm{z})}\right) \prec \chi(\mathrm{z}), \tag{26}$$

then

$$\left(\mathfrak{T}_{\sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu)f(\mathrm{z})\right)'\left(\frac{\mathrm{z}}{\mathfrak{K}_{\kappa,\, \sigma,\, q}^{m,\, \tau,\, \alpha}(\beta,\, \nu;\, \mathrm{z})}\right)^{\mu+1} \prec \varphi(\mathrm{z}) = exp\left(\int_0^{\mathrm{z}}\frac{\chi(s)}{s}ds\right), \tag{27}$$

with the function $\varphi(z)$ the best dominant.

Proof. Let

$$\mathscr{T}(\mathrm{z}) = (\mathfrak{T}^{m,\ \tau,\ \alpha}_{\sigma,\ q}(\beta,\ \nu)f(\mathrm{z}))' \left(\frac{\mathrm{z}}{\mathfrak{K}^{m,\ \tau,\ \alpha}_{\kappa,\ \sigma,\ q}(\beta,\ \nu;\ \mathrm{z})}\right)^{\mu+1}.$$

With this definition, $\mathscr{T}(\mathrm{z}) \neq 0$ in $\mathbb{D}$ and $\mathscr{T}(\mathrm{z}) \in \mathscr{H}(1,\ 1)$. Next, consider $\mathscr{D}(\mathrm{z}) = \log \mathscr{T}(\mathrm{z})$, which gives that $\mathscr{D}(\mathrm{z}) \in \mathscr{H}(0,\ 1)$. By differentiating in the logarithmic sense, we have

$$\frac{\mathrm{z}\mathscr{T}'(\mathrm{z})}{\mathscr{T}(\mathrm{z})} = \frac{\mathrm{z}\left(\mathfrak{T}^{m,\ \tau,\ \alpha}_{\sigma,\ q}(\beta,\ \nu)f(\mathrm{z})\right)''}{\left(\mathfrak{T}^{m,\ \tau,\ \alpha}_{\sigma,\ q}(\beta,\ \nu)f(\mathrm{z})\right)'} + (\mu+1)\left(1 - \frac{\mathrm{z}\left(\mathfrak{K}^{m,\ \tau,\ \alpha}_{\kappa,\ \sigma,\ q}(\beta,\ \nu;\ \mathrm{z})\right)'}{\mathfrak{K}^{m,\ \tau,\ \alpha}_{\kappa,\ \sigma,\ q}(\beta,\ \nu;\ \mathrm{z})}\right).$$

Thus, by (26), we have

$$z\mathscr{D}'(z) \prec \chi(z). \tag{28}$$

Using Lemma 2, we conclude that

$$\mathscr{D}(\mathrm{z}) \prec \xi(\mathrm{z}) = \int_0^{\mathrm{z}} \frac{\chi(s)}{s} ds,$$

and $\xi(\mathrm{z})$ is the best dominant of (28). Since $\xi(\mathrm{z}) \in \mathscr{S}$, it follows that $\mathscr{D}(\mathrm{z}) \in \mathscr{S}$ and

$$\mathscr{D}(\mathrm{z}) = exp\mathscr{T}(\mathrm{z}) \prec exp\xi(\mathrm{z}) = \varphi(\mathrm{z}),$$

concluding that $\mathscr{D}(\mathrm{z}) \prec \varphi(\mathrm{z})$. Knowing that $\xi(\mathrm{z})$ is the best dominant for (28), we conclude that $\varphi(\mathrm{z})$ is the best dominant for (27).

By considering $m = 0$ in Theorem 2, we obtain.

Corollary 7 [39] Suppose that $f \in \mathscr{A}$ with $f_\kappa(\mathrm{z})$ and $f'(\mathrm{z}) \neq 0$ for all $\mathrm{z} \in \mathbb{D} \setminus \{0\}$ and $\chi(\mathrm{z}) \in \mathscr{S}^*$ with $\chi(0) = 0$. If

$$\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})} + (\mu+1)\left(1 - \frac{\mathrm{z}f_\kappa'(\mathrm{z})}{f_\kappa(\mathrm{z})}\right) \prec \chi(\mathrm{z}),$$

then

$$f'(\mathrm{z})\left(\frac{\mathrm{z}}{f_\kappa(\mathrm{z})}\right)^{\mu+1} \prec \varphi(\mathrm{z}) = exp\left(\int_0^{\mathrm{z}} \frac{\chi(s)}{s} ds\right),$$

the function $\varphi(\mathrm{z})$ is the best dominant.

Taking $\mu = 0$ in Corollary 7, we obtain.

Corollary 8 Suppose that $f \in \mathscr{A}$ with $f_\kappa(\mathrm{z})$ and $f'(\mathrm{z}) \neq 0$ for all $\mathrm{z} \in \mathbb{D} \setminus \{0\}$ and $\chi(\mathrm{z}) \in \mathscr{S}^*$ with $\chi(0) = 0$. If

$$\frac{zf''(z)}{f'(z)}+1-\frac{zf'_\kappa(z)}{f_\kappa(z)}\prec\chi(z),$$

then

$$\frac{zf'(z)}{f_\kappa(z)}\prec\varphi(z)=exp\left(\int_0^z\frac{\chi(s)}{s}ds\right),$$

with the function $\varphi(z)$ the best dominant.

Theorem 3 Suppose that $f\in\mathscr{A}$ and $\mathbf{r}(z)\in\mathscr{S}$ satisfying $\mathbf{r}'(z)\neq 0$ in $\mathbb{D}$. If $\frac{z\mathbf{r}'(z)}{\mathbf{r}(z)}\in\mathscr{S}^*$ and

$$\frac{z\left(\mathfrak{T}^{m,\,\tau,\,\alpha}_{\sigma,\,q}(\beta,\,\nu)f(z)\right)''}{\left(\mathfrak{T}^{m,\,\tau,\,\alpha}_{\sigma,\,q}(\beta,\,\nu)f(z)\right)'}+(\mu+1)\left(1-\frac{z\left(\mathfrak{K}^{m,\,\tau,\,\alpha}_{\kappa,\,\sigma,\,q}(\beta,\,\nu;\,z)\right)'}{\mathfrak{K}^{m,\,\tau,\,\alpha}_{\kappa,\,\sigma,\,q}(\beta,\,\nu;\,z)}\right)\prec\frac{z\mathbf{r}'(z)}{\mathbf{r}(z)},$$

then

$$\left(\mathfrak{T}^{m,\,\tau,\,\alpha}_{\sigma,\,q}(\beta,\,\nu)f(z)\right)'\left(\frac{z}{\mathfrak{K}^{m,\,\tau,\,\alpha}_{\kappa,\,\sigma,\,q}(\beta,\,\nu;\,z)}\right)^{\mu+1}\prec\mathbf{r}(z),$$

with $\mathbf{r}(z)$ the best dominant.

Proof. Let

$$\mathscr{T}(z)=\left(\mathfrak{T}^{m,\,\tau,\,\alpha}_{\sigma,\,q}(\beta,\,\nu)f(z)\right)'\left(\frac{z}{\mathfrak{K}^{m,\,\tau,\,\alpha}_{\kappa,\,\sigma,\,q}(\beta,\,\nu;\,z)}\right)^{\mu+1}\quad(z\neq 0,\ z\in\mathbb{D}).$$

Putting $h(z)=\frac{d}{z}$, $d\neq 0$, it is simple to show that $h(z)$ is analytic in $\mathbb{C}-\{0\}$. Then

$$d\frac{z\mathscr{T}'(z)}{\mathscr{T}(z)}=d\left[\frac{z\left(\mathfrak{T}^{m,\,\tau,\,\alpha}_{\sigma,\,q}(\beta,\,\nu)f(z)\right)''}{\left(\mathfrak{T}^{m,\,\tau,\,\alpha}_{\sigma,\,q}(\beta,\,\nu)f(z)\right)'}+(\mu+1)\left(1-\frac{z\left(\mathfrak{K}^{m,\,\tau,\,\alpha}_{\kappa,\,\sigma,\,q}(\beta,\,\nu;\,z)\right)'}{\mathfrak{K}^{m,\,\tau,\,\alpha}_{\kappa,\,\sigma,\,q}(\beta,\,\nu;\,z)}\right)\right]\prec d\frac{z\mathbf{r}'(z)}{\mathbf{r}(z)}.$$

Now, an application of Lemma 4 leads to the theorem's assertion.

Putting $m=0$ in Theorem 3, we obtain.

Corollary 9 [39] Suppose that $f\in\mathscr{A}$ with f' and $f\neq 0$, $\forall z\in\mathbb{D}\setminus\{0\}$ and $\mathbf{r}(z)$ be univalent in $\mathbb{D}$, with $\mathbf{r}'(z)\neq 0$ in $\mathbb{D}$. If $\frac{z\mathbf{r}'(z)}{\mathbf{r}(z)}$ is starlike in $\mathbb{D}$ and

$$\frac{zf''(z)}{f'(z)}+(\mu+1)\left(1-\frac{zf'_\kappa(z)}{f_\kappa(z)}\right)\prec\frac{z\mathbf{r}'(z)}{\mathbf{r}(z)},$$

then

$$f'(z)\left(\frac{z}{f_{\kappa}(z)}\right)^{\mu+1} \prec \mathbf{r}(z),$$

and $\mathbf{r}(z)$ is the best dominant.

Putting $\mu = 0$ in Corollary 9, we obtain.

Corollary 10 Suppose that $f \in \mathscr{A}$ with f' and $f \neq 0, \forall z \in \mathbb{D} \setminus \{0\}$ and let $\mathbf{r}(z) \in \mathscr{S}$ satisfying $\mathbf{r}'(z) \neq 0$ in $\mathbb{D}$. If $\dfrac{z\mathbf{r}'(z)}{\mathbf{r}(z)} \in \mathscr{S}^*$ and

$$\frac{zf''(z)}{f'(z)} + 1 - \frac{zf_{\kappa}'(z)}{f_{\kappa}(z)} \prec \frac{z\mathbf{r}'(z)}{\mathbf{r}(z)},$$

then

$$\left(\frac{zf'(z)}{f_{\kappa}(z)}\right) \prec \mathbf{r}(z),$$

with $\mathbf{r}(z)$ the best dominant.

3. Conclusion

In this work, several important tools used for the research in geometric function theory are combined for obtaining new outcome related to a certain new class of non-Bazilevič functions with respect to k-symmetric points. The concept of convolution and means of q-operators are merged in order to generate a new linear q-differ-integral operator in the Definition 6. It is highlighted that special cases of this operator have been previously studied by other authors. The new class of non-Bazilevič functions with respect to k-symmetric point is provided in Definition 7 and the preparation of the new research involving differential subordination techniques is accomplished by listing four well-known results labeled as lemmas. Section 2 reveals the new outcome of this study. Differential subordinations for which the best dominants are also found are investigated in the theorems proved in this work. Particular functions exhibiting remarkable geometric properties such as convexity or starlikeness are applied as best dominants in the proved theorems providing noteworthy corollaries. Furthermore, other corollaries are obtained by particularizing certain parameters involved in the definitions of the convex functions used as best dominants. While some of those corollaries present previously established findings, others offer means for creating examples that illustrate how the new findings might be applied.

By showing that the results obtained here generalize previously known results, this paper adds knowledge to the investigations regarding differential subordination techniques employed on different classes of analytic functions. The new generalized linear q-differ-integral operator introduced in Definition 6 could be applied for defining additional classes of other types of analytic functions, similar to the work that involves bi-univalent functions in [40], multivalent functions linked with Janowski-type functions in [41, 42] or meromorphic functions in [43]. Furthermore, the means of the dual theory of differential superordination could be used for investigating the new class of non-Bazilevič functions with respect to k-symmetric point provided in Definition 7 linking that outcome to the results shown in this study through sandwich-type theorems as seen in [21], for example. The results presented here could be adapted to the differential subordination theory's variants such as strong differential subordination and fuzzy differential subordination theories as seen in [20, 44], respectively.

Conflict of interest

The authors declare no competing financial interest.

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