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Research Article

On the Role of Intuitionistic Fuzzy Preinvex Mappings in Fuzzy Optimization Using Variational-Like Inequalities

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Abstract: In this article, we generalize the notion of fuzzy ($\mathscr{F}$) invex sets and preinvex functions, and introduce the ideas of intuitionistic fuzzy ($\mathscr{I}\mathscr{F}$) invex sets and preinvex functions. These functions are also generalized forms of $\mathscr{F}$-convex functions when $\mathscr{I}\mathscr{F}$-invex functions are differentiable. With the support of some results, we explain the definitions like invex, incave, and quasi-$\mathscr{I}\mathscr{F}$-sets differently and $\mathscr{I}\mathscr{F}$-invex sets. Similarly, different preinvex functions are defined with the help of $\mathscr{I}\mathscr{F}$-mappings ($\mathscr{I}\mathscr{F}Ms$) on the invex set and characterize some properties. Finally, new proofs of some known important results are offered. These important outcomes are based on optimization in which the minimum of the $\mathscr{I}\mathscr{F}$-preinvex function is distinguished by $\mathscr{I}\mathscr{F}$-variational-like ($\mathscr{V}\mathscr{L}$) inequalities and $\mathscr{V}\mathscr{L}$-inequalities. These results define the relationship of $\mathscr{I}\mathscr{F}$-preinvex functions with $\mathscr{I}\mathscr{F}$-mixed-$\mathscr{V}\mathscr{L}$-inequalities. These concepts are more important than $\mathscr{F}$-nonconvexity because the $\mathscr{F}$-theory depend upon the membership function and $\mathscr{I}\mathscr{F}$-theory contains the membership and non-membership functions. Some of the non-trivial examples are also provided to validate the given results.

***Keywords*:** intuitionistic fuzzy numbers, intuitionistic fuzzy invex sets, preinvex function, intuitionistic fuzzy variational-like inequalities

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1. Introduction

In 1965, Zadeh [1] first introduced the idea of $\mathscr{F}$-sets and $\mathscr{F}$-numbers and explored their fundamental characteristics. Over time, many researchers have expanded on these concepts to address issues of uncertainty. A $\mathscr{F}$-number can be regarded as a generalization of an interval from classical set theory. After Zadeh's initial definition, Dubois and Prade [2] refined the concept by imposing additional conditions on $\mathscr{F}$-numbers. Later, Goetschel and Voxman [3] further advanced this framework by adjusting several of these conditions, making $\mathscr{F}$-numbers more practical to work with. For example, while Dubois and Prade required $\mathscr{F}$-numbers to be continuous functions, Goetschel and Voxman allowed them to be only upper semicontinuous. Such relaxations of conditions make it possible to establish a metric on the collection of $\mathscr{F}$-numbers, which in turn provides a basis for studying their topological properties.

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Similarly, with the help these $\mathscr{F}$-numbers we can easily handle the convexity of $\mathscr{F}$-set. Noor [4] initiated the notion of $\mathscr{F}$-preinvex function over the field R and this notion is function from $\mathscr{F}$-invex set to set of all $\mathscr{F}$-numbers and discussed the properties of $\mathscr{F}$-preinvex function. Furthermore, the optimal conditions are obtained and characterized by $\mathscr{V}\mathscr{L}$-inequalities. Subsequently, many authors contributed in this field, Syau [5–7], by using $\mathscr{F}$-mappings of one variable to introduced different types of generalized convexity and also explained the properties of these convex $\mathscr{F}$-mappings. For more information, related to $\mathscr{F}$-convex sets, $\mathscr{F}$-convex mappings, $\mathscr{F}$-optimization theory, $\mathscr{F}$-preinvex mappings and mixed-$\mathscr{V}\mathscr{L}$-inequalities [8–15]. As $\mathscr{F}$-number is not capable of dealing with any application where there is a lack of knowledge about membership degree. In the case when the degree of non-membership (or degree of rejection) is discussed simultaneously with degree of membership (or degree of acceptance) and if these degrees are not supportive to each other, then $\mathscr{I}\mathscr{F}$-sets are used to overcome the uncertainty. A collection of interval-valued function inequalities based on the Kulisch-Miranker order relation was then established by Costa et al. [16]. They used the improper integrals of Aumann and Kaleva to construct Gauss's inequality for interval functions. A family of log-s-convex $\mathscr{F}$-interval-valued functions was studied by Liu et al. [17]. They obtained some Jensen-and HH-Fejér-type inequalities using this kind of function. Srivastava et al. were the first to propose interval-valued preinvex functions [18]. In [19], the authors presented the idea of the interval-valued harmonic h-convex function. They developed many HH-type inequalities for interval Riemann integrals using this idea. Additionally, a number of applications of interval-valued functions in optimization theory are examined in studies like [20, 21]. The $\mathscr{F}$-and interval versions of these fractional integrals via exponential in the kernel over $\mathscr{F}$-and exponential trigonometric convex mappings were also provided by Khan et al. [22–27], along with its applications in inequality theory. Interval-valued functions have seen important recent advances; for further details, interested readers might consult [28–30] and the references listed therein.

On the other hand, Atanassove [31, 32], introduced $\mathscr{I}\mathscr{F}$-set and interval-valued $\mathscr{I}\mathscr{F}$-sets which are generalized form of $\mathscr{F}$-sets and explained this idea with the help of examples and compare with $\mathscr{F}$-set that is the special case of $\mathscr{I}\mathscr{F}$-set. In $\mathscr{F}$-sets, membership function defines the truth values of elements but $\mathscr{I}\mathscr{F}$-sets consist of two functions 1^{st} one is membership function and 2^{nd} one is non-membership function. The question is raised why we need to introduce $\mathscr{I}\mathscr{F}$-sets. To explain it we take an optimization problem because in this article, $\mathscr{I}\mathscr{F}$-preinvex functions are minimized and optimal conditions are fined. For instance, if we discuss $\mathscr{F}$-sets in an optimization problem [33–42], in general, optimization problem includes two major things like objective function and constraints but in $\mathscr{F}$-optimization problem all these relations are presented by $\mathscr{F}$-sets like if we take "Min" $\mathscr{F}$-optimization and "$\preceq$" represent the $\mathscr{F}$-inequality then $\mathscr{F}$-optimization problem is

$$\text{Min:}\, f_j(\varphi), \quad j = 1,\ 2,\ 3,\ \ldots,\ k, \tag{1}$$

such that

$$g_j(\varphi) \preceq 0, \quad j = 1,\ 2,\ 3,\ \ldots,\ n. \tag{2}$$

Similarly, when the degree of non-membership $\upsilon : \widetilde{X} \to [0,\ 1]$ (or degree of rejection) is considered together with the degree of membership $\Lambda : \widetilde{X} \to [0,\ 1]$ (or degree of acceptance), and these two degrees are not complementary to each other, $\mathscr{I}\mathscr{F}$-sets are employed to formulate an $\mathscr{I}\mathscr{F}$-problem. In this context, if $\Lambda_j(\varphi)$ and $\upsilon_j(\varphi)$ represent, respectively, the degree of satisfaction and the degree of rejection of an element φ with respect to the jth $\mathscr{I}\mathscr{F}$-set, then

$$\text{Max} : \Lambda_j(\varphi), \quad j = 1,\ 2,\ 3,\ \ldots,\ k+n, \tag{3}$$

$$\text{Min} : \upsilon_j(\varphi), \quad j = 1,\ 2,\ 3,\ \ldots,\ k+n, \tag{4}$$

such that

$$\upsilon_j(\varphi) \leq 0, \quad j = 1,\ 2,\ 3,\ \ldots,\ k+n, \tag{5}$$

$$\upsilon_j(\varphi) \leq \Lambda_j(\varphi), \text{ and } \upsilon_j(\varphi) + \Lambda_j(\varphi) \geq 1, \quad j = 1,\ 2,\ 3,\ \ldots,\ k+n. \tag{6}$$

$\mathscr{I}\mathscr{F}$-numbers are particularly useful when both membership and non-membership degrees are considered simultaneously. Burillo [43] introduced $\mathscr{I}\mathscr{F}$-numbers as a generalization of $\mathscr{F}$-numbers. An $\mathscr{I}\mathscr{F}$-number is characterized by a membership function and a non-membership function, and it is regarded as an $\mathscr{I}\mathscr{F}$-number when both functions exhibit the properties of $\mathscr{F}$-numbers. For further details on IF-numbers and $\mathscr{I}\mathscr{F}$-mappings, see [44–52].

Inspired by both above mentioned research work and by the importance of the idea of invexity and preinvexity, see [53]. In section 3, we generalized the notion of $\mathscr{F}$-invex sets and preinvex function and introduced the notion of $\mathscr{I}\mathscr{F}$-invex sets and preinvex function. With the support of some results, we explained some definitions like invex, incave and quasi-$\mathscr{I}\mathscr{F}$-sets in different way as compare to $\mathscr{I}\mathscr{F}$-invex sets. Similarly, different types of preinvex functions are defined with the help of $\mathscr{I}\mathscr{F}M$ on invex set and explained some properties. In section 4, some known important results is offered and these results based on optimization in which minimum of the $\mathscr{I}\mathscr{F}$-preinvex function is characterized by $\mathscr{I}\mathscr{F}$-$\mathscr{V}\mathscr{L}$-inequalities and mixed-$\mathscr{V}\mathscr{L}$-inequalities. These results define the relationship of $\mathscr{I}\mathscr{F}$-preinvex function with $\mathscr{I}\mathscr{F}$-$\mathscr{V}\mathscr{L}$-inequalities and mixed-$\mathscr{V}\mathscr{L}$-inequalities.

2. Preliminaries

This section introduces the essential definitions, notations, and preliminary results for the subsequent analysis.

Definition 1 A $\mathscr{F}$-set $\mathfrak{A}$ in a universe of discourse R is defined as

$$\mathfrak{A} = \{\langle \varphi,\ \Lambda_{\mathfrak{A}}(\varphi)\rangle : \varphi \in R\}, \tag{7}$$

where $\Lambda_{\mathfrak{A}} : R \to [0,\ 1]$ is the membership function of $\mathfrak{A}$, assigning to each element $\varphi \in R$ a degree of membership in the interval $[0,\ 1]$.

On the other hand, a $\mathscr{F}$-set on R can be expressed as a membership function $\Lambda : R \to [0,\ 1]$. For any $\tilde{\mathfrak{v}} \in (0,\ 1]$, the $\tilde{\mathfrak{v}}$-level set of Λ is defined as

$$\Lambda_{\tilde{\mathfrak{v}}} = \{\, \varphi \in R | \Lambda(\varphi) \geq \tilde{\mathfrak{v}} \,\}. \tag{8}$$

If $\tilde{\mathfrak{v}} = 0$, then

$$\Lambda_0 = \{\varphi \in R \mid \Lambda(\varphi) > 0\}, \tag{9}$$

which is called the *support* of Λ, and we denote the closure of this set by $\overline{\int\sqcap\surd(\Lambda)}$.

A $\mathscr{F}$-set is defined as normal if there exists $\varphi \in R$ such that $\Lambda(\varphi) = 1$.

Furthermore, a $\mathscr{F}$-set is convex if

$$\Lambda((1-\tau)\varphi+\tau\omega)\geq \min(\Lambda(\varphi),\ \Lambda(\omega)), \tag{10}$$

and concave if

$$\Lambda((1-\tau)\varphi+\tau\omega)\leq \max(\Lambda(\varphi),\ \Lambda(\omega)), \quad \forall\varphi,\ \omega\in R,\ \tau\in[0,\ 1]. \tag{11}$$

A $\mathscr{F}$-set $\Lambda : R \to [0,\ 1]$ is defined as a $\mathscr{F}$-number if it satisfies the following conditions:

1. (Normality) Λ is normal.
2. (Convexity) Λ is a convex $\mathscr{F}$-set.
3. (Compact Support) The support of Λ is a compact subset of R.

The set of all $\mathscr{F}$-numbers on R is denoted by $\mathscr{F}\mathscr{N}(R)$.

Definition 2 Let $\mathbf{E}$ be a fixed universe of discourse. An $\mathbf{I}\mathscr{F}$-set $\mathfrak{A}$ in $\mathbf{E}$ is defined as

$$\mathfrak{A}=\{\langle\varphi,\ \Lambda(\varphi),\ \upsilon(\varphi)\rangle \mid \varphi\in E\}, \tag{12}$$

where

- $\Lambda : \mathbf{E}\to[0,\ 1]$ is the membership function,
- $\upsilon : \mathbf{E}\to[0,\ 1]$ is the non-membership function,

satisfying the condition

$$0\leq\Lambda(\varphi)+\upsilon(\varphi)\leq 1, \quad \forall\varphi\in E. \tag{13}$$

In addition, the hesitation (or indeterminacy) degree of an element $\varphi\in E$ is defined as

$$\pi(\varphi)=1-\Lambda(\varphi)-\upsilon(\varphi). \tag{14}$$

An $\mathscr{I}\mathscr{F}$-set in E is characterized by a pair of mappings $(\Lambda,\ \upsilon)$. For $\tilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}}\in[0,\ 1]$, $(\tilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}})$-level set of $(\Lambda,\ \upsilon)$ is defined as

$$(\Lambda,\ \upsilon)_{(\tilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}})}=\Lambda_{\tilde{\mathfrak{v}}}\cap\upsilon_{\widetilde{\mathfrak{w}}}=\{\varphi\in R\mid\Lambda(\varphi)\geq\tilde{\mathfrak{v}} \text{ and } \upsilon(\varphi)\leq\widetilde{\mathfrak{w}}\}, \tag{15}$$

where

$$\Lambda_{\tilde{\mathfrak{v}}}=\{\varphi\in R\mid\Lambda(\varphi)\geq\tilde{\mathfrak{v}}\}, \quad \upsilon_{\widetilde{\mathfrak{w}}}=\{\varphi\in R\mid\upsilon(\varphi)\leq\widetilde{\mathfrak{w}}\}, \tag{16}$$

and $\tilde{\mathfrak{v}}+\widetilde{\mathfrak{w}}\leq 1$.

An $\mathscr{I}\mathscr{F}$-set $(\Lambda,\ \upsilon)$ is defined as:

1. Normal if there exist $\varphi,\ \omega\in E$ such that $\Lambda(\varphi)=1$ and $\upsilon(\omega)=1$.

2. Convex if Λ is a convex $\mathscr{F}$-set and υ is a concave $\mathscr{F}$-set, i.e.,

$$\Lambda((1-\tau)\varphi+\tau\omega) \geq \min(\Lambda(\varphi),\ \Lambda(\omega)), \tag{17}$$

and

$$\upsilon((1-\tau)\varphi+\tau\omega) \leq \max(\upsilon(\varphi),\ \upsilon(\omega)), \quad \forall\varphi,\ \omega \in R,\ \tau \in [0,\ 1]. \tag{18}$$

An $\mathscr{I}\mathscr{F}$-number is characterized by a pair of mappings $(\Lambda,\ \upsilon)$, where

$$\Lambda : R \to [0,\ 1] \quad \text{and} \quad \upsilon : R \to [0,\ 1], \tag{19}$$

satisfying the following conditions:

1. (Normality) $(\Lambda,\ \upsilon)$ is *normal*, i.e., there exists $\varphi \in R$ such that $\Lambda(\varphi) = 1$.
2. (Convexity/Concavity) Λ is a $\mathscr{F}$*-convex* function and υ is a $\mathscr{F}$*-concave* function.
3. (Continuity) Λ is *upper semicontinuous* and υ is *lower semicontinuous*.
4. (Bounded Support) The support of $(\Lambda,\ \upsilon)$, defined as

$$\int\sqcap_{\surd}(\Lambda,\ \upsilon) = \{\varphi \in R \mid \upsilon(\varphi) < 1\}, \tag{20}$$

is bounded.

The set of all $\mathscr{I}\mathscr{F}$-numbers on R is denoted by $\mathscr{I}\mathscr{F}\mathscr{N}(R)$.

It is worth noting that each $\mathscr{I}\mathscr{F}$-number can be regarded as a combination of two $\mathscr{F}$-numbers: one defined by the membership function Λ, and the other by the non-membership function υ.

For an $\mathscr{I}\mathscr{F}$-number, it is also convenient to describe its structure in terms of $(\tilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}})$-level sets, which provide a useful representation for analysis such that

$$\Lambda_{\tilde{\mathfrak{v}}} = \{\varphi \in R \mid \Lambda(\varphi) \geq \tilde{\mathfrak{v}}\}, \quad \upsilon_{\widetilde{\mathfrak{w}}} = \{\varphi \in R \mid \upsilon(\varphi) \leq \widetilde{\mathfrak{w}}\}. \tag{21}$$

From these definitions, we have

$$\Lambda_{\tilde{\mathfrak{v}}} = [\Lambda_*(\tilde{\mathfrak{v}}),\ \Lambda^*(\tilde{\mathfrak{v}})] \quad \text{and} \quad \upsilon_{\widetilde{\mathfrak{w}}} = [\upsilon_*(\widetilde{\mathfrak{w}}),\ \upsilon^*(\widetilde{\mathfrak{w}})]. \tag{22}$$

Where,

$$\Lambda_*(\tilde{\mathfrak{v}}) = \inf\{\varphi \in R \mid \Lambda(\varphi) \geq \tilde{\mathfrak{v}}\},\ \Lambda^*(\tilde{\mathfrak{v}}) = \sup\{\varphi \in R \mid \Lambda(\varphi) \geq \tilde{\mathfrak{v}}\}, \tag{23}$$

$$\upsilon_*(\widetilde{\mathfrak{w}}) = \inf\{\varphi \in R \mid \upsilon(\varphi) \leq \widetilde{\mathfrak{w}}\},\ \upsilon^*(\widetilde{\mathfrak{w}}) = \sup\{\varphi \in R \mid \upsilon(\varphi) \leq \widetilde{\mathfrak{w}}\}. \tag{24}$$

From the above definitions, an $\mathscr{I}\mathscr{F}$-number can also be expressed in terms of its $(\Lambda_{\tilde{\mathfrak{v}}},\ \upsilon_{\widetilde{\mathfrak{w}}})$-cuts. Specifically, let

$$(\Lambda_{\tilde{\mathfrak{v}}},\ \upsilon_{\widetilde{\mathfrak{w}}}) \subseteq [0,\ 1], \tag{25}$$

where $\widetilde{\mathfrak{S}}_{\Lambda_\varphi} : R \to [0,\ 1]$ and $\widetilde{\mathfrak{S}}_{\upsilon_\varphi} : R \to [0,\ 1]$ are referred to as the positive membership function and the negative membership function, respectively. These mappings satisfy the condition

$$0 \leq \widetilde{\mathfrak{S}}_{\Lambda_\varphi}(w) + \widetilde{\mathfrak{S}}_{\upsilon_\varphi}(w) \leq 1, \quad \forall w \in R, \text{ and for a } \varphi \in \mathfrak{D}. \tag{26}$$

For a given set $\mathfrak{D}$, the collection of all $\mathscr{I}\mathscr{F}$-mappings from $\mathfrak{D}$ to R is denoted by $\mathscr{I}\mathscr{F}\mathscr{M}(R)^{\mathfrak{D}}$. If we restrict the setting to Triangular $\mathscr{I}\mathscr{F}$-Numbers (TFNs), then the set of all such mappings is denoted by $T\mathscr{I}\mathscr{F}\mathscr{N}(R)^{\mathfrak{D}}$.

Furthermore, by considering the support of the $(\tilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}})$-cut, where $\tilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}} \in [0,\ 1]$ with $\tilde{\mathfrak{v}} + \widetilde{\mathfrak{w}} \leq 1$, we can formally define $\mathscr{I}\mathscr{F}$-mappings as

Definition 3 [20] An $\mathscr{I}\mathscr{F}$-set $\mathfrak{A}$ on R is defined as convex if

$$\Lambda_{\mathfrak{A}}((1-\tau)\varphi + \tau\omega) \geq (1-\tau)\Lambda_{\mathfrak{A}}(\varphi) + \tau\Lambda_{\mathfrak{A}}(\omega) \quad \forall \varphi,\ \omega \in R,\ \tau \in [0,\ 1], \tag{27}$$

$$\upsilon_{\mathfrak{A}}((1-\tau)\varphi + \tau\omega) \leq (1-\tau)\upsilon_{\mathfrak{A}}(\varphi) + \tau\upsilon_{\mathfrak{A}}(\omega) \quad \forall \varphi,\ \omega \in R,\ \tau \in [0,\ 1]. \tag{28}$$

Strictly convex $\mathscr{I}\mathscr{F}$-set $\mathfrak{A}$ on R if strict inequality holds for $\Lambda_{\mathfrak{A}}(\varphi) \neq \Lambda_{\mathfrak{A}}(\omega)$ and $\upsilon_{\mathfrak{A}}(\varphi) \neq \upsilon_{\mathfrak{A}}(\omega)$. An $\mathscr{I}\mathscr{F}$-set $\mathfrak{A}$ on R is defined as concave if

$$\Lambda_{\mathfrak{A}}((1-\tau)\varphi + \tau\omega) \leq (1-\tau)\Lambda_{\mathfrak{A}}(\varphi) + \tau\Lambda_{\mathfrak{A}}(\omega) \quad \forall \varphi,\ \omega \in R,\ \tau \in [0,\ 1], \tag{29}$$

$$\upsilon_{\mathfrak{A}}((1-\tau)\varphi + \tau\omega) \geq (1-\tau)\upsilon_{\mathfrak{A}}(\varphi) + \tau\upsilon_{\mathfrak{A}}(\omega) \quad \forall \varphi,\ \omega \in R,\ \tau \in [0,\ 1]. \tag{30}$$

Strictly concave $\mathscr{I}\mathscr{F}$-set $\mathfrak{A}$ on R if strict inequality holds for $\Lambda_{\mathfrak{A}}(\varphi) \neq \Lambda_{\mathfrak{A}}(\omega)$ and $\upsilon_{\mathfrak{A}}(\varphi) \neq \upsilon_{\mathfrak{A}}(\omega)$.

Definition 4 [20] An $\mathscr{I}\mathscr{F}$-set $\mathfrak{A}$ on R is defined as quasi-convex if, $\forall \varphi,\ \omega \in R,\ \tau \in [0,\ 1]$

$$\Lambda_{\mathfrak{A}}((1-\tau)\varphi + \tau\omega) \geq \min(\Lambda_{\mathfrak{A}}(\varphi),\ \Lambda_{\mathfrak{A}}(\omega)), \tag{31}$$

$$\upsilon_{\mathfrak{A}}((1-\tau)\varphi + \tau\omega) \leq \max(\upsilon_{\mathfrak{A}}(\varphi),\ \upsilon_{\mathfrak{A}}(\omega)). \tag{32}$$

Strictly quasi-convex $\mathscr{I}\mathscr{F}$-set $\mathfrak{A}$ on R if strict inequality holds for $\Lambda_{\mathfrak{A}}(\varphi) \neq \Lambda_{\mathfrak{A}}(\omega)$ and $\upsilon_{\mathfrak{A}}(\varphi) \neq \upsilon_{\mathfrak{A}}(\omega)$. An $\mathscr{I}\mathscr{F}$-set $\mathfrak{A}$ on R is defined as quasi-concave if, $\forall \varphi,\ \omega \in R,\ \tau \in [0,\ 1]$

$$\Lambda_{\mathfrak{A}}((1-\tau)\varphi + \tau\omega) \leq \max(\Lambda_{\mathfrak{A}}(\varphi),\ \Lambda_{\mathfrak{A}}(\omega)), \tag{33}$$

$$\upsilon_{\mathfrak{A}}((1-\tau)\varphi + \tau\omega) \geq \min(\upsilon_{\mathfrak{A}}(\varphi),\ \upsilon_{\mathfrak{A}}(\omega)). \tag{34}$$

Strictly quasi-concave $\mathscr{I}\mathscr{F}$-set $\mathfrak{A}$ on R if strict inequality holds for $\Lambda_{\mathfrak{A}}(\varphi) \neq \Lambda_{\mathfrak{A}}(\omega)$ and $\upsilon_{\mathfrak{A}}(\varphi) \neq \upsilon_{\mathfrak{A}}(\omega)$.

Definition 5 [18] Let $\mathfrak{D}$ be an invex set in $\mathscr{F}\mathscr{N}(R)$. Then $\mathscr{F}$-function $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{F}\mathscr{N}(R)$ is defined as preinvex at $\varphi \in \mathfrak{D}$ if

$$\widetilde{\mathfrak{S}}(\varphi + \tau\eta(\omega,\ \varphi)) \leq_{\Lambda} (1-\tau)\widetilde{\mathfrak{S}}(\varphi) + \tau\widetilde{\mathfrak{S}}(\omega) \quad \forall \omega \in \mathfrak{D},\ \tau \in [0,\ 1]. \tag{35}$$

Where, $\leq_{\Lambda}$ is a $\mathscr{F}$-order relation. $\widetilde{\mathfrak{S}}$ is defined as preinvex on $\mathfrak{D}$ if it is preinvex at each $\varphi \in \mathfrak{D}$. Strictly $\mathscr{F}$-preinvex mapping if strict inequality holds for $\widetilde{\mathfrak{S}}(\varphi) \neq_{\Lambda} \widetilde{\mathfrak{S}}(\omega)$. $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{F}\mathscr{N}(R)$ is defined as $\mathscr{F}$-preincave mapping at $\varphi \in \mathfrak{D}$, if $\widetilde{\mathfrak{S}}(\varphi + \tau\eta(\omega,\ \varphi)) \geq_{\Lambda} (1-\tau)\widetilde{\mathfrak{S}}(\varphi) + \tau\widetilde{\mathfrak{S}}(\omega) \quad \forall \omega \in \mathfrak{D},\ \tau \in [0,\ 1]$ and $\widetilde{\mathfrak{S}}$ is defined as preincave on $\mathfrak{D}$ if it is preincave at each $\varphi \in \mathfrak{D}$. Strictly $\mathscr{F}$-preincave mapping if strict inequality holds for $\widetilde{\mathfrak{S}}(\varphi) \neq_{\Lambda} \widetilde{\mathfrak{S}}(\omega)$.

Definition 6 [18] The $\mathscr{F}$-mapping $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{F}\mathscr{N}(R)$ is defined as quasi-preinvex at $\varphi \in \mathfrak{D}$ if

$$\widetilde{\mathfrak{S}}(\varphi + \tau\eta(\omega,\ \varphi)) \leq_{\Lambda} \max(\widetilde{\mathfrak{S}}(\varphi),\ \widetilde{\mathfrak{S}}(\omega)) \quad \forall \omega \in \mathfrak{D},\ \tau \in [0,\ 1], \tag{36}$$

and $\widetilde{\mathfrak{S}}$ is defined as quasi-preinvex on $\mathfrak{D}$ if it is preinvex at each $\varphi \in \mathfrak{D}$.

Strictly $\mathscr{F}$-quasi-preinvex mapping if strict inequality holds for $\widetilde{\mathfrak{S}}(\varphi) \neq_{\Lambda} \widetilde{\mathfrak{S}}(\omega)$. $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{F}\mathscr{N}(R)$ is defined as $\mathscr{F}$-quasi-preincave mapping at $\varphi \in \mathfrak{D}$, if $\widetilde{\mathfrak{S}}(\varphi + \tau\eta(\omega,\ \varphi)) \geq_{\Lambda} \min(\widetilde{\mathfrak{S}}(\varphi),\ \widetilde{\mathfrak{S}}(\omega))\ \forall \omega \in \mathfrak{D},\ \tau \in [0,\ 1]$ and $\widetilde{\mathfrak{S}}$ is defined as quasi-preincave on $\mathfrak{D}$ if it is quasi-preincave at each $\varphi \in \mathfrak{D}$. Strictly $\mathscr{F}$-quasi-preincave mapping if strict inequality holds for $\widetilde{\mathfrak{S}}(\varphi) \neq_{\Lambda} \widetilde{\mathfrak{S}}(\omega)$.

3. Related results

In this section, some properties of invex $\mathscr{I}\mathscr{F}$-sets are discussed over convexity and quasi convexity approaches. These $\mathscr{F}$-sets are also characterized by the $(\tilde{\mathfrak{v}},\ \tilde{\mathfrak{w}})$-cuts. Additionally, some new $\mathscr{I}\mathscr{F}$-mappings are introduced and with of help of these mappings, some interesting results are also obtained using generalized differentiability, where differentiable mappings are $\mathscr{I}\mathscr{F}$-mappings. Some new versions of $\mathscr{V}\mathscr{L}$-inequalities are also acquired.

3.1 *Invex intuitionistic fuzzy sets*

In this subsection, we generalized the notions of convex, quasi-convex $\mathscr{I}\mathscr{F}$-set and introduced the notions of invex, quasi-invex $\mathscr{I}\mathscr{F}$-sets and some basic results are proved with the help of $(\tilde{\mathfrak{v}},\ \tilde{\mathfrak{w}})$-cut.

Definition 7 An $\mathscr{I}\mathscr{F}$-set $\mathfrak{A}$ on R is defined as invex with respect to η if

$$\Lambda_{\mathfrak{A}}(\varphi + \tau\eta(\omega,\ \varphi)) \geq (1-\tau)\Lambda_{\mathfrak{A}}(\varphi) + \tau\Lambda_{\mathfrak{A}}(\omega) \quad \forall \varphi,\ \omega \in R,\ \tau \in [0,\ 1], \tag{37}$$

$$\upsilon_{\mathfrak{A}}(\varphi + \tau\eta(\omega,\ \varphi)) \leq (1-\tau)\upsilon_{\mathfrak{A}}(\varphi) + \tau\upsilon_{\mathfrak{A}}(\omega) \quad \forall \varphi,\ \omega \in R,\ \tau \in [0,\ 1], \tag{38}$$

where η is a bi-function from $R \times R$ to R. Strictly invex $\mathscr{I}\mathscr{F}$-set $\mathfrak{A}$ on R if strict inequality holds for $\Lambda_{\mathfrak{A}}(\varphi) \neq \Lambda_{\mathfrak{A}}(\omega)$ and $\upsilon_{\mathfrak{A}}(\varphi) \neq \upsilon_{\mathfrak{A}}(\omega)$. An $\mathscr{I}\mathscr{F}$-set $\mathfrak{A}$ on R is defined as incave if

$$\Lambda_{\mathfrak{A}}(\varphi + \tau\eta(\omega,\ \varphi)) \leq (1-\tau)\Lambda_{\mathfrak{A}}(\varphi) + \tau\Lambda_{\mathfrak{A}}(\omega) \quad \forall \varphi,\ \omega \in R,\ \tau \in [0,\ 1], \tag{39}$$

$$\upsilon_{\mathfrak{A}}(\varphi+\tau\eta(\omega,\ \varphi)) \geq (1-\tau)\upsilon_{\mathfrak{A}}(\varphi)+\tau\upsilon_{\mathfrak{A}}(\omega) \quad \forall\varphi,\ \omega\in R,\ \tau\in[0,\ 1]. \tag{40}$$

Strictly incave $\mathscr{I}\mathscr{F}$-set $\mathfrak{A}$ on R if strict inequality holds for $\Lambda_{\mathfrak{A}}(\varphi)\neq\Lambda_{\mathfrak{A}}(\omega)$ and $\upsilon_{\mathfrak{A}}(\varphi)\neq\upsilon_{\mathfrak{A}}(\omega)$.

Theorem 1 If $\mathfrak{A}$ be an $\mathscr{I}\mathscr{F}$-set R and invex with respect to η, then the level set $\mathfrak{A}_{(\tilde{\mathfrak{v}},\ \tilde{\mathfrak{w}})}=\{\varphi\in R\mid\Lambda_{\mathfrak{A}}(\varphi)\geq \tilde{\mathfrak{v}}$ and $\upsilon_{\mathfrak{A}}(\omega)\leq\tilde{\mathfrak{w}},\ \tilde{\mathfrak{v}},\ \tilde{\mathfrak{w}}\in[0,\ 1]\}$ is invex set.

Proof. Let $\varphi,\ \omega\in\mathfrak{A}_{\tilde{\mathfrak{v}}}$ then by definition of $\mathfrak{A}_{\tilde{\mathfrak{v}}}$, $\Lambda_{\mathfrak{A}}(\varphi)\leq\tilde{\mathfrak{v}}$ and $\upsilon_{\mathfrak{A}}(\omega)\geq\tilde{\mathfrak{w}}$. As $\varphi,\ \omega\in R$ and $\mathfrak{A}$ is invex then for $\tau\in[0,\ 1]$,

$$\Lambda_{\mathfrak{A}}(\varphi+\tau\eta(\omega,\ \varphi)) \geq (1-\tau)\Lambda_{\mathfrak{A}}(\varphi)+\tau\Lambda_{\mathfrak{A}}(\omega) \geq (1-\tau)\tilde{\mathfrak{v}}+\tau\tilde{\mathfrak{v}}=\tilde{\mathfrak{v}}, \tag{41}$$

and

$$\upsilon_{\mathfrak{A}}(\varphi+\tau\eta(\omega,\ \varphi)) \leq (1-\tau)\upsilon_{\mathfrak{A}}(\varphi)+\tau\upsilon_{\mathfrak{A}}(\omega) \leq (1-\tau)\tilde{\mathfrak{w}}+\tau\tilde{\mathfrak{w}}=\tilde{\mathfrak{w}}. \tag{42}$$

which implies that, $\varphi+\tau\eta(\omega,\ \varphi)\in\mathfrak{A}_{(\tilde{\mathfrak{v}},\ \tilde{\mathfrak{w}})}$, hence the level set $\mathfrak{A}_{(\tilde{\mathfrak{v}},\ \tilde{\mathfrak{w}})}$ is an invex set. □

Definition 8 An $\mathscr{I}\mathscr{F}$-set $\mathfrak{A}$ on R is defined as quasi-invex with respect to η if

$$\Lambda_{\mathfrak{A}}(\varphi+\tau\eta(\varphi,\ \omega)) \geq \min(\Lambda_{\mathfrak{A}}(\varphi),\ \Lambda_{\mathfrak{A}}(\omega)) \quad \forall\ \varphi,\ \omega\in R,\ \tau\in[0,\ 1] \tag{43}$$

$$\upsilon_{\mathfrak{A}}(\omega+\tau\eta(\omega,\ \varphi)) \leq \max(\upsilon_{\mathfrak{A}}(\varphi),\ \upsilon_{\mathfrak{A}}(\omega)) \quad \forall\ \varphi,\ \omega\in R,\ \tau\in[0,\ 1] \tag{44}$$

where η is a bi-function from $R\times R$ to R. Strictly quasi-invex $\mathscr{I}\mathscr{F}$-set $\mathfrak{A}$ on R if strict inequality holds for $\Lambda_{\mathfrak{A}}(\varphi)\neq\Lambda_{\mathfrak{A}}(\omega)$ and $\upsilon_{\mathfrak{A}}(\varphi)\neq\upsilon_{\mathfrak{A}}(\omega)$. An $\mathscr{I}\mathscr{F}$-set $\mathfrak{A}$ on R is defined as quasi-incave if

$$\Lambda_{\mathfrak{A}}((1-\tau)\varphi+\tau\omega) \leq \max(\Lambda_{\mathfrak{A}}(\varphi),\ \Lambda_{\mathfrak{A}}(\omega)) \quad \forall\ \varphi,\ \omega\in R,\ \tau\in[0,\ 1], \tag{45}$$

$$\upsilon_{\mathfrak{A}}((1-\tau)\varphi+\tau\omega) \geq \min(\upsilon_{\mathfrak{A}}(\varphi),\ \upsilon_{\mathfrak{A}}(\omega)) \quad \forall\ \varphi,\ \omega\in R,\ \tau\in[0,\ 1]. \tag{46}$$

Strictly quasi-incave $\mathscr{I}\mathscr{F}$-set $\mathfrak{A}$ on R if strict inequality holds for $\Lambda_{\mathfrak{A}}(\varphi)\neq\Lambda_{\mathfrak{A}}(\omega)$ and $\upsilon_{\mathfrak{A}}(\varphi)\neq\upsilon_{\mathfrak{A}}(\omega)$.

Theorem 2 If $\mathfrak{A}$ be an $\mathscr{I}\mathscr{F}$-set R and quasi-invex with respect to η if and only if the level set

$$\mathfrak{A}_{(\widetilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}})}=\left\{\varphi\in R|\Lambda_{\mathfrak{A}}(\omega)\geq\widetilde{\mathfrak{v}} \text{ and } \upsilon_{\mathfrak{A}}(\omega)\leq\widetilde{\mathfrak{w}},\ \widetilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}}\in[0,\ 1]\right\} \tag{47}$$

is quasi-invex set.

Proof. Let $\varphi,\ \omega\in\mathfrak{A}_{(\widetilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}})}$ then by definition of $\mathfrak{A}_{(\widetilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}})}$, $\Lambda_{\mathfrak{A}}(\varphi)\leq\widetilde{\mathfrak{v}}$ and $\upsilon_{\mathfrak{A}}(\omega)\geq\widetilde{\mathfrak{w}}$. As $\mathfrak{A}$ is invex, then

$$\begin{aligned}\Lambda_{\mathfrak{A}}(\varphi+\tau\eta(\omega,\ \varphi)) &\geq \min(\Lambda_{\mathfrak{A}}(\varphi),\ \Lambda_{\mathfrak{A}}(\omega)), \quad \forall\ \varphi,\ \omega\in R,\ \tau\in[0,\ 1],\\ &\geq \min\left(\widetilde{\mathfrak{v}},\ \widetilde{\mathfrak{v}}\right)=\widetilde{\mathfrak{v}},\end{aligned} \tag{48}$$

and

$$\upsilon_{\mathfrak{A}}\left(\varphi+\tau\eta(\omega,\ \varphi)\right)\leq\max\left(\upsilon_{\mathfrak{A}}(\varphi),\ \upsilon_{\mathfrak{A}}(\omega)\right),\quad\forall\ \varphi,\ \omega\in R,\ \tau\in[0,\ 1],$$

$$\leq\max\left(\widetilde{\mathfrak{w}},\ \widetilde{\mathfrak{w}}\right)=\widetilde{\mathfrak{w}},\tag{49}$$

from which we can note that $\varphi+\tau\eta(\omega,\ \varphi)\in\mathfrak{A}_{(\widetilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}})}$ is an invex set.

Conversely, assume that $\mathfrak{A}_{(\widetilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}})}$ be an invex set, to prove the $\mathscr{I}\mathscr{F}$-set $\mathfrak{A}$ is invex. As $\mathfrak{A}_{(\widetilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}})}$ be an invex set then for any $\varphi,\ \omega\in\mathfrak{A}_{(\widetilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}})}$ such that $\varphi+\tau\eta(\omega,\ \varphi)\in\mathfrak{A}_{(\widetilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}})}$ with $\tau\in[0,\ 1]$. We take $\min\left(\Lambda_{\mathfrak{A}}(\varphi),\ \Lambda_{\mathfrak{A}}(\omega)\right)=\widetilde{\mathfrak{v}}$ and $\Lambda_{\mathfrak{A}}(\varphi)\leq\Lambda_{\mathfrak{A}}(\omega)$, $\max\left(\upsilon_{\mathfrak{A}}(\varphi),\ \upsilon_{\mathfrak{A}}(\omega)\right)=\widetilde{\mathfrak{w}}$ and $\upsilon_{\mathfrak{A}}(\varphi)\geq\upsilon_{\mathfrak{A}}(\omega)$.

By the definition of invex set $\mathfrak{A}_{(\widetilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}})}$, we have

$$\Lambda_{\mathfrak{A}}\left(\varphi+\tau\eta(\omega,\ \varphi)\right)\geq\min\left(\Lambda_{\mathfrak{A}}(\varphi),\ \Lambda_{\mathfrak{A}}(\omega)\right)\geq\widetilde{\mathfrak{v}},\tag{50}$$

and

$$\upsilon_{\mathfrak{A}}\left(\varphi+\tau\eta(\omega,\ \varphi)\right)\leq\max\left(\upsilon_{\mathfrak{A}}(\varphi),\ \upsilon_{\mathfrak{A}}(\omega)\right)\leq\widetilde{\mathfrak{w}}.\tag{51}$$

The following result shows that the $\mathscr{I}\mathscr{F}$-set $\mathfrak{A}$ is invex. □

3.2 *Non-differentiable preinvex functions*

Two different kinds of preinvex function are defined like $\mathscr{I}\mathscr{F}$-preinvex function, $\mathscr{I}\mathscr{F}$-quasi-preinvex function and their properties are discussed. First, we start with classical definition of invex set.

Definition 9 [16] Let $\varphi\in\mathfrak{D}$. Then the set $\mathfrak{D}$ is defined as an invex set at φ if

$$\varphi+\tau\eta(\omega,\ \varphi)\in\mathfrak{D},\quad\forall\ \omega\in\mathfrak{D},\ \tau\in[0,\ 1],\tag{52}$$

where $\eta:\mathfrak{D}\times\mathfrak{D}\to R$ be bi-function.

$\mathfrak{D}$ would be an invex set if $\mathfrak{D}$ is an invex at each $\varphi\in\mathfrak{D}$.

Remark 1 Every invex set is convex set but the converse is not necessarily true because every convex set is invex set but converse is true.

Example 1 If $D=R-\left(\frac{-1}{2},\ \frac{1}{2}\right)$ then $\mathfrak{D}$ is an invex set but not convex because

$$\eta(\varphi,\ \omega)=\begin{cases}\omega-\varphi & \text{for } \omega>0,\ \varphi>0 \text{ or } \omega<0,\ \varphi<0,\\ \varphi-\omega & \text{for } \omega>0,\ \varphi<0 \text{ or } \omega<0,\ \varphi>0,\end{cases}\tag{53}$$

where $\eta:D\times D\to D$.

Theorem 3 If $\mathscr{K}$ be the collection of all invex sets, then the intersection of invex sets in $\mathscr{K}$ is an invex set.

Proof. Let $\mathscr{K}=\left\{D_j\right\}_{j\in I}$ be a family of invex sets, to prove $\bigcap_{j\in I}D_j$ is an invex set.

As $\mathscr{K} = \{D_j\}_{j\in I}$ be a family of invex sets in respect of some η, then for any $\varphi,\ \omega \in D_j\ \forall\ j\in I$ in respect of some η and $0\le\tau\le 1$ such that

$$\varphi+\tau\eta(\omega,\ \varphi)\in D_j \quad \forall\ j\in I, \tag{54}$$

implies

$$\varphi+\tau\eta(\omega,\ \varphi)\in\bigcap_{j\in I} D_j, \tag{55}$$

hence the intersection of invex sets in collection $\mathscr{K}$ is an $\mathscr{I}\mathscr{F}$-invex set. □

Note that in next work we take $\mathfrak{D}$ as an invex set over set of real numbers, otherwise mentioned.

Definition 10 A mapping $\widetilde{\mathfrak{S}}:\mathfrak{D}\to\mathscr{I}\mathscr{F}\mathscr{N}(R)$ is called $\mathscr{I}\mathscr{F}$-mapping. For each $\widetilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}}\in[0,\ 1]$, associated to $\widetilde{\mathfrak{S}}$, we define the family of interval valued functions $\widetilde{\mathfrak{S}}_{(\widetilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}})}:D\to\mathscr{K}$ defined by $\widetilde{\mathfrak{S}}_{(\widetilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}})}(\varphi)=_{(\Lambda,\ \upsilon)}\left[\widetilde{\mathfrak{S}}(\varphi)\right]^{(\widetilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}})}=_{(\Lambda,\ \upsilon)}\left[\widetilde{\mathfrak{S}}_*\left(\varphi,\ (\widetilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}})\right),\ \widetilde{\mathfrak{S}}^*\left(\varphi,\ (\widetilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}})\right)\right]$.

Definition 11 Let $\mathfrak{D}$ be an invex set in $\mathscr{I}\mathscr{F}\mathscr{N}(R)$. Then $\mathscr{I}\mathscr{F}M$ $\widetilde{\mathfrak{S}}:D\to\mathscr{I}\mathscr{F}\mathscr{N}(R)$ is defined as preinvex at $\varphi\in D$ if

$$\widetilde{\mathfrak{S}}\left(\varphi+\tau\eta(\omega,\ \varphi)\right)\le_{(\Lambda,\ \upsilon)}(1-\tau)\widetilde{\mathfrak{S}}(\varphi)+\tau\widetilde{\mathfrak{S}}(\omega), \quad \forall\ \omega\in D,\ \tau\in[0,\ 1]. \tag{56}$$

Where $\le_{(\Lambda,\ \upsilon)}$ is an $\mathscr{I}\mathscr{F}$-order relation. $\widetilde{\mathfrak{S}}$ is defined as preinvex on $\mathfrak{D}$ if it is preinvex at each $\varphi\in\mathfrak{D}$ and note that these inequalities "$(\Lambda,\ \upsilon)$" are used for $\mathscr{I}\mathscr{F}$-number which represents membership function Λ and non-membership function υ, respectively.

Strictly $\mathscr{I}\mathscr{F}$-preinvex mapping if strict inequality holds for $\widetilde{\mathfrak{S}}(\varphi)\neq_{(\Lambda,\ \upsilon)}\widetilde{\mathfrak{S}}(\omega)$. $\widetilde{\mathfrak{S}}:\mathfrak{D}\to\mathscr{I}\mathscr{F}\mathscr{N}(R)$ is defined as $\mathscr{I}\mathscr{F}$-preincave mapping at $\varphi\in\mathfrak{D}$, if $\tau\widetilde{\mathfrak{S}}(\varphi)+(1-\tau)\widetilde{\mathfrak{S}}(\omega)\ge_{(\Lambda,\ \upsilon)}\widetilde{\mathfrak{S}}\left(\varphi+\tau\eta(\omega,\ \varphi)\right),\forall\ \omega\in\mathfrak{D},\ \tau\in[0,\ 1]$ and $\widetilde{\mathfrak{S}}$ is defined as preincave on $\mathfrak{D}$ if it is preincave at each $\varphi\in\mathfrak{D}$. Strictly $\mathscr{I}\mathscr{F}$-preincave mapping if strict inequality holds for $\widetilde{\mathfrak{S}}(\varphi)\neq_{(\Lambda,\ \upsilon)}\widetilde{\mathfrak{S}}(\omega)$.

Remark 2 The $\mathscr{I}\mathscr{F}$-preinvex functions have some very nice properties similar to $\mathscr{I}\mathscr{F}$-convex functions.

1. if $\widetilde{\mathfrak{S}}$ is $\mathscr{I}\mathscr{F}$-preinvex function, then $\Upsilon\widetilde{\mathfrak{S}}$ is also preinvex for $\Upsilon\ge 0$.
2. if $\widetilde{\mathfrak{S}}$ and $\tilde{G}$ both are $\mathscr{I}\mathscr{F}$-preinvex functions, the $\widetilde{\mathfrak{S}}+\tilde{G}$ and $\max\left(\widetilde{\mathfrak{S}}(\varphi),\ \tilde{G}(\varphi)\right)$ are also preinvex functions.

Example 2 We consider the $\mathscr{I}\mathscr{F}$-mappings $\widetilde{\mathfrak{S}}:\mathfrak{D}\to\mathscr{I}\mathscr{F}\mathscr{N}(R)$ defined by,

$$\widetilde{\mathfrak{S}}(\varphi)(\sigma)=_{\Lambda}\begin{cases}\dfrac{4\sigma-\varphi}{3\varphi}\omega, & \sigma\in\left[\dfrac{1}{4}\varphi,\ \varphi\right],\\[2mm] \dfrac{3\varphi-2\sigma}{\varphi}\omega, & \sigma\in\left(\varphi,\ \dfrac{3}{2}\varphi\right],\\[2mm] 0, & \sigma\in\left[\dfrac{1}{4}\varphi,\ \dfrac{3}{2}\varphi\right],\end{cases} \tag{57}$$

$$\widetilde{\mathfrak{S}}(\varphi)(\sigma) =_{\Lambda} \begin{cases} \dfrac{4(\varphi-\sigma)+\mu(4\sigma-\varphi)}{3\varphi}, & \sigma \in \left[\dfrac{1}{4}\varphi,\ \varphi\right], \\ \dfrac{2(\sigma-\varphi)+\mu(3\varphi-2\sigma)}{\varphi}, & \sigma \in \left(\varphi,\ \dfrac{3}{2}\varphi\right], \\ 1, & \sigma \in \left[\dfrac{1}{4}\varphi,\ \dfrac{3}{2}\varphi\right], \end{cases} \tag{58}$$

For each $\widetilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}} \in [0,\ 1]$ and we have

$$\widetilde{\mathfrak{S}}_{(\widetilde{\mathfrak{v}},\ \widetilde{\mathfrak{w}})}(\varphi) = \left(\left[\frac{3\widetilde{\mathfrak{v}}+\omega}{4}\varphi,\ \frac{3\omega-\widetilde{\mathfrak{v}}}{2}\varphi\right],\ \left[\frac{4-3\widetilde{\mathfrak{w}}-\mu}{1-\mu}\varphi,\ \frac{2+\widetilde{\mathfrak{w}}-3\mu}{1-\mu}\varphi\right]\right) \tag{59}$$

we can easily see that the end point function are preinvex.

Note that if $\omega = 1,\ \varphi = 0$, then $\mathscr{I}\mathscr{F}$-preinvex function $\widetilde{\mathfrak{S}}$ becomes a $\mathscr{F}$-preinvex mapping.

Theorem 4 If $\widetilde{\mathfrak{S}}_j : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ is $\mathscr{I}\mathscr{F}$-preinvex and $\gamma_j \geq 0,\ j = 1,\ 2,\ 3,\ \ldots,\ m$. Then $\sum_{j=1}^{m} \gamma_j \widetilde{\mathfrak{S}}_j(\varphi)$ is $\mathscr{I}\mathscr{F}$-preinvex.

Proof. Let $\widetilde{\mathfrak{S}}_j : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ be $\mathscr{I}\mathscr{F}$-preinvex such that

$$\widetilde{\mathfrak{S}}_j(\varphi+\tau\eta(\omega,\ \varphi)) \leq_{(\Lambda,\ \upsilon)} (1-\tau)\widetilde{\mathfrak{S}}_j(\varphi)+\tau\widetilde{\mathfrak{S}}_j(\omega), \quad j = 1,\ 2,\ 3,\ \ldots,\ m \tag{60}$$

by the convex combination,

$$\tau\sum_{j=1}^{m}\gamma_j\widetilde{\mathfrak{S}}_j(\varphi)+(1-\tau)\sum_{j=1}^{m}\gamma_j\widetilde{\mathfrak{S}}_j(\omega) = \sum_{j=1}^{m}\gamma_j\left\{(1-\tau)\widetilde{\mathfrak{S}}_j(\varphi)+\tau\widetilde{\mathfrak{S}}_j(\omega)\right\} \leq_{(\Lambda,\ \upsilon)} \sum_{j=1}^{m}\gamma_j\widetilde{\mathfrak{S}}_j(\varphi+\tau\eta(\omega,\ \varphi)) \tag{61}$$

which implies that the $\sum_{j=1}^{m} \gamma_j \widetilde{\mathfrak{S}}_j(\varphi)$ is $\mathscr{I}\mathscr{F}$-preinvex. □

Definition 12 Let $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ be an $\mathscr{I}\mathscr{F}M$ then the level set of $\widetilde{\mathfrak{S}}$ is denoted by $\widetilde{\mathfrak{S}}_{\tilde{\gamma}}$ and defined as,

$$\widetilde{\mathfrak{S}}_{\tilde{\gamma}} = \left\{\varphi : \varphi \in \mathfrak{D},\ \widetilde{\mathfrak{S}}(\varphi) \leq_{(\Lambda,\ \upsilon)} \tilde{\gamma} \in \mathscr{I}\mathscr{F}\mathscr{N}(R)\right\}. \tag{62}$$

Theorem 5 Let $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ be an $\mathscr{I}\mathscr{F}$-preinvex. Then, $\widetilde{\mathfrak{S}}_{\tilde{\gamma}}$ is an invex set.

Proof. Let $\varphi,\ \omega \in \widetilde{\mathfrak{S}}_{\tilde{\gamma}}$ then $\widetilde{\mathfrak{S}}(\varphi) \leq_{(\Lambda,\ \upsilon)} \tilde{\gamma}$, and $\widetilde{\mathfrak{S}}(\omega) \leq_{(\Lambda,\ \upsilon)} \tilde{\gamma}$. Since $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ is an $\mathscr{I}\mathscr{F}$-preinvex so,

$$\widetilde{\mathfrak{S}}(\varphi+\tau\eta(\omega,\ \varphi)) \leq_{(\Lambda,\ \upsilon)} (1-\tau)\widetilde{\mathfrak{S}}(\varphi)+\tau\widetilde{\mathfrak{S}}(\omega) \leq_{(\Lambda,\ \upsilon)} (1-\tau)\tilde{\gamma}+\tau\tilde{\gamma} = \tilde{\gamma}, \tag{63}$$

hence $\widetilde{\mathfrak{S}}_{\tilde{\gamma}}$ is an invex set. □

Theorem 6 The $\mathscr{I}\mathscr{F}M$ $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ is preinvex if and only if

$$epi\left(\widetilde{\mathfrak{S}}\right)=\left\{(\varphi,\ \tilde{\gamma}):\varphi\in\mathfrak{D},\ \widetilde{\mathfrak{S}}(\varphi)\leq_{(\Lambda,\ \upsilon)}\tilde{\gamma},\ \tilde{\gamma}\in\mathscr{I}\mathscr{F}\mathscr{N}(R)\right\}, \tag{64}$$

is an invex set.

Proof. Let $(\varphi,\ \tilde{\gamma}_1)$, $(\omega,\ \tilde{\gamma}_1)\in D$ and $\widetilde{\mathfrak{S}}:\mathfrak{D}\to\mathscr{I}\mathscr{F}\mathscr{N}(R)$ is preinvex. To prove $epi(\widetilde{\mathfrak{S}})$ is an invex set. As $(\varphi,\ \tilde{\gamma}_1)$, $(\omega,\ \tilde{\gamma}_1)\in\mathfrak{D}$ so $\widetilde{\mathfrak{S}}(\varphi)\leq_{(\Lambda,\ \upsilon)}\tilde{\gamma}_1$, and $\widetilde{\mathfrak{S}}(\omega)\leq_{(\Lambda,\ \upsilon)}\tilde{\gamma}_2$.

Now $\widetilde{\mathfrak{S}}:\mathfrak{D}\to\mathscr{I}\mathscr{F}\mathscr{N}(R)$ is preinvex so $\forall\ \varphi,\ \omega\in\mathfrak{D}$ and $\tau\in(0,\ 1)$, we have

$$\widetilde{\mathfrak{S}}(\varphi+\tau\eta(\omega,\ \varphi))\leq_{(\Lambda,\ \upsilon)}(1-\tau)\widetilde{\mathfrak{S}}_j(\varphi)+\tau\widetilde{\mathfrak{S}}_j(\omega)\leq_{(\Lambda,\ \upsilon)}(1-\tau)\tilde{\gamma}_1+\tau\tilde{\gamma}_2 \tag{65}$$

which implies that

$$\widetilde{\mathfrak{S}}(\varphi+\tau\eta(\omega,\ \varphi))\leq_{(\Lambda,\ \upsilon)}(1-\tau)\tilde{\gamma}_1+\tau\tilde{\gamma}_2 \tag{66}$$

for which it follows that $(\varphi+\tau\eta(\omega,\ \varphi),\ (1-\tau)\tilde{\gamma}_1+\tau\tilde{\gamma}_2)\in epi(\widetilde{\mathfrak{S}})$ so $epi(\widetilde{\mathfrak{S}})$ is an invex set.

Conversely, let $epi(\widetilde{\mathfrak{S}})$ is an invex set to prove $\widetilde{\mathfrak{S}}:\mathfrak{D}\to\mathscr{I}\mathscr{F}\mathscr{N}(R)$ is preinvex. Assume that $\omega,\ \varphi\in\mathfrak{D}$ then, $\left(\varphi,\ \left(\widetilde{\mathfrak{S}}(\varphi),\ \widetilde{\mathfrak{S}}(\varphi)\right)\right)\in epi(\widetilde{\mathfrak{S}})$ and $\left(\omega,\ \left(\widetilde{\mathfrak{S}}(\omega),\ \widetilde{\mathfrak{S}}(\omega)\right)\right)\in epi(\widetilde{\mathfrak{S}})$. As $epi(\widetilde{\mathfrak{S}})$ is an invex set so

$$\left(\varphi+\tau\eta(\omega,\ \varphi),\ \left((1-\tau)\widetilde{\mathfrak{S}}(\varphi)+\tau\widetilde{\mathfrak{S}}(\omega),\ (1-\tau)\widetilde{\mathfrak{S}}(\varphi)+\tau\widetilde{\mathfrak{S}}(\omega)\right)\right)\in epi\left(\widetilde{\mathfrak{S}}\right), \tag{67}$$

so, by definition of $epi\left(\widetilde{\mathfrak{S}}\right)$,

$$\widetilde{\mathfrak{S}}(\varphi+\tau\eta(\omega,\ \varphi))\leq_{(\Lambda,\ \upsilon)}(1-\tau)\widetilde{\mathfrak{S}}(\varphi)+\tau\widetilde{\mathfrak{S}}(\omega), \tag{68}$$

Hence, $\mathscr{I}\mathscr{F}M$ $\widetilde{\mathfrak{S}}:\mathfrak{D}\to\mathscr{I}\mathscr{F}\mathscr{N}(R)$ is preinvex. □

Theorem 7 Let $\widetilde{\mathfrak{S}}:\mathfrak{D}\to\mathscr{I}\mathscr{F}\mathscr{N}(R)$ be an $\mathscr{I}\mathscr{F}$-preinvex, with $\tilde{\omega}=_{(\Lambda,\ \upsilon)}\inf_{\varphi\in\mathfrak{D}}\left(\widetilde{\mathfrak{S}}(\varphi)\right)$, then set $\$=\left\{\varphi:\varphi\in\mathfrak{D},\ \widetilde{\mathfrak{S}}(\varphi)=_{(\Lambda,\ \upsilon)}\tilde{\omega}\right\}$ is an invex set.

Proof. Since $\widetilde{\mathfrak{S}}$ an $\mathscr{I}\mathscr{F}$-preinvex function the by definition of $\widetilde{\mathfrak{S}}$, $\forall\ \varphi,\ \omega\in\$$ implies that $\varphi,\ \omega\in D$ and $\tau\in(0,\ 1)$, we have

$$\widetilde{\mathfrak{S}}(\varphi+\tau\eta(\omega,\ \varphi))\leq_{(\Lambda,\ \upsilon)}(1-\tau)\widetilde{\mathfrak{S}}(\varphi)+\tau\widetilde{\mathfrak{S}}(\omega)=_{(\Lambda,\ \upsilon)}(1-\tau)\tilde{\omega}+\tau\tilde{\omega}=_{(\Lambda,\ \upsilon)}\tilde{\omega}, \tag{69}$$

hence $\varphi+\tau\eta(\omega,\ \varphi)\in\$$ so $\$$ is an $\mathscr{I}\mathscr{F}$-invex set. □

Theorem 8 Let $\widetilde{\mathfrak{S}}:\mathfrak{D}\to\mathscr{I}\mathscr{F}\mathscr{N}(R)$ be a strictly $\mathscr{I}\mathscr{F}$-preinvex, with $\tilde{\omega}=_{(\Lambda,\ \upsilon)}\inf_{\varphi\in\mathfrak{D}}\left(\widetilde{\mathfrak{S}}(\varphi)\right)$, then $\$=\left\{\varphi:\varphi\in\mathfrak{D},\ \widetilde{\mathfrak{S}}(\varphi)=_{(\Lambda,\ \upsilon)}\tilde{\omega}\right\}$ is a singleton set.

Proof. Let $\widetilde{\mathfrak{S}}$ be a strictly $\mathscr{I}\mathscr{F}$-preinvex to prove $\$$ is a singleton set. Contrary suppose that $\widetilde{\mathfrak{S}}(\varphi)=_{(\Lambda,\ \upsilon)}\tilde{\omega}=_{(\Lambda,\ \upsilon)}\widetilde{\mathfrak{S}}(\omega)$. As $\widetilde{\mathfrak{S}}$ is strictly $\mathscr{I}\mathscr{F}$-preinvex so $\forall\ \varphi,\ \omega\in\mathfrak{D}$ and $\tau\in(0,\ 1)$, we have

$$\widetilde{\mathfrak{S}}\left(\varphi+\tau\eta(\omega,\ \varphi)\right) <_{(\Lambda,\ \upsilon)} (1-\tau)\widetilde{\mathfrak{S}}(\varphi)+\tau\widetilde{\mathfrak{S}}(\omega) =_{(\Lambda,\ \upsilon)} (1-\tau)\tilde{\omega}+\tau\tilde{\omega} =_{(\Lambda,\ \upsilon)} \tilde{\omega}, \tag{70}$$

which implies that $\tilde{\omega} \neq_{(\Lambda,\ \upsilon)} \inf_{\varphi\in\mathfrak{D}} \left(\widetilde{\mathfrak{S}}(\varphi)\right)$ so this is not possible. So $ is a singleton set. □

Definition 13 The $\mathscr{I}\mathscr{F}$-mapping $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ is defined as quasi-preinvex at $\varphi \in \mathfrak{D}$ if

$$\widetilde{\mathfrak{S}}\left(\varphi+\tau\eta(\omega,\ \varphi)\right) \leq_{(\Lambda,\ \upsilon)} \max\left(\widetilde{\mathfrak{S}}(\varphi),\ \widetilde{\mathfrak{S}}(\omega)\right), \quad \forall\ \omega \in \mathfrak{D},\ \tau \in [0,\ 1], \tag{71}$$

and $\widetilde{\mathfrak{S}}$ is defined as quasi-preinvex on $\mathfrak{D}$ if it is preinvex at each $\varphi \in \mathfrak{D}$. Strictly $\mathscr{I}\mathscr{F}$-quasi-preinvex mapping if strict inequality holds for $\widetilde{\mathfrak{S}}(\varphi) \neq_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}(\omega)$. $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ is defined as $\mathscr{I}\mathscr{F}$-quasi-preincave mapping at $\varphi \in \mathfrak{D}$, if $\widetilde{\mathfrak{S}}\left(\varphi+\tau\eta(\omega,\ \varphi)\right) \geq_{(\Lambda,\ \upsilon)} \min\left(\widetilde{\mathfrak{S}}(\varphi),\ \widetilde{\mathfrak{S}}(\omega)\right)$ $\forall\ \omega \in \mathfrak{D},\ \tau \in [0,\ 1]$ and $\widetilde{\mathfrak{S}}$ is defined as quasi-preincave on $\mathfrak{D}$ if it is quasi-preincave at each $\varphi \in \mathfrak{D}$. Strictly $\mathscr{I}\mathscr{F}$-quasi-preincave mapping if strict inequality holds for $\widetilde{\mathfrak{S}}(\varphi) \neq_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}(\omega)$.

Theorem 9 The $\mathscr{I}\mathscr{F}$-mapping $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ is defined as quasi-preinvex if and only if the level set $\widetilde{\mathfrak{S}}_{\tilde{\gamma}} = \left\{\varphi : \varphi \in \mathfrak{D},\ \widetilde{\mathfrak{S}}(\varphi) \leq_{(\Lambda,\ \upsilon)} \tilde{\gamma},\ \tilde{\gamma} \in \mathscr{I}\mathscr{F}\mathscr{N}(R)\right\}$ is an invex set.

Proof. Let $\varphi,\ \omega \in \widetilde{\mathfrak{S}}_{\tilde{\gamma}}$ then by definition of $\widetilde{\mathfrak{S}}_{\tilde{\gamma}}$, $\widetilde{\mathfrak{S}}(\varphi) \leq \tilde{\gamma}$. As $\varphi,\ \omega \in \mathfrak{D}$ and $\mathfrak{D}$ is invex so $\varphi+\tau\eta(\omega,\ \varphi) \in \mathfrak{D}$. Since $\widetilde{\mathfrak{S}}$ is quasi-preinvex then,

$$\begin{aligned}\widetilde{\mathfrak{S}}\left(\varphi+\tau\eta(\omega,\ \varphi)\right) &\leq_{(\Lambda,\ \upsilon)} \max\left(\widetilde{\mathfrak{S}}(\varphi),\ \widetilde{\mathfrak{S}}(\omega)\right), \quad \forall\ \varphi,\ \omega \in R,\ \tau \in [0,\ 1],\\ &\leq_{(\Lambda,\ \upsilon)} \max\left(\tilde{\gamma},\ \tilde{\gamma}\right) = \tilde{\gamma},\end{aligned} \tag{72}$$

□

From which we can note that $\varphi+\tau\eta(\omega,\ \varphi) \in \widetilde{\mathfrak{S}}_{\tilde{\gamma}}$ is an invex set.

Conversely, assume that $\mathfrak{D}_{\tilde{\gamma}}$ be an invex set to prove $\widetilde{\mathfrak{S}}$ is quasi-preinvex. As $\widetilde{\mathfrak{S}}_{\tilde{\gamma}}$ be an invex set then for any $\varphi,\ \omega \in \widetilde{\mathfrak{S}}_{\tilde{\gamma}}$ such that $\varphi+\tau\eta(\omega,\ \varphi) \in \widetilde{\mathfrak{S}}_{\tilde{\gamma}}$ with $\tau \in [0,\ 1]$. Now we take $\max\left(\widetilde{\mathfrak{S}}(\varphi),\ \widetilde{\mathfrak{S}}(\omega)\right) = \widetilde{\gamma}$ and $\widetilde{\mathfrak{S}}(\varphi) \geq_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}(\omega)$.

By the definition of invex set $\widetilde{\mathfrak{S}}_{\tilde{\gamma}}$, we have

$$\widetilde{\mathfrak{S}}\left(\varphi+\tau\eta(\omega,\ \varphi)\right) \leq_{(\Lambda,\ \upsilon)} \max\left(\widetilde{\mathfrak{S}}(\varphi),\ \widetilde{\mathfrak{S}}(\omega)\right) \leq_{(\Lambda,\ \upsilon)} \widetilde{\gamma} \tag{73}$$

Hence, $\mathscr{I}\mathscr{F}$-mapping $\widetilde{\mathfrak{S}}$ is quasi-preinvex.

Theorem 10 Let $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ be an $\mathscr{I}\mathscr{F}$-preinvex and $\forall\ \varphi,\ \omega \in \mathfrak{D}$ such that $\widetilde{\mathfrak{S}}(\varphi) >_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}(\omega)$, then $\widetilde{\mathfrak{S}}$ is strictly $\mathscr{I}\mathscr{F}$-quasi-preinvex.

Proof. Since $\widetilde{\mathfrak{S}}$ is $\mathscr{I}\mathscr{F}$-preinvex then, $\forall\ \tau \in (0,\ 1)$ and $\varphi,\ \omega \in \mathfrak{D}$ such that

$$\widetilde{\mathfrak{S}}\left(\varphi+\tau\eta(\omega,\ \varphi)\right) \leq_{(\Lambda,\ \upsilon)} (1-\tau)\widetilde{\mathfrak{S}}(\varphi)+\tau\widetilde{\mathfrak{S}}(\omega) <_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}(\varphi), \tag{74}$$

which implies $\widetilde{\mathfrak{S}}$ is strictly $\mathscr{I}\mathscr{F}$-quasi-preinvex. □

Theorem 11 Let $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ be an $\mathscr{I}\mathscr{F}$-mapping then $\mathscr{I}\mathscr{F}$-preinvexity $\Rightarrow$ $\mathscr{I}\mathscr{F}$-quasi-preinvexity.

Proof. Let $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ be an $\mathscr{I}\mathscr{F}$-preinvexity then, $\forall\ \tau \in (0,\ 1)$ and $\varphi,\ \omega \in \mathfrak{D}$ such that

$$\widetilde{\mathfrak{S}}\left(\varphi+\tau\eta(\omega,\ \varphi)\right) \leq_{(\Lambda,\ \upsilon)} \tau(1-\tau)\widetilde{\mathfrak{S}}(\varphi)+\tau\widetilde{\mathfrak{S}}(\omega) \leq_{(\Lambda,\ \upsilon)} \max\left(\widetilde{\mathfrak{S}}(\varphi),\ \widetilde{\mathfrak{S}}(\omega)\right), \tag{75}$$

Hence, it follows the required result. □

3.3 *Differentiable preinvex functions*

In this section, we discuss some properties of $\mathscr{I}\mathscr{F}$-preinvex function with the support of differential $\mathscr{I}\mathscr{F}M$. Further, the minimum of $\mathscr{I}\mathscr{F}$-preinvex function is characterized by mixed-$\mathscr{V}\mathscr{L}$-inequalities and $\mathscr{V}\mathscr{L}$-inequalities. With the help of some known important results, the relationship is developed between $\mathscr{I}\mathscr{F}M$ and mixed-$\mathscr{V}\mathscr{L}$-inequalities that are proved in [14].

Definition 14 Let $\widetilde{\mathfrak{S}}:\mathfrak{D}\to\mathscr{I}\mathscr{F}\mathscr{N}(R)$ be an intuionistic $\mathscr{F}$-function then $\widetilde{\mathfrak{S}}$ is called $\mathscr{I}\mathscr{F}$-invex if

$$\widetilde{\mathfrak{S}}(\omega)-\widetilde{\mathfrak{S}}(\varphi) \geq_{(\Lambda,\ \upsilon)} \left\langle \widetilde{\mathfrak{S}}'(\varphi),\ \eta(\omega,\ \varphi)\right\rangle. \tag{76}$$

Theorem 12 Let $\widetilde{\mathfrak{S}}:\mathfrak{D}\to\mathscr{I}\mathscr{F}\mathscr{N}(R)$ be a differentiable and $\mathscr{I}\mathscr{F}$-preinvex function then $\widetilde{\mathfrak{S}}$ is an $\mathscr{I}\mathscr{F}$-invex function.

Proof. Let $\widetilde{\mathfrak{S}}:\mathfrak{D}\to\mathscr{I}\mathscr{F}\mathscr{N}(R)$ be an inttionistic $\mathscr{F}$-differentiable and preinvex function to prove $\widetilde{\mathfrak{S}}$ is an $\mathscr{I}\mathscr{F}$-invex function. As $\widetilde{\mathfrak{S}}$ is preinvex so by definition of $\widetilde{\mathfrak{S}}$, for each $\varphi,\ \omega\in\mathfrak{D}$ and $t\in(1,\ 0)$, we have

$$\widetilde{\mathfrak{S}}\left(\varphi+\tau\eta(\omega,\ \varphi)\right) \leq_{(\Lambda,\ \upsilon)} (1-\tau)\widetilde{\mathfrak{S}}(\varphi)+\tau\widetilde{\mathfrak{S}}(\omega) =_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}(\varphi)+\tau\left(\widetilde{\mathfrak{S}}(\omega)-\widetilde{\mathfrak{S}}(\varphi)\right), \tag{77}$$

implies that

$$\widetilde{\mathfrak{S}}\left(\varphi+\tau\eta(\omega,\ \varphi)\right)-\widetilde{\mathfrak{S}}(\varphi) \leq_{(\Lambda,\ \upsilon)} \tau\left(\widetilde{\mathfrak{S}}(\omega)-\widetilde{\mathfrak{S}}(\varphi)\right), \tag{78}$$

$$\frac{\widetilde{\mathfrak{S}}\left(\varphi+\tau\eta(\omega,\ \varphi)\right)-\widetilde{\mathfrak{S}}(\varphi)}{\tau} \leq_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}(\omega)-\widetilde{\mathfrak{S}}(\varphi), \tag{79}$$

now taking $\lim_{\tau\to\infty}$, then

$$\left\langle \widetilde{\mathfrak{S}}'(\varphi),\ \eta(\omega,\ \varphi)\right\rangle \leq_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}(\omega)-\widetilde{\mathfrak{S}}(\varphi). \tag{80}$$

Note that, converse of this Theorem 12 is not true so this is only true when condition "C" is satisfying.

Condition C: So, we need Condition C introduced by Neogy and Mohen in 1995, see [16] and

$$\eta\left(\varphi,\ \varphi+\tau\eta(\omega,\ \varphi)\right)=-\tau\eta(\omega,\ \varphi) \quad \text{and} \quad \eta\left(\omega,\ \varphi+\tau\eta(\omega,\ \varphi)\right)=(1-\tau)\eta(\omega,\ \varphi). \tag{81}$$

So, the following result find the converse of Theorem 12. □

Theorem 13 Let $\widetilde{\mathfrak{S}}$ be a differentiable $\mathscr{I}\mathscr{F}M$ on the invex set $\mathfrak{D}$ and if Condition C holds, then the following are equivalent

(a) $\widetilde{\mathfrak{S}}$ is $\mathscr{I}\mathscr{F}$-preinvex function.

(b) $\widetilde{\mathfrak{S}}(\omega)-\widetilde{\mathfrak{S}}(\varphi) \geq_{(\Lambda,\ \upsilon)} \left\langle \widetilde{\mathfrak{S}}'(\varphi),\ \eta(\omega,\ \varphi) \right\rangle, \forall\ \varphi,\ \omega \in \mathfrak{D}$.

(c) $\left\langle \widetilde{\mathfrak{S}}'(\varphi),\ \eta(\omega,\ \varphi) \right\rangle + \left\langle \widetilde{\mathfrak{S}}'(\omega),\ \eta(\omega,\ \varphi) \right\rangle \leq_{(\Lambda,\ \upsilon)} 0, \forall\ \varphi,\ \omega \in \mathfrak{D}$.

Proof. (a) implies (b)

If $\widetilde{\mathfrak{S}}$ is $\mathscr{I}\mathscr{F}$-preinvex function then for each $\varphi,\ \omega \in \mathfrak{D}$ and $t \in [0,\ 1]$, we have

$$\widetilde{\mathfrak{S}}\left(\varphi+\tau\eta(\omega,\ \varphi)\right) \leq_{(\Lambda,\ \upsilon)} (1-\tau)\widetilde{\mathfrak{S}}(\varphi)+\tau\widetilde{\mathfrak{S}}(\omega) =_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}(\varphi)+\tau\left(\widetilde{\mathfrak{S}}(\omega)-\widetilde{\mathfrak{S}}(\varphi)\right), \tag{82}$$

implies that

$$\widetilde{\mathfrak{S}}\left(\varphi+\tau\eta(\omega,\ \varphi)\right)-\widetilde{\mathfrak{S}}(\varphi) \leq_{(\Lambda,\ \upsilon)} \tau\left(\widetilde{\mathfrak{S}}(\omega)-\widetilde{\mathfrak{S}}(\varphi)\right), \tag{83}$$

$$\frac{\widetilde{\mathfrak{S}}\left(\varphi+\tau\eta(\omega,\ \varphi)\right)-\widetilde{\mathfrak{S}}(\varphi)}{\tau} \leq_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}(\omega)-\widetilde{\mathfrak{S}}(\varphi), \tag{84}$$

now taking $\lim_{\tau\to\infty}$, then

$$\left\langle \widetilde{\mathfrak{S}}'(\varphi),\ \eta(\omega,\ \varphi) \right\rangle \leq_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}(\omega)-\widetilde{\mathfrak{S}}(\varphi). \tag{85}$$

By using same steps, we can prove $\widetilde{\mathfrak{S}}(\omega)-\widetilde{\mathfrak{S}}(\varphi) \leq_{\upsilon} \left\langle \widetilde{\mathfrak{S}}'(\varphi),\ \eta(\omega,\ \varphi) \right\rangle\ \forall\ \varphi,\ \omega \in \mathfrak{D}$.

(b) implies (c)

Replace ω by φ and φ by ω in, we get

$$\left\langle \widetilde{\mathfrak{S}}'(\omega),\ \eta(\varphi,\ \omega) \right\rangle \leq_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}(\varphi)-\widetilde{\mathfrak{S}}(\omega), \tag{86}$$

adding Equations (85) and (86), we have

$$\left\langle \widetilde{\mathfrak{S}}'(\varphi),\ \eta(\omega,\ \varphi) \right\rangle + \left\langle \widetilde{\mathfrak{S}}'(\omega),\ \eta(\varphi,\ \omega) \right\rangle \leq_{(\Lambda,\ \upsilon)} 0. \tag{87}$$

(c) implies (b)

Assume that (c) is hold then

$$\left\langle \widetilde{\mathfrak{S}}'(\varphi),\ \eta(\omega,\ \varphi) \right\rangle \leq_{(\Lambda,\ \upsilon)} -\left\langle \widetilde{\mathfrak{S}}'(\omega),\ \eta(\varphi,\ \omega) \right\rangle, \tag{88}$$

Since $\mathfrak{D}$ is an invex set so we have

$$\omega_\tau = \varphi + \tau\eta(\omega, \varphi) \in \mathfrak{D}, \quad \forall \varphi, \omega \in \mathfrak{D} \text{ and } t \in [0, 1]. \tag{89}$$

replacing ω by ω_τ in Equation (87), we get

$$\left\langle \widetilde{\mathfrak{S}}'(\varphi), \eta(\omega_\tau, \varphi) \right\rangle \leq_{(\Lambda, \upsilon)} -\left\langle \widetilde{\mathfrak{S}}'(\omega_\tau), \eta(\varphi, \omega_\tau) \right\rangle, \tag{90}$$

implies

$$\left\langle \widetilde{\mathfrak{S}}'(\varphi), \eta(\varphi + \tau\eta(\omega, \varphi), \varphi) \right\rangle \leq_{(\Lambda, \upsilon)} -\left\langle \widetilde{\mathfrak{S}}'(\varphi + \tau\eta(\omega, \varphi)), \eta(\varphi, \varphi + \tau\eta(\omega, \varphi)) \right\rangle, \tag{91}$$

by using Condition C

$$\left\langle \widetilde{\mathfrak{S}}'(\varphi), \tau\eta(\omega, \varphi) \right\rangle \leq_{(\Lambda, \upsilon)} -\left\langle \widetilde{\mathfrak{S}}'(\varphi + \tau\eta(\omega, \varphi)), -\tau\eta(\omega, \varphi) \right\rangle, \tag{92}$$

$$\left\langle \widetilde{\mathfrak{S}}'(\varphi + \tau\eta(\omega, \varphi)), \tau\eta(\omega, \varphi) \right\rangle \geq_{(\Lambda, \upsilon)} \left\langle \widetilde{\mathfrak{S}}'(\varphi), \tau\eta(\omega, \varphi) \right\rangle, \tag{93}$$

defining the auxiliary function

$$\widetilde{H}(\tau) = \widetilde{\mathfrak{S}}(\varphi + \tau\eta(\omega, \varphi)), \tag{94}$$

now taking derivative with respect to τ, we get

$$\widetilde{H}'(\tau) = \widetilde{\mathfrak{S}}'(\varphi + \tau\eta(\omega, \varphi)) \cdot \eta(\omega, \varphi) = \left\langle \widetilde{\mathfrak{S}}'(\varphi), \tau\eta(\omega, \varphi) \right\rangle, \tag{95}$$

by integrating Equation (95) over $[0, 1]$ with respect to τ, we get

$$\widetilde{H}(1) - \widetilde{H}(0) \geq_{(\Lambda, \upsilon)} \left\langle \widetilde{\mathfrak{S}}'(\varphi), \tau\eta(\omega, \varphi) \right\rangle, \tag{96}$$

$$\widetilde{\mathfrak{S}}(\varphi + \eta(\omega, \varphi)) - \widetilde{\mathfrak{S}}(\varphi) \geq_{(\Lambda, \upsilon)} \left\langle \widetilde{\mathfrak{S}}'(\varphi), \tau\eta(\omega, \varphi) \right\rangle, \tag{97}$$

since $\widetilde{\mathfrak{S}}$ is $\mathscr{I}\mathscr{F}$-preinvex fuction then for $\tau = 1$, we have

$$\widetilde{\mathfrak{S}}(\omega) - \widetilde{\mathfrak{S}}(\varphi) \geq_{(\Lambda, \upsilon)} \left\langle \widetilde{\mathfrak{S}}'(\varphi), \tau\eta(\omega, \varphi) \right\rangle. \tag{98}$$

(b) implies (c)

It can be easily proved so proof is omitted here. □

Definition 15 An element $\varphi \in \mathfrak{D}$ is called minimum of $\mathscr{I}\mathscr{F}M$ $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ if, $\forall\ \varphi \in \mathfrak{D}$ such that

$$\widetilde{\mathfrak{S}}(\varphi) \leq_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}(\omega). \tag{99}$$

Theorem 14 A differentiable $\mathscr{I}\mathscr{F}$-preinvex function $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ has minimum $\varphi \in \mathfrak{D}$ if and only if, $\varphi \in \mathfrak{D}$ satisfies

$$0 \leq_{(\Lambda,\ \upsilon)} \left\langle \widetilde{\mathfrak{S}}'(\varphi),\ \eta(\omega,\ \varphi) \right\rangle,\ \forall\ \omega \in \mathfrak{D}, \tag{100}$$

Proof. Let $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ be a differentiable $\mathscr{I}\mathscr{F}$-preinvex function and $\varphi \in \mathfrak{D}$ satisfies Equation (23) to prove $\varphi \in \mathfrak{D}$ is a minimum of $\widetilde{\mathfrak{S}}$.

As $\widetilde{\mathfrak{S}}$ is an $\mathscr{I}\mathscr{F}$-preinvex function so by Theorem 12, $\forall\ \varphi,\ \omega \in \mathfrak{D}$ we have

$$\widetilde{\mathfrak{S}}(\omega) - \widetilde{\mathfrak{S}}(\varphi) \geq_{(\Lambda,\ \upsilon)} \left\langle \widetilde{\mathfrak{S}}'(\varphi),\ \eta(\omega,\ \varphi) \right\rangle, \tag{101}$$

by Equation (99), implies that $\geq_{(\Lambda,\ \upsilon)} 0$, implies that $\widetilde{\mathfrak{S}}(\varphi) \leq_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}(\omega)$, $\forall\ \omega \in \mathfrak{D}$.

Conversely, let $\varphi \in \mathfrak{D}$ be a minimum of $\widetilde{\mathfrak{S}}$ to prove (7) satisfy. As $\varphi \in \mathfrak{D}$ be a minimum of $\widetilde{\mathfrak{S}}$ then by definition and $\forall\ \omega \in \mathfrak{D}$.

$$\widetilde{\mathfrak{S}}(\varphi) \leq_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}(\omega). \tag{102}$$

As $\mathfrak{D}$ is an invex set so $\omega_\tau = \varphi + \tau\eta(\omega,\ \varphi)$, $\forall\ \varphi,\ \omega \in \mathfrak{D}$ and $\tau \in (0,\ 1)$. First by solving Equation (102), replacing ω by ω_τ then, we have

$$\widetilde{\mathfrak{S}}(\varphi) \leq_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}(\omega_\tau),$$

$$\leq_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}(\varphi + \tau\eta(\omega,\ \varphi)), \tag{103}$$

$$0 \leq_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}(\varphi + \tau\eta(\omega,\ \varphi)) - \widetilde{\mathfrak{S}}(\varphi), \tag{104}$$

now dividing Equation (28) by "τ" and taking $\lim_{\tau \to \infty}$, we have

$$0 \leq_{(\Lambda,\ \upsilon)} \lim_{\tau \to \infty} \frac{\widetilde{\mathfrak{S}}(\varphi + \tau\eta(\omega,\ \varphi)) - \widetilde{\mathfrak{S}}(\varphi)}{\tau}, \tag{105}$$

$$0 \leq_{(\Lambda,\ \upsilon)} \left\langle \widetilde{\mathfrak{S}}'(\varphi),\ \eta(\omega,\ \varphi) \right\rangle, \tag{106}$$

$\forall\ \omega \in \mathfrak{D}$. $\square$

Theorem 15 A differentiable $\mathscr{I}\mathscr{F}$-invex function $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ has minimum $\varphi \in \mathfrak{D}$ if and only if $\varphi \in \mathfrak{D}$ satisfies

$$0 \leq_{(\Lambda,\ \upsilon)} \left\langle \widetilde{\mathfrak{S}}'(\varphi),\ \eta(\omega,\ \varphi) \right\rangle. \tag{107}$$

Proof. This result can easily be proved like Theorem 14. $\square$

Theorem 16 Let $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ be a differentiable $\mathscr{I}\mathscr{F}$-preinvex function and $\widetilde{\Omega} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ be an $\mathscr{I}\mathscr{F}$-preinvex function then the function $\widetilde{\xi}$ defined by $\widetilde{\xi}(\omega) = \widetilde{\mathfrak{S}}(\omega) + \widetilde{\Omega}(\omega)$ has minimum $\varphi \in \mathfrak{D}$ if and only if $\varphi \in \mathfrak{D}$ satisfies

$$0 \leq_{(\Lambda,\ \upsilon)} \left\langle \widetilde{\mathfrak{S}}'(\varphi),\ \eta(\omega,\ \varphi) \right\rangle + \widetilde{\Omega}(\omega) - \widetilde{\Omega}(\varphi),\ \forall\ \omega \in \mathfrak{D}. \tag{108}$$

Proof. Let $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ be a differentiable $\mathscr{I}\mathscr{F}$-preinvex function and $\widetilde{\Omega} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ be a non-differentiable $\mathscr{I}\mathscr{F}$-preinvex function and $\varphi \in \mathfrak{D}$ be the minimum of $\widetilde{\xi}$ to prove $\varphi \in \mathfrak{D}$ satisfies (11) $\forall\ \omega \in \mathfrak{D}$.

As $\varphi \in \mathfrak{D}$ be the minimum of $\widetilde{\xi}$ then by definition, $\forall\ \omega \in \mathfrak{D}$ we have

$$\widetilde{\xi}(\varphi) \leq_{(\Lambda,\ \upsilon)} \widetilde{\xi}(\omega), \tag{109}$$

as $\mathfrak{D}$ is an invex set so $\omega_\tau = \varphi + \tau\eta(\omega,\ \varphi)$, $\forall\ \varphi,\ \omega \in \mathfrak{D}$ and $\tau \in (0,\ 1)$. replacing ω by ω_τ in Equation (109), we get

$$\widetilde{\xi}(\varphi) \leq_{(\Lambda,\ \upsilon)} \widetilde{\xi}\left(\varphi + \tau\eta(\omega,\ \varphi)\right), \tag{110}$$

implies that

$$\widetilde{\mathfrak{S}}(\varphi) + \widetilde{\Omega}(\varphi) \leq_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}\left(\varphi + \tau\eta(\omega,\ \varphi)\right) + \widetilde{\Omega}\left(\varphi + \tau\eta(\omega,\ \varphi)\right), \tag{111}$$

since $\widetilde{\Omega}$ are $\mathscr{I}\mathscr{F}$-preinvex function then,

$$\widetilde{\mathfrak{S}}(\varphi) + \widetilde{\Omega}(\varphi) \leq_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}\left(\varphi + \tau\eta(\omega,\ \varphi)\right) + (1-\tau)\widetilde{\Omega}(\varphi) + \tau\widetilde{\Omega}(\omega), \tag{112}$$

implies that

$$0 \leq_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}\left(\varphi + \tau\eta(\omega,\ \varphi)\right) - \widetilde{\mathfrak{S}}(\varphi) + \tau\left(\widetilde{\Omega}(\omega) - \widetilde{\Omega}(\varphi)\right), \tag{113}$$

now dividing by "τ" and taking $\lim_{\tau\to\infty}$, we have

$$0 \leq_{(\Lambda,\ \upsilon)} \lim_{\tau \to \infty} \frac{\widetilde{\mathfrak{S}}(\varphi + \tau\eta(\omega,\ \varphi)) - \widetilde{\mathfrak{S}}(\varphi)}{\tau} + \widetilde{\Omega}(\omega) - \widetilde{\Omega}(\varphi), \tag{114}$$

implies that

$$0 \leq_{(\Lambda,\ \upsilon)} \left\langle \widetilde{\mathfrak{S}}'(\varphi),\ \eta(\omega,\ \varphi) \right\rangle + \widetilde{\Omega}(\omega) - \widetilde{\Omega}(\varphi). \tag{115}$$

Conversely, let Equation (108) satisfy to prove $\varphi \in \mathfrak{D}$ is a minimum of $\widetilde{\xi}$. Assume that, $\forall\ \omega \in \mathfrak{D}$ we have

$$\begin{aligned}
\widetilde{\xi}(\varphi) - \widetilde{\xi}(\omega) &= {}_{(\Lambda,\ \upsilon)} \widetilde{\mathfrak{S}}(\varphi) + \widetilde{\Omega}(\varphi) - \widetilde{\mathfrak{S}}(\omega) - \widetilde{\Omega}(\omega), \\
&= {}_{(\Lambda,\ \upsilon)} - \left(\widetilde{\mathfrak{S}}(\omega) - \widetilde{\mathfrak{S}}(\varphi) \right) + \widetilde{\Omega}(\varphi) - \widetilde{\Omega}(\omega), \\
&\leq {}_{(\Lambda,\ \upsilon)} - \left[\left\langle \widetilde{\mathfrak{S}}'(\varphi),\ \eta(\omega,\ \varphi) \right\rangle + \widetilde{\Omega}(\varphi) - \widetilde{\Omega}(\omega) \right], \quad \text{by Theorem 12}, \\
&\leq {}_{(\Lambda,\ \upsilon)} 0, \quad \text{by Equation (92)}
\end{aligned} \tag{116}$$

hence,

$$\widetilde{\xi}(\varphi) \leq_{(\Lambda,\ \upsilon)} \widetilde{\xi}(\omega). \tag{117}$$

Note that the Equation (108) is called $\mathscr{I}\mathscr{F}$-mixed-variational like-inequalities. This result shows the characterization of minimum of $\mathscr{I}\mathscr{F}$-preinvex function with the support of $\mathscr{I}\mathscr{F}$-mixed-variational like-inequalities. If $\widetilde{\Omega} = 0$ then Equation (108) becomes variational like-inequality. □

Theorem 17 Let $\widetilde{\mathfrak{S}} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ be a differentiable $\mathscr{I}\mathscr{F}$-invex function and $\widetilde{\Omega} : \mathfrak{D} \to \mathscr{I}\mathscr{F}\mathscr{N}(R)$ be a non-differentiable $\mathscr{I}\mathscr{F}$-preinvex function then the function $\widetilde{\xi}$ defined by $\widetilde{\xi}(\omega) = \widetilde{\mathfrak{S}}(\omega) + \widetilde{\Omega}(\omega)$ has minimum $\varphi \in \mathfrak{D}$ if and only if $\varphi \in \mathfrak{D}$ satisfies

$$0 \leq_{(\Lambda,\ \upsilon)} \left\langle \widetilde{\mathfrak{S}}'(\varphi),\ \eta(\omega,\ \varphi) \right\rangle + \widetilde{\Omega}(\omega) - \widetilde{\Omega}(\varphi),\ \forall\ \omega \in \mathfrak{D}. \tag{118}$$

Proof. The proof proceeds in a manner analogous to Theorem 16 and is omitted for conciseness. □

4. Conclusion

In this paper, new concepts have been presented over $\mathscr{I}\mathscr{F}$-theory. Firstly, the definitions of $\mathscr{I}\mathscr{F}$-invex and quai-invex sets are given. After that, we proposed the concept of $\mathscr{I}\mathscr{F}$-preinvex function which is generalized form of $\mathscr{F}$-preinvex function and discussed its properties. Further, some optimization conditions are fined. During finding minimum of preinvex function, we fined relationship between $\mathscr{I}\mathscr{F}$-preinvex function and $\mathscr{I}\mathscr{F}$-variational like-inequalities and mixed-$\mathscr{IF}$-variational like-inequalities. We conclude that these results give rise to duality and optimality results for $\mathscr{IF}$-nonlinear programing. So, further research is required to expose the best detailed relationship. In future, we will try to explore these concepts in integral theory as well as game theory.

Conflict of interest

The author declares no conflict of interest.

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