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Research Article

A Novel Fixed Point Framework for Positive Solutions to Coupled Undamped Sturm-Liouville Systems

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Abstract: This study addresses the existence of at least one positive and nondecreasing solution to a system of second-order undamped Sturm-Liouville Boundary Value Problems (BVPs). By carefully analyzing the properties of the associated Green functions and applying a fixed-point theorem in a Banach space, we establish new existence results under broadly nonlinear and coupled boundary settings. To the best of our knowledge, this is the first work to derive such findings for coupled undamped Sturm-Liouville systems using this innovative analytical strategy. The proposed approach extends classical methods to effectively accommodate nonlinear interdependencies and atypical boundary conditions, thereby providing a novel perspective for the theory of BVPs. Unlike previous studies that mainly address damped systems or single-equation models, this work fills an important gap by considering the more challenging undamped coupled case. The findings extend classical methods and contribute to a deeper understanding of nonlinear BVPs under nonstandard and interdependent boundary structures.

***Keywords*:** kernel, boundary value problem, positive solution, Banach space

MSC: 34B15, 34B18, 46B25

1. Introduction

Boundary Value Problems (BVPs) governed by Differential Equations (DEs) occupy a foundational position in both pure and applied mathematics, especially in mathematical physics, engineering, and the life sciences. Among these, Sturm-Liouville problems are particularly notable due to their robust theoretical underpinnings, analytical solvability, and broad spectrum of applications, ranging from quantum mechanics and signal processing to mechanical vibrations and thermal analysis [1]. The classical framework for linear Sturm-Liouville problems with regular boundary conditions has been thoroughly studied in the works of Levitan and Sargsjan [2] as well as Zettl [3]. In contrast, Agarwal et al. [4, 5] have addressed nonlinear extensions, with a focus on establishing the existence of positive solutions.

A key subclass within this domain is the family of undamped Sturm-Liouville BVPs, which describe conservative systems where energy dissipation is absent. These problems arise naturally in idealized models of mechanical systems, lossless electromagnetic propagation, and inviscid fluid flows. The absence of damping not only reflects certain physical

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idealizations but also introduces mathematical difficulties due to the loss of stabilizing effects, thereby necessitating more sophisticated analytical techniques to study solution behavior and establish existence results. Pioneering work by Lazer and McKenna [6] examined resonance phenomena in second-order DEs, laying the groundwork for subsequent studies. Chu et al. [7] and O'Regan [8] further contributed by employing truncation and topological approaches to treat Neumann-type Boundary Conditions (BCs). Additional progress was made by Wang and Zhang [9] using cone-theoretic Fixed Point Theorems (FPTs) to address parameter-dependent nonlinearities. This methodology was extended to semipositone problems by Henderson and Kosmatov [10] through the application of Krasnosel'skii's FPT [11], building on earlier advances by Anderson and Avery [12] using cone compression and expansion techniques.

The undamped BVP setting continues to attract attention due to its analytical complexity and the pivotal role that nonlinear effects play in solution behavior. While the classical Sturm-Liouville theory for scalar linear cases is well established [13, 14], system-level generalizations remain relatively underdeveloped. Researchers have employed a range of mathematical strategies such as variational formulations, topological arguments, and FPTs to investigate nonlinear DEs [15]. Seminal contributions in this direction include the nonlinear alternative of Leray-Schauder [16] and the deployment of degree-theoretic methods [12]. DEs are essential for modeling and analyzing real-world phenomena in science and engineering, enabling prediction, control, and optimization of dynamic systems. Turab and Sintunavarat [17] solved iterative BVPs using FPT methods. In [18], authors modeled avoidance learning behavior via integral DEs, and Turab et al. [19] investigated stability and numerical solutions in mechanical systems, highlighting the practical importance of DEs in real-world problems. Recent developments have extended the Levinson theorem to higher-dimensional and non-local interaction systems using the Sturm-Liouville framework, including considerations of half-bound states [20–22]. These generalizations provide a rigorous foundation for analyzing the spectral properties of both nonrelativistic and relativistic quantum systems.

Nonetheless, most existing studies focus on scalar problems or systems constrained by symmetry, leaving a significant gap in the general theory of coupled, nonlinear, undamped Sturm-Liouville systems with flexible BCs. This gap is particularly important given the relevance of such models to real-world phenomena, including thermal diffusion in stratified materials [23], vibration of composite structures [11], and biological transport processes. For instance, Avery et al. [24] developed a six-functional fixed-point theorem for scalar BVPs, whereas [25, 26] established the existence of multiple positive solutions for Neumann-type problems in single-equation settings. Although these contributions advanced the theory of scalar and damped systems, they did not address the analytical complexities of the coupled undamped case, where nonlinear boundary conditions induce intricate interdependencies that complicate Green's function analysis and positivity arguments.

To address these challenges, the present study employs fixed-point methods in cones. Variational approaches are unsuitable here because they presuppose a variational structure that does not exist in this setting, and monotone operator techniques rely on strong order-preserving properties that may fail under nonlinear coupling. In contrast, fixed-point theory provides the necessary flexibility to establish the existence of positive, non-decreasing solutions in coupled, undamped Sturm-Liouville systems.

In the simpler setting where $m = 0$, significant research has focused on proving the existence of positive solutions to second-order ordinary DEs, typically using cone-theoretic or multifunctional FPTs [24, 27].

Chu et al. [28] looked into the prospects of positive solutions for the BVP in 2008:

$$v'' + m^2 v = f(z,\ v) + e(z),$$

$$v'(0) = v'(1) = 0,$$

where $m \in \left(0,\ \dfrac{\pi}{2}\right)$ and $e \in C[0,\ 1]$, employing a truncation technique and the Leray-Schauder nonlinear alternative principle. Additionally, Wang et al. [9] studied a second-order Neumann-type BVP:

$$-v'' + m^2 v = h(z)f(z,\ v), \quad 0 < z < 1,$$

$$v'(0) = v'(1) = 0,$$

where $m > 0$, using the FPT of cone compression and expansion to derive conditions for the existence of positive solutions. In 2015, Henderson and Kosmatov [29] examined the existence of positive solutions for the semipositone Neumann BVP:

$$-v'' = f\big(z,\ v(z)\big), \quad 0 < z < 1,$$

$$v'(0) = v'(1) = 0,$$

by applying Krasnosel'skii's FPT.

DEs with initial or BCs are widely used across industries such as pharmaceuticals, electrical engineering, healthcare, and automotive design to model complex phenomena involved in advanced product development. In these applications, the existence of positive solutions is fundamental. Consequently, BVPs have attracted significant attention due to their deep theoretical foundations and broad practical relevance [5].

The primary objective of this article is to investigate the existence of at least one positive and nondecreasing solution for a system of second-order Sturm-Liouville BVPs:

$$-v_1''(z) + m_1^2 v_1(z) = f_1\big(z,\ v_1(z),\ v_2(z)\big), \quad z \in (0,\ 1), \tag{1}$$

$$-v_2''(z) + m_2^2 v_2(z) = f_2\big(z,\ v_1(z),\ v_2(z)\big), \quad z \in (0,\ 1), \tag{2}$$

$$\left.\begin{aligned} &\beta_1 v_1(0) - \beta_2 v_1'(0) = 0, \\ &\beta_3 v_1(1) + \beta_4 v_1'(1) = 0, \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} &\mu_1 v_2(0) - \mu_2 v_2'(0) = 0, \\ &\mu_3 v_2(1) + \mu_4 v_2'(1) = 0, \end{aligned}\right\} \tag{4}$$

where $\beta_k,\ \mu_k,\ k = \overline{1,\ 4}$ are constants and $m_1,\ m_2 > 0$. To overcome the analytical challenges caused by the nonlinear interactions among system components, we develop an appropriate function space equipped with a suitable norm. Various existence results are obtained under different structural assumptions on the nonlinearities. This research notably advances previous work by examining a coupled system of Sturm-Liouville type equations and employing a generalized version of the six-functionals method introduced by Avery and Henderson [24], which guarantees at least one positive solution. Our results offer new contributions to the theory of undamped BVPs and expand the range of known findings. In particular, we fill a gap in the literature by studying a coupled, nonlinear, undamped system with BCs. Through this approach, classical fixed-point methods are extended to incorporate coupling effects, complementing recent developments on singular Neumann problems by Sun et al. [30] and broadening the theoretical frameworks from [31–33] to the undamped context.

Throughout the paper, we assume the following conditions:

($\mathbf{B}_1$) $\beta_k \in \mathbb{R}^+,\ k = \overline{1,\ 4}$ s.t. either $\beta_1^2 + \beta_2^2 > 0$ or $\beta_3^2 + \beta_4^2 > 0$,

($\mathbf{B}_2$) $\Delta_1 = m_1^2(\beta_1\beta_4 + \beta_2\beta_3)\cosh(m_1) + m_1(\beta_1\beta_3 + \beta_2\beta_4 m_1^2)\sinh(m_1) > 0$,

($\mathbf{B}_3$) $\mu_k \in \mathbb{R}^+,\ k = \overline{1,\ 4}$ s.t. either $\mu_1^2 + \mu_2^2 > 0$ or $\mu_3^2 + \mu_4^2 > 0$,

($\mathbf{B}_4$) $\Delta_2 = m_2^2(\mu_1\mu_4 + \mu_2\mu_3)\cosh(m_2) + m_2(\mu_1\mu_3 + \mu_2\mu_4 m_2^2)\sinh(m_2) > 0$,

($\mathbf{B}_5$) $f_i : [0,\ 1] \times \mathbb{R}^2 \to \mathbb{R}^+$ are continuous, for $i = \overline{1,\ 2}$.

The paper is structured as follows. Section 2 provides essential preliminary material that forms the foundation for the subsequent analysis. In section 3, we state and prove our main results. Section 4 presents an illustrative example to validate the theoretical findings. The concluding remarks are given in the final section.

2. Essential preliminaries

We transform the BVP into an equivalent integral equation based on kernels. Also, we provide bounds for the Kernel that are essential to our key conclusions.

Let us take $I_1 = [0,\ 1]$ and $I_2 = \left[\frac{1}{4},\ \frac{3}{4}\right]$.

Lemma 1 Suppose ($\mathbf{B}_1$), ($\mathbf{B}_2$) hold. If $h_1(z) \in C[I_1]$, then the DE

$$-v_1''(z) + m_1^2 v_1(z) = h_1(z), \qquad z \in (0,\ 1), \tag{5}$$

with BCs (3) has a unique solution:

$$v_1(z) = \int_0^1 H_{m_1}(z,\ s)h_1(s)ds,$$

where

$$H_{m_1}(z,\ s) = \begin{cases} \dfrac{1}{\Delta_1}\Theta_1(z)\Theta_2(s), & z \le s, \\[2ex] \dfrac{1}{\Delta_1}\Theta_2(z)\Theta_1(s), & s \le z, \end{cases} \tag{6}$$

$$\Theta_1(z) = \beta_1 \sinh(m_1 z) + \beta_2 m_1 \cosh(m_1 z),$$

$$\Theta_2(z) = \beta_3 \sinh\big(m_1(1-z)\big) + \beta_4 m_1 \cosh\big(m_1(1-z)\big).$$

Proof. By algebraic calculations, we can establish the result. □

Lemma 2 The Kernel $H_{m_1}(z,\ s)$ given in (6) is nonnegative, for all $z,\ s \in I_1$.

Proof. From the definition of the Kernel $H_{m_1}(z,\ s)$, it is clear that $H_{m_1}(z,\ s) \ge 0$ for all $z,\ s \in I_1$. □

Lemma 3 The Kernel $H_{m_1}(z,\ s)$ given in (6) satisfies:

$$k_{m_1} H_{m_1}(s,\ s) \leq H_{m_1}(z,\ s) \leq H_{m_1}(s,\ s),\ \text{for}\ z,\ s \in I_2, \tag{7}$$

where

$$k_{m_1} = \min\left\{ \frac{\Theta_1\left(\frac{1}{4}\right)}{\Theta_1\left(\frac{3}{4}\right)}, \quad \frac{\Theta_2\left(\frac{3}{4}\right)}{\Theta_2\left(\frac{1}{4}\right)} \right\}.$$

Proof. For $z,\ s \in I_1$,

$$\frac{H_{m_1}(z,\ s)}{H_{m_1}(s,\ s)} = \begin{cases} \dfrac{\Theta_1(z)\Theta_2(s)}{\Theta_1(s)\Theta_2(s)} = \dfrac{\Theta_1(z)}{\Theta_1(s)} \leq 1, & z \leq s, \\[2ex] \dfrac{\Theta_2(z)\Theta_1(s)}{\Theta_2(s)\Theta_1(s)} = \dfrac{\Theta_2(z)}{\Theta_2(s)} \leq 1, & s \leq z, \end{cases}$$

which proves $H_{m_1}(z,\ s) \leq H_{m_1}(s,\ s)$. Finally for $z,\ s \in I_1$,

$$\frac{H_{m_1}(z,\ s)}{H_{m_1}(s,\ s)} = \begin{cases} \dfrac{\Theta_1(z)}{\Theta_1(s)} \geq \dfrac{\beta_2 m_1}{\beta_1 \sinh(m_1) + \beta_2 m_1 \cosh(m_1)}, & z \leq s, \\[2ex] \dfrac{\Theta_2(z)}{\Theta_2(s)} \geq \dfrac{\beta_4 m_1}{\beta_3 \sinh(m_1) + \beta_4 m_1 \cosh(m_1)}, & s \leq z, \end{cases}$$

that implies $k_{m_1} H_{m_1}(s,\ s) \leq H_{m_1}(z,\ s)$, and the lemma follows. □

Lemma 4 Suppose $(\mathbf{B}_3)$, $(\mathbf{B}_4)$ hold. If $h_2(z) \in C[I_1]$, then the DE

$$-v_2''(z) + m_2^2 v_2(z) = h_2(z), \quad z \in (0,\ 1), \tag{8}$$

with BCs (3) has a unique solution:

$$v_2(z) = \int_0^1 H_{m_2}(z,\ s) h_2(s) ds,$$

where

$$H_{m_2}(z,\ s) = \begin{cases} \dfrac{1}{\Delta_2} \Psi_1(z) \Psi_2(s), & z \leq s, \\[2ex] \dfrac{1}{\Delta_2} \Psi_2(z) \Psi_1(s), & s \leq z, \end{cases} \tag{9}$$

$$\Psi_1(z)=\mu_1\sinh(m_2 z)+\mu_2 m_2\cosh(m_2 z),$$

$$\Psi_2(z)=\mu_3\sinh\big(m_2(1-z)\big)+\mu_4 m_2\cosh\big(m_2(1-z)\big).$$

Lemma 5 The Kernel $H_{m_2}(z,\ s)$ given in (9) is nonnegative, for all $z,\ s\in I_1$.
Lemma 6 For $z,\ s\in I_2$, then the Kernel $H_{m_2}(z,\ s)$ given in (9) satisfies:

$$k_{m_2}H_{m_2}(s,\ s)\le H_{m_2}(z,\ s)\le H_{m_2}(s,\ s), \tag{10}$$

where

$$k_{m_2}=\min\left\{\frac{\Psi_1\left(\frac{1}{4}\right)}{\Psi_1\left(\frac{3}{4}\right)},\ \frac{\Psi_2\left(\frac{3}{4}\right)}{\Psi_2\left(\frac{1}{4}\right)}\right\}.$$

Property 1 Let Λ be a bounded open subset of B with $0\in\Lambda$, and let P be a cone in a Banach space B. A continuous functional θ is maps from P to $[0,\ \infty)$ is then considered to satisfy Property 1 if one of the subsequent assertions is gratified:

(Q_1) θ is convex, $\theta(0)=0,\ \theta(z)\neq 0$ whenever $\inf\limits_{z\in P\cap\partial\Lambda}\theta(z)>0$ along with $z\neq 0$,

(Q_2) θ is sublinear, $\theta(0)=0,\ \theta(z)\neq 0$ whenever $\inf\limits_{z\in P\cap\partial\Lambda}\theta(z)>0$ along with $z\neq 0$,

(Q_3) θ is concave and unbounded.

Property 2 Let Λ be a bounded open subset of B with $0\in\Lambda$, and let P be a cone in a Banach space B. A continuous functional ϑ is maps from P to $[0,\ \infty)$ is said to satisfy Property 2 if one of the subsequent assertions is fulfilled:

(Q_4) ϑ is convex, $\vartheta(0)=0,\ \vartheta(z)\neq 0$ whenever $z\neq 0$,

(Q_5) ϑ is sublinear, $\vartheta(0)=0,\ \vartheta(z)\neq 0$ whenever $z\neq 0$,

(Q_6) $\vartheta(z+y)\geq\vartheta(z)+\vartheta(y),\ \forall\ z,\ y\in P,\ \vartheta(0)=0,\ \vartheta(z)\neq 0$ whenever $z\neq 0$.

The existence condition for the BVP (1)-(4) will be established using the subsequent fixed point hypothesis of cone advancement and restrictive of functional type resulting from Avery et al. [15], which broadly applies the fixed point theorems of [12, 27].

Theorem 1 [15] Let $\Lambda_1,\ \Lambda_2$ be two bounded open sets in a Banach Space B such that $0 \in \Lambda_1$ and $\overline{\Lambda}_1 \subset \Lambda_2$ in B. Suppose $A : P \cap (\overline{\Lambda}_2 \backslash \Lambda_1) \to P$ is completely continuous operator, θ and ϑ are non-negative continuous functional on P, and one of the two conditions:

(i) θ satisfies Property 1 with $\theta(Az) \geq \theta(z),\, z \in P \cap \partial\Lambda_1$ and ϑ satisfies Property 2 with $\vartheta(Az) \leq \vartheta(z),\, z \in P \cap \partial\Lambda_2$.

(ii) ϑ satisfies Property 2 with $\vartheta(Az) \leq \vartheta(z),\, z \in P \cap \partial\Lambda_1$ and θ satisfies Property 1 with $\theta(Az) \geq \theta(z),\, z \in P \cap \partial\Lambda_2$, is fulfilled.

Then A has at least one fixed point in $P \cap (\overline{\Lambda}_2 \backslash \Lambda_1)$.

3. Existence results

For computational convenience, we introduce the following notations:

$$\left.\begin{aligned} k_l^{\star} &= \min\left\{k_{m_1},\ k_{m_2}\right\},\\ M_1 &= \min\left\{\int_0^1 H_{m_1}(s,\ s)ds,\ \int_0^1 H_{m_2}(s,\ s)ds\right\},\\ M_2 &= \max\left\{\int_{\frac{1}{4}}^{\frac{3}{4}} k_l^{\star} H_{m_1}(s,\ s)ds,\ \int_{\frac{1}{4}}^{\frac{3}{4}} k_l^{\star} H_{m_2}(s,\ s)ds\right\}. \end{aligned}\right\} \tag{11}$$

Consider the Banach space $B = E \times E$, where $E = \{v_1 : v_1 \in C[I_1]\}$ endowed with the norm $\|(v_1(z),\ v_2(z))\| = \|v_1\|_0 + \|v_2\|_0,$ for $(v_1(z),\ v_2(z)) \in B$ and we denote the norm,

$$\|v_1\|_0 = \max_{z \in I_1} |v_1(z)|.$$

The cone $P \subset B$ is defined as

$$P = \Big\{(v_1(z),\ v_2(z)) \in B : v_1(z),\ v_2(z) \text{ are non-negative and increasing on } I_1$$

$$\text{and } \min_{z \in I_2} \sum_{i=1}^{2} \left[v_i(z)\right] \geq k_l^{\star} \|(v_1(z),\ v_2(z))\|\Big\}.$$

On the cone P, define the continuous functionals θ and ϑ as follows:

$$\theta(v_1(z),\ v_2(z)) = \min_{z \in I_2} \sum_{i=1}^{2} |v_i| \ \text{ and}$$

$$\vartheta(v_1(z),\ v_2(z)) = \max_{z \in I_1} \sum_{i=1}^{2} |v_i| = \sum_{i=1}^{2} \left[v_i(1)\right] = \left\|(v_1(z),\ v_2(z))\right\|.$$

It is evident that $\theta(v_1(z),\ v_2(z)) \leq \vartheta(v_1(z),\ v_2(z)),\ \forall (v_1(z),\ v_2(z)) \in P$.

Let $A_1,\ A_2 : P \to E$ be the operators defined as

$$
\begin{cases}
A_1\big(v_1(z),\ v_2(z)\big) = \int_0^1 H_{m_1}(z,\ s) f_1\big(s,\ v_1(s),\ v_2(s)\big) ds, \\
A_2\big(v_1(z),\ v_2(z)\big) = \int_0^1 H_{m_2}(z,\ s) f_2\big(s,\ v_1(s),\ v_2(s)\big) ds.
\end{cases}
$$

Consider the operator $A : P \to B$, which is defined as

$$
A(v_1(z),\ v_2(z)) = \Big(A_1(v_1(z),\ v_2(z)),\ A_2(v_1(z),\ v_2(z))\Big),\ \text{ for } (v_1(z),\ v_2(z)) \in B. \tag{12}
$$

The fact that the Fractional Boundary Value Problem (FBVP) (1)-(4) solution is a fixed point of A. We seek a fixed point of A.

Lemma 7 Let $(v_1(z),\ v_2(z)),\ (u_1(z),\ u_2(z)) \in P$ with $(v_1(z),\ v_2(z)) \leq (u_1(z),\ u_2(z))$ in the cone order for all $z \in [0,\ 1]$. Then the operator A is monotone with respect to the cone order, that is,

$$
A(v_1(z),\ v_2(z)) \leq A(u_1(z),\ u_2(z)), \quad \forall\ z \in [0,\ 1].
$$

Moreover, every element of P is nonnegative and nondecreasing on $[0,\ 1]$, and A maps nondecreasing functions in P into nondecreasing functions.

Proof. By the definition (12), for $i = 1,\ 2$ we have

$$
A_i(v_1(z),\ v_2(z)) = \int_0^1 H_{m_i}(z,\ s)\ f_i\big(s,\ v_1(s),\ v_2(s)\big)\ ds.
$$

Since $H_{m_i}(z,\ s) \geq 0$ for all $z,\ s \in [0,\ 1]$, and by $(\mathbf{B}_1)$-$(\mathbf{B}_5)$ the nonlinearities f_i are nondecreasing in their second and third arguments, it follows that

$$
(v_1(z),\ v_2(z)) \leq (u_1(z),\ u_2(z)) \ \Rightarrow\ f_i\big(s,\ v_1(s),\ v_2(s)\big) \leq f_i\big(s,\ u_1(s),\ u_2(s)\big), \quad \forall s \in [0,\ 1].
$$

Therefore,

$$
A_i(v_1(z),\ v_2(z)) \leq A_i(u_1(z),\ u_2(z)), \quad \forall z \in [0,\ 1],\ \ i = 1,\ 2.
$$

Hence,

$$
A(v_1(z),\ v_2(z)) \leq A(u_1(z),\ u_2(z)),
$$

which shows that A is monotone with respect to the cone order. In particular, since elements of P are nonnegative and increasing, A maps nondecreasing functions in P into nondecreasing functions. □

Theorem 2 Assume that $(\mathbf{B}_1)$-$(\mathbf{B}_5)$ hold and suppose there exist positive real numbers $\omega_1,\ \omega_2$ with $\omega_1 < k_l^\star \omega_2$ such that $f_i,\ i=1,\ 2$ satisfies:

$$(\mathrm{K}_1)\ f_i\big(z,\ v_1(z),\ v_2(z)\big) \geq \frac{1}{2}\frac{\omega_1}{k_l^\star M_2},\ \forall\, z \in I_2 \text{ and } \big(v_1(z),\ v_2(z)\big) \in \big[\omega_1,\ \omega_2\big],$$

$$(\mathrm{K}_2)\ f_i\big(z,\ v_1(z),\ v_2(z)\big) \leq \frac{1}{2}\frac{\omega_2}{M_1},\ \ \forall\, z \in I_1 \text{ and } \big(v_1(z),\ v_2(z)\big) \in \big[0,\ \omega_2\big].$$

Then the BVP (1)-(4) has at least one nondecreasing and positive solution, $(v_1^\star,\ v_2^\star)$ satisfying $\omega_1 \leq \theta(v_1^\star,\ v_2^\star)$ with $\vartheta(v_1^\star,\ v_2^\star) \leq \omega_2$.

Proof. With standard arguments, we can establish that the operator A is completely continuous, providing us with the verification of $A(P) \subset P$. Let $(v_1(z),\ v_2(z)) \in P$. Clearly, $A_1(v_1(z),\ v_2(z)) \geq 0$ and $A_2(v_1(z),\ v_2(z)) \geq 0$ for $z \in I_1$. Also, for $(v_1(z),\ v_2(z)) \in P$,

$$\begin{cases} \|A_1(v_1(z),\ v_2(z))\|_0 \leq \displaystyle\int_0^1 H_{m_1}(s,\ s) f_1\big(s,\ v_1(s),\ v_2(s)\big) ds, \\ \\ \|A_2(v_1(z),\ v_2(z))\|_0 \leq \displaystyle\int_0^1 H_{m_2}(s,\ s) f_2\big(s,\ v_1(s),\ v_2(s)\big) ds, \end{cases}$$

and

$$\begin{aligned} \min_{z\in I_2} A_1(v_1(z),\ v_2(z))(z) &= \min_{z\in I_2}\left[\int_0^1 H_{m_1}(z,\ s) f_1\big(s,\ v_1(s),\ v_2(s)\big) ds\right] \\ &\geq k_{m_1}\int_0^1 H_{m_1}(s,\ s) f_1\big(s,\ v_1(s),\ v_2(s)\big) ds \\ &\geq k_l^\star \|A_1(v_1(z),\ v_2(z))\|_0. \end{aligned}$$

Similarly $\min\limits_{z\in I_2} A_2(v_1(z),\ v_2(z))(z) \geq k_l^\star \|A_2(v_1(z),\ v_2(z))\|_0$. Therefore

$$\begin{aligned} \min_{z\in I_2}\Big\{A_1(v_1(z),\ v_2(z))(z) + A_2(v_1(z),\ v_2(z))(z)\Big\} &\geq k_l^\star \|A_1(v_1(z),\ v_2(z))\|_0 + k_l^\star \|A_2(v_1(z),\ v_2(z))\|_0 \\ &= k_l^\star \|(A_1(v_1(z),\ v_2(z)),\ A_2(v_1(z),\ v_2(z)))\| \\ &= k_l^\star \|A(v_1(z),\ v_2(z))\|. \end{aligned}$$

Thus, $A(P) \subset P$. This inclusion follows rigorously from the positivity and monotone kernel properties of the associated Green functions, ensuring that A preserves positivity and maps the cone P into itself. According to conventional reasoning based on the Arzela-Ascoli theorem, A is therefore a completely continuous operator.

Recall that $B = E \times E$ with $E = C[I_1]$ endowed with the norm $\|(v_1(z),\ v_2(z))\| = \|v_1\|_0 + \|v_2\|_0$, and that the cone $P \subset B$ and the functionals

$$\theta(v_1(z),\ v_2(z)) = \min_{z\in I_2}\sum_{i=1}^{2}\left|v_i(z)\right|,\ \vartheta(v_1(z),\ v_2(z)) = \max_{z\in I_1}\sum_{i=1}^{2}\left|v_i(z)\right|$$

are defined on P. We then define

$$\begin{cases}\Lambda_1 := \left\{(v_1(z),\ v_2(z)) \in P:\ \theta(v_1(z),\ v_2(z)) < \omega_1\right\},\\ \\ \Lambda_2 := \left\{(v_1(z),\ v_2(z)) \in P:\ \vartheta(v_1(z),\ v_2(z)) < \omega_2\right\}.\end{cases}$$

It is easy to see that $0 \in \Lambda_1$, and Λ_1, Λ_2 are bounded open subsets of B. Let $(v_1(z),\ v_2(z)) \in \Lambda_1$, then we have

$$\omega_1 > \theta(v_1(z),\ v_2(z)) = \min_{z\in I_2}\sum_{i=1}^{2}\left[v_i(z)\right] \geq k_l^{\star}\sum_{i=1}^{2}\|v_i\| = k_l^{\star}\vartheta(v_1(z),\ v_2(z)).$$

Thus $\omega_2 > \dfrac{\omega_1}{k_l^{\star}} > \vartheta(v_1(z),\ v_2(z))$, i.e., $(v_1(z),\ v_2(z)) \in \Lambda_2$, so $\Lambda_1 \subseteq \Lambda_2$.

Claim 1: $\theta\big(A(v_1(z),\ v_2(z))\big) \geq \theta(v_1(z),\ v_2(z))$, for $(v_1(z),\ v_2(z)) \in P\cap\partial\Lambda_1$. To show this let $(v_1(z),\ v_2(z)) \in P\cap\partial\Lambda_1$ then $\omega_2 = \vartheta(v_1(z),\ v_2(z)) \geq \sum_{i=1}^{2}\left[v_i(s)\right] \geq \theta(v_1(z),\ v_2(z)) = \omega_1$, for $s \in I_2$. Thus it follows from (K_1), Lemma 3 and Lemma 6 that

$$\begin{aligned}\theta\Big(A\big(v_1(z),\ v_2(z)\big)\Big) &= \min_{z\in I_2}\sum_{i=1}^{2}\left[\int_0^1 H_{m_i}(z,\ s)f_i\big(s,\ v_1(s),\ v_2(s)\big)ds\right]\\ &\geq \sum_{i=1}^{2}\left[\int_{\frac{1}{4}}^{\frac{3}{4}} k_l^{\star}H_{m_i}(s,\ s)f_i\big(s,\ v_1(s),\ v_2(s)\big)ds\right]\\ &\geq \frac{1}{2}\frac{\omega_1}{k_l^{\star}M_2}\int_{\frac{1}{4}}^{\frac{3}{4}} k_l^{\star}H_{m_1}(s,\ s)ds + \frac{1}{2}\frac{\omega_1}{k_l^{\star}M_2}\int_{\frac{1}{4}}^{\frac{3}{4}} k_l^{\star}H_{m_2}(s,\ s)ds\\ &= \frac{\omega_1}{2} + \frac{\omega_1}{2} = \omega_1 = \theta(v_1(z),\ v_2(z)).\end{aligned}$$

Claim 2: $\vartheta\big(A(v_1(z),\ v_2(z))\big) \leq \vartheta(v_1(z),\ v_2(z))$, for $(v_1(z),\ v_2(z)) \in P\cap\partial\Lambda_2$. To show this, let $(v_1(z),\ v_2(z)) \in P\cap\partial\Lambda_2$ then $\sum_{i=1}^{2}\left[v_i(s)\right] \leq \vartheta(v_1(z),\ v_2(z)) = \omega_2$, for $s \in I_1$. Thus it follows from (K_2), Lemma 3 and Lemma 6 yields

$$\vartheta\Big(A\big(v_1(z),\ v_2(z)\big)\Big) = \max_{z\in I_1}\sum_{i=1}^{2}\left[\int_0^1 H_{m_i}(z,\ s)f_i\big(s,\ v_1(s),\ v_2(s)\big)ds\right]$$

$$\leq \sum_{i=1}^{2}\left[\int_0^1 H_{m_i}(s,\ s)f_i\big(s,\ v_1(s),\ v_2(s)\big)ds\right]$$

$$\leq \frac{1}{2}\frac{\omega_2}{M_1}\int_0^1 H_{m_1}(s,\ s)ds+\frac{1}{2}\frac{\omega_2}{M_1}\int_0^1 H_{m_2}(s,\ s)ds$$

$$\leq \frac{\omega_2}{2}+\frac{\omega_2}{2}=\omega_2=\vartheta(v_1(z),\ v_2(z)).$$

Evidently, θ meets Property $1(Q_3)$ and ϑ meets Property $2(Q_4)$. Therefore the condition (i) of Theorem 1 is fulfilled and thus A has at least one fixed point $(v_1^\star,\ v_2^\star)\in P\cap(\overline{\Lambda}_2\backslash\Lambda_1)$. By Lemma 7, the operator A is monotone with respect to the cone order, and therefore the fixed point obtained corresponds to a nondecreasing function. Hence the BVP (1)-(4) has at least one nondecreasing and positive solution $(v_1^\star,\ v_2^\star)$ satisfying $\omega_1\leq\theta(v_1^\star,\ v_2^\star)$ with $\vartheta(v_1^\star,\ v_2^\star)\leq\omega_2$. □

Theorem 3 Assume that $(\mathbf{B}_1)$-$(\mathbf{B}_5)$ hold and suppose there exist positive real numbers $\omega_1,\ \omega_2$ with $\omega_1<\omega_2$ such that $f_i,\ i=1,\ 2$ satisfies:

$$(\mathrm{K}_3)\ f_i\big(z,\ v_1(z),\ v_2(z)\big)\leq\frac{1}{2}\frac{\omega_1}{M_2},\ \forall\, z\in I_1 \text{ and } \big(v_1(z),\ v_2(z)\big)\in\big[0,\ \omega_1\big],$$

$$(\mathrm{K}_4)\ f_i\big(z,\ v_1(z),\ v_2(z)\big)\geq\frac{1}{2}\frac{\omega_2}{k_l^\star M_1},\ \forall\, z\in I_2 \text{ and } \big(v_1(z),\ v_2(z)\big)\in\left[\omega_2,\ \frac{\omega_2}{k_l^\star}\right].$$

Then the BVP (1)-(4) has at least one nondecreasing and positive solution, $(v_1^\star,\ v_2^\star)$ satisfying $\omega_1\leq\vartheta(v_1^\star,\ v_2^\star)$ with $\theta(v_1^\star,\ v_2^\star)\leq\omega_2$.

Proof. Let $\Lambda_3=\big\{(v_1(z),\ v_2(z)):\vartheta(v_1(z),\ v_2(z))<\omega_1\big\}$ and $\Lambda_4=\big\{(v_1(z),\ v_2(z)):\theta(v_1(z),\ v_2(z))<\omega_2\big\}$. We have $0\in\Lambda_3$ and $\Lambda_3\subseteq\Lambda_4$ with Λ_3 and Λ_4 are bounded open subsets of B.

Claim 1: $\vartheta\big(A(v_1(z),\ v_2(z))\big)\leq\vartheta(v_1(z),\ v_2(z))$, $(v_1(z),\ v_2(z))\in P\cap\partial\Lambda_3$. To establish this, let $(v_1(z),\ v_2(z))\in P\cap\partial\Lambda_3$ then $\sum_{i=1}^{2}\big[v_i(s)\big]\leq\vartheta(v_1(z),\ v_2(z))=\omega_1$, for $s\in I_1$, and so it follows from the condition (K_3), Lemma 3 and Lemma 6 that yields

$$\vartheta\Big(A\big(v_1(z),\ v_2(z)\big)\Big) = \max_{z\in I_1}\sum_{i=1}^{2}\left[\int_0^1 H_{m_i}(z,\ s)f_i\big(s,\ v_1(s),\ v_2(s)\big)ds\right]$$

$$\leq \sum_{i=1}^{2}\left[\int_0^1 H_{m_i}(s,\ s)f_i\big(s,\ v_1(s),\ v_2(s)\big)ds\right]$$

$$\leq \frac{1}{2}\frac{\omega_1}{M_2}\int_0^1 H_{m_1}(s,\ s)ds+\frac{1}{2}\frac{\omega_1}{M_2}\int_0^1 H_{m_2}(s,\ s)ds$$

$$= \frac{\omega_1}{2} + \frac{\omega_1}{2} = \omega_1 = \vartheta(v_1(z),\ v_2(z)).$$

Claim 2: If $(v_1(z),\ v_2(z)) \in P \cap \partial\Lambda_4$ then $\theta\big(A(v_1(z),\ v_2(z))\big) \geq \theta(v_1(z),\ v_2(z))$. To see this let $(v_1(z),\ v_2(z)) \in P \cap \partial\Lambda_4$ then $\frac{\omega_2}{k_l^\star} = \frac{\theta(v_1(z),\ v_2(z))}{k_l^\star} \geq \vartheta(v_1(z),\ v_2(z)) \geq \sum_{i=1}^{2}\big[v_i(s)\big] \geq \theta(v_1(z),\ v_2(z)) = \omega_2$, for $s \in I_2$. Thus it follows from (K_4), Lemma 3 and Lemma 6 that

$$\theta\Big(A\big(v_1(z),\ v_2(z)\big)\Big) = \min_{z\in I_2}\sum_{i=1}^{2}\left[\int_0^1 H_{m_i}(z,\ s)f_i\big(s,\ v_1(s),\ v_2(s)\big)ds\right]$$

$$\geq \sum_{i=1}^{2}\left[\int_{\frac{1}{4}}^{\frac{3}{4}} k_l^\star H_{m_i}(s,\ s)f_i\big(s,\ v_1(s),\ v_2(s)\big)ds\right]$$

$$\geq \frac{1}{2}\frac{\omega_2}{k_l^\star M_1}\int_{\frac{1}{4}}^{\frac{3}{4}} k_l^\star H_{m_1}(s,\ s)ds + \frac{1}{2}\frac{\omega_2}{k_l^\star M_1}\int_{\frac{1}{4}}^{\frac{3}{4}} k_l^\star H_{m_2}(s,\ s)ds$$

$$= \frac{\omega_2}{2} + \frac{\omega_2}{2} = \omega_2 = \theta(v_1(z),\ v_2(z)).$$

Thus it is shown that θ and ϑ satisfy Property $1(Q_3)$ and Property $2(Q_4)$ respectively. Thus, A has at least one fixed point $(v_1^\star,\ v_2^\star) \in P \cap (\overline{\Lambda}_4 \backslash \Lambda_3)$ since the hypothesis (ii) of Theorem 1 is satisfied.

At this stage, it remains to justify rigorously that the integral operator A preserves the monotonicity of functions in P. This requires a lemma establishing that A is monotone with respect to the cone order, thereby ensuring that the obtained fixed point corresponds to a nondecreasing solution.

As a result, BVP (1)-(4) has at least one nondecreasing and positive solution $(v_1^\star,\ v_2^\star)$ satisfying $\omega_1 \leq \vartheta(v_1^\star,\ v_2^\star)$ with $\theta(v_1^\star,\ v_2^\star) \leq \omega_2$. □

4. Illustrative example

To demonstrate the applicability of our main results, consider the following coupled system of second-order Sturm-Liouville BVPs:

$$-v_1''(z) + \frac{1}{9}v_1(z) = f_1\big(z,\ v_1(z),\ v_2(z)\big), \quad z \in (0,\ 1), \tag{13}$$

$$-v_2''(z) + \frac{1}{16}v_2(z) = f_2\big(z,\ v_1(z),\ v_2(z)\big), \quad z \in (0,\ 1), \tag{14}$$

subject to the BCs

$$\left.\begin{aligned} &11v_1(0)-5v_1'(0)=0,\\ &13v_1(1)+7v_1'(1)=0, \end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} &12v_2(0)-6v_2'(0)=0,\\ &14v_2(1)+7v_2'(1)=0, \end{aligned}\right\} \tag{16}$$

where the nonlinearities are given by

$$\begin{cases} f_1\big(z,\ v_1(z),\ v_2(z)\big)=\dfrac{22}{147}\ z^2\big(v_1(z)+v_2(z)\big)+\dfrac{29}{30}\ e^{-\left(v_1(z)+v_2(z)\right)^2},\\[2ex] f_2\big(z,\ v_1(z),\ v_2(z)\big)=\dfrac{1}{11}\big(z^2+3\big)\big(v_1(z)+v_2(z)\big)+\dfrac{323}{300}\ \sin z+\dfrac{92}{103}. \end{cases}$$

By direct computation, we obtain

$$k_l^\star=0.57588,\qquad M_1=27.90433,\qquad M_2=151.0836.$$

Setting $\omega_1=8$ and $\omega_2=137$, we verify that the inequality $\omega_1<k_l^\star\omega_2$ holds true. Furthermore, the nonlinear functions f_i for $i=1,\ 2$ fulfill the growth conditions required by Theorem 2, specifically:

$$f_i(z,\ v_1(z),\ v_2(z))\geq 0.04597=\frac{1}{2}\cdot\frac{\omega_1}{k_l^\star M_2},\qquad \forall(z,\ v_1(z),\ v_2(z))\in\left[\frac{1}{4},\ \frac{3}{4}\right]\times[8,\ 137]^2,$$

$$f_i(z,\ v_1(z),\ v_2(z))\leq 2.45481=\frac{1}{2}\cdot\frac{\omega_2}{M_1},\qquad \forall(z,\ v_1(z),\ v_2(z))\in[0,\ 1]\times[0,\ 137]^2.$$

Consequently, all assumptions of Theorem 2 are fulfilled. Therefore, by Theorem 1, the BVP (13)-(16) admits at least one positive and nondecreasing solution $(v_1^\star,\ v_2^\star)$.

5. Conclusion

This work investigates the existence of positive and nondecreasing solutions for a coupled system of second-order undamped Sturm-Liouville BVPs, characterized by nonlinear interactions and nonstandard BCs. By analyzing the structure of the associated Green functions and employing a cone-theoretic FPT framework within a Banach space, we develop a rigorous analytical approach that effectively captures the complexity of such nonlinear coupled systems. While the analysis guarantees at least one positive solution, the method has limitations, including open questions regarding uniqueness, multiplicity, and quantitative behavior of solutions. The scalability of the approach to large-scale or highdimensional systems, stability under parameter variations, and applicability to real-world data with uncertainties require further evaluation.

The proposed methodology extends classical techniques by incorporating monotonicity constraints into the solution space and adapting fixed-point principles to the undamped and fully coupled setting. Beyond establishing new existence results, this approach offers a flexible and robust strategy for addressing boundary phenomena that lie beyond the reach of traditional linear analysis. It further opens promising directions for future exploration, including multipoint BCs, higher-order systems, and fractional-order formulations within the Sturm-Liouville paradigm. Our analysis establishes the existence of at least one positive solution. It is worth noting that the framework may also be adapted to study the uniqueness and multiplicity of positive solutions through cone-theoretic approaches. A comprehensive treatment of these aspects will be pursued in future work. Additionally, development of numerical schemes and adaptation to practical applications will enhance the method's stability and real-world relevance.

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Conflict of interest

The authors declare no competing financial interest.

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