Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Global Solvability of the Generalized Boussinesq System with Linear or Nonlinear Buoyancy Force

Shiwei Cao, Huiyang Zhang, Qinghua Zhang*

School of Mathematics and Statistics, Nantong University, Nantong, Jiangsu, 226019, China
E-mail: zhangqh@ntu.edu.cn

Received: 06 July 2025; **Revised:** 26 August 2025; **Accepted:** 01 September 2025

Abstract: This paper is devoted to the global solvability of the Boussinesq system with fractional Laplacian $(-\Delta)^{\alpha}$ in $\mathbb{R}^n$ for $n \geq 3$, where the buoyancy force has the form $|\theta|^{m-1}\theta e_n$ with $m \geq 1$. By establishing estimates for the difference $|\theta_1|^{m-1}\theta_1 - |\theta_2|^{m-1}\theta_2$ in Besov spaces and employing the maximal regularity property of $(-\Delta)^{\alpha}$ in Lorentz spaces, we prove the following results: under some reasonable assumptions on the exponents α, m, p, r and ρ, if the small initial data of velocity and temperature (or salinity) fall in $\dot{B}^{1+n/p-2\alpha}_{p,r,\sigma} \times \dot{B}^{1+n/p_i-4\alpha}_{p_i,r}$ (where $p_1 = p$ for $1 < p < n$, and $p_2 = p/2$ for $n \leq p < 2n$) when $m = 1$, and in $\dot{B}^{1+n/p-2\alpha}_{p,r,\sigma} \times \dot{B}^{n/p-(4\alpha-1)/m}_{p,r}$ when $m > 1$, then the generalized Boussinesq system admits a unique global strong solution (u, θ) in $L^{\rho,r}(0, \infty; \dot{B}^{1+n/p+2\alpha/\rho-2\alpha}_{p,1,\sigma}) \times L^{\rho/2,r}(0, \infty; \dot{B}^{1+n/p_i+4\alpha/\rho-4\alpha}_{p_i,1})$ (with $i = 1, 2$ corresponding to the definition of p_1, p_2) for $m = 1$ and in $L^{\rho,r}(0, \infty; \dot{B}^{1+n/p+2\alpha/\rho-2\alpha}_{p,1,\sigma}) \times L^{m\rho/2,r}(0, \infty; \dot{B}^{n/p-(4\alpha-1)/m+4\alpha/m\rho}_{p,1})$ for $m > 1$, respectively.

***Keywords*:** global solvability, Boussinesq system, fractional Laplacian, linear or nonlinear buoyancy force

MSC: 35Q30, 76D05

1. Introduction

This paper aims at global solvability of the Boussinesq system with fractional power of the negative Laplacian in $\mathbb{R}^n$ ($n \geq 3$), i.e.

$$\begin{cases} \partial_t u + \mu(-\Delta)^{\alpha} u + u \cdot \nabla u + \nabla \pi = \kappa J_m(\theta) e_n, \quad t > 0, \ x \in \mathbb{R}^n; \\ \partial_t \theta + \nu(-\Delta)^{\beta}\theta + u \cdot \nabla \theta = 0, \quad t > 0, \ x \in \mathbb{R}^n; \\ \nabla \cdot u = 0, \quad t > 0, \ x \in \mathbb{R}^n; \\ \theta(0, x) = \theta_0(x), \ u(0, x) = u_0(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1}$$

DOI: https://doi.org/10.37256/cm.6520257734

where $J_m(\theta) = |\theta|^{m-1}\theta$ for $m \geq 1$ and $\alpha,\, \beta > 0$. Boussinesq system is a simplified model to motivate the motion of the ocean or the atmosphere. Here $u(x,\, t) = (u_1(x,\, t),\, u_2(x,\, t),\, \cdots,\, u_n(x,\, t))$, $\theta(x,\, t)$ and $\pi(x,\, t)$ indicate respectively the velocity of the flow, the temperature (or salinity) and interior pressure of the fluid at the place $x \in \mathbb{R}^n$ and the time $t > 0$. Correspondingly, the initial temperature (or salinity) and initial velocity are respectively denoted by $\theta_0(x)$ and $u_0(x)$. Here $e_n = (0,\, \cdots,\, 0,\, 1)$ is the unit vertical vector, and κ is the proportion coefficient.

It is known that the buoyancy force (per volume) F increases with salinity, as higher salinity leads to greater density of the liquid, and buoyancy is proportional to the density of the liquid (by Archimedes' principle). If conditions such as temperature and pressure remain unchanged, the relationship between liquid density and salinity is approximately linear, and buoyancy can also be approximated as a linear function of salinity, i.e. $F = \kappa\theta e_n$. As for the thermal buoyancy, thermodynamic principles indicate that it stems from temperature differences within a liquid. For small temperature differences, it can be approximated as a linear function of temperature. However, when temperature differences are large, especially near the freezing or boiling points, the relationship between thermal buoyancy and temperature becomes nonlinear, which can be approximated by a polynomial, i.e. $F = (a_0 + a_1\theta + \cdots + a_k\theta^k)e_n$. When the liquid temperature stays above a certain point, thermal buoyancy can also be modeled as a power-law function, i.e. $F = \kappa J_m(\theta)e_n$ for some $m > 1$. For the sake of simplicity, here the viscosity coefficients $\mu,\, \nu$ and the proportion coefficient κ are all assumed to 1.

Danchin and Paicu [1] established global existence of the weak solution of the Standard Boussinesq system ($\alpha = \beta = m = 1$) with partial viscosity $\nu = 0$ under the initial condition $u_0 \in L^2_\sigma(\mathbb{R}^n)$ and $\theta_0 \in L^p(\mathbb{R}^n)$ for some $2n/(n+2) < p \leq 2$. Brandolese and Schonbek [2] and Han [3] studied the time-decaying profiles of the weak solutions in energy space under the extra assumption $\theta_0 \in L^1$. Danchin and Paicu [4] re-addressed the global existence of the weak solutions by employing the initial assumption: $u_0 \in L^{n,\,\infty}$ and $\theta_0 \in L^{n/3,\,\infty} \cap L^{p,\,\infty}$ for some $p > n/3$ with small data $\mu^{-1}\|\theta_0\|_{L^{n/3,\,\infty}} + \|u\|_{L^{n,\,\infty}} \leq \varepsilon\mu$, where $L^{r,\,\infty} = (L^1,\, L^\infty)_{1-1/r,\,\infty}$ is the real interpolation space between L^1 and L^∞, called Lorentz space.

Strong solvability is also a main topic in the study of the Standard Boussinesq system. It was proved that if $n \leq p < \infty$ and $(u_0,\, \theta_0) \in \dot{B}^{n/p-1}_{p,\,1} \times \dot{B}^0_{n,\,1}$ for $\nu = 0$, or $(u_0,\, \theta_0) \in \dot{B}^{n/p-1}_{p,\,1} \times \dot{B}^{s-1}_{p,\,1}$ for some $-n/p < s \leq n/p$ and $\nu > 0$, then the Boussinesq system admits a unique local strong solution. This solution is globally exists under the additional assumption $\mu^{-1}\|\theta_0\|_{L^{n/3,\,\infty}} + \|u\|_{L^{n,\,\infty}} \leq \varepsilon\mu$, or respectively $\min\{\mu,\, \nu\}^{-1}\|\theta_0\|_{L^{n/3}} + \|u\|_{L^n} \leq \varepsilon \min\{\mu,\, \nu\}$, see [1, 5] for references. Similar condition $\|\theta_0\|_{L^1} + \|\theta_0\|_{L^\infty(\mathbb{R}^3,\, |x|^3dx)} + \|u_0\|_{L^\infty(\mathbb{R}^3,\, |x|dx)} \leq \varepsilon$ was applied in [2] to deal with the globally existence of the strong solution to $3D$ Boussinesq system with $\mu = \nu = 1$ in weighted $L^\infty-$space.

In recent years, Navier-Stokes equations with fractional Laplacian $(-\Delta)^\alpha$, known as the Generalized Navier-Stokes equations (GNS), have attracted growing interests. Recall that global regular solution of (GNS) exists only for $\alpha \geq (n+2)/4$ (cf. [6, 7]). When $\alpha < (n+2)/4$, the situation becomes more complex. Well-posedness for (GNS) holds only under small-data assumptions (either small initial norm or small lifespan). By virtue of the analytic semigroup method in Besov spaces and employment of Gevrey regularity, Chen [8] established local existence of the strong solution u in the class $L^\infty(0,\, T;\, \dot{B}^{1-2\alpha+n/p}_{p,\,\infty})$ for $\alpha > 1/2$, $1 < p < \infty$ and $2 - 2\alpha + n/p > 0$, and in $C([0,\, T];\, \dot{B}^{n/p}_{p,\,\infty})$ for $\alpha = 1/2$ and $1 < p < \infty$. Duan [9] proved global existence of the strong solution to $3D$ (GNS) in homogeneous Sobolev spaces under the small-norm assumption on $\|u_0\|_{\dot{H}^{(5-4\alpha)/2}}$ for $1/2 < \alpha < 5/4$. Wu [10] investigated the lower bounds for an integral involving $(-\Delta)^\alpha$, and further established global solvability of $3D$ (GNS) in homogeneous Besov spaces $\dot{B}^{1-2\alpha+n/p}_{p,\,q}$ under a small initial data assumption for $\alpha > 1/2$ and $p = 2$ or $1/2 < \alpha \leq 1$ and $2 < p < \infty$. Global solvability of $3D$ (GNS) in the largest critical space $\dot{B}^{1-2\alpha}_{\infty,\,\infty}$ with $1/2 < \alpha \leq 1$ can be found in Yu-Zhai [11], while ill-posedness results in the largest critical space $\dot{B}^{1-2\alpha}_{\infty,\,\infty}$ with $1 \leq \alpha < 5/4$ were studied in Cheskidov-Shvydkoy [12]. Liu [13] addressed the same topic for $1/2 < \alpha \leq 1$. In his work, though the initial data $\|u_0\|_{\dot{B}^{1-2\alpha+n/p}_{p,\,1}}$ may be large, a small-norm assumption is imposed on the convection term $e^{-t(-\Delta)^\alpha}u_0 \cdot \nabla e^{-t(-\Delta)^\alpha}u_0$ in $L^1(0,\, \infty;\, \dot{B}^{1-2\alpha+n/p}_{p,\,1})$.

Return to the Generalised Boussinesq System (or (GBS) symbolically). We first summarize the research on the 2D (GBS). The global existence and uniqueness of the strong solution was proved by Miao and Xue [14] for $\alpha \in ((6 - \sqrt{6})/4,\, 1)$ and $\beta \in (1-\alpha,\, 2(1-\alpha))$, by Jiu, Miao, Wu and Zhang [15] for $0 < \alpha,\, \beta < 1$ and $\alpha + \beta = 1$, and by Yang, Jiu and Wu [16] for $0 < \alpha,\, \beta < 1$ with $\alpha/2 + \beta > 1$ and $\beta \geq (2+\alpha)/3$. In comparison, the literature on the 3D (GBS) is far more limited. Here we only mention the work of Jiu and Yu [17], where global well-posedness of the 3D (GBS) with partial viscosity ($\nu = 0$) in fractional Sobolev spaces for $\alpha \geq 5/4$ was addressed. By employment of the log Sobolev inequality to bound $\|\nabla u\|_\infty$ by $\|\nabla u\|_{\mathrm{BMO}}$ and $\|\nabla u\|_{H^s}$ for $s > 5/2$, together with commutator estimates in H^s spaces, they proved global existence and uniqueness of the strong solution to $3D$ (GBS) with $\nu = 0$ under the assumption $(u_0, \theta_0) \in H^s \times H^{s-1}$ for $s > 5/4$.

This paper focuses on the global solvability for the generalised Boussinesq system (1) with full viscosity ($\mu, \nu \neq 0$) and linear or nonlinear buoyancy force $J_m(\theta)e_n$ in $\mathbb{R}^n$ ($n \geq 3$). It is easy to check that, this system is invariant under the following scaling transformations

$$\begin{aligned} &u_\lambda(x, t) = \lambda^{2\alpha-1} u(\lambda x, \lambda^{2\alpha} t), \ \pi_\lambda(x, t) = \lambda^{4\alpha-2}\pi(\lambda x, \lambda^{2\alpha} t) \text{ and} \\ &\theta_\lambda(x, t) = \lambda^{(4\alpha-1)/m}\theta(\lambda x, \lambda^{2\alpha} t) \end{aligned} \tag{2}$$

if and only if $\beta = \alpha$. So throughout this paper, condition $\beta = \alpha$ is assumed. We also assume $\kappa = \mu = \nu = 1$ as mentioned in the first paragraph.

Our research will be carried out under the framework of maximal regularity of $(-\Delta)^\alpha$ in Lorenz-Besov spaces, developed by Kozono and Shimizu [18, 19] based on the L^p-regularity of Stokes operators established by Giga and Sohr [20]. Its key idea is to reduce the (GBS) system (1) to an abstract system (75), meanwhile, the global solution to (1) turns out to be a fixed point of an abstract operator Γ (see p. 16. line 11 for the definition). To ensure the existence of the fixed point, we need estimates for the semigroup $e^{-t(-\Delta)^\alpha}$, the nonlinear term $J_m(\theta)$, and the difference $J_m(\theta_1) - J_m(\theta_2)$ (for $m > 1$) in homogeneous Besov spaces. The first one is a natural generalization of the heat semigroup $e^{t\Delta}$, the second one can be obtained in a way very similar to that in [21, §1.5] for $|\theta|^m$. However, for the final one, previous estimates rely on the boundedness of θ_i, a property unavailable in most Besov spaces. Initially, we will give an auxiliary estimate for $J_m(\theta)$ when $0 < m < 1$, then successively derive new estimates for $J_m(\theta_1) - J_m(\theta_2)$ and $J_m(\theta)$ when $m > 1$. These estimates seem optimal as they apply to a large number of critical homogeneous Besov spaces.

It should also be pointed out that, to offer bounds for the Lorenz-Besov norm of the solution to the abstract system (75), we will carry out investigations on a type of high-order polynomial containing the term λ^m, which emerges from the a priori estimates, and search for initial value assumptions to determine the polynomial's first root, this is to enable the resolvent operator Γ to become a contractive map on some carefully selected function spaces. After that, by applying the contraction principle, we will prove the global existence of the strong solution to system (1) in both linear and nonlinear cases. To our best knowledge, this is the first attempt to deal with the Boussinesq system with nonlinear buoyancy force. The main results of this paper read.

Theorem 1 Assume that $m > 1$, $1 \leq r < \infty$, and p, α, ρ satisfy hypothesis H_3 or H_4. Let

$$s = 1 + \frac{n}{p} + \frac{2\alpha}{\rho} - 4\alpha, \quad s_0 = 1 + \frac{n}{p} - 2\alpha, \tag{3}$$

and

$$s_m = \frac{n}{p} - \frac{4\alpha - 1}{m} + \frac{4\alpha}{m\rho} - 2\alpha, \quad \varsigma_m = \frac{n}{p} - \frac{4\alpha - 1}{m}. \tag{4}$$

Under this setting, there exists a small number $c > 0$ and a constant $C > 0$ depending on m, α, p, r and n such that if $(u_0, \theta_0) \in \dot{B}^{s_0}_{p,r,\sigma} \times \dot{B}^{\varsigma_m}_{p,r}$ satisfies

$$\|u_0\|_{\dot{B}^{s_0}_{p,r}} + \|\theta_0\|_{\dot{B}^{\varsigma_m}_{p,r}} \le c, \tag{5}$$

then Boussinesq system (1) has a unique strong solution $(u, \nabla\pi, \theta)$ on the whole interval $[0, \infty)$ such that

$$u \in L^{\rho, r}(0, \infty; \dot{B}^{s+2\alpha}_{p, 1, \sigma}), \;\; u' \in L^{\rho, r}(0, \infty; \dot{B}^{s}_{p, 1}), \;\; \nabla\Pi \in L^{\rho, r}(0, \infty; \dot{B}^{s}_{p, 1}), \tag{6}$$

and

$$\theta \in L^{m\rho/2, r}(0, \infty; \dot{B}^{s_m+2\alpha}_{p, 1}), \;\; \theta' \in L^{m\rho/2, r}(0, \infty; \dot{B}^{s_m}_{p, 1}). \tag{7}$$

Moreover, this solution complies with the following estimates

$$\begin{aligned} &\|u\|_{L^{\rho, r}(0, \infty; \dot{B}^{s+2\alpha}_{p, 1, \sigma})} + \|u'\|_{L^{\rho, r}(0, \infty; \dot{B}^{s}_{p, 1})} + \|\nabla\Pi\|_{L^{\rho, r}(0, \infty; \dot{B}^{s}_{p, 1})} \\ &+ \|\theta\|_{L^{m\rho/2, r}(0, \infty; \dot{B}^{sm+2\alpha}_{p, 1})} + \|\theta'\|_{L^{m\rho/2, r}(0, \infty; \dot{B}^{sm}_{p, 1})} \\ \le &C(\|u_0\|_{\dot{B}^{s_0}_{p, r}} + \|\theta_0\|_{\dot{B}^{\varsigma_m}_{p, r}}). \end{aligned} \tag{8}$$

Theorem 2 Assume that $m = 1$, $1 \le r < \infty$, p, α, ρ satisfy hypothesis H_1 or H_2. Let s and s_0 take the values in (3), and

$$\sigma_i = 1 + \frac{n}{p_i} + \frac{4\alpha}{\rho} - 6\alpha, \;\; \text{and} \;\; \omega_i = 1 + \frac{n}{p_i} - 4\alpha, \;\; i = 1, 2, \tag{9}$$

where $p_1 = p$ if $1 < p < n$, and $p_2 = p/2$ if $n \le p < 2n$. Under these assumptions, there exists a small number $c > 0$ a constant $C > 0$ depending on α, p, r and n such that if $(u_0, \theta_0) \in \dot{B}^{s_0}_{p, r, \sigma} \times \dot{B}^{\omega_i}_{p_i, r}$ and

$$\|u_0\|_{\dot{B}^{s_0}_{p, r}} + \|\theta_0\|_{\dot{B}^{\omega_i}_{p_i, r}} \le c, \tag{10}$$

then Boussinesq system (1) admits a unique strong solution $(u, \nabla\pi, \theta)$ on $[0, \infty)$. Here u and $\nabla\pi$ verify the inclusion (6), while θ satisfies

$$\theta \in L^{\rho/2, r}(0, \infty; \dot{B}^{\sigma_i+2\alpha}_{p_i, 1}), \;\; \theta' \in L^{\rho/2, r}(0, \infty; \dot{B}^{\sigma_i}_{p_i, 1}),$$

and all of them are subject to the following estimates

$$\|u\|_{L^{\rho,\,r}(0,\,\infty;\,\dot{B}^{s+2\alpha}_{p,\,1,\,\sigma})}+\|u'\|_{L^{\rho,\,r}(0,\,\infty;\,\dot{B}^{s}_{p,\,1})}+\|\nabla\Pi\|_{L^{\rho,\,r}(0,\,\infty;\,\dot{B}^{s}_{p,\,1})}$$

$$+\|\theta\|_{L^{\rho/2,\,r}(0,\,\infty;\,\dot{B}^{\sigma_i+2\alpha}_{p_i,\,1})}+\|\theta'\|_{L^{\rho/2,\,r}(0,\,\infty;\,\dot{B}^{\sigma_i}_{p_i,\,1})} \tag{11}$$

$$\le C(\|u_0\|_{\dot{B}^{s_0}_{p,\,r}}+\|\theta_0\|_{\dot{B}^{\omega_i}_{p_i,\,r}}).$$

Here $i=1$ if H_1 hold, and $i=2$ if H_2 hold.

Remark 3 Since the norm $\|u(\lambda\cdot)\|_{\dot{B}^{s}_{p,\,r}}$ is equivalent to $\lambda^{s-n/p}\|u\|_{\dot{B}^{s}_{p,\,r}}$ for all $\lambda>0$, one can easily check that $\|u_0\|_{\dot{B}^{s_0}_{p,\,r}}$ is invariant under the transformation $u_0\mapsto\lambda^{2\alpha-1}u_0(\lambda\cdot)$, while $\|u\|_{L^{\rho,\,r}(0,\,\infty;\,\dot{B}^{s+2\alpha}_{p,\,1,\,\sigma})}$ is invariant under the transformation $u\mapsto u_\lambda$. In this sense, $\dot{B}^{s_0}_{p,\,r}$ is called critical initial space, and $L^{\rho,\,r}(0,\,\infty;\,\dot{B}^{s+2\alpha}_{p,\,1,\,\sigma})$ is called critical temporal-spatial space. We can also check that $\dot{B}^{\omega_i}_{p_i,\,r}$ and $\dot{B}^{\varsigma_m}_{p,\,r}$ are both critical initial spaces associated with θ_0 for $m=1$ and $m>1$ respectively, while $L^{\rho/2,\,r}(0,\,\infty;\,\dot{B}^{\sigma_i+2\alpha}_{p_i,\,1})$ and $L^{m\rho/2,\,r}(0,\,\infty;\,\dot{B}^{s_m+2\alpha}_{p,\,1})$ are both critical temporal-spatial spaces associated with θ for $m=1$ and $m>1$ respectively.

Remark 4 Evidently, when $m=1$, the exponent α can assume the value of 1 if, whithin hypothesis H_1, we specify $1<p<n/2$. This phenomenon also occurs when $m\ge 2$. When $m_0\le m<2$, we set $\max\{m(2-m)n,\,n/m\}\le p<(m+2)n/(6-m)$ in H_3 so that α can take the value 1, where $4/3<m_0<3/2$ is the unique real root of the polynomial $m^3-8m^2+11m-2$ within $(1,\,2)$. This indicates that global solvability of the Boussinesq system with $\alpha=1$ and the buoyancy $\kappa J_m(\theta)e_n$ in critical homogeneous Besov spaces is maintained through the careful selection of $p,\,\rho$ when $m=1$ or $m\ge m_0$. We wonder whether this fact remains valid for $1<m<m_0$.

2. Preliminaries on Besov spaces and fractional Laplacian

Throughout this paper, we employ $L^p(\mathbb{R}^n)$ $(1\le p\le\infty)$ to signify the common Lebesgue space of scalar or vector type whose norm is denoted by $\|\cdot\|_p$. Additionally, we utilize $C_0^\infty(\mathbb{R}^n)$ for the collection of all smooth functions with compact supports, and take advantage of $\mathscr{S}(\mathbb{R}^n)$ to denote the space of all rapidly decreasing smooth functions, while its dual, the collection of all tempered generalised functions, is designated by $\mathscr{S}'(\mathbb{R}^n)$. For $\mathscr{S}'_h(\mathbb{R}^n)$, we denote the subset of $\mathscr{S}'(\mathbb{R}^n)$, whose members satisfy $\|\mathscr{F}^{-1}(\theta(\lambda\xi)\mathscr{F}f)\|_{L^\infty}\to 0$ as $\lambda\to\infty$ for all $\theta\in C_0^\infty(\mathbb{R}^n)$, where $\mathscr{F}$ represents the Fourier transformation on $\mathscr{S}'(\mathbb{R}^n)$, while $\mathscr{F}^{-1}$ represents its inverse. For the sake of convenience, hereinafter, the part $(\mathbb{R}^n)$ in all the notations of function spaces, wether scalar or vector type, is omitted.

We begin with the brief introduction to the theory of Littlewood-Paley decomposition and homogeneous Besov spaces, for the detailed discussions, please refer to [22, §2.3]. Take $\chi\in C_0^\infty$ such that $0\le\chi\le 1$, supp $\chi\subseteq\{|\xi|\le 4/3\}$ and $\chi(\xi)=1$ on the ball $\{|\xi|\le 3/4\}$. Let $\varphi(\xi)=\chi(2^{-1}\xi)-\chi(\xi)$, then we have $\chi\in C_0^\infty$ satisfying supp $\varphi\subseteq\{3/4\le|\xi|\le 8/3\}$,

$$\chi(\xi)+\sum_{q=0}^{\infty}\varphi(2^{-j}\xi)=1 \text{ for all } \xi\in\mathbb{R}^n, \tag{12}$$

and

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1 \text{ for all } \xi\in\mathbb{R}^n\setminus\{0\}. \tag{13}$$

Given $f \in \mathscr{S}'$, for each $q \in \mathbb{Z}$, define

$$\begin{aligned}\dot{\Delta}_j f &= \mathscr{F}^{-1}(\varphi(2^{-j}\cdot)\mathscr{F}f) = 2^{jn}(\mathscr{F}^{-1}\varphi)(2^j\cdot) * f,\\ \dot{S}_j f &= \mathscr{F}^{-1}(\chi(2^{-j}\cdot)\mathscr{F}f) = 2^{jn}(\mathscr{F}^{-1}\chi)(2^j\cdot) * f.\end{aligned} \tag{14}$$

By Young's inequality, it is easy to see that $\|\dot{\Delta}_j f\|_{L^p} \le C\|f\|_{L^p}$ and $\|\dot{S}_j f\|_{L^p} \le C\|f\|_{L^p}$ for all $1 \le p \le \infty$. Taking any $f \in \mathscr{S}'$, by the forgoing definitions, one can easily check the following equality

$$\dot{S}_{-1} f + \sum_{j=0}^{\infty} \dot{\Delta}_j f = f \ \text{ in } \ \mathscr{S}'. \tag{15}$$

Moreover, if additionally $f \in \mathscr{S}'_h$, then we have

$$\sum_{j\in\mathbb{Z}} \dot{\Delta}_q f = f \ \text{ and } \ \dot{S}_j f = \sum_{q'\le q-1} \dot{\Delta}_q f \ \text{ in } \ \mathscr{S}'. \tag{16}$$

Given $s \in \mathbb{R}$ and $p,\, r \in [1,\, \infty]$, the homogeneous Besov space $\dot{B}^s_{p,\,r}$ is defined by

$$\dot{B}^s_{p,\,r} = \left\{ f \in \mathscr{S}'_h \colon \|f\|_{\dot{B}^s_{p,\,r}} = \|(2^{qs}\|\dot{\Delta}_q f\|_{L^p})\|_{l^r} < \infty \right\}, \tag{17}$$

where $\|(a_q)\|_{l^r} = (\sum_{q\in\mathbb{Z}} |a_q|^r)^{1/r}$ if $1 \le r < \infty$, and $\|(a_q)\|_{l^r} = \sup_{q\in\mathbb{Z}} |a_q|$ if $r = \infty$. Evidently, attached to the norm $\|\cdot\|_{\dot{B}^s_{p,\,r}}$, $\dot{B}^s_{p,\,r}$ is a normed space. Additionally if

$$s < \frac{n}{p}, \ \text{ or } \ s = \frac{n}{p} \ \text{ and } \ r = 1, \tag{18}$$

then $\dot{B}^s_{p,\,r}$ becomes a Banach space. There are some embedding relations between Besov and Lebesgue spaces. If $1 \le p_1 \le p_2 \le \infty$ and $1 \le r_1 \le r_2 \le \infty$, then we have $\dot{B}^s_{p_1,\,r_1} \hookrightarrow \dot{B}^{s-n(1/p_1-1/p_2)}_{p_2,\,r_2}$. Besides, for all $1 \le p \le \infty$, we have $\dot{B}^0_{p,\,1} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,\,\infty}$.

Let $\alpha \in \mathbb{R}$, define the fractional power of the negative Laplacian by $(-\Delta)^\alpha f = \mathscr{F}^{-1}(|\xi|^{2\alpha}\mathscr{F}f)$. Since $|\xi|^{2\alpha}$ is $2|\alpha|-$order homogeneous in $\mathbb{R}^n \setminus \{0\}$, we know that $(-\Delta)^\alpha$ is bounded from $\dot{B}^s_{p,\,r}$ to $\dot{B}^{s-2\alpha}_{p,\,r}$ provided $s-2\alpha,\, p,\, r$ fulfill the condition (18) (cf. [22, Proposition 2.30]). Especially, the norms $\|(-\Delta)^{1/2} f\|_{\dot{B}^s_{p,\,r}}$ and $\|\nabla f\|_{\dot{B}^s_{p,\,r}}$ are equivalent. Without further specification, in the coming discussions, we always assume that hypothesis (18) is fulfilled by $s,\, p,\, r$, unless there is any confusion arising.

Denote by $B_\alpha = (-\Delta)^\alpha$ for $\alpha \ge 0$, and define the generalised heat semigroup as follows:

$$e^{-tB_\alpha} = \mathscr{F}^{-1} e^{-t|\xi|^{2\alpha}} \mathscr{F}, \ t \ge 0. \tag{19}$$

Lemma 5 ([21], §1.1, §2.3) Given an annulus $\widetilde{\mathscr{C}}$ in $\mathbb{R}^n$, there are corresponding two constants $c, C > 0$ such that for any $1 \le p \le \infty$ and $f \in \mathscr{S}'$ with supp$\mathscr{F} f$ contained in $\lambda\widetilde{\mathscr{C}}$ for some $\lambda > 0$, it holds

$$\|e^{-tB_\alpha} f\|_{L^p} \le Ce^{-ct\lambda^{2\alpha}} \|f\|_{L^p},\ \forall\, t > 0. \tag{20}$$

Similar to the case $\alpha = 1$, by taking advantage of this lemma, we can deduce the uniform boundedness and strong continuity of the semigroup e^{-tB_α} on $\dot{B}^s_{p,\,r}$ provided $s,\ p,\ r$ satisfy (18) and $r < \infty$. More precisely, for all $a \in \dot{B}^s_{p,\,r}$, the function $e^{-tB_\alpha} a$ lies in $C([0, \infty), \dot{B}^s_{p,\,r})$, and

$$\|e^{-tB_\alpha} a\|_{\dot{B}^s_{p,\,r}} \le C\|a\|_{\dot{B}^s_{p,\,r}}. \tag{21}$$

Herein after, $C > 0$ represents a universal constant, it may vary from line to line, but does not depends on the involved functions and the time t.

Lemma 6 ([22], Lemma 2.35) For all $a \ge 1$ and $b > 0$, it holds that

$$\sup_{t>0} \sum_{j\in\mathbb{Z}} (a^j t)^b e^{-cta^j} < \infty. \tag{22}$$

The following corollary is an extension of that presented in [19].

Corollary 7 Let $0 \le \gamma \le \alpha$, $1 \le r \le \infty$, and let $s_0,\ s \in \mathbb{R}$, $1 < \rho < \infty$ and $1 \le p_0 \le p \le \infty$ such that

$$s_0 - \frac{n}{p_0} - 2\alpha < s - \frac{n}{p} < s_0 - \frac{n}{p_0} \ \text{ and }\ s - \frac{n}{p} - \frac{2\alpha}{\rho} = s_0 - \frac{n}{p_0} - 2\alpha. \tag{23}$$

If $a \in \dot{B}^{s_0}_{p_0,\,r}$, then $B_\gamma e^{-tB_\alpha} a \in L^{\rho,\,r}(0, \infty; \dot{B}^{s+2(\alpha-\gamma)}_{p,\,1})$, and

$$\|B_\gamma e^{-tB_\alpha} a\|_{L^{\rho,\,r}(0,\,\infty;\, \dot{B}^{s+2(\alpha-\gamma)}_{p,\,1})} \le C\|a\|_{\dot{B}^{s_0}_{p_0,\,r}}, \tag{24}$$

where the constant $C > 0$ is independent of a and γ.

Proof. By Lemma 5 and Bernstein's inequality, we have

$$\|\dot{\Delta}_j B_\gamma e^{-tB_\alpha} a\|_{L^p} = \|e^{-tB_\alpha} B_\gamma \dot{\Delta}_j a\|_{L^p} \le C2^{j(2\gamma+n/p_0-n/p)} e^{-ct2^{2\alpha j}} \|\dot{\Delta}_j a\|_{L^{p_0}}. \tag{25}$$

Consequently,

$$\begin{aligned} \|B_\gamma e^{-tB_\alpha} a\|_{\dot{B}^{s+2(\alpha-\gamma)}_{p,\,1}} &\le Ct^{-1/\rho} \sum_{j\in\mathbb{Z}} (t2^{2\alpha j})^{1/\rho} e^{-ct2^{2\alpha j}} 2^{js_0} \|\dot{\Delta}_j a\|_{L^{p_0}} \\ &\le Ct^{-1/\rho} \|a\|_{\dot{B}^{s_0}_{p_0,\,\infty}}, \end{aligned} \tag{26}$$

where inequality (22) is applied. Thus we have $B_\gamma e^{-tB_\alpha}a \in L^{\rho,\,\infty}(0,\,\infty;\,\dot{B}^{s+2(\alpha-\gamma)}_{p,\,1})$, and

$$\|B_\gamma e^{-tB_\alpha}a\|_{L^{\rho,\,\infty}(0,\,\infty;\,\dot{B}^{s+2(\alpha-\gamma)}_{p,\,1})} \le C\|a\|_{\dot{B}^{s_0}_{p_0,\,\infty}}. \tag{27}$$

Now, we choose $s_1 < s_0 < s_2$ in such a manner that both s_1 and s_2 fulfill condition (23), and let $\rho_i = 2\alpha/(s - s_i + 2\alpha + n/p_0 - n/p)$ with $i = 1,\,2$. Subsequently, the estimate (27) remains valid for ρ_i and s_i in place of ρ and s_0 respectively. Suppose that $s_0 = (1-\varepsilon)s_1 + \varepsilon s_2$ for some $\varepsilon \in (0,\,1)$, then it follows that $1/\rho = (1-\varepsilon)/\rho_1 + \varepsilon/\rho_2$. Thus by means of interpolation and the facts (refer to [23, §1.18.6, §2.4.1]

$$\begin{aligned}
&(L^{\rho_1,\,q_1}(0,\,\infty;\,X),\,L^{\rho_2,\,q_2}(0,\,\infty;\,X))_{\varepsilon,\,r} = L^{\rho,\,r}(0,\,\infty;\,X),\\
&(\dot{B}^{s_1}_{p_0,\,q_1},\,\dot{B}^{s_2}_{p_0,\,q_2})_{\varepsilon,\,r} = \dot{B}^{s_0}_{p_0,\,r},
\end{aligned} \tag{28}$$

where X denotes an arbitrary Banach space and $1 \le q_1,\,q_2 \le \infty$, we can derive the desired estimate (24). □

Lemma 8 If $1 < p < \infty$, then B_α is a BIP type operator on L^p. Specifically, $B_\alpha^{iy} \in \mathscr{L}(L^p)$ for all $y \in \mathbb{R}$, and for every $\varepsilon > 0$, there is correspondingly a constant $M_\varepsilon \ge 1$ such that

$$\|B_\alpha^{iy}\|_{\mathscr{L}(L^p)} \le M_\varepsilon e^{\varepsilon|y|},\ \ \forall\, y \in \mathbb{R}. \tag{29}$$

Proof. For each $\eta > 0$, we know that the sum $B_{\alpha,\,\eta} = \eta I + B_\alpha$ is a positive operator (cf. [24, §4.1]). So its complex power $B^z_{\alpha,\,\eta}$ with $\mathrm{Re}z < 0$ can be defined as follows:

$$B^z_{\alpha,\,\eta} = \frac{1}{2\pi i}\int_\gamma \lambda^z(\lambda I - B_{\alpha,\,\eta})^{-1}d\lambda. \tag{30}$$

Here the integration path γ can be divided into three parts: γ_1, γ_2 and γ_3, where γ_1 and γ_3 are both half-lines from $\infty e^{\kappa i}$ to $\delta e^{\kappa i}$ and from $\delta e^{-\kappa i}$ to $\infty e^{-\kappa i}$ respectively, while γ_2 is the major arc of the circle centered at 0 with the radius δ from $\delta e^{\kappa i}$ to $\delta e^{-\kappa \mathrm{i}}$, where $\pi/2 < \kappa < \pi$ and $0 < \delta < 1$ such that the spectrum of $B_{\alpha,\,\eta}$ is located on the right side of γ. Furthermore, the function λ^z takes the prime-value branch.

Taking any $f \in L^p$, we have

$$\mathscr{F}(B^z_{\alpha,\,\eta}f) = \frac{1}{2\pi i}\int_\gamma \lambda^z(\lambda - \eta - |\xi|^{2\alpha})^{-1}\mathscr{F}f d\lambda = (\eta + |\xi|^{2\alpha})^z\mathscr{F}f, \tag{31}$$

which imfers that

$$B^z_{\alpha,\,\eta} = \mathscr{F}^{-1}[(\eta + |\xi|^{2\alpha})^z\mathscr{F}] = (\eta I + (-\Delta)^\alpha)^z. \tag{32}$$

As a straight consequence of (32), it follows that

$$B_{\alpha,\,\eta}^{iy} = B_{\alpha,\,\eta}B_{\alpha,\,\eta}^{-1+iy} = (\eta I + (-\Delta)^{\alpha})^{iy}. \tag{33}$$

Direct calculation shows that for each $\beta \in \mathbb{N}^n$, the partial derivative of $(\eta + |\xi|^{2\alpha})^{iy}$ of the order β obeys the following inequality

$$|\partial_{\xi}^{\beta}(\eta + |\xi|^{2\alpha})^{iy}| \le \frac{C_{\alpha,\,\beta}}{|\xi|^{|\beta|}} \sum_{k=1}^{|\beta|} \frac{|\xi|^{2\alpha k}}{(\eta + |\xi|^{2\alpha})^k} \le \frac{|\beta| C_{\alpha,\,\beta}}{|\xi|^{|\beta|}},\ \ \forall\ \xi \neq 0. \tag{34}$$

Then by invoking Michlin's multiplier theorem (cf. [23, §2.2.4]), we can assert that $B_{\alpha,\,\eta}^{iy} \in \mathscr{L}(L^p)$ and

$$\|B_{\alpha,\,\eta}^{iy}\|_{\mathscr{L}(L^p)} \le C,\ \ \forall\, y \in \mathbb{R} \tag{35}$$

for some $C = C(n,\, p,\, \alpha) > 0$, which in turn yields (29) for $B_{\alpha,\,\eta}^{iy}$ immediately. Since B_{α} is an injective operator whose domain $D(B_{\alpha})$ and range $R(B_{\alpha})$ are both dense in L^p, from [25, Theorem 7.64, Proposition 7.47], we have $B_{\alpha}^{iy} \in \mathscr{L}(L^p)$, and

$$B_{\alpha}^{iy} = s - \lim_{\eta \to 0^+} B_{\alpha,\,\eta}^{iy}, \tag{36}$$

which, combined with (35), yields (29) eventually. □

Since for $1 < p < \infty$, L^p is a $\zeta-$convex space, we can deduce from Lemma 8 and [20] that operator B_{α} has the maximal $L^{\rho}-$regularity on L^p. Concretely, for every $1 < \rho < \infty$, $0 < T \le \infty$ and $f \in L^{\rho}(0,\, T;\, L^p)$, evolution equation

$$\frac{du}{dt} + B_{\alpha}u = f(t),\ \ t > 0,\ \ u(0) = 0 \tag{37}$$

has a unique solution $u \in C([0,\, T);\, L^p)$ such that both u' and $B_{\alpha}u$ lie in $L^{\rho}(0,\, T;\, L^p)$, and

$$\int_0^T \|u'(t)\|_{L^p}^{\rho} dt + \int_0^T \|B_{\alpha}u(t)\|_{L^p}^{\rho} dt \le C^{\rho} \int_0^T \|f(t)\|_{L^p}^{\rho} dt, \tag{38}$$

where the constant $C > 0$ is independent of T and f.

Consider the Sobolev space $\dot{H}_p^s = \{f \in \mathscr{S}' \colon |\xi|^s \mathscr{F} f \in L^p\}$. We know when $s < n/p$, $\dot{H}_p^s$ is complete according to the norm $\|f\|_{\dot{H}_p^s} = \||\xi|^s \mathscr{F} f\|_{L^p}$, and $(-\Delta)^{s/2}$ is an isomorphism from $\dot{H}_p^s$ onto L^p. Hence the maximal $L^{\rho}-$regularity of B_{α} reserves if we regard B_{α} as a closed operator in $\dot{H}_p^s$ with the domain and range restricted in $\dot{H}_p^s$. Furthermore, by virtue of the equivalence $(\dot{H}_p^{s_1},\, \dot{H}_p^{s_2})_{\varepsilon,\,q} = \dot{B}_{p,\,q}^s$, where $s_1 < s < s_2 < n/p$ and $s = s_1(1-\varepsilon) + s_2\varepsilon$, together with the method of interpolation, we conclude that B_{α} has the maximal $L^{\rho}-$regularity on $\dot{B}_{p,\,q}^s$. Finally, on the basis of (28), we can derive the maximal $L^{\rho,\,r}-$regularity of B_{α} on $\dot{B}_{p,\,q}^s$. Combining this fact with Corollary 7, we obtain a proposition similar to [19, Theorem 1], that is

Proposition 9 Under the hypothesis (23) with $1 < p_0 \le p < \infty$, for all $0 < T \le \infty$, all $a \in \dot{B}_{p_0,\,r}^{s_0}$ and $f \in L^{\rho,\,r}(0,\, T;\, \dot{B}_{p,\,q}^s)$, the Cauchy problem

$$\frac{du}{dt} + B_\alpha u = f(t), \ t > 0, \ u(0) = a \tag{39}$$

has a unique solution $u \in L^{\rho, r}(0, T; \dot{B}^{s+2\alpha}_{p, q})$ such that both u' and $B_\alpha u$ are lie in the class $L^{\rho, r}(0, T; \dot{B}^{s}_{p, q})$, and

$$\|u'\|_{L^{\rho, r}(0, T; \dot{B}^s_{p, q})} + \|B_\alpha u\|_{L^{\rho, r}(0, T; \dot{B}^s_{p, q})} \le C\big(\|a\|_{\dot{B}^{s_0}_{p_0, r}} + \|f\|_{L^{\rho, r}(0, T; \dot{B}^s_{p, q})}\big), \tag{40}$$

where the constant $C > 0$ is independent of a, T and f.

Evidently, each solution of (39) has the following representation

$$u(t) = e^{-t(-\Delta)^\alpha} a + \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} f(\tau) d\tau =: u_L(t) + (Sf)(t), \tag{41}$$

where the function Sf solves the problem (37), and satisfies the following properties.

Proposition 10 Assume that $s, \tilde{s} \in \mathbb{R}$, $1 \le r \le \infty$, $1 < \tilde{p} \le p < \infty$ and $1 < \tilde{\rho} < \rho < \infty$ verifying

$$\tilde{s} - \frac{n}{\tilde{p}} - 2\alpha < s - \frac{n}{p} < \tilde{s} - \frac{n}{\tilde{p}} \ \text{ and } \ s - \frac{n}{p} - \frac{2\alpha}{\rho} = \tilde{s} - \frac{n}{\tilde{p}} - \frac{2\alpha}{\tilde{\rho}}, \tag{42}$$

then for each $0 \le \gamma \le \alpha$, we have

$$\|B_\gamma(Sf)\|_{L^{\rho, r}(0, T; \dot{B}^{s+2(\alpha-\gamma)}_{p, 1})} \le C\|f\|_{L^{\tilde{\rho}, r}(0, T; \dot{B}^{\tilde{s}}_{\tilde{p}, \infty})}, \tag{43}$$

where the constant $C > 0$ is independent of γ, f and T.

Proof. In light of (26), we have

$$\|B_\gamma(Sf)(t)\|_{\dot{B}^{s+2(\alpha-\gamma)}_{p, 1}} \le C\int_0^t (t-\tau)^{-1/\rho} \|f(\tau)\|_{\dot{B}^{\tilde{s}}_{\tilde{p}, \infty}}, \tag{44}$$

where $\rho = 2\alpha/(s - \tilde{s} + n/\tilde{p} - n/p + 2\alpha) \in (1, \infty)$. Noting that $1 + 1/\rho = 1/\rho + 1/\tilde{\rho}$, by invoking [19, Proposition 3.1], we can derive (43). □

Let $\dot{B}^s_{p, q, \sigma}$ be the collection of all solenoidal fields whose components are all members of $\dot{B}^s_{p, q}$. It is easy to see that $\dot{B}^s_{p, q, \sigma}$ is a closed subspace of $\dot{B}^s_{p, q}$. Recall the Helmholtz projection $P = I + \nabla(-\Delta)^{-1}\mathrm{div}$, it is bounded from L^p onto L^p_σ for $1 < p < \infty$, and as a consequence, it is also bounded from $\dot{B}^s_{p, q}$ onto $\dot{B}^s_{p, q, \sigma}$. Define the Stokes operator of fractional type by $A_\alpha := PB_\alpha$. Since $PB_\alpha = B_\alpha P$ on $S'(\mathbb{R}^n)$, it follows that A_α has the same properties as B_α has, and particularly $e^{-tA_\alpha} = e^{-tB_\alpha}P = e^{-tB_\alpha}$ on $\dot{B}^s_{p, q, \sigma}$. These equalities and properties will be employed in the coming paragraphs without further specification.

3. Estimate for the nonlinear powers of functions

We first recall the estimates for the products of two functions in Besov spaces.

Lemma 11 ([21 (§3.1), 22 (§2.6)]) Let $s > 0$, p, p_i, q_i, $r \in [1, \infty]$, $i = 1, 2$ such that $1/p = 1/p_1 + 1/p_2 = 1/q_1 + 1/q_2$, and s, p, r satisfy (18), then it holds

$$\|fg\|_{\dot{B}^s_{p,r}} \le C(\|f\|_{\dot{B}^{s+\delta_1}_{p_1,r}}\|g\|_{\dot{B}^{-\delta_1}_{p_2,\infty}} + \|f\|_{\dot{B}^{-\delta_2}_{q_1,\infty}}\|g\|_{\dot{B}^{s+\delta_2}_{q_2,r}}) \tag{45}$$

for all $\delta_i > 0$, $i = 1, 2$, or

$$\|fg\|_{\dot{B}^s_{p,r}} \le C(\|f\|_{\dot{B}^s_{p_1,r}}\|g\|_{p_2} + \|f\|_{q_1}\|g\|_{\dot{B}^s_{q_2,r}}). \tag{46}$$

The following two lemmas can be verified by simple calculations.

Lemma 12 For all a, $b \in \mathbb{R}$,

$$|J_m(a) - J_m(b)| \le \begin{cases} m(|a|^{m-1} + |b|^{m-1})|a-b|, & m > 1, \\ 2|a-b|^m, & 0 < m \le 1. \end{cases} \tag{47}$$

Lemma 13 For all $m > 0$ and all a, $b \in \mathbb{R}$,

$$\begin{aligned} J_m(a) - J_m(b) &= m\int_0^1 |(1-\lambda)a + \lambda b|^{m-1} d\lambda \cdot (a-b) \\ &= \int_0^1 J'_m((1-\lambda)a + \lambda b) d\lambda \cdot (a-b). \end{aligned} \tag{48}$$

Recall that if $0 < s < 1$, then for all p, $r \in [1, \infty]$ and all $f \in S'_h$, the norm $\|f\|_{\dot{B}^s_{p,q}}$ is equivalent to

$$\Big(\int_{\mathbb{R}^n} \frac{\|\Delta_y f\|_p^r}{|y|^{sr+n}} dy\Big)^{1/r} \text{ if } r < \infty, \text{ or } \sup_{y\in\mathbb{R}^n} \frac{\|\Delta_y f\|_p}{|y|^s} \text{ if } r = \infty, \tag{49}$$

where $\Delta_y f(x) = \tau_{-y} f(x) - f(x)$ (cf. [22, §2.3]).

Proposition 14 Assume that $0 < s < m \le 1$ and $1/m \le p$, $r \le \infty$, then for all $f \in \dot{B}^{s/m}_{mp,mr}$, we have $J_m(f) \in \dot{B}^s_{p,r}$, and

$$\|J_m(f)\|_{\dot{B}^s_{p,r}} \le C\|f\|^m_{\dot{B}^{s/m}_{mp,mr}} \tag{50}$$

for some constant $C > 0$ independent of f.

Proof. We only consider the case $m < 1$ and $r < \infty$, the case $m = 1$ is obvious and $r = \infty$ can be dealt with in a similar way.

First of all, since $\dot{B}^{s/m}_{mp,mr} \subseteq L^{mp}_{\text{loc}}$, we have $J_m(f) \in L^p_{\text{loc}}$. For each $j \in \mathbb{Z}$, let $F_j = J_m(\dot{S}_{j+1}f) - J_m(\dot{S}_j f)$ and consider the double-sides series $\sum_{j\in\mathbb{Z}} F_j$. On the one hand, for two positive integers N and M, we have

$$\sum_{j=-N}^{-1} F_j = J_m(\dot{S}_0 f) - J_m(\dot{S}_{-N} f) \text{ and } \sum_{j=0}^{M} F_j = J_m(\dot{S}_{M+1} f) - J_m(\dot{S}_0 f). \tag{51}$$

Noting that $f \in \mathscr{S}'_h$, we have $\|\dot{S}_{-N} f\|_\infty \to 0$ as $N \to \infty$. Thus $\|J_m(\dot{S}_{-N} f)\|_\infty \to 0$ as $N \to \infty$, since $\|J_m(\dot{S}_{-N} f)\|_\infty \leq 2\|\dot{S}_{-N} f\|_\infty^m$ by Lemma 12. Consequently the single-side series $\sum_{j=-\infty}^{-1} F_j$ converges to $J_m(\dot{S}_0 f)$ in L^∞.

On the other hand, for any $k \in \mathbb{Z}^+$, by lemma 12 again, it comes

$$\begin{aligned} \|J_m(\dot{S}_{M+k+1} f) - J_m(\dot{S}_M f)\|_p &\leq 2\|\dot{S}_{M+k+1} f - \dot{S}_M f\|_{mp}^m \leq 2\big(\sum_{j=M}^{M+k} \|\dot{\Delta}_j f\|_{mp}\big)^m \\ &\leq 2\|(2^{-js/m})_{j \geq M}\|_{l^{r'}} \big(\sum_{j=M}^{M+k} 2^{jsr/m} \|\dot{\Delta}_j f\|_{mp}^r\big)^{m/r} \\ &\leq C 2^{-Ms/m} \big(\sum_{j=M}^{M+k} 2^{jsr/m} \|\dot{\Delta}_j f\|_{mp}^r\big)^{m/r}, \end{aligned} \tag{52}$$

from which we can deduce the convergence of both $\{\dot{S}_M f\}$ and $\{J_m(\dot{S}_M f)\}$ in L^{mp} and L^p respectively. Recall that $\dot{S}_M f \to f$ in L^{mp}_{loc} (cf. [22, §2.9]). Hence there is a subsequence, say $\{\dot{S}_M f\}$ itself, such that $\lim_{M\to\infty} \dot{S}_M f(x) = f(x)$ and $\lim_{M\to\infty} J_m(\dot{S}_M f(x)) = J_m(f(x))$ for a.e. $x \in \mathbb{R}^n$. In the second limits, continuity of the function J_m is applied. Thus we can assert that the limits of $\{\dot{S}_M f\}$ and $\{J_m(\dot{S}_M f)\}$ are f and $J_m(f)$ respectively. Consequently the single-side series $\sum_{j=0}^{\infty} F_j$ converges to $J_m(f) - J_m(\dot{S}_0 f)$ in L^p.

Summing up, we have $\sum_{j\in\mathbb{Z}} F_j \to J_m(f)$ in $L^\infty + L^p$, hence $\mathscr{S}'$. Performing the same arguments as in the proof of [22, Lemma 2.12], we can also show that $\lim_{N\to\infty} \|\dot{S}_N J_m(f)\|_\infty = 0$, which infers that $J_m(f) \in S'_h$.

For all $x, y \in \mathbb{R}^n$, thanks to Lemma 12, we get

$$|\Delta_y J_m(f)(x)| = |J_m(f(x+y)) - J_m(f(x))| \leq 2|\Delta_y f(x)|^m. \tag{53}$$

Consequently,

$$\big(\int_{\mathbb{R}^n} \frac{\|\Delta_y J_m(f)\|_p^r}{|y|^{sr}} \frac{dy}{|y|^n}\big)^{1/r} \leq 2\big(\int_{\mathbb{R}^n} \frac{\|\Delta_y f\|_{mp}^{mr}}{|y|^{sr}} \frac{dy}{|y|^n}\big)^{1/r}, \tag{54}$$

which, together the equivalent characterization of the norm of $\dot{B}^s_{p,r}$, yields (50) immediately. □

Remark 15 Suppose that $m \geq 1$, $0 < s < m$, $1 < p < \infty$ and $1 \leq r \leq \infty$, then similar to [21, §1.5, Lemma 5.6], we can derive the following estimate

$$\|J_m(f)\|_{\dot{B}^s_{p,r}} \leq C\|f\|_{p_1}^{m-1} \|f\|_{\dot{B}^s_{p_2,r}}, \tag{55}$$

where $\max\{1, p(m-1)\} < p_1 < \infty$ and $p < p_2 < \infty$ verifying $1/p = (m-1)/p_1 + 1/p_2$.

Theorem 16 Assume that $1 < p, m < \infty$, $\max\{1/(m-1), m/(m-1)\} \le r \le \infty\}$ and

$$0 < s < \begin{cases} \min\{m-1, (m-1)^2 n/p\}, & \text{if } 1 < m < 2, \\ \min\{1-1/m, (m-1)^2 n/(m^2-m+1)p\}, & \text{if } m \ge 2. \end{cases} \tag{56}$$

Then there exists a constant $C > 0$ such that

$$\|J_m(f) - J_m(g)\|_{\dot{B}^s_{p,r}} \le C(\|f\|^{m-1}_{\dot{B}^{s/m+(1-1/m)n/p}_{p,1}} + \|g\|^{m-1}_{\dot{B}^{s/m+(1-1/m)n/p}_{p,1}})\|f-g\|_{\dot{B}^{s/m+(1-1/m)n/p}_{p,1}}. \tag{57}$$

Proof. For any $x, y \in \mathbb{R}^n$, in light of Lemma 13, we can derive that

$$J_m(f(x)) - J_m(g(x)) = \int_0^1 J'_m((1-\lambda)f(x) + \lambda g(x))d\lambda \cdot (f(x) - g(x)). \tag{58}$$

Thus by employing (46), we obtain

$$\begin{aligned} \|J_m(f) - J_m(g)\|_{\dot{B}^s_{p,r}} \le C\Big(&\int_0^1 \|J'_m((1-\lambda)f + \lambda g)\|_{\dot{B}^s_{p_1,r}} d\lambda \cdot \|f-g\|_{p_2} \\ &+ \int_0^1 \|J'_m((1-\lambda)f + \lambda g)\|_{q_1} d\lambda \cdot \|f-g\|_{\dot{B}^s_{q_2,r}}\Big). \end{aligned} \tag{59}$$

In the case $1 < m < 2$, we can use (50) to deduce that

$$\begin{aligned} \|J_m(f) - J_m(g)\|_{\dot{B}^s_{p,r}} \le\ & C\int_0^1 \|(1-\lambda)f + \lambda g\|^{m-1}_{\dot{B}^{s/(m-1)}_{(m-1)p_1,(m-1)r}} d\lambda \cdot \|f-g\|_{p_2} \\ &+ C\int_0^1 \|(1-\lambda)f + \lambda g\|^{m-1}_{(m-1)q_1} d\lambda \cdot \|f-g\|_{\dot{B}^s_{q_2,r}} \\ \le\ & C(\|f\|^{m-1}_{\dot{B}^{s/(m-1)}_{(m-1)p_1,1}} + \|g\|^{m-1}_{\dot{B}^{s/(m-1)}_{(m-1)p_1,1}})\|f-g\|_{p_2} \\ &+ C(\|f\|^{m-1}_{(m-1)q_1} + \|g\|^{m-1}_{(m-1)q_1})\|f-g\|_{\dot{B}^s_{q_2,1}}. \end{aligned} \tag{60}$$

Take $p/(m-1) \le p_1, q_1 < \infty$ such that

$$\frac{n}{p_1} = \big(1 - \frac{1}{m}\big)\frac{n}{p} + \frac{s}{m}, \quad \frac{n}{p_2} = \frac{1}{m}\big(\frac{n}{p} - s\big), \quad \text{and} \tag{61}$$

$$\frac{n}{q_1} = \left(1 - \frac{1}{m}\right)\left(\frac{n}{p} - s\right), \quad \frac{n}{q_2} = \frac{n}{mp} + \left(1 - \frac{1}{m}\right)s. \tag{62}$$

Under this setting, we have $p \le (m-1)p_1 \le p_2$, $p \le q_2 \le (m-1)q_1$, and then

$$\dot{B}^{s/m+(1-1/m)n/p}_{p,\,1} \hookrightarrow \dot{B}^{s/(m-1)}_{(m-1)p_1,\,1} \hookrightarrow L^{p_2}, \quad \dot{B}^{s/m+(1-1/m)n/p}_{p,\,1} \hookrightarrow \dot{B}^{s}_{q_2,\,1} \hookrightarrow L^{(m-1)q_1}. \tag{63}$$

Inserting these imbedding relations into (60), we obtain (57).

In the case $m \ge 2$, we can take $p \le p_i,\, q_i < \infty$, $i = 1, 2$ as in (61) and (62) respectively, and $p_1 \le p_3,\ p_4 < \infty$ such that

$$\frac{n}{p_3} = \frac{1}{m}\left(\frac{n}{p_1} - s\right), \quad \frac{n}{p_4} = \frac{n}{mp_1} + \left(1 - \frac{1}{m}\right)s. \tag{64}$$

Under this setting, we have $p \le (1-1/m)p_4 \le (1-1/m)p_3 = p_2$, $p \le q_2 \le (m-1)q_1$, and $(m-1)/p_3 + 1/p_4 = 1/p_1$. Consequently,

$$\begin{aligned} &\dot{B}^{s/m+(1-1/m)n/p}_{p,\,1} \hookrightarrow \dot{B}^{sm/(m-1)}_{(1-1/m)p_4,\,1} \hookrightarrow L^{(1-1/m)p_3}, \\ &\dot{B}^{s/m+(1-1/m)n/p}_{p,\,1} \hookrightarrow \dot{B}^{s}_{q_2,\,1} \hookrightarrow L^{(m-1)q_1}. \end{aligned} \tag{65}$$

Using these imbedding relations, with the aid of (55) and (50), we can derive from (59) that

$$\begin{aligned} \|J_m(f) - J_m(g)\|_{\dot{B}^s_{p,\,r}} \le{}& C\int_0^1 \left\| \left(|(1-\lambda)f + \lambda g|^{1-1/m}\right)^m \right\|_{\dot{B}^s_{p_1,\,r}} d\lambda \cdot \|f-g\|_{p_2} \\ &+ C\int_0^1 \|(1-\lambda)f + \lambda g\|^{m-1}_{(m-1)q_1} d\lambda \cdot \|f-g\|_{\dot{B}^s_{q_2,\,r}} \\ \le{}& C\int_0^1 \left\||(1-\lambda)f + \lambda g|^{1-1/m}\right\|^{m-1}_{p_3} \left\||(1-\lambda)f + \lambda g|^{1-1/m}\right\|_{\dot{B}^s_{p_4,\,r}} d\lambda \cdot \|f-g\|_{p_2} \\ &+ C\left(\|f\|^{m-1}_{(m-1)q_1} + \|g\|^{m-1}_{(m-1)q_1}\right)\|f-g\|_{\dot{B}^s_{q_2,\,r}} \\ \le{}& C\left(\|f\|^{(m-1)^2/m}_{(1-1/m)p_3} + \|g\|^{(m-1)^2/m}_{(1-1/m)p_3}\right)\left(\|f\|^{1-1/m}_{\dot{B}^{sm/(m-1)}_{(1-1/m)p_4,\,(1-1/m)r}} + \|g\|^{1-1/m}_{\dot{B}^{sm/(m-1)}_{(1-1/m)p_4,\,(1-1/m)r}}\right) \\ &\cdot \|f-g\|_{p_2} + C\left(\|f\|^{m-1}_{(m-1)q_1} + \|g\|^{m-1}_{(m-1)q_1}\right)\|f-g\|_{\dot{B}^s_{q_2,\,r}} \end{aligned}$$

$$\leq C(\|f\|^{m-1}_{\dot{B}^{s/m+(1-1/m)n/p}_{p,\,1}} + \|g\|^{m-1}_{\dot{B}^{s/m+(1-1/m)n/p}_{p,\,1}})\|f-g\|_{\dot{B}^{s/m+(1-1/m)n/p}_{p,\,1}}. \tag{66}$$

Thus we reobtain (57) and complete the proof. □

Remark 17 If we take $g = 0$ in (57) with hypothesis (56) unchanged, then we obtain an inequality for $\|J_m(f)\|_{\dot{B}^s_{p,\,r}}$ immediately, i.e.

$$\|J_m(f)\|_{\dot{B}^s_{p,\,r}} \leq C\|f\|^m_{\dot{B}^{s/m+(1-1/m)n/p}_{p,\,1}}. \tag{67}$$

The following lemma can be proved by means of interpolation

Lemma 18 Let $1 < \rho \leq \rho_1,\, \rho_2 < \infty$, $1 \leq r,\, r_1 \leq \infty$ such that $1/\rho = 1/\rho_1 + 1/\rho_2$, then we have

$$\|fg\|_{L^{\rho,\,r}(\mathbb{R})} \leq C\|f\|_{L^{\rho_1,\,r_1}(\mathbb{R})}\|g\|_{L^{\rho_2,\,r}(\mathbb{R})}, \tag{68}$$

where the constant $C > 0$ is independent f and g.

Proposition 19 Let $1 < \rho < \infty$, $1 \leq r \leq \infty$ and $m \geq 1$, then there exists a constant $C > 0$ such that

$$\frac{1}{C}\|\varphi\|^m_{L^{m\rho,\,mr}(\mathbb{R})} \leq \||\varphi|^m\|_{L^{\rho,\,r}(\mathbb{R})} \leq C\|\varphi\|^m_{L^{m\rho,\,mr}(\mathbb{R})}. \tag{69}$$

Proof. For each $t > 0$, define

$$\varphi_*(t) = |\{s \in \mathbb{R}\colon |\varphi(s)| > t\}| \text{ and } \varphi^*(t) = \inf\{\tau > 0\colon \varphi_*(\tau) \leq t\}. \tag{70}$$

Then the norm of $L^{\rho,\,r}(\mathbb{R})$ is equivalent to the quasi-norm (refer to [26, §7.25])

$$\Big\{\int_0^\infty (t^{1/\rho}\varphi^*(t))^r \frac{dt}{t}\Big\}^{1/r} \text{ for } 1 \leq r < \infty \text{ and } \sup_{t>0} t^{1/\rho}\varphi^*(t) \text{ for } r = \infty. \tag{71}$$

For each $t > 0$, it is easy to check that $(|\varphi|^m)_*(t) = \varphi_*(t^{1/m})$ and then $(|\varphi|^m)^*(t) = \varphi^*(t)^m$. Consequently,

$$\begin{aligned} &\Big\{\int_0^\infty (t^{1/\rho}(|\varphi|^m)^*(t))^r \frac{dt}{t}\Big\}^{1/r} = \Big\{\int_0^\infty (t^{1/m\rho}\varphi^*(t))^{mr} \frac{dt}{t}\Big\}^{1/r} \text{ for } 1 \leq r < \infty, \text{ and} \\ &\sup_{t>0} t^{1/\rho}(|\varphi|^m)^*(t) = \big(\sup_{t>0} t^{1/m\rho}\varphi^*(t)\big)^m \text{ for } r = \infty, \end{aligned} \tag{72}$$

which leads to the desired equivalence. □

4. Global existence and uniqueness of the strong solution

We first give a definition for the strong solution of the (GBS) (1). Given $s, s_0, s_1, \varsigma_0 \in \mathbb{R}$, $r, q \in [1, \infty)$, and $(u_0, \theta_0) \in \dot{B}^{s_0}_{p, r, \sigma} \times \dot{B}^{\varsigma_0}_{p, r}$. A function triple $(u, \nabla\pi, \theta)$ is said to be the strong solution of (1) on the interval $[0, T]$, if

$$\begin{aligned} & u \in L^1(0, T; \dot{B}^{s+2\alpha}_{p, q, \sigma}) \cap C([0, T); \dot{B}^{s_0}_{p, r, \sigma} + \dot{B}^{s}_{p, q, \sigma}),\ u', \nabla\pi \in L^1(0, T; \dot{B}^{s}_{p, q}), \\ & \theta \in L^1(0, T; \dot{B}^{s_1+2\alpha}_{p, q}) \cap C([0, T); \dot{B}^{\varsigma_0}_{p, r, \sigma} + \dot{B}^{s_1}_{p, q, \sigma}),\ \theta' \in L^1(0, T; \dot{B}^{s_1}_{p, q}), \end{aligned} \tag{73}$$

with $u(0) = u_0$, $\theta(0) = \theta_0$, and both the equations in (1) are verified by $u(t)$, $\nabla\pi(t)$ and $\theta(t)$ in $\dot{B}^{s}_{p, q}$ and $\dot{B}^{s_1}_{p, q}$ respectively for a.e. $t \in [0, T]$.

By performing Helmholtz projection $\mathscr{P}$ on the first equation of (1), we obtain an abstract evolution system, i.e.

$$\begin{cases} u'(t) + A_\alpha u + \mathscr{P}(u \cdot \nabla u) = \mathscr{P}(J_m(\theta) e_n),\ t > 0, \\ \theta'(t) + B_\alpha \theta + u \cdot \nabla\theta = 0,\ t > 0, \\ u(0) = u_0,\ \theta(0) = \theta_0. \end{cases} \tag{74}$$

Evidently, the function pair (u, θ) arising from the strong solution $(u, \nabla\pi, \theta)$ of (1) constitutes the strong solution to (74). In other words, it solves the following system

$$\begin{cases} u = u_L + \Phi(u, u) + M(\theta), \\ \theta = \theta_L + \Psi(u, \theta). \end{cases} \tag{75}$$

Hence (u, θ) is a fixed point of the nonlinear operator Γ defined by

$$\Gamma(u, \theta) = (u_L + \Phi(u, u) + M(\theta), \theta_L + \Psi(u, \theta)), \tag{76}$$

where $\Phi(u, v) = -S(P(u \cdot \nabla v))$, $M(\theta) = S(P(J_m(\theta) e_n))$ and $\Psi(u, \theta) = -S(u \cdot \nabla\theta)$. Estimates for these operators are arranged below.

Let $2 < \rho < \infty$, and take

$$s = 1 + \frac{n}{p} + \frac{2\alpha}{\rho} - 4\alpha,\ \tilde{s} = s + \frac{2\alpha}{\rho}. \tag{77}$$

Direct calculation shows that, under the assumption $\alpha > 1/2$ and $1/\rho \leq 1 - 1/2\alpha$, it comes $s + 2\alpha \leq n/p$. Then thanks to Proposition 10, we have

$$\|\Phi(u, v)\|_{L^{\rho, r}(0, T; \dot{B}^{s+2\alpha}_{p, 1})} + \|(\Phi(u, v))'\|_{L^{\rho, r}(0, T; \dot{B}^{s}_{p, 1})} \leq C\|u \cdot \nabla v\|_{L^{\rho/2, r}(0, T; \dot{B}^{\tilde{s}}_{p, \infty})}. \tag{78}$$

In the case $m = 1$, there is $J_m(\theta) = \theta$ evidently. So by employing Proposition 10 again, we obtain

$$\|M(\theta)\|_{L^{\rho,\,r}(0,\,T;\,\dot{B}^{s+2\alpha}_{p,\,1})} + \|(M(\theta))'\|_{L^{\rho,\,r}(0,\,T;\,\dot{B}^{s}_{p,\,1})} \leq C_1\|\theta\|_{L^{\rho/2,\,r}(0,\,T;\,\dot{B}^{\tilde{s}}_{p,\,\infty})} \tag{79}$$

for $1 < p < \infty$, and

$$\|M(\theta)\|_{L^{\rho,\,r}(0,\,T;\,\dot{B}^{s+2\alpha}_{p,\,1})} + \|(M(\theta))'\|_{L^{\rho,\,r}(0,\,T;\,\dot{B}^{s}_{p,\,1})} \leq C_1\|\theta\|_{L^{\rho/2,\,r}(0,\,T;\,\dot{B}^{\tilde{s}+n/p}_{p/2,\,\infty})} \tag{80}$$

for $2 < p < \infty$, where the constant $C_1 > 0$ is independent of θ and T.

In the case $1 < m < 2$, we let

$$\frac{1}{4}\max\left\{2-m+\frac{n}{p},\, 1+\frac{m(2-m)n}{p}\right\} < \alpha < \frac{1}{2}\left(1+\frac{n}{p}\right) \tag{81}$$

and

$$1-\frac{1}{4\alpha}\left(1+\frac{n}{p}\right) < \frac{1}{\rho} < 1-\frac{1}{4\alpha}\left(1+\frac{n}{p}\right) + \frac{m-1}{4\alpha}\min\left\{1,\, \frac{(m-1)n}{p}\right\} \tag{82}$$

to assure $\rho > 2$ and $0 < \tilde{s} < \min\{m-1,\, (m-1)^2 n/p\}$. In the case $m \geq 2$, we let

$$\frac{1}{4}\max\left\{\frac{n}{p}+\frac{1}{m},\, 1+\frac{mn}{(m^2-m+1)p}\right\} < \alpha < \frac{1}{2}\left(1+\frac{n}{p}\right) \tag{83}$$

and

$$1-\frac{1}{4\alpha}\left(1+\frac{n}{p}\right) < \frac{1}{\rho} < 1-\frac{1}{4\alpha}\left(1+\frac{n}{p}\right) + \frac{m-1}{4\alpha}\min\left\{\frac{1}{m},\, \frac{(m-1)n}{(m^2-m+1)p}\right\} \tag{84}$$

to assure $\rho > 2$ and $0 < \tilde{s} < \min\{1-1/m,\, (m-1)^2 n/(m^2-m+1)p\}$. Thus in both cases, we can use Proposition 10, 19, together with inequality (67) to deduce that

$$\begin{aligned}
&\|M(\theta)\|_{L^{\rho,\,r}(0,\,T;\,\dot{B}^{s+2\alpha}_{p,\,1})} + \|(M(\theta))'\|_{L^{\rho,\,r}(0,\,T;\,\dot{B}^{s}_{p,\,1})} \\
&\leq C\|J_m(\theta)\|_{L^{\rho/2,\,r}(0,\,T;\,\dot{B}^{\tilde{s}}_{p,\,\infty})} \leq C\Big\|\|\theta\|^m_{\dot{B}^{\tilde{s}/m+(1-1/m)n/p}_{p,\,1}}\Big\|_{L^{\rho/2,\,r}(0,\,T)} \\
&\leq C\|\theta\|^m_{L^{m\rho/2,\,mr}(0,\,T;\,\dot{B}^{\tilde{s}/m+(1-1/m)n/p}_{p,\,1})} \leq C_m\|\theta\|^m_{L^{m\rho/2,\,r}(0,\,T;\,\dot{B}^{sm+2\alpha}_{p,\,1})},
\end{aligned} \tag{85}$$

where the constant $C_m > 0$ is free from θ and T,

$$s_m = \frac{\tilde{s}}{m} + \big(1 - \frac{1}{m}\big)\frac{n}{p} - 2\alpha. \tag{86}$$

Estimates for $\Psi(u, \theta)$ are also made in two cases. In the case $m = 1$, restriction $3 < \rho < \infty$ is required. If we assume $1 < p < \infty$, then we have

$$\|\Psi(u, \theta)\|_{L^{\rho/2, r}(0, T; \dot{B}^{\sigma_1+2\alpha}_{p, 1})} + \|(\Psi(u, \theta))'\|_{L^{\rho/2, r}(0, T; \dot{B}^{\sigma_1}_{p, 1})} \leq C\|u \cdot \nabla\theta\|_{L^{\rho/3, r}(0, T; \dot{B}^{\tilde{\sigma}_1}_{p, \infty})} \tag{87}$$

where

$$\sigma_1 = \tilde{s} - 2\alpha = 1 + \frac{n}{p} + \frac{4\alpha}{\rho} - 6\alpha \ \text{ and } \ \tilde{\sigma}_1 = \sigma_1 + \frac{2\alpha}{\rho} \tag{88}$$

verifying $\sigma_1 + 2\alpha = \tilde{s} \leq n/p$ provided $\alpha > 1/4$ and $1/\rho \leq 1 - 1/4\alpha$. If we assume further $2 < p < \infty$, then we have

$$\|\Psi(u, \theta)\|_{L^{\rho/2, r}(0, T; \dot{B}^{\sigma_2+2\alpha}_{p/2, 1})} + \|(\Psi(u, \theta))'\|_{L^{\rho/2, r}(0, T; \dot{B}^{\sigma_2}_{p/2, 1})} \leq C\|u \cdot \nabla\theta\|_{L^{\rho/3, r}(0, T; \dot{B}^{\tilde{\sigma}_2}_{p/2, \infty})}, \tag{89}$$

where

$$\sigma_2 = \tilde{s} + \frac{n}{p} - 2\alpha = 1 + \frac{2n}{p} + \frac{4\alpha}{\rho} - 6\alpha \ \text{ and } \ \tilde{\sigma}_2 = \sigma_2 + \frac{2\alpha}{\rho} \tag{90}$$

verifying $\sigma_2 + 2\alpha \leq n/p$, or equivalently $\tilde{s} \leq 0$, provided $\alpha > (1 + n/p)/4$ and $1/\rho \leq 1 - (1 + n/p)/4\alpha$.

In the case $m > 1$, we let $\alpha > 1/4$ and $\rho > \max\{(m+2)/m, 4\alpha/(4\alpha - 1)\}$. Under this setting, we have $s_m + 2\alpha \leq n/p$, i.e. $\tilde{s} \leq n/p$. Consequently,

$$\|\Psi(u, \theta)\|_{L^{m\rho/2, r}(0, T; \dot{B}^{s_m+2\alpha}_{p, 1})} + \|(\Psi(u, \theta))'\|_{L^{m\rho/2, r}(0, T; \dot{B}^{s_m}_{p, 1})} \leq C\|u \cdot \nabla\theta\|_{L^{m\rho/(m+2), r}(0, T; \dot{B}^{\tilde{s}_m}_{p, \infty})}, \tag{91}$$

where $\tilde{s}_m = s_m + 2\alpha/\rho$.

We next use Lemma 11 to derive the estimates for the terms $u \cdot \nabla v$ and $u \cdot \nabla\theta$.

Proposition 20 Let $1 < p < \infty$, α satisfy

$$\frac{1}{2} < \alpha < 1 + \frac{n}{2p} \tag{92}$$

and let ρ be a parameter with $2 < \rho < \infty$ such that

$$1-\frac{1}{2\alpha}\left(1+\frac{n}{2p}\right)<\frac{1}{\rho}<1-\frac{1}{2\alpha}. \tag{93}$$

Then for all $u,\, v\in L^{\rho,\, r}(0,\, T;\, \dot{B}^{s+2\alpha}_{p,\, 1})$ with $divu(t)=0$ for a.e. $t\in(0,\, T)$, the following inequality holds:

$$\|u\cdot\nabla v\|_{L^{\rho/2,\, r}(0,\, T;\, \dot{B}^{\tilde{s}}_{p,\, \infty})}\le C\|u\|_{L^{\rho,\, r}(0,\, T;\, \dot{B}^{s+2\alpha}_{p,\, 1})}\|v\|_{L^{\rho,\, r}(0,\, T;\, \dot{B}^{s+2\alpha}_{p,\, 1})}, \tag{94}$$

where $C>0$ is is a constant independent of u, v and T.

Proof. Thanks to Lemma 11, for a.e. $t\in[0,\, T]$, we first have

$$\begin{aligned}
&\|u(t)\cdot\nabla v(t)\|_{\dot{B}^{\tilde{s}}_{p,\, \infty}}\le C\|u(t)\otimes v(t)\|_{\dot{B}^{\tilde{s}+1}_{p,\, \infty}}\\
&\le C(\|u(t)\|_{\dot{B}^{\tilde{s}+1+\delta}_{p,\, \infty}}\|v(t)\|_{\dot{B}^{-\delta}_{\infty,\, \infty}}+\|v(t)\|_{\dot{B}^{\tilde{s}+1+\delta}_{p,\, \infty}}\|u(t)\|_{\dot{B}^{-\delta}_{\infty,\, \infty}})\\
&\le C(\|u(t)\|_{\dot{B}^{s+2\alpha}_{p,\, 1}}\|v(t)\|_{\dot{B}^{-\delta+n/p}_{p,\, 1}}+\|v(t)\|_{\dot{B}^{s+2\alpha}_{p,\, 1}}\|u(t)\|_{\dot{B}^{-\delta+n/p}_{p,\, 1}})\\
&\le C\|u(t)\|_{\dot{B}^{s+2\alpha}_{p,\, 1}}\|v(t)\|_{\dot{B}^{s+2\alpha}_{p,\, 1}},
\end{aligned} \tag{95}$$

where $-1<\tilde{s}\le n/p$, $\delta=2\alpha(1-1/\rho)-1>0$ stem from the assumptions (93) and (94). Then by invoking Lemma 18, we can derive that

$$\begin{aligned}
\|u\cdot\nabla v\|_{L^{\rho/2,\, r}(0,\, T;\, \dot{B}^{\tilde{s}}_{p,\, \infty})}&\le C\|u\|_{L^{\rho,\, 2r}(0,\, T;\, \dot{B}^{s+2\alpha}_{p,\, 1})}\|v\|_{L^{\rho,\, 2r}(0,\, T;\, \dot{B}^{s+2\alpha}_{p,\, 1})}\\
&\le C\|u\|_{L^{\rho,\, r}(0,\, T;\, \dot{B}^{s+2\alpha}_{p,\, 1})}\|v\|_{L^{\rho,\, r}(0,\, T;\, \dot{B}^{s+2\alpha}_{p,\, 1})}.
\end{aligned} \tag{96}$$

This completes the proof. □

All the constraints placed on the exponents p, α, ρ for $m=1$ are realised under the hypotheses proposed in the next two propositions.

Proposition 21 Suppose that hypothesis H_1 is as follows: there exist parameters p and ρ with $1<p<n$ and $3<\rho<\infty$ such that

$$\frac{1}{2}<\alpha<\frac{1}{2}\left(1+\frac{n}{2p}\right)\ \text{ and }\ 1-\frac{1}{3\alpha}\left(1+\frac{n}{2p}\right)<\frac{1}{\rho}<1-\frac{1}{2\alpha}. \tag{97}$$

Then for all functions u and θ with $u\in L^{\rho,\, r}(0,\, T;\, \dot{B}^{s+2\alpha}_{p,\, 1,\, \sigma})$ and $\theta\in L^{\rho/2,\, r}(0,\, T;\, \dot{B}^{\sigma_1+2\alpha}_{p,\, q})$, the following inequality holds:

$$\|u\cdot\nabla\theta\|_{L^{\rho/3,\,r}(0,\,T;\,\dot{B}^{\tilde{\sigma}_1}_{p,\,\infty})}\leq C\|u\|_{L^{\rho,\,r}(0,\,T;\,\dot{B}^{s+2\alpha}_{p,\,1})}\|\theta\|_{L^{\rho/2,\,r}(0,\,T;\,\dot{B}^{\sigma_1+2\alpha}_{p,\,1})} \tag{98}$$

for some constant $C>0$ independent of u, v and T.

Proof. Similar to (95), for a.e. $t\in[0,\,T]$, we have

$$\begin{aligned}\|u(t)\cdot\nabla\theta(t)\|_{\dot{B}^{\tilde{\sigma}_1}_{p,\,\infty}}&\leq C\|\theta(t)u(t)\|_{\dot{B}^{\tilde{\sigma}_1+1}_{p,\,\infty}}\\&\leq C(\|u(t)\|_{\dot{B}^{\tilde{\sigma}_1+1+\delta_0}_{p,\,\infty}}\|\theta(t)\|_{\dot{B}^{-\delta_0}_{\infty,\,\infty}}+\|\theta(t)\|_{\dot{B}^{\tilde{\sigma}_1+1+\delta}_{p,\,\infty}}\|u(t)\|_{\dot{B}^{-\delta}_{\infty,\,\infty}})\\&\leq C(\|u(t)\|_{\dot{B}^{s+2\alpha}_{p,\,1}}\|\theta(t)\|_{\dot{B}^{-\delta_0+n/p}_{p,\,1}}+\|\theta(t)\|_{\dot{B}^{\sigma_1+2\alpha}_{p,\,1}}\|u(t)\|_{\dot{B}^{-\delta+n/p}_{p,\,1}})\\&\leq C\|u(t)\|_{\dot{B}^{s+2\alpha}_{p,\,1}}\|\theta(t)\|_{\dot{B}^{\sigma_1+2\alpha}_{p,\,1}},\end{aligned} \tag{99}$$

where $-1<\tilde{\sigma}_1\leq n/p$ and $\delta_0=n/p-\sigma_1-2\alpha>0$ come from the assumption (97). Again by Lemma 18, we can derive inequality (98). □

Proposition 22 Suppose that hypothesis H_2 is as follows: $n\leq p<2n$ and $3<\rho<\infty$ fulfill

$$\frac{1}{2}<\alpha<\frac{1}{2}\big(1+\frac{n}{p}\big)\ \text{ and }\ 1-\frac{1}{3\alpha}\big(1+\frac{n}{p}\big)<\frac{1}{\rho}<1-\frac{1}{2\alpha}. \tag{100}$$

Then for all functions u and θ with $u\in L^{\rho,\,r}(0,\,T;\,\dot{B}^{s+2\alpha}_{p,\,1,\,\sigma})$ and $\theta\in L^{\rho/2,\,r}(0,\,T;\,\dot{B}^{\sigma_2+2\alpha}_{p/2,\,1})$, we have

$$\|u\cdot\nabla\theta\|_{L^{\rho/3,\,r}(0,\,T;\,\dot{B}^{\tilde{\sigma}_2}_{p/2,\,\infty})}\leq C\|u\|_{L^{\rho,\,r}(0,\,T;\,\dot{B}^{s+2\alpha}_{p,\,1})}\|\theta\|_{L^{\rho/2,\,r}(0,\,T;\,\dot{B}^{\sigma_2+2\alpha}_{p/2,\,1})}, \tag{101}$$

where the constant $C>0$ is independent of u, v and T.

Proof. For a.e. $t\in[0,\,T]$, it holds that

$$\begin{aligned}\|u(t)\cdot\nabla\theta(t)\|_{\dot{B}^{\tilde{\sigma}_2}_{p/2,\,\infty}}&\leq C\|\theta(t)u(t)\|_{\dot{B}^{\tilde{\sigma}_2+1}_{p/2,\,\infty}}\\&\leq C(\|u(t)\|_{\dot{B}^{\tilde{\sigma}_2+1+\tilde{\delta}_0}_{p,\,\infty}}\|\theta(t)\|_{\dot{B}^{-\tilde{\delta}_0}_{p,\,\infty}}+\|\theta(t)\|_{\dot{B}^{\tilde{\sigma}_2+1+\delta}_{p/2,\,\infty}}\|u(t)\|_{\dot{B}^{-\delta}_{\infty,\,\infty}})\\&\leq C(\|u(t)\|_{\dot{B}^{s+2\alpha}_{p,\,1}}\|\theta(t)\|_{\dot{B}^{-\tilde{\delta}_0+n/p}_{p/2,\,1}}+\|\theta(t)\|_{\dot{B}^{\sigma_2+2\alpha}_{p/2,\,1}}\|u(t)\|_{\dot{B}^{-\delta+n/p}_{p,\,1}})\\&\leq C\|u(t)\|_{\dot{B}^{s+2\alpha}_{p,\,1}}\|\theta(t)\|_{\dot{B}^{\sigma_2+2\alpha}_{p/2,\,1}},\end{aligned} \tag{102}$$

where $-1 < \tilde{\sigma}_2 \leq n/p$ and $\tilde{\delta}_0 = n/p - \sigma_2 - 2\alpha = -\tilde{s} > 0$ are realised by the assumption (100). By invoking Lemma 18 the third time, we can derive inequality (101). □

Proposition 23 Assume $m > 1$, $1 < p < mn$ and $(m+2)/m < \rho < \infty$ satisfy the conditions

$$\frac{1}{2} < \alpha < \frac{1}{4}\big(m+1+\frac{mn}{p}\big) \tag{103}$$

and

$$1 - \frac{1}{2\alpha(m+2)}\big(m+1+\frac{mn}{p}\big) < \frac{1}{\rho} < 1 - \frac{1}{2\alpha}. \tag{104}$$

Then for all functions $u \in L^{\rho,\,r}(0,\,T;\,\dot{B}^{s+2\alpha}_{p,\,1,\,\sigma})$ and $\theta \in L^{m\rho/2,\,r}(0,\,T;\,\dot{B}^{s_m+2\alpha}_{p,\,1})$, we have

$$\|u\cdot\nabla\theta\|_{L^{m\rho/(m+2),\,r}(0,\,T;\,\dot{B}^{\tilde{s}_m}_{p,\infty})} \leq C\|u\|_{L^{\rho,\,r}(0,\,T;\,\dot{B}^{s+2\alpha}_{p,\,1})}\|\theta\|_{L^{m\rho/2,\,r}(0,\,T;\,\dot{B}^{s_m+2\alpha}_{p,\,1})} \tag{105}$$

for some constant $C > 0$ independent of u, v and T.

Proof. Similar to (95), for a.e. $t \in [0,\,T]$, we have

$$\begin{aligned}
\|u(t)\cdot\nabla\theta(t)\|_{\dot{B}^{\tilde{s}_m}_{p,\infty}} &\leq C\|\theta(t)u(t)\|_{\dot{B}^{\tilde{s}_m+1}_{p,\infty}} \\
&\leq C(\|u(t)\|_{\dot{B}^{\tilde{s}_m+1+\delta_i}_{p,\,\infty}}\|\theta(t)\|_{\dot{B}^{-\delta_m}_{\infty,\,\infty}} + \|\theta(t)\|_{\dot{B}^{\tilde{s}_m+1+\delta}_{p,\,\infty}}\|u(t)\|_{\dot{B}^{-\delta}_{\infty,\,\infty}}) \\
&\leq C(\|u(t)\|_{\dot{B}^{s+2\alpha}_{p,\,1}}\|\theta(t)\|_{\dot{B}^{-\delta_m+n/p}_{p,\,1}} + \|\theta(t)\|_{\dot{B}^{s_m+2\alpha}_{p,\,1}}\|u(t)\|_{\dot{B}^{-\delta+n/p}_{p,\,1}}) \\
&\leq C\|u(t)\|_{\dot{B}^{s+2\alpha}_{p,\,1}}\|\theta(t)\|_{\dot{B}^{s_m+2\alpha}_{p,\,1}},
\end{aligned} \tag{106}$$

where the restrains $-1 < \tilde{s}_m \leq n/p$ and $\delta_m = (n/p - \tilde{s})/m > 0$ are assured by (103) and (104). By using Lemma 18 the last time, we can derive inequality (105). □

Remark 24 Under the additional assumption $p < n$, we can put (81), (82) and (103), (104) together to yield the following hypothesis H_3: $1 < m < 2$, $\max\{m(2-m)n,\, n/m\} \leq p < n$ and $(m+2)/m < \rho < \infty$ such that

$$\frac{1}{2} < \alpha < \frac{m+2}{8}\big(1+\frac{n}{p}\big) \ \text{ and } \ 1 - \frac{1}{4\alpha}\big(1+\frac{n}{p}\big) < \frac{1}{\rho} < 1 - \frac{1}{2\alpha}. \tag{107}$$

We can also put (83), (84) and (103), (104) together to yield the following hypothesis H_4: $m \geq 2$, $mn/(2m-1) \leq p < n$ and $2 < \rho < \infty$ such that

$$\frac{1}{2} < \alpha < \frac{1}{2}\big(1+\frac{n}{p}\big) \ \text{ and } \ 1 - \frac{1}{4\alpha}\big(1+\frac{n}{p}\big) < \frac{1}{\rho} < 1 - \frac{1}{2\alpha}. \tag{108}$$

We are ready to give proofs of the main theorems.

Proof of Theorem 1 Let

$$X = \{u \in L^{\rho,\, r}(0,\, \infty;\, \dot{B}^{s+2\alpha}_{p,\,1,\,\sigma}):\ u' \in L^{\rho,\, r}(0,\, \infty;\, \dot{B}^{s}_{p,\,1})\} \tag{109}$$

with the norm

$$\|u\|_X = \|u\|_{L^{\rho,\, r}(0,\, \infty;\, \dot{B}^{s+2\alpha}_{p,\,1})} + \|u'\|_{L^{\rho,\, r}(0,\, \infty;\, \dot{B}^{s}_{p,\,1})}, \tag{110}$$

and let

$$Y = \{\theta \in L^{m\rho/2,\, r}(0,\, \infty;\, \dot{B}^{s_m+2\alpha}_{p,\,1}):\ \theta' \in L^{m\rho/2,\, r}(0,\, \infty;\, \dot{B}^{s_m}_{p,\,1})\} \tag{111}$$

with the norm

$$\|\theta\|_Y = \|\theta\|_{L^{m\rho/2,\, r}(0,\, \infty;\, \dot{B}^{s_m+2\alpha}_{p,\,1})} + \|\theta'\|_{L^{m\rho/2,\, r}(0,\, \infty;\, \dot{B}^{s_m}_{p,\,1})}. \tag{112}$$

Finally, let $W = X \times Y$ with the norm $\|(u,\, \theta)\|_W = \|u\|_X + \|\theta\|_Y$. It is easy to see that, all the spaces defined above are complete according to their own norms.

Given $u_0 \in \dot{B}^{s_0}_{p,\, r,\, \sigma}$ and $\theta_0 \in \dot{B}^{\varsigma_m}_{p,\, r}$, where

$$s_0 = s + 2\alpha\left(1 - \frac{1}{\rho}\right) = 1 + \frac{n}{p} - 2\alpha, \tag{113}$$

and

$$\varsigma_m = s_m + 2\alpha\left(1 - \frac{2}{m\rho}\right) = \frac{n}{p} - \frac{4\alpha - 1}{m}. \tag{114}$$

Under this setting, it follows from Corollary 7 that $u_L \in X$, $\theta_L \in Y$, and

$$\|u_L\|_X \le C\|u_0\|_{\dot{B}^{s_0}_{p,\, r}}, \quad \|\theta_L\|_Y \le C\|\theta_0\|_{\dot{B}^{\varsigma_m}_{p,\, r}}. \tag{115}$$

Taking any $(u,\, \theta) \in W$, by H_3 for $1 < m < 2$, and H_4 for $m \ge 2$, we can plug (95) and (105) into (78) and (91) respectively to show that $\Phi(u,\, u) \in X$, $\Psi(u,\, \theta) \in Y$ and

$$\|\Phi(u,\, u)\|_X \le k_1\|u\|_X^2,\ \ \|\Psi(u,\, \theta)\|_Y \le k_2\|u\|_X\|\theta\|_Y \tag{116}$$

for some constants $k_i > 0$ independent of u and θ, which, combined with (85), produces

$$\|\Gamma(u,\,\theta)\|_W \le \|(u_L,\,\theta_L)\|_W + k_1\|u\|_X^2 + C_m\|\theta\|_Y^m + k_2\|u\|_X\|\theta\|_Y, \tag{117}$$

where the constant $C_m > 0$ comes from (85). Thus if we set

$$\|\theta\|'_Y = (2C_m)^{1/m}\|\theta\|_Y \text{ and } \|(u,\,\theta)\|'_W = \|u\|_X + \|\theta\|'_Y, \tag{118}$$

then we have

$$\begin{aligned} \|\Gamma(u,\,\theta)\|'_W &\le \|(u_L,\,\theta_L)\|'_W + k_1\|u\|_X^2 + \frac{1}{2}(\|\theta\|'_Y)^m + k_2\|u\|_X\|\theta\|'_Y \\ &\le \|(u_L,\,\theta_L)\|'_W + (k_1+k_2)(\|(u,\,\theta)\|'_W)^2 + \frac{1}{2}(\|(u,\,\theta)\|'_W)^m. \end{aligned} \tag{119}$$

Taking two points $(u_i,\,\theta_i) \in W$, $i = 1,\,2$ such that $\|(u_i,\,\theta_i)\|_W \le \lambda_1$ for some $\lambda_1 > 0$ specified later, we have

$$\begin{aligned} \Gamma(u_1,\,\theta_1) - \Gamma(u_2,\,\theta_2) &= (\Phi(u_1,\,u_1-u_2) + M(\theta_1) - M(\theta_2),\,\Psi(u_1,\,\theta_1-\theta_2)) \\ &\quad + (\Phi(u_1-u_2,\,u_2),\,\Psi(u_1-u_2,\,\theta_2)). \end{aligned} \tag{120}$$

Now combining (78), (91) and (95), (105) again, we have

$$\begin{aligned} &\|\Phi(u_1,\,u_1-u_2)\|_X \le k_1\|u_1\|_X\|u_1-u_2\|_X, \\ &\|\Phi(u_1-u_2,\,u_2)\|_X \le k_1\|u_1-u_2\|_X\|u_2\|_X, \\ &\|\Psi(u_1,\,\theta_1-\theta_2)\|_Y \le k_2\|u_1\|_X\|\theta_1-\theta_2\|_Y, \\ &\|\Psi(u_1-u_2,\,\theta_2)\|_Y \le k_2\|u_1-u_2\|_X\|\theta_2\|_Y. \end{aligned} \tag{121}$$

Additionally, from Proposition 10, inequality (57), Lemma 18 and Proposition 19 in turn, it follows immediately

$$\|M(\theta_1)-M(\theta_2)\|_X \le C\|J_m(\theta_1)-J_m(\theta_2)\|_{L^{\rho/2,\,r}(0,\,\infty;\,\dot{B}^{\tilde{s}}_{p,\,\infty})}$$

$$\le C\Big\|(\|\theta_1\|^{m-1}_{\dot{B}^{\tilde{s}/m+(1-1/m)n/p}_{p,\,1}} + \|\theta_2\|^{m-1}_{\dot{B}^{\tilde{s}/m+(1-1/m)n/p}_{p,\,1}})\|\theta_1-\theta_2\|_{\dot{B}^{\tilde{s}/m+(1-1/m)n/p}_{p,\,1}}\Big\|_{L^{\rho/2,\,r}(0,\,\infty)}$$

$$\leq C_m\big(\|\theta_1\|^{m-1}_{L^{m\rho/2,\,mr}(0,\,\infty;\,B^{sm+2\alpha}_{p,\,1})}+\|\theta_2\|^{m-1}_{L^{m\rho/2,\,mr}(0,\,\infty;\,B^{sm+2\alpha}_{p,\,1})}\big)\|\theta_1-\theta_2\|_{L^{m\rho/2,\,r}(0,\,\infty;\,B^{sm+2\alpha}_{p,\,1})}$$

$$\leq C_m(\|\theta_1\|^{m-1}_Y+\|\theta_2\|^{m-1}_Y)\|\theta_1-\theta_2\|_Y. \tag{122}$$

Putting all the foregoing estimates into (120), we end up with

$$\begin{aligned}\|\Gamma(u_1,\,\theta_1)-\Gamma(u_2,\,\theta_2)\|'_W &\leq k_1(\|u_1\|_X+\|u_2\|_X)\|u_1-u_2\|_X\\ &\quad+\frac{1}{2}\big((\|\theta_1\|'_Y)^{m-1}+(\|\theta_2\|'_Y)^{m-1}\big)\|\theta_1-\theta_2\|'_Y\\ &\quad+k_2(\|u_1\|_X\|\theta_1-\theta_2\|'_Y+\|\theta_2\|'_Y\|u_1-u_2\|_X).\end{aligned} \tag{123}$$

Next task is to search for suitable assumptions on k_1+k_2 and $K_0=\|(u_L,\,\theta_L)\|'_W$ to assure the operator Γ having a fixed point. For this purpose, we divided our discussions into two cases.

Case 1 $m\geq 2$. In this case, we assume

$$K_0<\frac{1}{16(k_1+k_2)}, \tag{124}$$

and denote by

$$\lambda_1=\frac{1-\sqrt{1-16K_0(k_1+k_2)}}{4(k_1+k_2)}, \tag{125}$$

the first real root of the quadratic equation $(k_1+k_2)\lambda^2-\lambda/2+K_0=0$. Without loss of generality, assume that $k_1+k_2\geq 1$. By this assumption, we have $\lambda_1<1$. Indeed, from (124), it holds that

$$\lambda_1\leq 4K_0<\frac{1}{4(k_1+k_2)}\leq\frac{1}{4}. \tag{126}$$

Hence under the assumption $\|(u,\,\theta)\|'_W\leq\lambda_1$, it follows from (119) that

$$\|\Gamma(u,\,\theta)\|'_W\leq K_0+(k_1+k_2)\lambda_1^2+\frac{1}{2}\lambda_1^m\leq\lambda_1. \tag{127}$$

Furthermore, if we let $\|(u_i,\,\theta_i)\|'_W\leq\lambda_1$, $i=1,\,2$ additionally, then from (123), we obtain

$$\|\Gamma(u_1, \theta_1)-\Gamma(u_2, \theta_2)\|'_W \leq [2(k_1+k_2)\lambda_1+\lambda_1^{m-1}]\|(u_1-u_2, \theta_1-\theta_2)\|'_W$$
$$\leq [2(k_1+k_2)\lambda_1+\lambda_1]\|(u_1-u_2, \theta_1-\theta_2)\|'_W. \tag{128}$$

Since $2(k_1+k_2)\lambda_1+\lambda_1 \leq 3/4$, we conclude that Γ is a contraction on the ball $B_1 = \{(u, \theta) \in W\colon \|(u, \theta)\|'_W \leq \lambda_1\}$. Hence it has a unique fixed point in B_1.

Case 2 $1 < m < 2$. In this case, let

$$H(\lambda) = (k_1+k_2)\lambda^2 - \lambda + \frac{\lambda^m}{2} + K_0 \text{ and } h(\lambda) = 2(k_1+k_2)\lambda + \lambda^{m-1}. \tag{129}$$

Given $0 < \varepsilon < 1$, the number

$$\lambda_\varepsilon = \Big(\frac{\varepsilon}{4(k_1+k_2)}\Big)^{1/(m-1)} \tag{130}$$

lies in $(0, 1)$ such that $h(\lambda) \leq \varepsilon$ for all $\lambda \in [0, \lambda_\varepsilon]$. Consequently

$$H'(\lambda) = h(\lambda) - 1 \leq \varepsilon - 1, \quad \forall\, \lambda \in [0, \lambda_\varepsilon]. \tag{131}$$

So if we assume

$$K_0 < (1-\varepsilon)\lambda_\varepsilon, \tag{132}$$

then we have

$$H(\lambda_\varepsilon) \leq K_0 - (1-\varepsilon)\lambda_\varepsilon < 0. \tag{133}$$

Noting that $H(0) = K_0 > 0$, we assert that H has a unique real root, say λ_1, in the interval $(0, \lambda_\varepsilon]$. Hence, if we let $\|(u, \theta)\|'_W \leq \lambda_1$, then we have

$$\|\Gamma(u, \theta)\|'_W \leq K_0 + (k_1+k_2)\lambda_1^2 + \frac{1}{2}\lambda_1^m = \lambda_1. \tag{134}$$

Moreover, if in addtion $\|(u_i, \theta_i)\|'_W \leq \lambda_1$, $i = 1, 2$, then it comes

$$\|\Gamma(u_1, \theta_1)-\Gamma(u_2, \theta_2)\|'_W \leq h(\lambda_1)\|(u_1-u_2, \theta_1-\theta_2)\|'_W \leq \varepsilon\|(u_1-u_2, \theta_1-\theta_2)\|'_W, \tag{135}$$

which shows that Γ is also a contractive operator on B_1. Consequently it has a unique fixed point in B_1.

Denote by (u, θ) the fixed point of Γ in both cases. Since $(u, \theta) \in W$, it follows from (85), (94) and (105) that $J_m(\theta)$, $u \cdot \nabla u \in L^{\rho/2, r}(0, \infty; \dot{B}^{\tilde{s}}_{p, \infty})$ and $u \cdot \nabla\theta \in L^{m\rho/2(m+2), r}(0, \infty; \dot{B}^{\tilde{s}_m}_{p, \infty})$, from which, it is easy to check that (u, θ) is the strong solution to (74). Furthermore, the function $\nabla\pi$ comes from the Helmholtz decomposition, that is

$$\nabla\pi = (I-P)(J_m(\theta)e_n - u \cdot \nabla u) = \nabla(-\Delta)^{-1}\mathrm{div}(J_m(\theta)e_n - u \cdot \nabla u). \tag{136}$$

As a direct consequence, we can easily see that $\nabla\pi \in L^{\rho/2, r}(0, \infty; \dot{B}^{\tilde{s}}_{p, \infty})$, and the function triple $(u(t), \nabla\pi(t), \theta(t))$ verify both the equations in (1) in $\dot{B}^{s}_{p, \infty}$ and $\dot{B}^{s_m}_{p, \infty}$ respectively for a.e. $t \in [0, T]$. Thus, existence of the strong solution of the (GBS) (1) has been reached. Uniqueness of the strong solution in the class W comes from the uniqueness of the fixed point of Γ.

Continuity of u and θ of the strong solution can be checked directly. Since the semigroup e^{-tB_α} is continuous on $\dot{B}^{s}_{p, r}$ provided $r < \infty$, we have $u_L \in C([0, \infty); \dot{B}^{s_0}_{p, r})$ and $\theta_L \in C([0, \infty); \dot{B}^{\varsigma_m}_{p, r})$. Furthermore, it is easy to see that $\Phi(u, u), M(\theta) \in C([0, \infty); \dot{B}^{s}_{p, 1})$ and $\Psi(u, \theta) \in C([0, \infty); \dot{B}^{s_m}_{p, 1}$. Putting them together, we have $u \in C([0, \infty); \dot{B}^{s_0}_{p, r} + \dot{B}^{s}_{p, 1})$ and $\theta \in C([0, \infty); \dot{B}^{\varsigma_m}_{p, r} + \dot{B}^{s_m}_{p, 1})$.

Finally, according to the estimate (115), we can find a small number $c > 0$ such that under the initial hypothesis (5), condition (124) for $m \geq 2$, or (132) for $1 < m < 2$ is realised. Moreover, by the definition of λ_1, we have

$$\lambda_1 \leq \left\{ \begin{array}{lll} 4K_0 & \text{if} & m \geq 2, \\ & & \\ K_0/(1-\varepsilon), & \text{if} & 1 < m < 2 \end{array} \right\} \leq C(\|u_0\|_{\dot{B}^{s_0}_{p, r}} + \|\theta_0\|_{\dot{B}^{\varsigma_m}_{p, r}}), \tag{137}$$

which in turn yields (8) for the above-obtained solution in both cases. Thus the proof has been completed. □

Proof of Theorem 2 In this proof, we need only redefine

$$Y = \left\{\theta \in L^{\rho/2, r}(0, \infty; \dot{B}^{\sigma_i+2\alpha}_{p_i, 1}):\ \theta' \in L^{\rho/2, r}(0, \infty; \dot{B}^{\sigma_i}_{p_i, 1})\right\} \tag{138}$$

with the norm

$$\|\theta\|_Y = \|\theta\|_{L^{\rho/2, r}(0, \infty; \dot{B}^{\sigma_i+2\alpha}_{p_i, 1})} + \|\theta'\|_{L^{\rho/2, r}(0, \infty; \dot{B}^{\sigma_i}_{p_i, 1})} \tag{139}$$

and assume $\theta_0 \in \dot{B}^{\omega_i}_{p, r, \sigma}$ with the choice

$$\omega_i = \sigma_i + 2\alpha\big(1 - \frac{2}{\rho}\big) = \left\{ \begin{array}{ll} 1 + n/p - 4\alpha, & i = 1, \\ & \\ 1 + 2n/p - 4\alpha, & i = 2. \end{array} \right. \tag{140}$$

Now by following the same arguments as in the $m \geq 2$ part of the previous proof with minor revisions, we can complete the proof. □

Remark 25 Some examples of the initial values for the case $n = 3$ are provided below to verify the reasonableness of the assumptions established in this study.

In the case $m > 1$, we take two scalar functions $a, b \in L^p$, and let

$$u_0 = \varepsilon(-\Delta)^{-\beta}\mathrm{rot}P(ae), \quad \theta_0 = \varepsilon(-\Delta)^{\gamma}b, \tag{141}$$

where e denotes a unit vector, $\varepsilon > 0$ is a small number, $\beta = 1+3/2p-\alpha$ and $\gamma = (4\alpha-1)/2m-3/2p$. Evidently, under this setting, we have $u_0 \in \dot{B}^{s_0}_{p,\,p}$ with $\mathrm{div}u_0 = 0$ and $\theta_0 \in \dot{B}^{\varsigma_m}_{p,\,p}$, and

$$\|u_0\|_{\dot{B}^{s_0}_{p,\,p}} + \|\theta_0\|_{\dot{B}^{\varsigma_m}_{p,\,p}} \le C\varepsilon \tag{142}$$

for some constant $C > 0$ independent of a and b. In the case $m = 1$, we then take $b \in L^{p_i}$, and $\gamma = 2\alpha - 1/2 - n/2p_i$ with a and β unchanged. Under this setting, we have $\theta_0 \in \dot{B}^{\omega_i}_{p_i,\,p}$, and

$$\|u_0\|_{\dot{B}^{s_0}_{p,\,p}} + \|\theta_0\|_{\dot{B}^{\omega_i}_{p_i,\,p}} \le C\varepsilon \tag{143}$$

for some constant $C > 0$ independent of a and b.

5. Conclusions and prospects

In this paper, we study the global solvability of the Boussinesq system with the fractional Laplacian $(-\Delta)^{\alpha}$ in $\mathbb{R}^n$ ($n \ge 3$), where the buoyancy force is defined as $|\theta|^{m-1}\theta e_n$ and $m \ge 1$. To address the problem, two key technical approaches are adopted: One is the estimates for the difference $|\theta_1|^{m-1}\theta_1 - |\theta_2|^{m-1}\theta_2$ within the framework of Besov spaces, and the other is the maximal regularity property of the fractional Laplacian $(-\Delta)^{\alpha}$ in Lorentz spaces.

By implementing these methods, and setting reasonable assumptions on the exponents α, m, p, r, and ρ (see $(H_1)-(H_4)$), we prove that under the small norm assumption on the initial values, the generalized Boussinesq system admits a unique global strong solution $(u,\,\theta)$ in the critical temporal-spatial spaces. Concretely, in the case $m = 1$, if the initial data of velocity u_0 and temperature (or salinity) θ_0 belong to $\dot{B}^{s_0}_{p,\,r,\,\sigma} \times \dot{B}^{\omega_i}_{p,\,r}$ with small norm, then the solution $(u,\,\theta)$ exists in $L^{\rho,\,r}(0,\,\infty;\,\dot{B}^{s+2\alpha}_{p,\,1,\,\sigma}) \times L^{\rho/2,\,r}(0,\,\infty;\,\dot{B}^{\sigma_i}_{p_i,\,1})$, where $p_1 = p$ if $1 < p < n$, and $p_2 = p/2$ if $n \le p < 2n$, s_0, s and ω_i, σ_i take the values as in (3) and (9) respectively. In the case $m > 1$, if the initial value $(u_0,\,\theta_0)$ fall into $\dot{B}^{s_0}_{p,\,r,\,\sigma} \times \dot{B}^{\varsigma_m}_{p,\,r}$. The solution $(u,\,\theta)$ exists in $L^{\rho,\,r}(0,\,\infty;\,\dot{B}^{s+2\alpha}_{p,\,1,\,\sigma}) \times L^{m\rho/2,\,r}(0,\,\infty;\,\dot{B}^{s_m+2\alpha}_{p,\,1})$, where ς_m, s_m take the values as in (4). See Theorem 1, 2 for the details.

We conclude that global solvability of the generalized Boussinesq systems under small initial value assumptions holds for $m = 1$ and $1/2 < \alpha < 1/2+n/4$, $1 < m < 2$ and $1/2 < \alpha < (m+1)(m+2)/8$, and $m \ge 2$ and $1/2 < \alpha < (3m-1)/2m$.

Jiu and Wu in [17] showed that for the 3D Generalised Bounssinesq System (GBS) with partial viscosity ($\nu = 0$), a global strong solution exists without small initial data assumption when $\alpha \ge 5/4$. We wonder in the case where $\nu > 0$ and $\alpha = \beta$, what conditions on α and m ensure the existence of a global strong solution to the 3D (GBS). Another interesting topic is: if we assume $\alpha \ne \beta$, what additional conditions are required to maintain the global solvability of the 3D (GBS) under the following three cases: $\alpha+\beta < 1$, $\alpha+\beta = 1$ and $\alpha+\beta > 1$? Furthermore, what are the differences in the conditions for ensuring global solvability between the 2D and 3D versions of the (GBS)?

Funding

This research received no specific grant from public, commercial, or non-profit funding agencies.

Authors' contributions

Qinghua Zhang: Conceptualization, Methodology and Formal analysis. Shiwei Cao: Writing-Original draft preparation. Huiyang Zhang: Writing-Reviewing and Editing. All authors reviewed and supported the final version of the manuscript.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Danchin R, Paicu M. The Leray and Fujita-Kato theorems for the Boussinesq system with partial viscosity. *Bulletin of the French Mathematical Society*. 2008; 136(2): 261-309. Available from: https://doi.org/10.24033/bsmf.2557.

[2] Brandolese L, Schonbek ME. Large time decay and growth for solutions of a viscous Boussinesq system. *Transactions of the American Mathematical Society*. 2012; 364: 5057-5090. Available from: https://doi.org/10.1090/S0002-9947-2012-05432-8.

[3] Han P. Algebraic L^2 decay for weak solutions of a viscous Boussinesq system in exterior domains. *Journal of Differential Equations*. 2012; 252: 6306-6323. Available from: https://doi.org/10.1016/j.jde.2012.03.002.

[4] Danchin R, Paicu M. Existence and uniqueness results for the Boussinesq system with data in Lorentz spaces. *Physica D: Nonlinear Phenomena*. 2008; 237(10-12): 1444-1460. Available from: https://doi.org/10.1016/j.physd.2008.03.034.

[5] Yan S, Wang L, Zhang Q. Local and global solvability for the Boussinesq system in Besov spaces. *Open Mathematics*. 2024; 22: 20230182. Available from: https://doi.org/10.1515/math-2023-0182.

[6] Lions JL. *Quelques Méthodes De Résolution Des Problèmes Aux Limites Non Linéaires [Some Methods for Solving Nonlinear Boundary Value Problems], vol 1*. Dunod; 1969.

[7] Katz NH, Pavlović N. A cheap Caffarelli-Kohn-Nirenberg inequality for Navier-Stokes equations with hyperdissipation. *Geometric and Functional Analysis*. 2001; 12(2): 355-379. Available from: https://doi.org/10.1007/s00039-002-8250-z.

[8] Chen ZM. Analitic semigroup approach to generalized Navier-Stokes flows in Besov spaces. *Journal of Mathematical Fluid Mechanics*. 2017; 19: 709-724. Available from: https://doi.org/10.1007/s00021-016-0302-5.

[9] Duan N. Well-posedness and decay of solutions for three dimensional generalized Navier-Stokes equations. *Computers and Mathematics with Applications*. 2018; 76: 1026-1033. Available from: https://doi.org/10.1016/j.camwa.2018.05.038.

[10] Wu J. Lower bounds for an integral involving fractional Laplacians and the generalized Navier-Stokes equations in Besov spaces. *Communications in Mathematical Physics*. 2005; 263: 803-831. Available from: https://doi.org/10.1007/s00220-005-1483-6.

[11] Yu X, Zhai Z. Well-posedness for fractional Navier-Stokes equations in the largest critical spaces $\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)$. *Mathematical Methods in the Applied Sciences*. 2014; 35: 676-683. Available from: https://doi.org/10.1002/mma.1582.

[12] Cheskidov A, Shvydkoy R. Ill-posedness for subcritical hyperdissipative Navier-Stokes equations in the largest critical spaces. *Journal of Mathematical Physics*. 2012; 53: 115620. Available from: https://doi.org/10.1063/1.4765332.

[13] Liu Q. Global well-posedness of the generalized incompressible Navier-Stokes equations with large initial data. *Bulletin of the Malaysian Mathematical Sciences Society*. 2020; 43(2): 2549-2564. Available from: https://doi.org/10.1007/s40840-019-00818-5.

[14] Miao CX, Xue LT. On the global well-posedness of a class of Boussinesq-Navier-Stokes systems. *NoDEA Nonlinear Differential Equations and Applications*. 2011; 18: 707-735. Available from: https://doi.org/10.1007/s00030-011-0114-5.

[15] Jiu QS, Miao C, Wu J, Zhang Z. The 2D incompressible Boussinesq equations with general critical dissipation. *SIAM Journal on Mathematical Analysis*. 2014; 46: 3426-3454. Available from: https://doi.org/10.2139/ssrn.1763167.

[16] Yang W, Jiu QS, Wu J. Global well-posedness for a class of 2D Boussinesq systems with fractional dissipation. *Journal of Differential Equations*. 2014; 257(11): 4188-4213. Available from: https://doi.org/10.1016/j.jde.2014.08.006.

[17] Jiu QS, Yu H. Global well-posedness for 3D generalized Navier-Stokes-Boussinesq equations. *Acta Mathematicae Applicatae Sinica, English Series*. 2016; 32: 1-16. Available from: https://doi.org/10.1007/s10255-016-0539-z.

[18] Kozono H, Shimizu S. Navier-Stokes equations with external forces in Lorentz spaces and its application to the self-similar solutions. *Journal of Mathematical Analysis and Applications*. 2018; 458: 1693-1708. Available from: https://doi.org/10.1016/j.jmaa.2017.10.048.

[19] Kozono H, Shimizu S. Strong solutions of the Navier-Stokes equations based on the maximal Lorentz regularity theorem in Besov spaces. *Journal of Functional Analysis*. 2019; 276: 896-931. Available from: https://doi.org/10.1016/j.jfa.2018.06.006.

[20] Giga Y, Sohr H. Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains. *Journal of Functional Analysis*. 1991; 102: 72-94. Available from: https://doi.org/10.1016/0022-1236(91)90136-S.

[21] Miao C, Wu J, Zhang Z. *Littlewood-Paley Li Lun Ji Qi Zai Liu Ti Dong Li Xue Fang Cheng Zhong De Ying Yong [Littlewood-Paley Decomposition Theory and Its Application in Fluid Dynamics Equations], vol 142*. Foundational Series of Modern Mathematics. Science Press; 2012.

[22] Bahouri H, Chemin JY, Danchin R. Fourier analysis methods for the compressible Navier-Stokes equations. In: *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, vol 343*. Springer Berlin, Heidelberg; 2011. p.2196-9701.

[23] Triebel H. *Interpolation Theory, Function Spaces, Differential Operators, vol 18*. North-Holland Mathematics Library. Amsterdam: North-Holland; 1978.

[24] Lunardi A. *Interpolation Theory, vol. 9*. 2nd ed. Scuola Normale Superiore Pisa (N.S.); 2009.

[25] Martínez C, Sanz M. *The Theory of Fractional Powers of Operators, vol 187*. North-Holland Mathematics Studies. Elsevier Science; 2001.

[26] Adams RA, Fournier JJF. *Sobolev Spaces, vol. 140*. 2nd ed. Pure and Applied Mathematics. Academic Press; 2003.