Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Nature Set: Qualitative and Quantitative Representation of Elements

Sovan Samanta[1,2,3*], Vivek Kumar Dubey[1], Leo Mrsic[1,4], Antonios Kalampakas[5]

[1] Department of Technical Sciences, Algebra Bernays University, Gradiscanska 24, Zagreb, 10000, Croatia
[2] Research Center of Performance and Productivity Analysis, Istinye University, Istanbul, Turkey
[3] Department of Mathematics, Tamralipta Mahavidyalaya, Tamluk, W.B-721636, India
[4] Rudolfovo Science and Technology Centre, Podbreznik 15, Novo Mesto, 8000, Slovenia
[5] College of Engineering and Technology, American University of the Middle East, Egaila, 54200, Kuwait
E-mail: ssamantavu@gmail.com

Received: 8 July 2025; **Revised:** 4 September 2025; **Accepted:** 10 September 2025

Abstract: Classical sets are characterized by 'well-defined' objects or elements, but this property is relative and linguistic. Classical sets, like the set of natural numbers, are primarily used for counting and measuring quantitative data. However, the literature does not address the differences in the representation of cardinalities between infinite sets and their infinite subsets. This study investigates the cardinalities of infinite sets and their finite versions (cut cardinalities) within the finite approximation of the real number system. Traditional fuzzy sets measure only belongingness or membership values, thus capturing qualitative data. In the second part, this study introduces nature sets, which describe both the quantitative and qualitative aspects of elements. Quality measurement is provided by various techniques. Several properties of nature sets are examined, and potential applications are highlighted.

***Keywords*:** nature set, qualitative property, quantitative property, cut cardinality

MSC: 03E72

1. Introduction

While infinity is a powerful mathematical concept representing unbounded quantities, its direct application in the real world is limited [1]. Our finite universe, with measurable boundaries and discrete building blocks like photons and electrons, excludes the existence of truly infinite systems. Even seemingly "infinite" entities like disease agent variations, though theoretically boundless, are practically approached through finite sampling and analysis due to resource limitations and continuous evolution [2]. Therefore, understanding the distinction between theoretical and practical considerations is crucial when directing seemingly infinite concepts in the real world.

The cardinality of infinite sets of real numbers is such a case. Several works of literature involve the countability of real numbers. Georg Cantor's diagonalization argument is one of the most well-known ways to show that the real number system ($\mathbb{R}$) can't be counted. This elegant argument was not even Cantor's first proof of this theorem! Cantor published a different proof of the uncountability of $\mathbb{R}$ more than 15 years before the diagonalization argument came out [3]. There are several proofs by other scientists, too, that real numbers are not countable [4]. This uncountability property of real numbers is an obstacle to the finite world.

DOI: https://doi.org/10.37256/cm.6520257742

Leopold Kronecker was a German mathematician who was active in the fields of number theory and algebra during his lifetime (1823-1891). He is regarded as well-known intuitionist who rejected Cantor's set theory 2 [5]. This study introduces a finite approximation of $\mathbb{R}$ to capture the cardinalities of infinite sets in the finite approximation of $\mathbb{R}$.

The question of whether real numbers can be finitely represented has been a longstanding theme in mathematics and theoretical computer science. Tsirelson [6] examined the possibility of describing each number by a finite text. His work highlights fundamental limitations of finite descriptions in capturing the uncountable nature of the real number system, emphasizing the inherent gap between the theoretical continuum of real numbers and their practical finite approximations. In parallel, Chen et al. [7] proposed a fast division algorithm for finite real numbers, providing a computational method tailored for truncated or discretized versions of the real line. Their algorithm demonstrates how operations on finite real approximations can be optimized for efficiency in practical computing environments. As shown in Masáková et al. [8], the finiteness property of Cantor real numeration systems provides an alternative framework for representing real numbers with finite expansions. Similarly, Aggarwal et al. [9] analyzed the accuracy of finite volume approximations to nonlocal conservation laws, demonstrating how discredited schemes can reliably approximate continuous models.

When irrationals are written as square root operations or as an infinitely expandable series, it implies that an infinite number of steps are needed to reach the true value of the number. Thus, the number cannot be explicitly written. We can only put it between a known predecessor and a known successor and say that it is between the two. The conclusion is that the irrational number is 'fixed' in its position in the real line but can't be located at the exact place in the real line. In a finite world, the notion of an infinite representation of something is not acceptable or fathomable by the human mind. This study simplifies the concept of the usability of irrational numbers and their countability. Further, it addresses the general issue of representing an interval (of real space) by a finite set of numbers, thus making it more amenable to human conceptualization.

In current literature, fuzzy sets capture the memberships of elements perfectly within [0, 1] [10]. A bipolar fuzzy set is introduced by Zhang to include positive and negative memberships of each element, representing the bipolarity of certain properties [11]. There are several types of extensions of set theory to represent different natures of elements. To include true membership, falsity, and indeterminacy, neutrosophic sets have been introduced [12]. Several existing concepts of set theory can be found in the literature. But the nature of an element (single parameter) may be positive or negative for each element.

The cardinality of a set indicates only a quantitative measurement. But, to include qualitative measurement along with quantitative, the definition of the fuzzy set was introduced. However, the cardinality of fuzzy sets aggregates the membership values of the elements. This concept is modified here to include qualitative and quantitative measures. Let S be the set of all students in a class ($S = \{s_1, s_2, \ldots, s_{40}\}$). $|S| = 40$. Each student has some membership values based on their quality. Let us represent them as $S' = \{(s_1, \text{good}), (s_2, \text{bad}), \ldots, (s_{40}, \text{average})\}$. We have divided the categories of students into good, average, and bad. The aggregate of the set in the qualitative aspect depends on a certain rule base. If it is 'good' on aggregate output, then $|S'| = 40 \times 0.95 = 38$. This study develops such a set that characterizes the nature of an element to find the quantitative and qualitative measurements.

1.1 *Research gap*

While significant progress has been made in the field of set theory and its extensions, there is a clear gap in the literature concerning the practical application of these concepts in finite, real-world scenarios. Existing theories, such as classical sets, fuzzy sets, bipolar fuzzy sets, and neutrosophic sets, primarily focus on either quantitative or qualitative aspects, but not both simultaneously. Additionally, the concept of infinity, while theoretically strong, poses challenges when applied to finite systems, leading to a need for a more practical approach.

This research aims to fill this gap by introducing the concept of nature sets, which describes both quantitative and qualitative aspects of elements. By proposing a finite approximation of the real number system, this study seeks to provide a more practical framework for handling real-world data. The development of nature sets and bunch sets offers a novel approach to data representation, addressing the limitations of existing theories and providing a foundation for future research in this area.

1.2 *Motivation*

The motivation behind this article originates from the need to bridge the gap between theoretical mathematical concepts and their practical applications in the real world. Traditional set theories and the concept of infinity, while powerful, often fall short when applied to finite, observable systems. This discrepancy creates challenges in various fields, such as data analysis, decision-making, and computational mathematics, where both quantitative and qualitative aspects of data need to be accurately represented and measured. By introducing the concept of nature sets and a finite approximation of the real number system, this study aims to provide a more practical and comprehensive framework for handling real-world data.

1.3 *Objectives*

The primary objective of this article is to address the limitations of applying the concept of infinity in practical, real-world scenarios by proposing a finite approximation of the real number system. This study aims to introduce nature sets, a novel concept that captures both quantitative and qualitative aspects of elements, thereby providing a more comprehensive approach to data representation. By examining the properties of nature sets, including operations such as union, intersection, and complement, the article seeks to establish a solid theoretical foundation. Additionally, the development of bunch sets extends the concept of nature sets to groups of elements, facilitating the analysis of complex systems. The exploration of cut cardinalities within the finite approximation of real number system offers a new perspective on measuring the size of infinite sets. Furthermore, the article highlights the practical applications of nature sets and bunch sets in various fields, demonstrating their utility in real-world scenarios. Ultimately, this study aims to lay the groundwork for future research in the area of finite approximation of real number systems and nature sets, encouraging further exploration and development of these concepts.

The paper is organized as follows: Section 2 investigates the concept of cut cardinalities and the finite approximation of real number system, providing a detailed explanation and examples. Section 3 introduces the conceptual basis for nature sets, explaining their definition and significance. Section 4 discusses the properties of nature sets, including union, intersection, and complement operations. Section 5 presents the concept of bunch sets, which extend the idea of nature sets to groups of elements. Section 6 explores the power set of nature sets and their cut cardinalities, along with relevant properties. Section 7 highlights potential applications of nature sets in various fields. Finally, Section 8 concludes the paper by summarizing the key findings and suggesting directions for future research.

2. Cut cardinalities and finite approximation of real number systems

The universe is an absolutely infinite, unending structure, but we have a finite observable universe. The extent of this observable part varies with the latest technology. Thus, the finite version of an infinite system is the only useful and important aspect for the real world. The real number system is uncountable due to the highly dense nature of real numbers. For infinite sets, cardinalities are assumed as Aleph Null (for countable infinite sets) and Aleph One (for uncountable sets). However, this concept of cardinality is not applicable in the finite world. In this section, the concept of cut cardinalities is introduced.

The cardinality of infinite sets cannot be directly compared. Therefore, we are introducing a system of a finite approximation of the real number system as follows.

The following properties are assumed to define a finite world $R_F(\partial)$ from the real number system:

- $|x| \leq \aleph(\partial)\ \forall x \in R_F(\partial)$ where $\aleph(\partial)$ is the maximum possible finite number.
- The decimal representation of every number of $R_F(\partial)$ can be arranged as $a.a_1a_2a_3\ldots a_\partial$ where $a_i \in \{0, 1, 2, \ldots, 9\},\ i = 1, 2, \ldots, \partial$ (last possible step).

Note: Choosing of $\aleph(\partial)$ in $R_F(\partial)$

This system is a relative system depending on ∂. In the latest computers, any large numbers can be computed. But for a certain algorithm, with the specific computation time, there is a limit. For that specific problem, the rounding of numbers is done, thus the real number system is less important there. Suppose one computer with the latest technology can compute 10 trillion decimal digits, then ∂ may be assumed as 10 trillion. In a similar way, the maximum possible number will be selected depending on ∂. Thus, ∂ depends on the latest technology of computation.

In this way, sets are always countable in $R_F(\partial)$ and cardinalities of a set can be defined as a finite number. Note that $R_F(\partial)$ is not strictly a number system in the algebraic sense, since it is not closed under addition, multiplication, and other operations. Rather, it is a finite approximation of $\mathbb{R}$, useful for computational and applied purposes.

2.1 *Counting in $R_F(\partial)$*

Consider the following method for counting rational numbers of $[0,\ 1]$ and irrational numbers in the finite approximation of real number system $R_F(\partial)$ (see Figure 1).

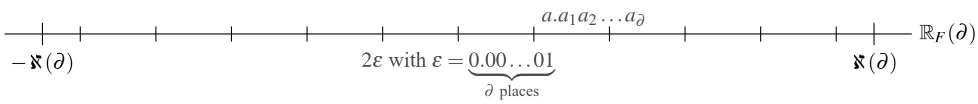


Figure 1. Finite approximation of real number system $R_F(\partial)^*$: each representable number carries a smallest interval of length 2ε

Method 1: Take any subset of real numbers. For example, let $A = [0,\ 1]$. We are to count all real numbers between 0 and 1 in $R_F(\partial)$.

1. Count all numbers with a gap of 0.1, starting from 0 and ending at 1. It is finite, and count = $2+9$.

2. Count all numbers with a gap of 0.01, starting from 0 and ending at 1. It is finite, and count = $2+10^0\times 9+10\times 9$.

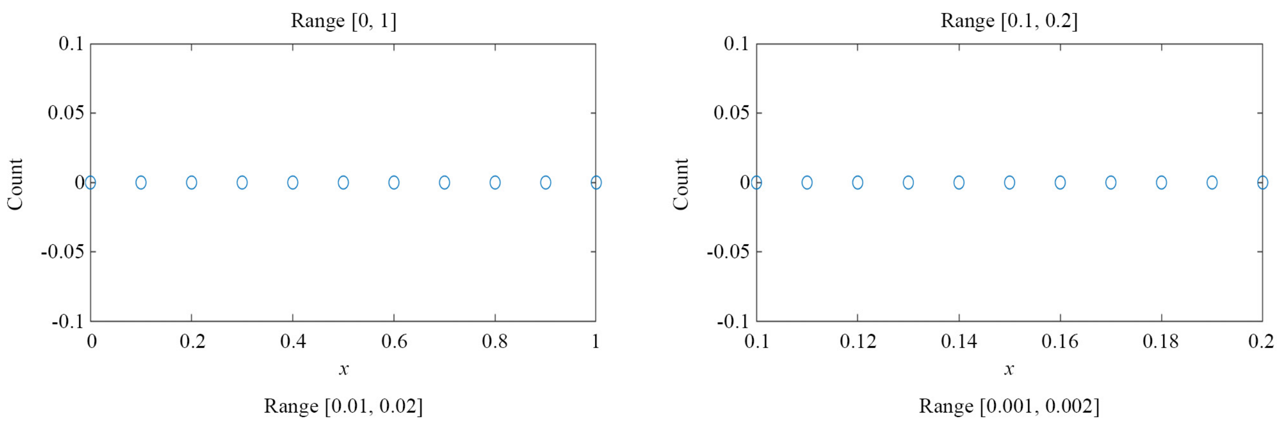
3. Count all numbers with a gap of 0.001, starting from 0 and ending at 1. It is finite, and count = $2+10^0\times 9+10\times 9+10^2\times 9$.

4. Count all numbers with a gap of 0.0001, starting from 0 and ending at 1. It is finite, and count = $2+10^0\times 9+10\times 9+10^2\times 9+10^3\times 9$.

5. And so on.

6. Step n: Count all numbers with a gap of 0.000...01 (with $n-1$ zeroes after decimal), starting from 0 and ending at 1. It is finite, and count = $2+10^0\times 9+10\times 9+10^2\times 9+10^3\times 9+\ldots+10^{(n-1)}\times 9$.

This process can be continued up to a large number of times (maximum possible) to include all possible real numbers between 0 and 1 in $R_F(\partial)$.



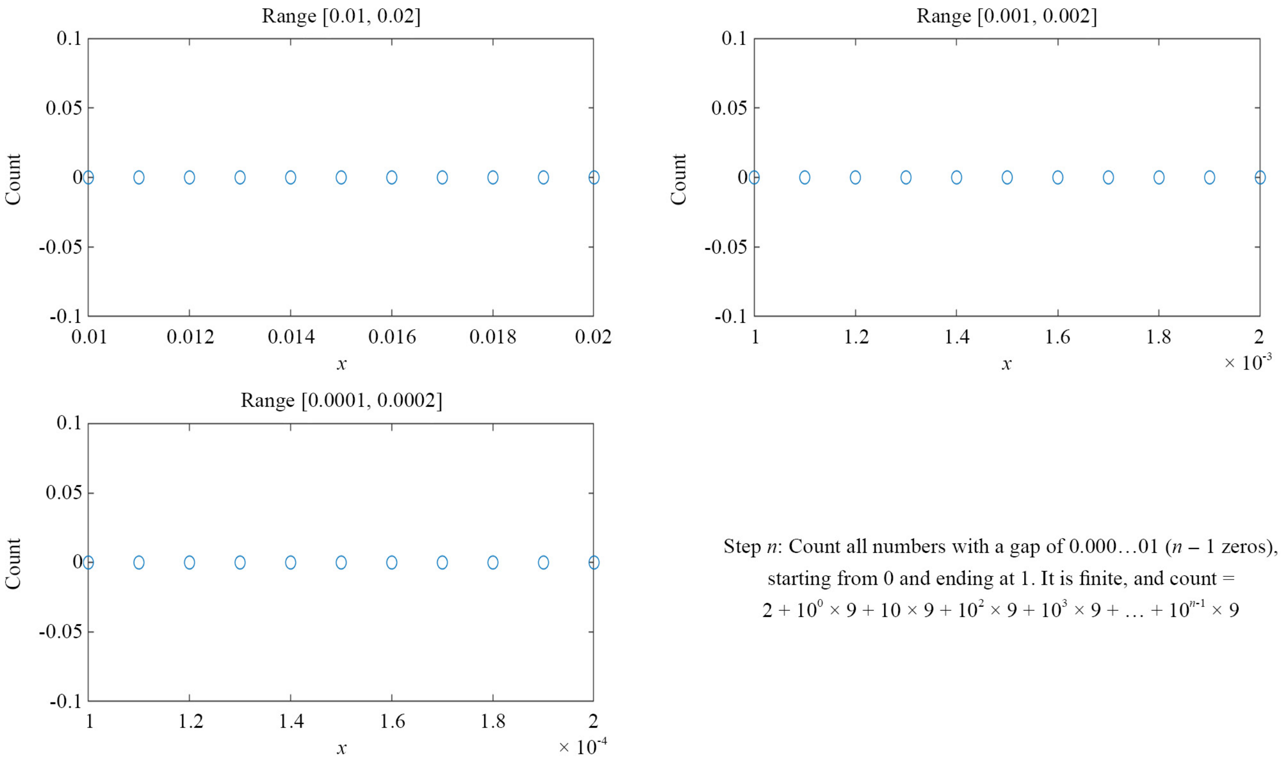


Figure 2. Counting of numbers between 0 and 1 in finite approximation of real number system

Example 1: As per the calculation $\sqrt{2} = 1.4142135623730950488\ldots$ the 19th decimal place can be captured in the 19th step of the proposed method for the counting of real numbers between 1 and 2. Then the irrational number up to the 19th digit after the decimal can be reached within the step $2+10^0\times 9+10\times 9+10^2\times 9+10^3\times 9+\ldots+10^{(19-1)}\times 9$ (See Figure 2).

2.2 *Cut cardinalities*

Basically, the cardinalities of finite sets have no ambiguities. The cardinalities of infinite sets are not comparable. It is a paradox that for infinite sets $A \subset B \subset \mathbb{R}$ but still both the sets have the same cardinality symbol $\aleph_0$ or $\aleph_1$. We mention two cases, the sets $\mathbb{Z}$ and $5\mathbb{Z} = \{\ldots, -15, -10, -5, 0, 5, 10, 15, \ldots\}$ both sets have the same cardinality $\aleph_0$ and for another case $[0, 1]$ and $[0, 10]$ have the same cardinality symbol $\aleph_1$. This is because of the dense nature of real numbers. Here, all the numbers are assumed with the maximum possible (finite) number of decimal places, thus every set in the system $R_F(\partial)$ is countable.

Cut cardinalities are the cardinalities of sets in $R_F(\partial)$. The following statements are obvious in $R_F(\partial)$ for any sets:

- **Case 1:** $A \subset B \subset R_F(\partial)$

Cut Cardinality of B = Cut Cardinality of A + Cut Cardinality of $(B-A)$.

- **Case 2:** $B \subset A \subset R_F(\partial)$

Cut Cardinality of A = Cut Cardinality of B + Cut Cardinality of $(A-B)$.

- **Case 3:** $B, A \subset R_F(\partial)$

Cut Cardinality of $A \cup B$ = Cut Cardinality of A + Cut Cardinality of B-Cut Cardinality of $(A \cap B)$.

Alternatively, the Cut cardinalities can be compared as follows. Let $B \subset A \subset R_F(\partial)$. Let us assume that $A = \{a_1, a_2, \ldots, a_n, \ldots\}$ and $B = \{b_1, b_2, \ldots, b_n, \ldots\}$ are two sets such that between any two elements of B there are elements of A. Suppose, $b_i \le a_{m1} < a_{m2} < \ldots < a_{mk} \le b_{(i+1)}$ for all $i = 1, 2, \ldots$, then Cut Cardinality of $A < (k-1)\times$ Cut Cardinality of B.

The finite approximation of real number system $R_F(\partial)$ is extended by the following added terms as follows:

• $|x| \leq \aleph(\partial) \quad \forall x \in R_F$, here $\aleph(\partial)$ is the maximum possible finite number.

• Every number of $R_F(\partial)$ can be arranged as a number with a shortest possible tail. For the number $a.a_1a_2a_3 \ldots a_{\partial-1}$ we assume the tail as $\langle a.a_1a_2a_3 \ldots a_{\partial-1} - \varepsilon,\ a.a_1a_2a_3 \ldots a_{\partial-1} + \varepsilon\rangle$ where $a_i \in \{0, 1, 2, \ldots, 9\},\ i = 1, 2, \ldots, \partial$ (last step). Here $\varepsilon = 10^{-\partial}$. We are considering the smallest interval of length ε to construct the extended version of the finite approximation of the real number system as

$$\bigcup_{a \text{ is any integer and } a_i \in \{0, \pm 1, \pm 2, \ldots, \pm 9\}} (a.a_1a_2a_3 \ldots a_{\partial-1} - \varepsilon,\ a.a_1a_2a_3 \ldots a_{\partial+1} + \varepsilon) = [-\aleph(\partial),\ \aleph(\partial)] = R_F(\partial)^*.$$

Note 1 Addition of two numbers in $R_F(\partial)$

Numbers without tails (rational numbers whose decimal representation ends before ∂ steps) are added with the same rule as $\mathbb{R}$, if the resultant has less than ∂ decimal places. If the resultant has equal or creates more steps than ∂, then it will be a number with a tail whose first $\partial - 1$ steps will be equal to that of the resulting number. Here $a.a_1a_2a_3 \ldots a_\partial + b.b_1b_2b_3 \ldots b_\partial \in (c.c_1c_2c_3 \ldots c_{\partial-1} - \varepsilon,\ c.c_1c_2c_3 \ldots c_{\partial-1} + \varepsilon)$ where $a.a_1a_2a_3 \ldots a_\partial + b.b_1b_2b_3 \ldots b_\partial = c.c_1c_2c_3 \ldots c_{\partial-1}$ (correct to first $\partial - 1$ steps).

Note 2 The concept of infinite sets is present in the real world, such as the set of viruses, bacteria, etc. But, human technology only detects/can handle a finite number of species of viruses or bacteria. In other words, the set of virus variants (depending on type, toxicity, infectiousness, etc.) is infinite, but the detected number of variants is finitely many in nature. Thus, infinite cardinalities are rarely used in the physical world. The concept of cut cardinalities is useful in the finite world.

Cardinalities are the quantitative measurements of a set. In the next section, the quantitative and qualitative measurements are introduced.

3. Conceptual basis for nature set

Let X be a non-empty set, and μ is a function capturing the quality of elements of the set, ranging $[-1, 1]$. Then (X, q, μ) is called nature sets where q: $X \to \mathbb{R}$ represents a proportionate number to quantify the elements of X. And μ: $X \to [-1, 1]$ represents the quality of the elements of X.

Measure of quality is captured in different ways. Each element has some quality measurement represented as a number between -1 to 1. Here q represents the quantity measurement (cardinality of set). To represent both quantity and quality in a figure, the quantities of elements are normalized.

Example 1: Let X be a set of researchers, q quantifies the researchers' characteristics, let q be the number of publications, μ represents the quality of the publication, and let μ be a number in $[-1, 1]$ based on the number of citations compared to standard citations of an article in the related topic (see Figure 3). The corresponding values of such a nature set are illustrated in Table 1.

Table 1. Values of a nature set

Researchers	Publications (q)	Normalized value of q	Citations (x)	(μ)-values
Researcher 1	12	1	112	1
Researcher 2	7	0.58	210	1
Researcher 3	4	0.33	57	0.46
Researcher 4	10	0.83	12	-0.4

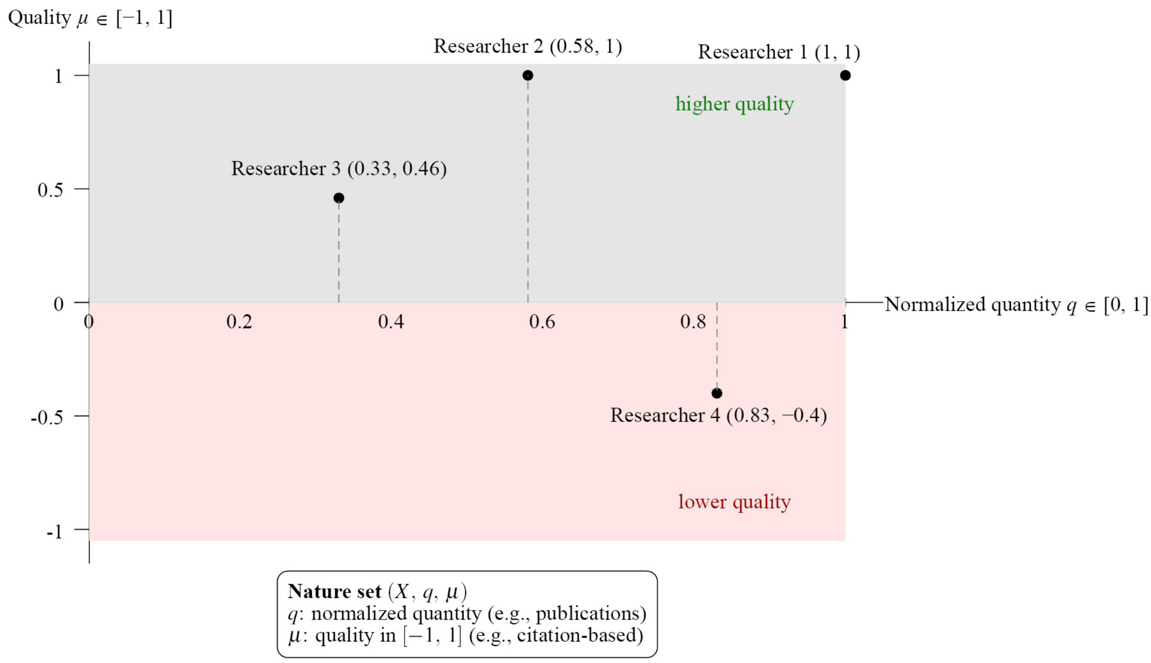


Figure 3. Nature set visualization: each element $x \in X$ is mapped to a point $(q(x), \mu(x))$ with $q \in [0, 1]$ and $\mu \in [-1, 1]$. Example shows researchers with normalized quantity (publications) and quality (citation-based score).

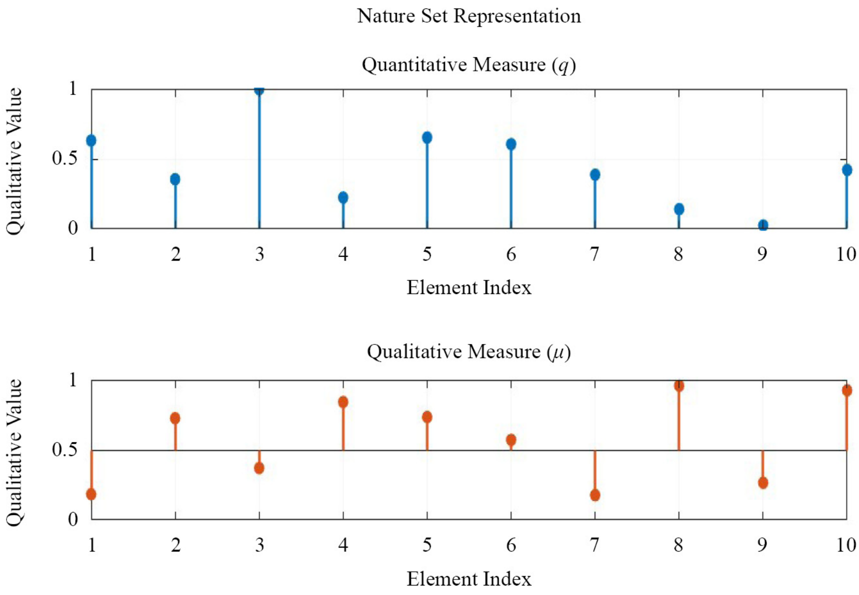
3.1 *Nature set*

Definition 1: Let X be a non-empty set, and μ is a function capturing the quality of elements of the set, ranging $[-1, 1]$. Then (X, q, μ, ν) is called nature sets where q: $X \to \mathbb{R}$ represents a proportionate number to quantify the elements of X. And μ: $X \to [-1, 1]$ represents the quality of the elements of X. And $\nu(\mu(x)) \in [-1, 1]$ represents the quality of the decision maker who assesses the value $\mu(x)$.

All linguistic terms defining μ such as good, bad, etc., can be mapped with similarly defined functions. Measure of quality is captured in different ways. Here q represents the quantity measurement. To represent both quantity and quality in a figure, the quantities of elements are normalized (see Figures 4, 5). The decision-maker may/may not be able to assess the quality ($\mu(x)$) well, depending on the knowledge of the subject (x).

Example 2: A few cases which allow us to measure the quality of decision-makers (ν) are mentioned below:

- Different school teachers assess students differently if there is no standardization training.
- The standardization process is difficult for University Professors.
- Risk profile managers are different for each and are harder to assess.
- A common householder may not have any standardization training (due to limited schooling and exposure), and hence their decision-making would reveal significant idiosyncratic decision-making characteristics.
- The Decision-making process is difficult to assess, such as complex processes where the outcomes have natural variations. Customers and suppliers may have difficulty (both have different assessments) in assessing the process and process outcomes.



Nature Set:

$X = x1, x2, ..., x10$

q = [0.63, 0.36, 1.00, 0.22, 0.65, 0.60, 0.39, 0.14, 0.03, 0.42]

μ = [-0.63, 0.45, -0.26, 0.68, 0.47, 0.14, -0.65, 0.91, -0.47, 0.85]

v: Quality of decision maker

Figure 4. Graphical representation of the elements of nature set

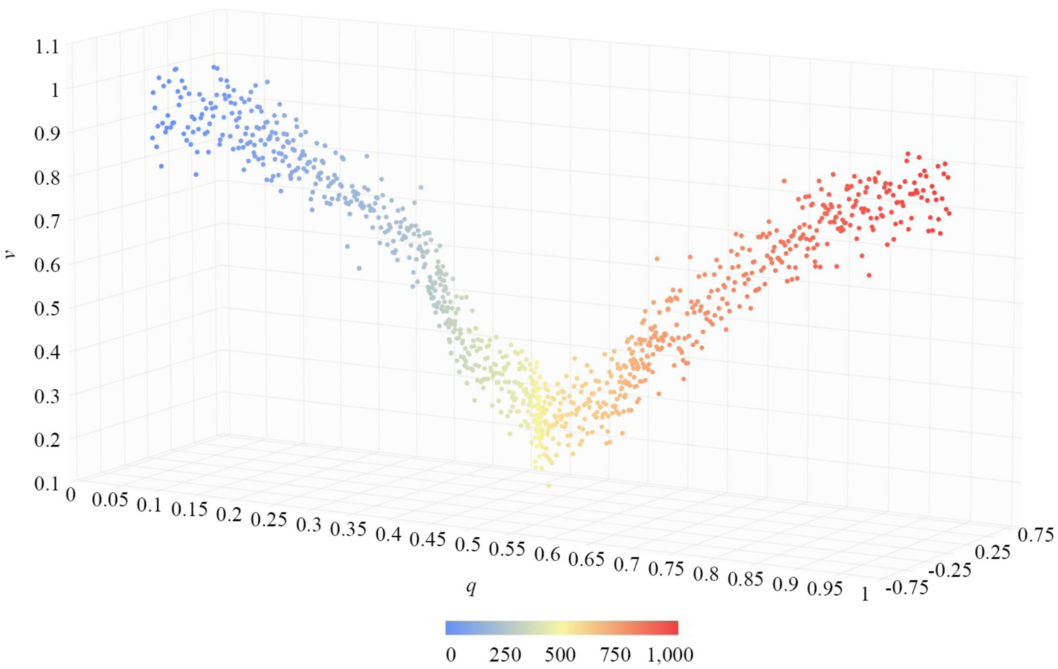


Figure 5. Example of nature set (data set of 1,000 points of a nature set)

4. Properties of nature set

In this section, we have assumed for simplicity that ν is equal for all the cases, and hence ν is ignored in this part.

4.1 *Union of two nature sets*

Let (X_1, q_1, μ_1) and (X_2, q_2, μ_2) be two nature sets. Now the union (quantitative) of two nature sets is defined as $(X_1 \cup X_2, q, \mu)$ where $q = \max\{q_1, q_2\}$, $k \in \mathbb{R}$ and $\mu(x) = \mu_1(x)$ if $q_1 \geq q_2$ and $\mu_2(x)$ otherwise. Now the union (qualitative) of two nature sets is defined as $(X_1 \cup X_2, q, \mu)$ where $q = q_1$ if $\mu_1(x) \geq \mu_2(x)$ and q_2 otherwise, $k \in \mathbb{R}$ and $\mu(x) = \max(\mu_1(x), \mu_2(x))$.

4.2 *Intersection of two nature sets*

Let (X_1, q_1, μ_1) and (X_2, q_2, μ_2) be two nature sets. Now the intersection (quantitative) of two nature sets is defined as $(X_1 \cap X_2, q, \mu)$ where $q = \min\{q_1, q_2\}$, $k \in \mathbb{R}$ and $\mu(x) = \mu_1(x)$ if $q_1(x) \leq q_2(x)$ and $\mu_2(x)$ otherwise. Now the intersection (qualitative) of two nature sets is defined as $(X_1 \cap X_2, q, \mu)$ where $q = q_1$ if $\mu_1(x) \leq \mu_2(x)$ and q_2 otherwise, $k \in \mathbb{R}$ and $\mu(x) = \min(\mu_1(x), \mu_2(x))$.

Example 3: Let us consider two sets, as shown in Table 2 and in Table 3. The union and intersection of the two sets have been shown in Table 4.

Table 2. Data of first nature set

Researchers (X_1)	Publications in Indexing 1 (q_1)	Normalized value of q_1	Citations (x)	(μ_1)-values
Researcher 1	0	0	0	0
Researcher 2	5	0.83	110	1
Researcher 3	2	0.33	37	0.2125
Researcher 4	6	1	6	-0.7

Table 3. Data of second nature set

Researchers (X_2)	Publications in Indexing 2 (q_2)	Normalized value of q_2	Citations (x)	(μ_2)-values
Researcher 1	12	1	112	1
Researcher 2	2	0.17	100	1
Researcher 3	4	0.33	20	0
Researcher 4	8	0.67	6	-0.7

Table 4. Union and intersection of two nature sets

Researchers	Union (quantitative)	Intersection (quantitative)	Union (qualitative)	Intersection (qualitative)
Researcher 1	(1, 1)	(0, 0)	(1, 1)	(0, 0)
Researcher 2	(0.83, 1)	(0.17, 1)	(0.17, 1)	(0.17, 1)
Researcher 3	(0.33, 0.21)	(0.33, 0)	(0.33, 0.21)	(0.33, 0)
Researcher 4	(1, -0.7)	(0.67, -0.7)	(1, -0.7)	(0.67, -0.7)

4.3 *Complement of a nature set*

The complement of a nature set $A = (X, q, \mu)$ is defined as $A^c = (X, q^c, \mu^c)$ where $q^c(x) = -q(x)$, and $\mu^c(x) = -\mu(x)$.
Example 4: The complement set of the nature set shown in Table 2 is shown in Table 5.

Table 5. Complement of the nature set of Table 2

Researchers (X_1)	Normalized value of q_1	(μ_1)-values
Researcher 1	0	0
Researcher 2	-0.83	-1
Researcher 3	-0.33	-0.2125
Researcher 4	-1	0.7

Nature set representation covers group concept. To represent the group items, the following definition of bunch set is introduced.

5. Bunch set

Let U be a universal set. Now, the bunch set is defined as follows $B = \{(X, q_X, \mu_X, \nu_X): X \subset U\}$ where q_X quantifies X, μ_X represents the nature of X and ν_X is the quality of decision-makers who provide the nature of the elements of X.

Example 5: U is the set of all students of a high school. B is a collection of subsets of different classes of the school, say, class V, class VI, class VII, class VIII, class IX, class X (see Table 6). Now each class has a different number of students and different quality (merit rank) of each student. To represent the whole students' sets of the school, a bunch set is essential. Here one way to measure the class characteristic (q_X) is to count the cardinality of the class sets (number of students in the class). Now, the nature of the class (merit ranking as assessed by the teacher objectively) can be represented as $\mu_X = \{\min \mu(x), (\sum \mu(x))/|X|, \max \mu(x) | x \in X\}$. Also, the inherent quality of the students as assessed by the teachers subjectively (such as honesty, obedience, promptness, etc.) as $\nu_X = \{\min \nu(x), (\sum \nu(x))/|X|, \max \nu(x) | x \in X\}$.

Table 6. An example of a bunch set

Class (X)	$W_X(q)$	$N_X(\mu)$	Quality of decision makers ($D_X(\nu)$)	
V	70	-0.4	0.2	0.6
VI	68	-0.5	0.1	0.7
VII	78	-0.3	0.4	0.8
VIII	64	-0.7	-0.2	0.4
IX	60	-0.1	0.5	0.9
X	62	-0.4	0	0.5

Example 6: Let $X = \{(\text{Science Stream}, 80, [0, 1, 0.5]), (\text{Arts Stream}, 150, [-0.7, 0.6, 0.1]), \text{Commerce Stream}, 10, [-0.9, 0, -0.4])\}$ be a set of students of higher secondary level in a school. The first element 'Science Stream' has 80 as quantity/number of students belonging to the stream and nature/membership of the element belongs to $[0, 1]$ and 0.5 is the average of all memberships of the inside quantity of "Science". Similarly, "Arts Stream" has 150 as quantity and with

nature $[-0.7, 0.6]$ and average value is 0.1. And the last case, Commerce Stream has 10 as its quantity and $[-0.9, 0]$ as nature of the stream with -0.4 as the average membership of the element (see Figure 6).

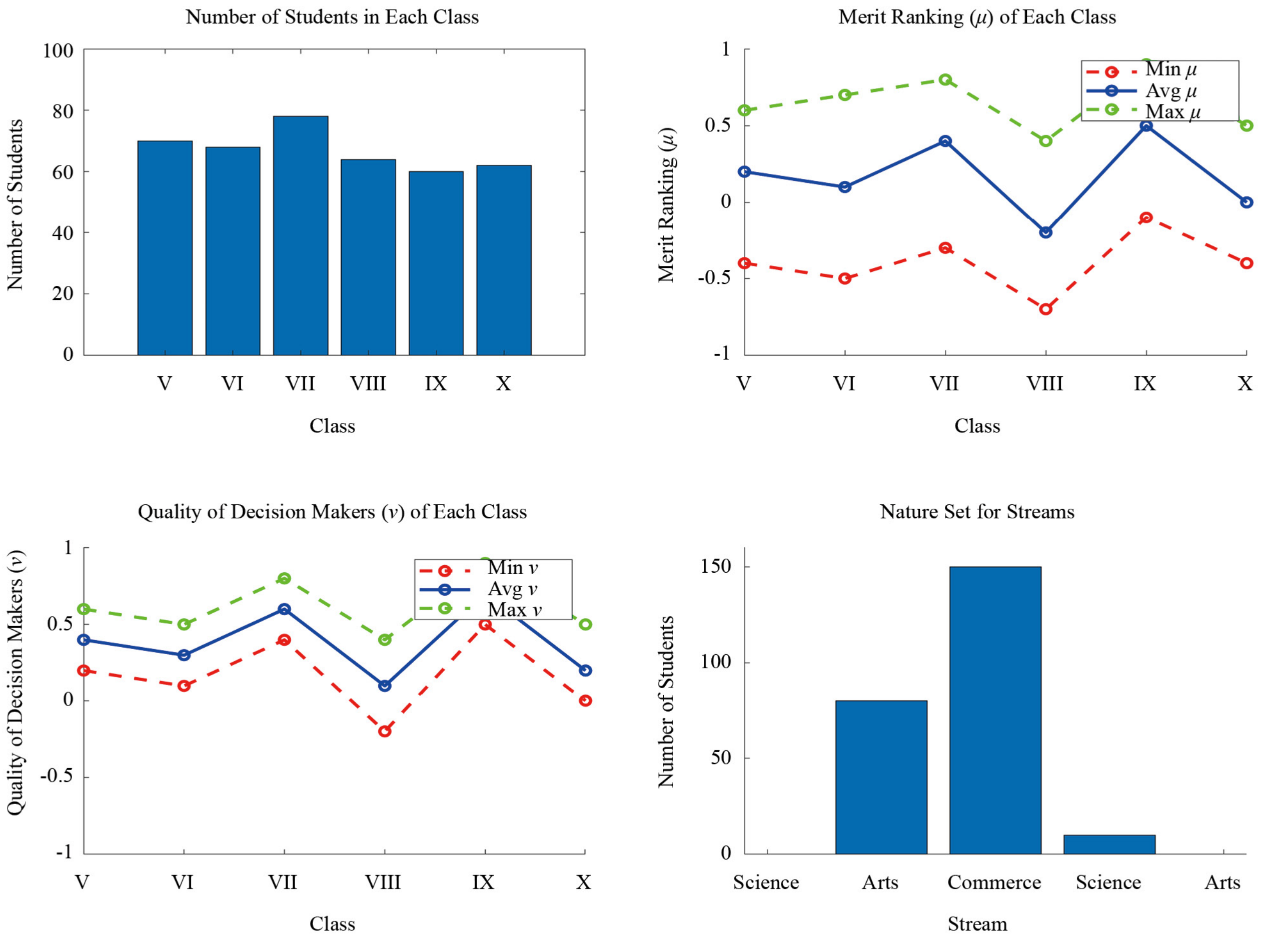


Figure 6. An example of bunch set

6. Power set and cut cardinality of nature set

Let $N = (X, q, \mu, \nu)$ be a nature set where $X = \{x_1, x_2, x_3, \ldots, x_n\}$ be a finite set. Now the power set of N is denoted as $P(N)$ and defined as

$$P(N) = \{(x_1, q(x_1), \mu(x_1), \nu(x_1)), (x_2, q(x_2), \mu(x_2), \nu(x_2)), \ldots, (x_n, q(x_n), \mu(x_n), \nu(x_n)),$$

$$((x_1, x_2), f_2(q(x_1), q(x_2)), g_2(\mu(x_1), \mu(x_2)), h_2(\nu(x_1), \nu(x_2))),$$

$$((x_1, x_3), f_2(q(x_1), q(x_3)), g_2(\mu(x_1), \mu(x_3)), h_2(\nu(x_1), \nu(x_3))), \ldots,$$

$$((x_1, x_2, x_3), f_3(q(x_1), q(x_2), q(x_3)), g_3(\mu(x_1), \mu(x_2), \mu(x_3)), h_3(\nu(x_1), \nu(x_2), \nu(x_3))), \ldots,$$

$$((x_1, x_2, \ldots, x_n), f_n(q(x_1), q(x_2), \ldots, q(x_n)), g_n(\mu(x_1), \mu(x_2), \ldots, \mu(x_n)), h_n(\nu(x_1), \nu(x_2), \ldots, \nu(x_n))\}.$$

Here F_i: $[0, 1] \times [0, 1] \times \ldots \times [0, 1] \to [0, 1]$ where F_i could represent f_i, g_i, h_i; F_i could represent functions such as min, max, or average.

6.1 *Cut cardinality of nature set*

'The number of elements' is a crisp term. But, if someone asks about the number of good elements, number of talented boys, etc., these are very complex/fuzzy to count. Here cut cardinality of nature set is denoted as the functional value involving q, μ, ν.

Example: Cardinality (qualitative measure) of a nature set can be formulated as follows:

$$C(N) = \frac{1}{|X|} \sum_{x \in N} q(x) \times (\mu(x) \wedge \nu(x))$$

$$C(P(N)) = \frac{1}{|X|} \sum_{x \in N} q(x) \times (\mu(x) \wedge \nu(x))$$

$$+ \frac{1}{|P(X) - X - \{\emptyset\}|} \sum_{x \in N} f_i(q(x_1), q(x_2), \ldots, q(x_i))$$

$$\times (g_i(\mu(x_1), \mu(x_2), \ldots, \mu(x_i)) \wedge h_i(\nu(x_1), \nu(x_2), \ldots, \nu(x_i))).$$

6.2 *Properties*

Cardinalities $P(N)$ of a power set of a nature set $N = (X, q, \mu, \nu)$ is given as follows when f_i, g_i, h_i are increasing:

$$q(x_1) \le q(x_2) \le q(x_3) \le \ldots \le q(x_n)$$

$$\mu(x_1) \le \mu(x_2) \le \mu(x_3) \le \ldots \le \mu(x_n)$$

$$\nu(x_1) \le \nu(x_2) \le \nu(x_3) \le \ldots \le \nu(x_n).$$

And $f_i(x, y) = x \wedge y$, $g_i(x, y) = x \wedge y$, $h_i(x, y) = x \wedge y$

$$C(P(N)) = \frac{1}{|X|} \sum_{x \in N} q(x) \times (\mu(x) \wedge \nu(x)) + \frac{1}{|P(X) - X - \{\emptyset\}|} \sum_{x \in N} f_i(q(x_1), q(x_2), \ldots, q(x_i))$$

$$\times (g_i(\mu(x_1), \mu(x_2), \ldots, \mu(x_i)) \wedge h_i(\nu(x_1), \nu(x_2), \ldots, \nu(x_i)))$$

$$C(P(N)) = \frac{1}{|X|} \sum_{x \in N} q(x) \times (\mu(x) \wedge \nu(x)) + \frac{1}{\binom{|X|}{2}} [(q(x_1) \times (\mu(x_1) \wedge \nu(x_1))$$

$$+ (q(x_2) \times (\mu(x_1) \wedge \nu(x_1))) + \frac{1}{|P(X) - X - \{\emptyset\}|} \sum_{x \in N} f_i(q(x_1), q(x_2), \ldots, q(x_i))$$

$$\times (g_i(\mu(x_1), \mu(x_2), \ldots, \mu(x_i)) \wedge h_i(\nu(x_1), \nu(x_2), \ldots, \nu(x_i))).$$

Similar proofs can be done for the following cases:
- when f_i, g_i, h_i are decreasing functions;
- when f_i is increasing and g_i, h_i are decreasing;
- when f_i is decreasing and g_i, h_i are increasing.

7. Application in health management

In this section, we analyze the performance and quality of different departments in a hospital using the bunch set framework. The data for this analysis was collected from a hospital in India, providing a comprehensive overview of healthcare performance in different regions. This dataset includes information on patient numbers, treatment outcomes, and the quality of healthcare providers, offering valuable insights into the operational efficiency and service quality of these hospitals.

7.1 *Patient data analysis*

We consider the performance and quality of three departments: A, B, and C as shown in Table 7. The analysis includes the number of patients, treatment outcomes (μ_X), and the quality of healthcare providers (ν_X).

Table 7. Performance and quality of different departments in a hospital

Characteristic	Department A	Department B	Department C
Number of Patients	50	40	60
Min Treatment Outcome	-0.3	-0.5	-0.2
Mean Treatment Outcome	0.275	0.15	0.375
Max Treatment Outcome	0.7	0.6	0.8
Min Quality of Providers	0.6	0.7	0.8
Mean Quality of Providers	0.767	0.733	0.9
Max Quality of Providers	0.9	0.8	1.0

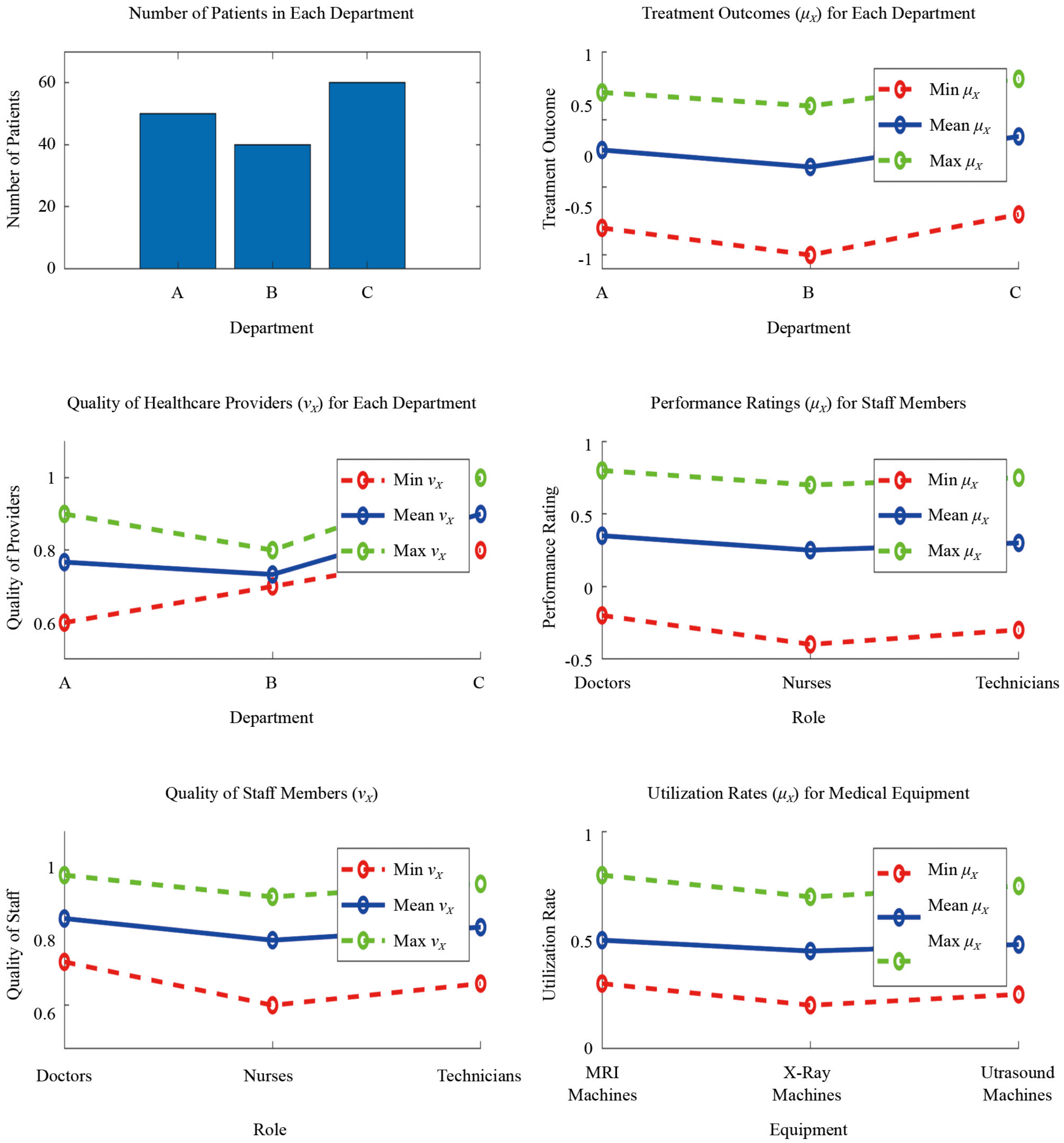


Figure 7. Application of bunch sets

Analysis

• **Number of Patients:** Department C handles the highest number of patients (60), followed by Department A (50) and Department B (40).

• **Treatment Outcomes:** Department C has the highest mean treatment outcome (0.375), indicating better treatment effectiveness.

• **Quality of Providers:** Department C also has the highest mean quality rating (0.9), suggesting superior healthcare providers (see Figure 7).

7.2 Staff performance evaluation

We evaluate the performance and quality of staff members in three roles: Doctors, Nurses, and Technicians (see Table 8).

Table 8. Example data for staff performance evaluation

Characteristic	Doctors	Nurses	Technicians
Number of Staff (q_X)	300	500	200
Min Performance Rating (μ_X)	-0.2	-0.4	-0.3
Mean Performance Rating (μ_X)	0.35	0.25	0.3
Max Performance Rating (μ_X)	0.8	0.7	0.75
Min Quality of Staff (ν_X)	0.7	0.6	0.65
Mean Quality of Staff (ν_X)	0.8	0.75	0.78
Max Quality of Staff (ν_X)	0.9	0.85	0.88

Analysis

• **Number of Staff:** Nurses form the largest group (500), followed by Doctors (300) and Technicians (200).

• **Performance Ratings:** Doctors have the highest mean performance rating (0.35), indicating better overall performance.

• **Quality of Staff:** Doctors also have the highest mean quality rating (0.8), suggesting superior staff quality.

7.3 Equipment utilization

We analyze the utilization and quality of different types of medical equipment: MRI Machines, X-Ray Machines, and Ultrasound Machines (see Table 9).

Table 9. Example data for equipment utilization

Characteristic	MRI Machines	X-Ray Machines	Ultrasound Machines
Number of Equipment (q_X)	200	400	400
Min Utilization Rate (μ_X)	0.3	0.2	0.25
Mean Utilization Rate (μ_X)	0.5	0.45	0.48
Max Utilization Rate (μ_X)	0.8	0.7	0.75
Min Quality of Equipment (ν_X)	0.7	0.6	0.65
Mean Quality of Equipment (ν_X)	0.8	0.75	0.78
Max Quality of Equipment (ν_X)	0.9	0.85	0.88

• **Number of Equipment:** X-Ray and Ultrasound Machines are the most numerous (400 each), followed by MRI Machines (200).

• **Utilization Rates:** MRI Machines have the highest mean utilization rate (0.5), indicating higher usage.

• **Quality of Equipment:** MRI Machines also have the highest mean quality rating (0.8), suggesting better equipment quality.

8. Conclusion

Traditionally, the cardinality of any infinite set was considered to be infinite, with all infinite sets assumed to be equivalent in size. This article challenges and refines this concept by introducing a framework that allows for the classification of infinite sets of real numbers in terms of cardinality. Additionally, we have introduced the concepts of nature sets, bunch sets, and proved several properties of nature sets. These innovations started the way for a new area of algebra that can be applied across various fields, including network science, decision-making, and other branches of algebra.

By proposing a finite approximation of the real number system, this study provides a practical approach to understanding and utilizing the cardinality of infinite sets. This new perspective not only enhances theoretical understanding but also offers practical applications for researchers focused on finitism. The exploration of these concepts opens up new avenues for applying set theory to real-world problems, making complex mathematical results more accessible and useful in practical scenarios.

Author credit statements

1. Sovan Samanta: Conceptualization, Original Draft Writing, Visualization.
2. Vivek Kumar Dubey: Conceptualization, Methodology, Supervision, Review and Editing.
3. Leo Mrsic: Data Curation, Formal Analysis, Funding Acquisition Project Administration.
4. Antonios Kalampakas: Investigation, Resource Provision, Validation.

Data availability statement

The data that support the findings of this study are available from the corresponding author, Sovan Samanta, upon request.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Müller-Stach S. *Richard Dedekind: What Are and What Should the Numbers Be? Continuity and Irrational Numbers*. Springer Nature; 2024.

[2] Pahune S, Venkateswaranaidu K, Mathur S. *Adaptive Intelligence: Evolutionary Computation for Nextgen AI*. Notion Press; 2025.

[3] Rioux JW. Cantor, finitism, and twentieth-century controversies. In: *Thomas Aquinas' Mathematical Realism*. Springer International Publishing; 2023. p.165-216.

[4] Ross DA. Dedekind cuts, moving markers, and the uncountability of $\mathbb{R}$. *Mathematical Magazine*. 2024; 97(3): 286-292.
[5] Liu X. Infinity. In: *Mathematics in Programming*. Springer Nature Singapore; 2024. p.225-272.
[6] Tsirelson B. Can each number be specified by a finite text? *WikiJournal of Science*. 2020; 3(1): 1-53.
[7] Chen T, Li F, Lin J. Fast division algorithm for finite real number. In: *2021 16th International Conference on Intelligent Systems and Knowledge Engineering (ISKE)*. IEEE; 2021. p.147-151.
[8] Masáková Z, Pelantová E, Studeničová K. Finiteness property in Cantor real numeration systems. *Proceedings of the Royal Academy of Exact, Physical and Natural Sciences Series A: Mathematics*. 2025; 119(2): 30.
[9] Aggarwal A, Holden H, Vaidya G. On the accuracy of the finite volume approximations to nonlocal conservation laws. *Numerical Mathematics*. 2024; 156(1): 237-271.
[10] Zadeh LA. Fuzzy sets. *Information and Control*. 1965; 8: 338-353.
[11] Al-Quran A, Jamil N, Tehrim ST, Riaz M. Cubic bipolar fuzzy VIKOR and ELECTRE-II algorithms for efficient freight transportation in Industry 4.0. *AIMS Mathematics*. 2023; 8(10): 24484-24514.
[12] Cruz RA. Neutrosophic soft cubic refined sets. *Neutrosophic Sets and Systems*. 2024; 73: 244.