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Research Article

A New Modified Extended Generalized Inverted Exponential (NMEGIEx) Distribution: A Distribution for Flexible and Accurate Data Analysis

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Abstract: This study proposes and investigates a New Modified Extended Generalised Inverse Exponential (NMEGIEx) distribution, a novel distribution. The basic one-parameter inverse exponential distribution is extended in the new model. Four additional positive shape parameters are added to an extended Topp-Leone exponentiated a generalised family of distributions to create the new model, which simultaneously controls the centre and tail weights. The model's asymptotic behaviour, explicit formulations for ordinary moments, mean, quantile function, hazard function, survival function, median, Moment Generating Function (MGF), and Probability Density Function (PDF) of lowest and highest order statistics were among the many statistical properties derived. Monte Carlo simulation is used to test the estimators of the proposed distribution. As predicted, as the sample size increases, the Root Mean Square Error (RMSE) and biases approach zero, and the estimated parameter values approach the true values of the parameters. Maximum Likelihood Estimation (MLE) is used to determine the values of the unknown parameters that make the observed data most likely under the assumed model. The superiority of the proposed NMEGIEx distribution is demonstrated through application to two real-world quality control engineering datasets, and it is clear that the proposed model fits the datasets better than competing distributions.

***Keywords*:** akaike information criteria, bayesian information criteria, log-likelihood, simulation, reliability function, flexibility

MSC: 60E05, 60K10

1. Introduction

In the field of statistics, data analysis is a fundamental and important subject. By establishing the model of variation, probability distributions help to classify the variability and uncertainty seen in the data. The creation of appropriate probability distributions that adequately describe a set of data collected through research, observation, testing, experimentation, etc. is the ultimate goal of statistical modelling. Accordingly, new generalized families of distributions

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have been introduced and many new methods have been provided to generate these new distributions, leading to a considerable development in probability distribution theory. Several attempts have also been made in the field of distribution theory and probability to develop and improve the flexibility and fit of existing distributions [1–5]. This improvement of the existing distributions came in handy as the existing or classical distributions do not fit the recent trend in real-life phenomena. This has prompted researchers in the field of distribution theory to extend, generalise and modify the classical distributions. The generalisation is done by adding extra shape parameter(s) to the existing distributions to improve their fit and flexibility and also to make them more robust [6–9].

The constant failure rate in the exponential distribution led Keller et al. [10] to propose an inverse exponential distribution with an inverse bathtub hazard rate, since it is almost impossible to have a constant failure rate in real life, as pointed out in [11]. Therefore, many researchers have extended the inverse exponential distribution to improve its fit and flexibility, such as [12–19].

Inverse distributions have practical applications in various fields, including life testing, especially in engineering, life sciences, environmental sciences, medical sciences, econometrics, etc.

Al-Fattah et al. [20] proposed the Inverse Kumaraswamy distribution (IKw) using the transformation $Y = \dfrac{1}{x-1}$. Later [21, 22], generalised the distribution using the transformation $T = X^{\gamma}$ to start the IKw family of distributions and then proposed the generalised inverted Kumaraswamy distribution.

This paper aims to introduce flexibility and a better fit to the inverse exponential distribution functions by introducing shape parameters to the base distribution and subsequently fitting it to the quality control data set to demonstrate its application and flexibility.

The adoption of the Generalised Inverse Exponential (GIE) distribution is driven by its ability to overcome the limitations of simpler models. Common distributions, such as exponential or gamma, often struggle to capture the complexities of real-world data [23]. This includes data with heavy tails (where extreme values are more common) or skewed data that doesn't follow a symmetrical pattern [24, 25]. GIE distributions address this by offering greater flexibility. They introduce additional parameters compared to simpler models, allowing them to adapt to a wider range of data shapes, including those with heavy tails or skewness.

The advantage of the proposed New Modified Extended Generalised Inverse Exponential (NMEGIEx) distribution over the existing same family of distributions is its ability to overcome the limitations of simpler models. This increased flexibility leads to improved data modelling and broader applicability in different research areas.

The remainder of this manuscript is presented as follows: the derivation of Probability Density Function (PDF), Cumulative Distribution Function (CDF) and expansion of densities are explored in section 2. In section 3, several statistical properties, moment, reliability function, hazard function, quantile function, median, order statistic, and maximum likelihood estimation were derived. A simulation study was carried out to determine the efficiency of Maximum Likelihood Estimators (MLEs) in section 4. Section 5 demonstrates the practical importance and flexibility of our proposed NMEGIEx by applying it to a real dataset, section 6 summarises the results and section 7 is the conclusion.

2. The proposed NMEGIEx distribution

The PDF and CDF are features of the classical inverse exponential distribution with a single scale parameter, given as:

$$H(x) = e^{-\left(\frac{\alpha}{x}\right)} \tag{1}$$

$$h(x) = \left(\frac{\alpha}{x^2}\right) e^{-\left(\frac{\alpha}{x}\right)},\ x > 0,\ \alpha > 0, \tag{2}$$

where α is the scale parameter.

A family of continuous distributions called the extended Topp-Leone exponentiated generalized distributions, introduced by Sule et al. [26], and based on the idea of the T-X family pioneered by Alzaatreh et al. [23], with CDF and PDF given as:

$$F(x) = \left[1 - \left[1 - [1 - [1 - H(x)^{\sigma}]^{\omega}]^{\rho}\right]^2\right]^{\psi} \tag{3}$$

$$f(x) = 2\sigma\omega\rho\psi h(x)\,[H(x)]^{\sigma-1}\,[1 - H(x)^{\sigma}]^{\omega-1}\,\left[1 - [1 - H(x)^{\sigma}]^{\omega}\right]^{\rho-1}$$

$$\left[1 - [1 - [1 - H(x)^{\sigma}]^{\omega}]^{\rho}\right]\left[1 - \left[1 - [1 - [1 - H(x)^{\sigma}]^{\omega}]^{\rho}\right]^2\right]^{\psi-1}. \tag{4}$$

The CDF and PDF of the new distribution are obtained by substituting equation (1) into (3) and equation (2) into (4). The resulting PDF and CDF are as follows:

$$F(x;\,\sigma,\,\psi,\,\alpha,\,\rho,\,\omega) = \left[1 - \left[1 - \left[1 - \left[1 - e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{\omega}\right]^{\rho}\right]^2\right]^{\psi} \tag{5}$$

$$f(x,\,\sigma,\,\psi,\,\alpha,\,\rho,\,\omega) = 2\sigma\omega\rho\psi\left(\frac{\alpha}{x^2}\right)e^{-\sigma\left(\frac{\alpha}{x}\right)}\left[1 - e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{\omega-1}$$

$$\left[1 - \left[1 - e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{\omega}\right]^{\rho-1}\left[1 - \left[1 - \left[1 - e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{\omega}\right]^{\rho}\right]$$

$$\left[1 - \left[1 - \left[1 - \left[1 - e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{\omega}\right]^{\rho}\right]^2\right]^{\psi-1}. \tag{6}$$

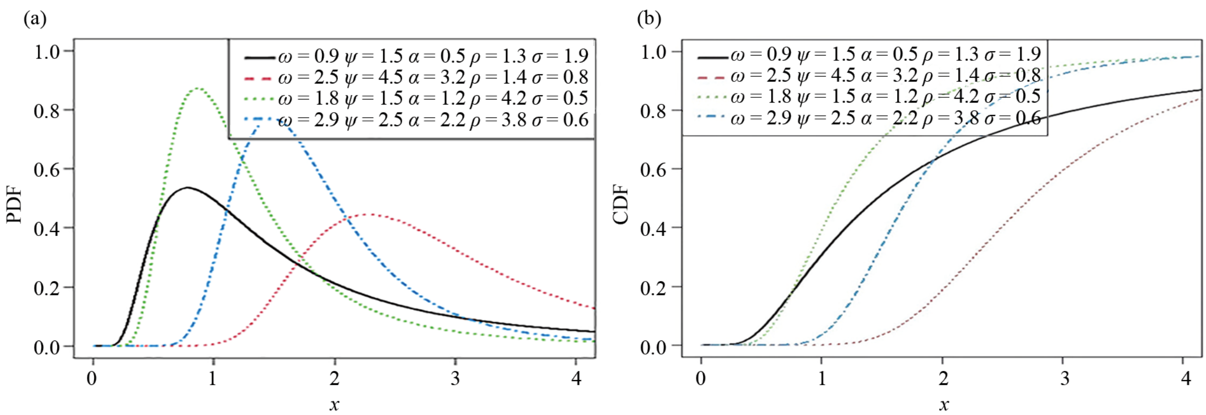

Figure 1. NMEGIEx distribution PDF and CDF plots at various parameter values

This new distribution is known as the New Modified Extended Generalized Inverse Exponential (NMEGIEx) distribution, which has been shown to be more flexible than other families of distributions; see [27, 28]. The PDF and CDF plot is show in Figure 1.

2.1 *Expansion of density*

Using the series expansion

$$(1-x)^{p-1}=\sum_{i=0}^{\infty}(-1)^i\binom{p-1}{i}[x]^{2i}. \tag{7}$$

Applying equation (7) to equations (5) and (6), the density expansions for CDF and PDF are as follows:

$$f(x;\ \sigma,\ \psi,\ \alpha,\ \rho,\ \omega)$$

$$=2\sigma\omega\rho\psi\sum_{i,\ j,\ k,\ m=0}^{\infty}(-1)^{i+j+k+m}\binom{\psi-1}{i}\binom{2i+1}{j}\binom{\rho(j+1)-1}{k}\binom{\omega(k+1)-1}{m}\left(\frac{\alpha}{x^2}\right)\left[e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{m+1}.$$

Resulting to

$$f(x;\ \sigma,\ \psi,\ \alpha,\ \rho,\ \omega)=\Omega\sum_{i,\ j,\ k,\ m=0}^{\infty}\left[e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{m+1}, \tag{8}$$

where

$$\Omega=2\sigma\omega\rho\psi(-1)^{i+j+k+m}\binom{\psi-1}{i}\binom{2i+1}{j}\binom{\rho(j+1)-1}{k}\binom{\omega(k+1)-1}{m}\left(\frac{\alpha}{x^2}\right)$$

$$F(x)=\sum_{w=0}^{h}\sum_{q,\ s,\ z=0}^{\infty}(-1)^{w+q+s+z}\binom{\psi h}{w}\binom{2w}{q}\binom{\rho q}{s}\binom{\omega s}{z}\left[e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{z}.$$

Therefore

$$F(x;\ \sigma,\ \psi,\ \alpha,\ \rho,\ \omega)=\Lambda\sum_{w=0}^{h}\left[e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{z}, \tag{9}$$

where

$$\Lambda=\sum_{q,\ s,\ z=0}^{\infty}(-1)^{w+q+s+z}\binom{\psi h}{w}\binom{2w}{q}\binom{\rho q}{s}\binom{\omega s}{z}.$$

Equation (8) is the expansion of the PDF, while equation (9) is the expansion of the CDF. These expansions are used to derive some of the properties of the new model.

3. Properties of the NMEGIEx distribution

3.1 *Moment*

A moment plays an important role in the analysis of several significant distribution characteristics, such as kurtosis, dispersion, central tendency and skewness [29]. The probability weighted moment of the parent distribution can be used to describe the r^{th} moment as a weighted sum, as follows:

$$E\left(X^r\right)=\int_0^{\infty} x^r f(x) dx. \tag{10}$$

The r^{th} moment of the NMEGIEx distribution is obtained by inserting equation (9) into equation (10), yielding the following expression:

$$E\left(X^r\right)=\Omega \sum_{i,\ j,\ k,\ m=0}^{\infty} \int_0^{\infty} x^r\left[e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{m+1} dx$$

$$\text{let } y=\sigma(m+1)\left(\frac{\alpha}{x}\right) \Longrightarrow x=\sigma(m+1)\left(\frac{\alpha}{y}\right) \Longrightarrow dx=\frac{dyx^2}{\alpha\sigma(m+1)} \tag{11}$$

$$\int_0^{\infty}\left[\sigma(m+1)\left(\frac{\alpha}{y}\right)\right]^r e^{-y} \frac{dyx^2}{\alpha\sigma(m+1)}=\int_0^{\infty} y^r e^{-y} dy=\Gamma(1-r)$$

$$E\left(X^r\right)=(m+1)^{r-1} \sigma^r \alpha^r \Omega \sum_{i,\ j,\ k,\ m=0}^{\infty} \Gamma(1-r),$$

where

$$\Omega=2\omega\rho\psi(-1)^{i+j+k+m}\binom{\psi-1}{i}\binom{2i+1}{j}\binom{\rho(j+1)-1}{k}\binom{\omega(k+1)-1}{m}.$$

3.2 *MGF*

At time t, the Moment Generating Function (MGF) of a random variable X is given as:

$$M_{(x)}(t)=\int_0^{\infty} e^{tx} f(x) dx. \tag{12}$$

By substituting equation (9) into equation (12), the MGF of the NMEGIEx distribution is obtained as: since the series expansion for e^{tx} is given as

$$e^{tx} = \sum_{w=0}^{\infty} \frac{(tx)^w}{w!}. \tag{13}$$

Then,

$$M_x(t) = \sum_{i,\ j,\ k,\ m=0}^{\infty} \sum_{w=0}^{\infty} \frac{t^w (m+1)^{w-1} \sigma^w \alpha^w (-1)^i \Omega \Gamma(1-w)}{w!}. \tag{14}$$

3.3 *Reliability function*

The measure of reliability of industrial components is very important, especially in engineering [30]. The reliability of a product is the probability that it will perform its intended function up to a given time when used in its normal condition. Therefore, the reliability function of NMEGIEx is given as:

$$\begin{aligned} R(x;\ \sigma,\ \psi,\ \alpha,\ \rho,\ \omega) &= 1 - F(x;\ \sigma,\ \psi,\ \alpha,\ \rho,\ \omega) \\ R(x;\ \sigma,\ \psi,\ \alpha,\ \rho,\ \omega) &= 1 - \left[1 - \left[1 - \left[1 - \left[1 - e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{\omega}\right]^{\rho}\right]^{2}\right]^{\psi}. \end{aligned} \tag{15}$$

3.4 *Hazard function*

The following is the NMEGIEx hazard function:

$$T(x;\ \sigma,\ \psi,\ \alpha,\ \rho,\ \omega) = \frac{f(x;\ \sigma,\ \psi,\ \alpha,\ \rho,\ \omega)}{R(x;\ \sigma,\ \psi,\ \alpha,\ \rho,\ \omega)}$$

$$s(x,\ \sigma,\ \psi,\ \alpha,\ \rho,\ \omega) = \frac{\begin{array}{c} 2\sigma\omega\rho\psi\left(\frac{\alpha}{x^2}\right) e^{-\sigma\left(\frac{\alpha}{x}\right)} \left[1 - e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{\omega-1} \left[1 - \left[1 - e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{\omega}\right]^{\rho-1} \\ \left[1 - \left[1 - \left[1 - e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{\omega}\right]^{\rho}\right] \left[1 - \left[1 - \left[1 - \left[1 - e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{\omega}\right]^{\rho}\right]^{2}\right]^{\psi-1} \end{array}}{1 - \left[1 - \left[1 - \left[1 - \left[1 - e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{\omega}\right]^{\rho}\right]^{2}\right]^{\psi}}. \tag{16}$$

Figure 2 displays the survival and hazard functions for the NMEGIEx distribution with different parameter values.

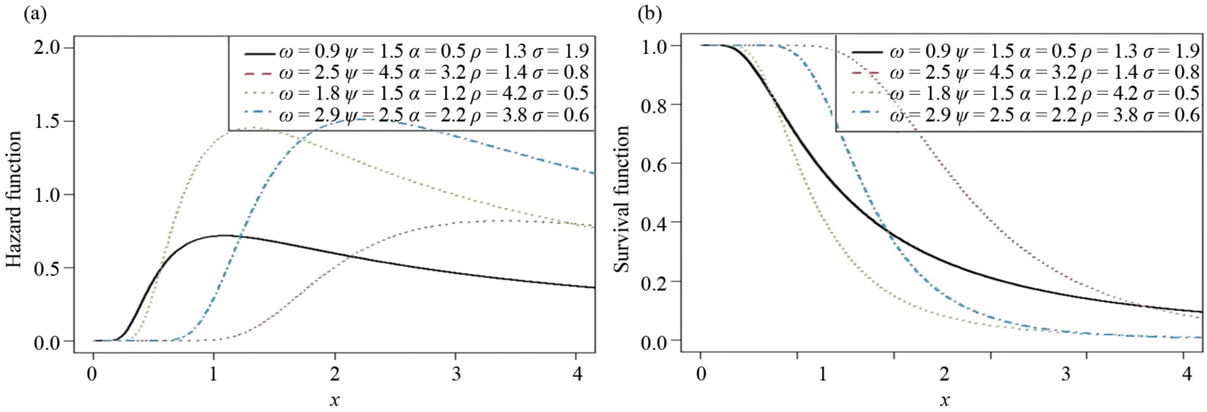

Figure 2. Plots of the NMEGIEx distribution's survival and hazard functions with varying parameter values

3.5 *Quantile function*

The NMEGIEx's quantile function is as follows:

$$x = Q(u) = \frac{\sigma\alpha}{\left[-\log\left(1-\left[1-\left[1-\left[1-U^{\frac{1}{\psi}}\right]^{\frac{1}{2}}\right]^{\frac{1}{\rho}}\right]^{\frac{1}{\omega}}\right]\right]}. \tag{17}$$

3.6 *Median*

The median is obtained by setting $u = 0.5$, in equation (17) as

$$\text{Median} = Q(0.5) = \frac{\sigma\alpha}{\left[-\log\left(1-\left[1-\left[1-\left[1-0.5^{\frac{1}{\psi}}\right]^{\frac{1}{2}}\right]^{\frac{1}{\rho}}\right]^{\frac{1}{\omega}}\right]\right]}. \tag{18}$$

3.7 *Order statistics*

The r^{th} order statistics' PDF is provided as:

$$f_{r:n}(x) = \frac{f(x)}{B(r,\ n-r+1)} \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} F(x)^{v+r-1}. \tag{19}$$

Using equation (19), the PDF of the r^{th} order statistic for the NMEGIEx model is gotten by replacing h with $v+r-1$ in equation (9) as

$$f_{r:n}(x) = \frac{\Omega\Lambda\left[e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{m+1}}{B(r,\ n-r+1)} \sum_{v=0}^{n-r} \sum_{i,\ j,\ k,\ m=0}^{\infty} \sum_{w=0}^{v+r-1} \binom{n-r}{v} \left[\left[e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{z}\right]^{v+r-1}. \tag{20}$$

The NMEGIEx distribution's r^{th} order statistic is as presented in equation (20). Setting $r = 1$ in equation (20) results in the PDF of the NMEGIEx distribution's minimal order statistic.

$$f_{1:n}(x) = n\Omega\Lambda \sum_{v=0}^{n-1} \sum_{i,\ j,\ k,\ m=0}^{\infty} \sum_{w=0}^{v} \binom{n-1}{v} \left[e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{zv+m+1}, \tag{21}$$

where

$$\Omega = 2\sigma\omega\rho\psi(-1)^{i+j+k+m} \binom{\psi-1}{i} \binom{2i+1}{j} \binom{\rho(j+1)-1}{k} \binom{\omega(k+1)-1}{m} \left(\frac{\alpha}{x}\right),$$

and

$$\Lambda = \sum_{q,\ s,\ z=0}^{\infty} (-1)^{w+q+s+z} \binom{\psi v}{w} \binom{2w}{q} \binom{\rho q}{s} \binom{\omega s}{z}.$$

Set r to be equal to n in equation (20) result in the PDF of the distribution's maximum order statistic as:

$$f_{r:n}(x) = n\Omega\Lambda \sum_{i,\ j,\ k,\ m=0}^{\infty} \sum_{w=0}^{v+n-1} \left[e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{z(v+n-1)+m+1}, \tag{22}$$

where

$$\Lambda = \sum_{q,\ s,\ z=0}^{\infty} (-1)^{w+q+s+z} \binom{\psi(v+n-1)}{w} \binom{2w}{q} \binom{\rho q}{s} \binom{\omega s}{z}.$$

3.8 *Maximum likelihood estimation method*

Suppose $x_1,\ x_2,\ \ldots\ x_n$ represent a random sample of size n from the NMEGIEx $(\alpha,\ \psi,\ \rho,\ \omega,\ \sigma)$ distribution. The sample log-likelihood function of the NMEGIEx $(\alpha,\ \psi,\ \rho,\ \omega,\ \sigma)$ distribution is obtained as:

$$\begin{aligned}
\log L = {} & n\log(2) + n\log(\sigma) + n\log(\omega) + n\log(\rho) + n\log(\psi) + n\log(\alpha) + \sum_{i=1}^{n} \log\left(\frac{1}{x_i}\right) \\
& - \sigma\sum_{i=1}^{n}\left(\frac{\alpha}{x_i}\right) + (\omega-1)\sum_{i=1}^{n}\log\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right] + (\rho-1)\sum_{i=1}^{n}\log\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right] \\
& + \sum_{i=1}^{n}\log\left[1-\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]^{\rho}\right] + (\psi-1)\sum_{i=1}^{n}\log\left[1-\left[1-\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]^{\rho}\right]^{2}\right].
\end{aligned} \tag{23}$$

To obtain the estimate of each parameter, equation (23) is differentiated with respect to each parameter and equate them zero.

$$\begin{aligned}\frac{\partial L}{\partial \omega} =& \frac{n}{\omega} + \sum_{i=1}^{n} \log\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right] - (\rho-1)\sum_{i=1}^{n} \frac{\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\log\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]}{\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]} \\ &+ \sum_{i=1}^{n} \frac{\rho\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]^{\rho-1}\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\log\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]}{\left[1-\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]^{\rho}\right]} \\ &- (\psi-1)\sum_{i=1}^{n} \frac{2\left[1-\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]^{\rho}\right]\rho\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]^{\rho-1}\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\log\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]}{\left[1-\left[1-\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]^{\rho}\right]^{2}\right]}.\end{aligned} \tag{24}$$

$$\begin{aligned}\frac{\partial L}{\partial \rho} =& \frac{n}{\rho} + \sum_{i=1}^{n} \log\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right] - \sum_{i=1}^{n} \frac{\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]^{\rho}\log\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]}{\left[1-\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]^{\rho}\right]} \\ &+ (\psi-1)\sum_{i=1}^{n} \frac{2\left[1-\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]^{\rho}\right]\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{\omega}\right]^{\rho}\log\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]}{\left[1-\left[1-\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]^{\rho}\right]^{2}\right]}.\end{aligned} \tag{25}$$

$$\frac{\partial L}{\partial \psi} = \frac{n}{\psi} + \sum_{i=1}^{n} \log\left[1-\left[1-\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]^{\rho}\right]^{2}\right]. \tag{26}$$

$$\frac{\partial L}{\partial \sigma}=\frac{n}{\sigma}-\sum_{i=1}^{n}\left(\frac{\alpha}{x_i}\right)+(\omega-1)\sum_{i=1}^{n}\frac{e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\sigma\left(\frac{\alpha}{x_i}\right)\log\left(\frac{\alpha}{x_i}\right)}{\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]}$$

$$+(\rho-1)\sum_{i=1}^{n}\frac{\omega\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega-1}e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\sigma\left(\frac{\alpha}{x_i}\right)\log\left(\frac{\alpha}{x_i}\right)}{\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]}$$

$$+\sum_{i=1}^{n}\frac{\rho\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]^{\rho-1}\omega\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega-1}e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\sigma\left(\frac{\alpha}{x_i}\right)\log\left(\frac{\alpha}{x_i}\right)}{\left[1-\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]^{\rho}\right]} \tag{27}$$

$$-(\psi-1)\sum_{i=1}^{n}\frac{\left(\begin{array}{c}2\left[1-\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]^{\rho}\right]\rho\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]^{\rho-1}\\ \omega\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega-1}e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\sigma\left(\frac{\alpha}{x_i}\right)\log\left(\frac{\alpha}{x_i}\right)\end{array}\right)}{\left[1-\left[1-\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]^{\rho}\right]^{2}\right]}.$$

$$\frac{\partial L}{\partial \alpha}=\frac{n}{\alpha}-\sigma\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)+(\omega-1)\sum_{i=1}^{n}\frac{e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\sigma\left(\frac{\alpha}{x_i}\right)\left(\frac{1}{x_i}\right)}{\left[1-e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]}$$

$$+(\rho-1)\sum_{i=1}^{n}\frac{\omega\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega-1}e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\sigma\left(\frac{\alpha}{x_i}\right)\left(\frac{1}{x_i}\right)}{\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]}$$

$$+\sum_{i=1}^{n}\frac{\rho\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]^{\rho-1}\omega\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega-1}e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\sigma\left(\frac{\alpha}{x_i}\right)\left(\frac{1}{x_i}\right)}{\left[1-\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x_i}\right)}\right]^{\omega}\right]^{\rho}\right]}$$

$$-(\psi-1)\sum_{i=1}^{n}\frac{\left(\begin{array}{l}2\left[1-\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{\omega}\right]^{\rho}\right]\rho\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{\omega}\right]^{\rho-1}\\ \omega\left[1-e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{\omega-1}e^{-\sigma\left(\frac{\alpha}{x}\right)}\sigma\left(\frac{\alpha}{x_i}\right)\left(\frac{1}{x_i}\right)\end{array}\right)}{\left[1-\left[1-\left[1-\left[1-e^{-\sigma\left(\frac{\alpha}{x}\right)}\right]^{\omega}\right]^{\rho}\right]^{2}\right]}. \tag{28}$$

These equations (24)-(28) are non-linear and cannot be solved analytically. Therefore, statistical software such as *R* with iterative numerical techniques is required to obtain the value of the unknown parameters.

4. Simulation study

To assess the performance of Maximum Likelihood Estimators (MLEs), a simulation-based investigation was undertaken. This study involved the generation of 1,000 synthetic datasets from the NMEGIEx distribution, employing the quantile function specified in Equation (17). By taking different samples and adjusting for different parameter values, the bias and Root Mean Square Error (RMSE) are used to examine the precision of the MLEs. The first example uses (0.5, 1, 0.5, 0.8 and 0.7) for the initial values of the model parameters ($\alpha,\ \rho,\ \sigma,\ \psi$ and ω), while the second example uses (2.5, 0.5, 3.1, 3 and 4) for the initial values of the parameters.

We generated $N = 1{,}000$ samples of sizes $n = 20$, 50, 100, 250, 500 and 1,000 from NMEGIEx distribution with its Quantile Function (QF). Then we computed the empirical means, biases and Root Mean Squared Errors (RMSE) of the MLE with

$$\text{Bias}_{\hat{\eta}} = \frac{1}{N}\sum_{i=1}^{N}(\hat{\eta}_i - \eta_i),$$

and

$$\text{RMSE}_{\hat{\eta}} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\eta}_i - \eta_i)^2},$$

where $\eta = (\alpha,\ \rho,\ \sigma,\ \psi,\ \omega)$.

To examine the performance of the MLEs for the NMEGIEx distribution, we perform a simulation study as follows:

1. Generate N samples of size n from the NMEGIEx distribution with its Quantile Function (QF).
2. Compute the MLEs for the N samples, say $\eta = (\alpha,\ \rho,\ \sigma,\ \psi,\ \omega)$, for $i = 1,\ 2,\ \ldots,\ N$.
3. Compute the MLEs for N samples.
4. Compute the biases and RMSE.

We repeat these steps for $N = 1{,}000$ and $n = 20$, 50, 100, 250, 500 and 1,000 with different values for $\eta = (\alpha,\ \rho,\ \sigma,\ \psi,\ \omega)$.

Table 1 shows the results of the simulation.

Table 1. Performance of MLEs: Bias and RMSE analysis for different parameter values

N	Parameters	(0.5, 1.0, 5, 0.8, 0.7)			(2.5, 0.5, 3.1, 3.4)		
		Estimated values	Bias	RMSE	Estimated values	Bias	RMSE
20	σ	0.8453	0.3453	0.4307	3.2395	0.7395	1.1641
	ρ	0.9432	-0.0568	0.2711	0.6007	0.1007	0.2780
	α	0.5283	0.0283	0.1723	3.1143	0.0143	0.5125
	ψ	0.8453	0.0453	0.2614	3.2395	0.2395	0.9303
	ω	0.8640	0.1640	0.3762	4.3484	0.3484	0.9180
50	σ	0.8263	0.3263	0.3776	3.1218	0.6218	0.8903
	ρ	0.9843	-0.0157	0.1952	0.5323	0.0323	0.1637
	α	0.5005	0.0005	0.1128	3.1514	0.0514	0.3559
	ψ	0.8263	0.0263	0.1918	3.1218	0.1218	0.6487
	ω	0.7629	0.0629	0.1924	4.2099	0.2099	0.6813
100	σ	0.5252	0.0252	0.3572	3.1357	0.6357	0.7915
	ρ	1.0031	0.0031	0.1504	0.5028	0.0028	0.1024
	α	0.5003	0.0003	0.0857	3.1686	0.0686	0.2432
	ψ	0.8252	0.0252	0.1499	3.1357	0.1357	0.4907
	ω	0.7279	0.0279	0.1159	4.1235	0.1235	0.5071
250	σ	0.5197	0.0197	0.3352	3.0736	0.5736	0.6486
	ρ	1.0189	0.0189	0.1045	0.5015	0.0015	0.0611
	α	0.5001	0.0001	0.0547	3.1618	0.0618	0.1484
	ψ	0.8197	0.0197	0.1027	3.0736	0.0736	0.3116
	ω	0.7083	0.0083	0.0658	4.0889	0.0889	0.3522
500	σ	0.5157	0.0157	0.3218	3.0652	0.5652	0.6015
	ρ	1.0188	0.0188	0.0681	0.5003	0.0003	0.0433
	α	0.5000	0.0000	0.0381	3.1484	0.0484	0.1138
	ψ	0.8157	0.0157	0.0646	3.0652	0.0652	0.2159
	ω	0.7032	0.0032	0.0445	4.0567	0.0567	0.2443
1,000	σ	0.5155	0.0155	0.3183	3.0564	0.5564	0.5680
	ρ	1.0136	0.0136	0.0424	0.5000	0.0000	0.0325
	α	0.5000	0.0000	0.0276	3.1456	0.0456	0.0790
	ψ	0.8155	0.0155	0.0448	3.0564	0.0564	0.1272
	ω	0.7016	0.0016	0.0309	4.0524	0.0524	0.1490

Table 1 clearly shows a trend: the bias of the estimates decreases with increasing sample size, resulting in a significant decrease in the Mean Squared Error (MSE). This suggests that the estimates are getting closer to the true parameter values as the sample size increases. It can be concluded that the NMEGIEx distribution has good parameter stability.

5. Applications

This section explores the practical application of the NMEGIEx distribution by evaluating its flexibility, robustness and goodness of fit in modelling real (quality control) data sets. We compare the performance of the NMEGIEx distribution with established models (Exponentiated Generalized Frechet (EGF), Frechet Weibull (FW), Exponentiated Generalised Inverse Exponential (EGIEx) and Inverse Exponential (IEx)) that have a similar basis and potentially provide good fits to the selected data sets. Model selection is done using the Bayesian Information Criterion (BIC) and the Akaike Information Criterion (AIC). The distribution that best fits the data set is the one with the lowest AIC and BIC.

The first results were glass fibre strengths measured at 1.5 cm collected by staff at the UK National Physical Laboratory. This information was used by Smith and Naylor [31]. The dataset is shown below: 0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.24.

Nichols and Padgett [32] provide a dataset of 100 observations on the breaking stress of carbon fibers (Gba), presented as follows: 3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43,2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

The third data set represents the lifetime data relating to relief times (in minutes) of patients receiving an analgesic as reported by Gross and Clark [33]. The data set consists of twenty (20) observations and it is as follows: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.

The first step is to determine whether the data fit the newly developed NMEGIEx distribution. The Quantile-Qantile (Q-Q), Probability-Probability (P-P), as well as theoretical and empirical density function plots were used to determine whether the data fit the theoretical distribution (see Figures 3, 4 and 5). The NMEGIEx distributionwas found to be well fitted by the acceptable results of the goodness-of-fittests from Figures 3 and 4.

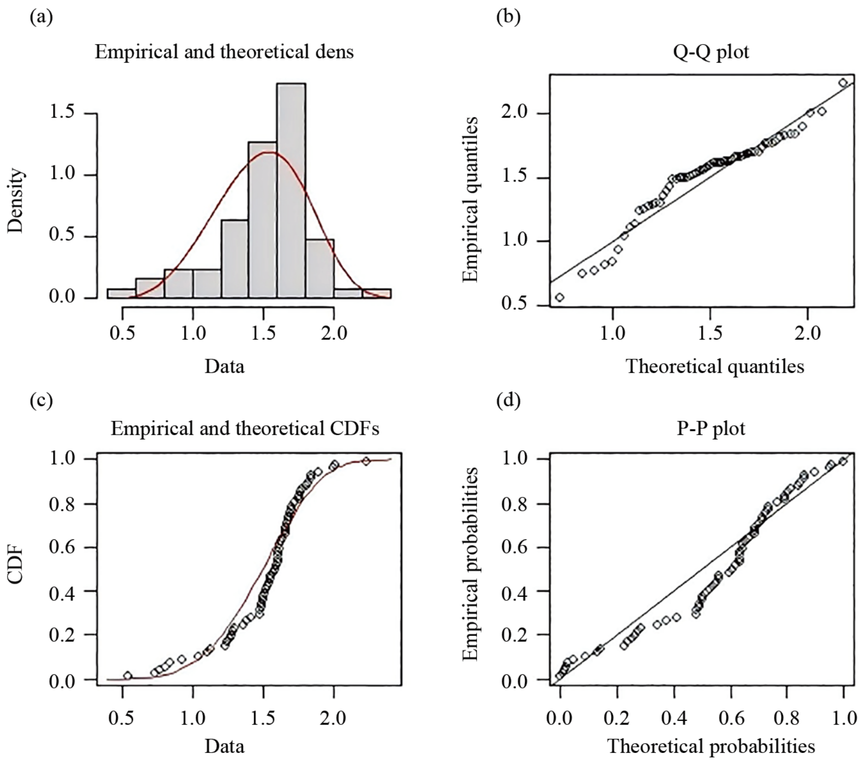

Figure 3. Fitted distribution plots for data set 1 (PDF, CDF, QQ, PP)

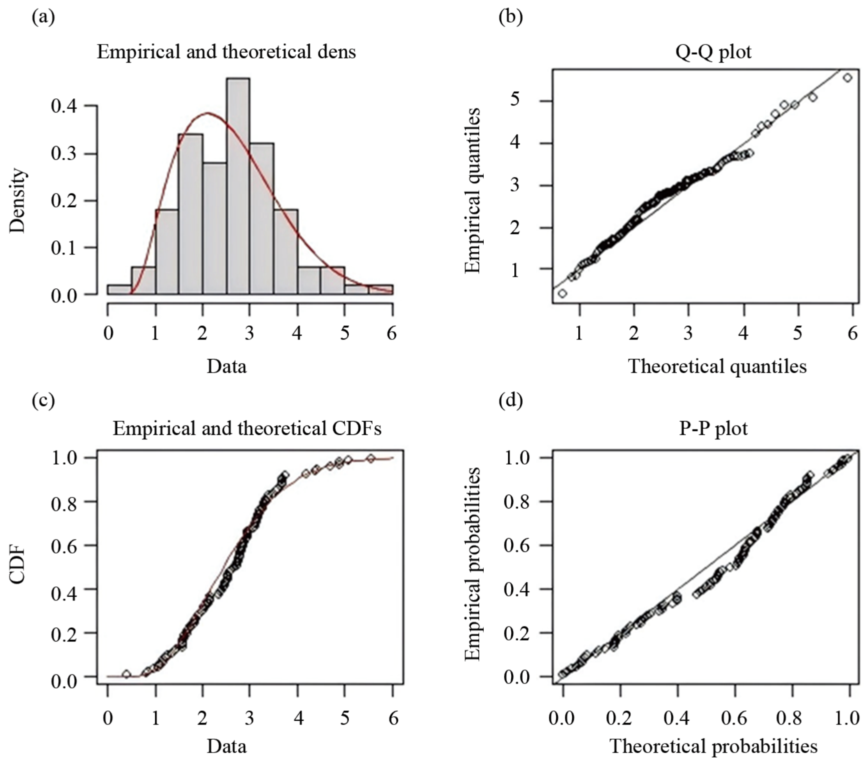

Figure 4. Fitted distribution plots for data set 2 (PDF, CDF, QQ, PP)

Table 2. MLEs and model performance on data set 1

Models	$\hat{\omega}$	$\hat{\psi}$	$\hat{\alpha}$	$\hat{\rho}$	$\hat{\sigma}$	ll	AIC	BIC
NMEGIEx	42.3725	0.2259	0.9329	4.7214	6.7455	-21.2421	52.4843	63.1999
EGF	4.0501	0.2584	1.2874	13.4939	1.8831	-31.5783	73.1566	83.8723
FW	6.5081	0.4436	2.2217	0.2091	-	-46.8533	101.7066	110.2792
EGIEx	7.2402	5.4895	-	0.1727	55.6955	-38.1951	84.3903	92.9617
IEx	-	-	-	1.4034	-	-89.4392	180.8784	183.0215

Table 2 shows that the NMEGIEx distribution provides the best fit compared to its competing models (EGIEx, EGF, FW, and IEx). This conclusion is based on its achieving the lowest values for several goodness-of-fit tests, including Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC), with the highest value for log-likelihood. Visual representations of the fitted densities and the histogram are shown in Figure 3.

Table 3. Maximum Likelihood Estimates (MLEs) and model performance on data set 2

Models	$\hat{\omega}$	$\hat{\psi}$	$\hat{\alpha}$	$\hat{\rho}$	$\hat{\sigma}$	ll	AIC	BIC
NMEGIEx	22.4477	0.3085	3.8316	1.0646	3.6980	-144.3791	298.7582	311.7614
EGF	2.8278	1.8678	0.8202	2.0946	2.2032	-155.7254	321.4507	334.4766
FW	6.6887	0.2646	2.0135	0.1343	-	-173.1440	354.2879	362.8605
EGIEx	0.7929	4.2856	-	0.6608	8.7159	-151.6408	311.2816	321.2023
IEx	-	-	-	2.1399	-	-199.3956	400.7912	403.3964

When NMEGIEx is compared to other competing distributions (EGIEx, EGF, FW, EGIEx and IEx), the results in Table 3 also show that the proposed NMEGIEx distribution provides the best fit to the quality control data sets.

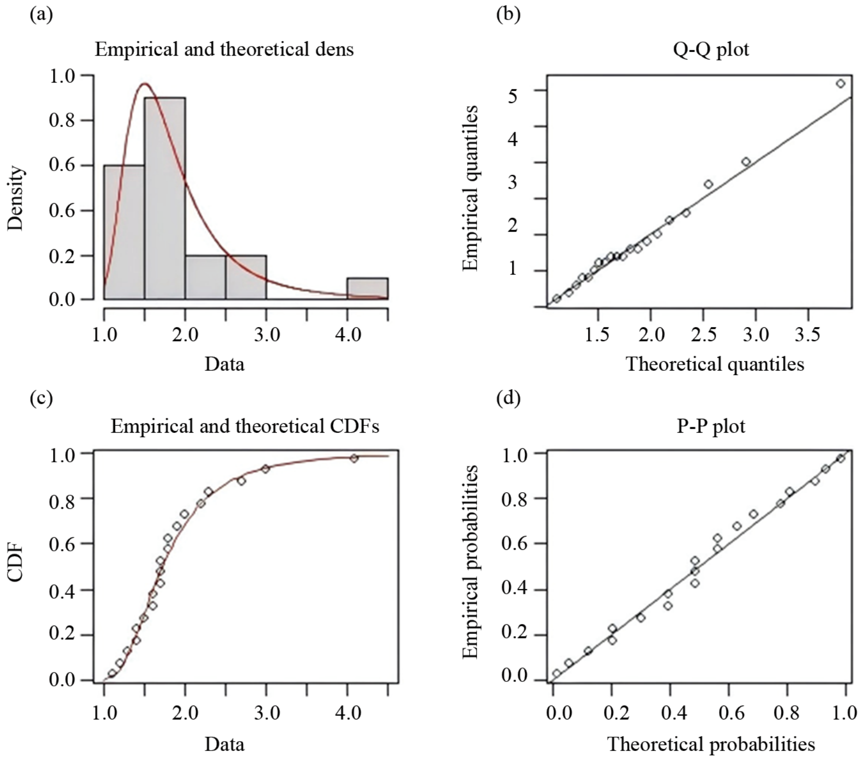

Figure 5. Fitted distribution plots for data set 3 (PDF, CDF, QQ, PP)

Table 4. Maximum Likelihood Estimates (MLEs) and model performance on data set 3

Models	$\hat{\omega}$	$\hat{\psi}$	$\hat{\alpha}$	$\hat{\rho}$	$\hat{\sigma}$	ll	AIC	BIC
NMEGIEx	0.7352	6.6184	2.8322	0.4781	2.7186	-14.2113	38.4223	43.4013
EGF	0.4612	0.5523	5.1898	3.6415	2.4749	-15.0133	40.0226	44.3083
FW	0.0497	80.8623	9.0964	1.5212	-	-16.6408	41.2816	45.2645
EGIEx	2.1509	90.0851	-	0.0688	0.09029	-18.9832	45.9664	49.9493
IEx	-	-	-	1.7287	-	-32.6687	67.3374	68.3331

Table 4 shows that the NMEGIEx distribution provides the best fit compared to its competing models (EGIEx, EGF, FW, and IEx). This conclusion is based on it achieving the lowest values for several goodness-of-fit tests, including Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC), with the highest value for log-likelihood. Visual representations of the fitted densities and the histogram are shown in Figure 5.

6. Discussions

In addition to the theoretical aspects, we also investigated the practical application of the NMEGIEx distribution by evaluating its flexibility, stability and goodness of fit in real-life reliability assurance situations. We evaluated its performance against four existing models that can fit the data well: the exponentiated generalized Frechet, Frechet Weibull, Inverse Exponential (IEx) and the Exponentiated Generalised Inverse Exponential (EGIEx). The model with the lowest values for these metrics was considered the better fit. The selection criteria for the best fitting model were based on the values of the log likelihood, the Bayesian Information Criterion (BIC) and the Akaike Information Criterion (AIC).

We used several metrics, such as Quantile-Quantile (Q-Q) and Probability-Probability (P-P) plots, as well as theoretical and empirical density function plots (Figures 3, 4 and 5), to assess the applicability and flexibility of the NMEGIEx distribution. The values of the three goodness-of-fit tests (ll, AIC and BIC) indicate that the NMEGIEx distribution is a good fit for this specific dataset, which is supported by the satisfactory results of these tests. This confirms the work of Ibrahim et al. [18], whose previously established Exponentiated Generalised Inverse Exponential distribution failed to model negatively skewed data sets. Instead, the NMEGIEx model can be successfully used to model life data sets and real-life phenomena that are positively skewed, negatively skewed, with bathtub hazard functions or failure rates.

7. Conclusions

This study successfully introduces a new family of distributions, called NMEGIEx, to incorporate real-world applications, thus increasing the flexibility of the inverse exponential distribution. In this study, several probability densities and distribution functions were derived and various statistical properties of the new distribution were investigated. A simulation study was conducted to determine the parameter stability of the Maximum Likelihood Estimation (MLE) approach to evaluate its performance. Data sets with bathtub hazard functions, failure rates and positive or negative skewness can all be modelled using the NMEGIEx distribution. When the novel distribution is applied to the three real-world data sets, Goodness of Fit (GoF) measurements show that it outperforms competing distributions.

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Conflict of interest

This manuscript is the original work of the authors and there are no competing interests. No external funding was received for writing assistance.

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Appendix

R-Codes

```
##pdf plots
x=seq(0,15,0.0008)
omega=0.9
psi=1.5
alpha=0.5
rho=1.3
sigma=1.9
f=2*sigma*omega*psi*alpha*x**(-2)*rho*exp(-sigma*(alpha/x))*(1-exp(-sigma*(alpha/x)))**(omega-1)*(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho-1)*(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))*(1-(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))**(2))**(psi-1)
lower.X=0; upper.X=4; lower.Y=0; upper.Y=1
windows()
par(new=TRUE)
plot(x,f,lty=1,type="l", lwd = 2.5,xlim =c(lower.X, upper.X), ylim = c(lower.Y, upper.Y),xlab="x",ylab="pdf", col="1")
x=seq(0,15,0.0008)
omega=2.5
psi=4.5
alpha=3.2
rho=1.4
sigma=0.8
f=2*sigma*omega*psi*alpha*x**(-2)*rho*exp(-sigma*(alpha/x))*(1-exp(-sigma*(alpha/x)))**(omega-1)*(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho-1)*(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))*(1-(1-(1-(1-exp(-sigma*(alpha/ x)))**(omega))**(rho))**(2))**(psi-1)
lines(x,f,lty=3,lwd = 2.5, type="l",col="2")
x=seq(0,15,0.0008)
omega=1.8
psi=1.5
alpha=1.2
rho=4.2
sigma=0.5
f=2*sigma*omega*psi*alpha*x**(-2)*rho*exp(-sigma*(alpha/x))*(1-exp(-sigma*(alpha/x)))**(omega-1)*(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho-1)*(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))*(1-(1-(1-(1-exp (-sigma*(alpha/x)))**(omega))**(rho))**(2))**(psi-1)
lines(x,f,lty=3,lwd = 2.5, type="l",col="3")
x=seq(0,15,0.0008)
omega=2.9
psi=2.5
alpha=2.2
rho=3.8
sigma=0.6
f=2*sigma*omega*psi*alpha*x**(-2)*rho*exp(-sigma*(alpha/x))*(1-exp(-sigma*(alpha/x)))**(omega-1)*(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho-1)*(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))*(1-(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))**(2))**(psi-1)
lines(x,f,lty=4,lwd = 2.5, type="l",col="4")
```

```
legend(("topright"),legend=c(expression (omega == 0.9 ~~ psi == 1.5 ~~ alpha == 0.5 ~~ rho == 1.3 ~~ sigma == 1.9), expression (omega == 2.5 ~~ psi == 4.5 ~~ alpha == 3.2 ~~ rho == 1.4 ~~ sigma == 0.8),expression (omega == 1.8 ~~ psi == 1.5 ~~ alpha == 1.2 ~~ rho == 4.2 ~~ sigma == 0.5),expression (omega == 2.9 ~~ psi == 2.5 ~~ alpha == 2.2 ~~ rho == 3.8 ~~ sigma == 0.6)), cex = 1,lty=c(1,2,3,4),col=("1":"2":"3":"4"), lwd = 2)
#HF
x=seq(0,15,0.0008)
omega=0.9
psi=1.5
alpha=0.5
rho=1.3
sigma=1.9
f=(2*sigma*omega*psi*alpha*x**(-2)*rho*exp(-sigma*(alpha/x))*(1-exp(-sigma*(alpha/x)))**(omega-1)*(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho-1)*(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))*(1-(1-(1-(1-exp(-sigma (alpha/x)))**(omega))**(rho))**(2))**(psi-1))/(1-(1-(1-(1-(1-exp(-sigma*(alpha/x)))** (omega))**(rho))**(2))**(psi))
lower.X=0; upper.X=4; lower.Y=0; upper.Y=2
windows()
par(new=TRUE)
plot(x,f,lty=1,type="l", lwd = 2.5,xlim =c(lower.X, upper.X), ylim = c(lower.Y, upper.Y),xlab="x",ylab="Hazard function",col="1")
x=seq(0,15,0.0008)
omega=2.5
psi=4.5
alpha=3.2
rho=1.4
sigma=0.8
f=(2*sigma*omega*psi*alpha*x**(-2)*rho*exp(-sigma*(alpha/x))*(1-exp(-sigma*(alpha/x)))**(omega-1)*(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho-1)*(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))*(1-(1-(1-(1-exp(-sigma (alpha/ x)))**(omega))**(rho))**(2))**(psi-1))/(1-(1-(1-(1-(1-exp(-sigma*(alpha/x)))** (omega))**(rho))**(2))**(psi))
lines(x,f,lty=3,lwd = 2.5, type="l",col="2")
x=seq(0,15,0.0008)
omega=1.8
psi=1.5
alpha=1.2
rho=4.2
sigma=0.5
f=(2*sigma*omega*psi*alpha*x**(-2)*rho*exp(-sigma*(alpha/x))*(1-exp(-sigma*(alpha/x)))**(omega-1)* (1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho-1)*(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))*(1-(1-(1-(1-exp(-sigma (alpha/ x)))**(omega))**(rho))**(2))**(psi-1))/(1-(1-(1-(1-(1-exp(-sigma* (alpha/x)))**(omega))**(rho))**(2))**(psi))
lines(x,f,lty=3,lwd = 2.5, type="l",col="3")
x=seq(0,15,0.0008)
omega=2.9
psi=2.5
alpha=2.2
rho=3.8
sigma=0.6
f=(2*sigma*omega*psi*alpha*x**(-2)*rho*exp(-sigma*(alpha/x))*(1-exp(-sigma*(alpha/x)))**(omega-1)*(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho-1)*(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))*(1-(1-(1-(1-exp(-
```

```
sigma*(alpha/x)))**(omega))**(rho))**(2))**(psi-1))/(1-(1-(1-(1-(1-exp(-sigma*(alpha/ x)))**(omega))** (rho))**(2))
*(psi))
lines(x,f,lty=4,lwd = 2.5, type="l",col="4")
legend(("topright"),legend=c(expression (omega == 0.9 ~~ psi == 1.5 ~~ alpha == 0.5 ~~ rho == 1.3 ~~ sigma
== 1.9), expression (omega == 2.5 ~~ psi == 4.5 ~~ alpha == 3.2 ~~ rho == 1.4 ~~ sigma == 0.8),expression (omega
== 1.8 ~~ psi == 1.5 ~~ alpha == 1.2 ~~ rho == 4.2 ~~ sigma == 0.5),expression (omega == 2.9 ~~ psi == 2.5 ~~
alpha == 2.2 ~~ rho == 3.8 ~~ sigma == 0.6)),cex = 1,lty=c(1,2,3,4),col=("1":"2":"3":"4"), lwd = 2)
q#SF
x=seq(0,15,0.0008)
omega=0.9
psi=1.5
alpha=0.5
rho=1.3
sigma=1.9
f=(1-(1-(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))**(2))**(psi))
lower.X=0; upper.X=5; lower.Y=0; upper.Y=1
windows()
par(new=TRUE)
plot(x,f,lty=1,type="l", lwd = 2.5,xlim =c(lower.X, upper.X), ylim = c(lower.Y, upper.Y),xlab="x",ylab="Survival
function",col="1")
x=seq(0,15,0.0008)
omega=2.5
psi=4.5
alpha=3.2
rho=1.4
sigma=0.8
f=(1-(1-(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))**(2))**(psi))
lines(x,f,lty=3,lwd = 2.5, type="l",col="2")
x=seq(0,15,0.0008)
omega=1.8
psi=1.5
alpha=1.2
rho=4.2
sigma=0.5
f=(1-(1-(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))**(2))**(psi))
lines(x,f,lty=3,lwd = 2.5, type="l",col="3")
x=seq(0,15,0.0008)
omega=2.9
psi=2.5
alpha=2.2
rho=3.8
sigma=0.6
f=(1-(1-(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))**(2))**(psi))
lines(x,f,lty=4,lwd = 2.5, type="l",col="4")
legend(("topright"),legend=c(expression (omega == 0.9 ~~ psi == 1.5 ~~ alpha == 0.5 ~~ rho == 1.3 ~~ sigma
== 1.9), expression (omega == 2.5 ~~ psi == 4.5 ~~ alpha == 3.2 ~~ rho == 1.4 ~~ sigma == 0.8),expression (omega
== 1.8 ~~ psi == 1.5 ~~ alpha == 1.2 ~~ rho == 4.2 ~~ sigma == 0.5),expression (omega == 2.9 ~~ psi == 2.5 ~~
alpha == 2.2 ~~ rho == 3.8 ~~ sigma == 0.6)),cex = 1,lty=c(1,2,3,4),col=("1":"2":"3":"4"), lwd = 2)
```

```
#CDF
x=seq(0,15,0.0008)
omega=0.9
psi=1.5
alpha=0.5
rho=1.3
sigma=1.9
f=(1-(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))**(2))**(psi)
lower.X=0; upper.X=4; lower.Y=0; upper.Y=1
windows()
par(new=TRUE)
plot(x,f,lty=1,type=”l”, lwd = 2.5,xlim =c(lower.X, upper.X), ylim = c(lower.Y, upper.Y),xlab=”x”,ylab=”cdf”,
col=”1”)
x=seq(0,15,0.0008)
omega=2.5
psi=4.5
alpha=3.2
rho=1.4
sigma=0.8
f=(1-(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))**(2))**(psi)
lines(x,f,lty=3,lwd = 2.5, type=”l”,col=”2”)
x=seq(0,15,0.0008)
omega=1.8
psi=1.5
alpha=1.2
rho=4.2
sigma=0.5
f=(1-(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))**(2))**(psi)
lines(x,f,lty=3,lwd = 2.5, type=”l”,col=”3”)
x=seq(0,15,0.0008)
omega=2.9
psi=2.5
alpha=2.2
rho=3.8
sigma=0.6
f=(1-(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))**(2))**(psi)
lines(x,f,lty=4,lwd = 2.5, type=”l”,col=”4”)
legend((”topleft”),legend=c(expression (omega == 0.9 ~~ psi == 1.5 ~~ alpha == 0.5 ~~ rho == 1.3 ~~ sigma
== 1.9), expression (omega == 2.5 ~~ psi == 4.5 ~~ alpha == 3.2 ~~ rho == 1.4 ~~ sigma == 0.8),expression (omega
== 1.8 ~~ psi == 1.5 ~~ alpha == 1.2 ~~ rho == 4.2 ~~ sigma == 0.5),expression (omega == 2.9 ~~ psi == 2.5 ~~
alpha == 2.2 ~~ rho == 3.8 ~~ sigma == 0.6)),cex = 1,lty=c(1,2,3,4),col=(”1”:”2”:”3”:”4”), lwd = 2)
#####MLE NMEGIEx distribution data 1
x=c(0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48,
1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67,
1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01,
2.24)
length(x)
```

```
dnmegiex <- function(x,omega,psi,alpha,rho,sigma) 2*sigma*omega*psi*alpha*x**(-2)*rho*exp(-sigma*(alpha/x)) *(1-exp(-sigma*(alpha/x)))**(omega-1)*(1-(1-exp(-sigma* (alpha/x)))**(omega))**(rho-1)*(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))*(1-(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))**(2))**(psi-1)
pnmegiex <- function(q,omega,psi,alpha,rho,sigma) (1-(1-(1-(1-exp(-sigma*(alpha/q)))**(omega))**(rho))**(2))**(psi)
qnmegiex<- function(p,omega,psi,alpha,rho,sigma) (sigma*alpha)/(-log(1-(1-(1-(1-p**(1/psi))**(1/2))**(1/rho))**(1/omega)))
nmegiexMLE<-
fitdist(x,"nmegiex",start=list(omega=0.005,psi=0.05,alpha=0.05,rho=0.005,sigma=0.005),lower = 0, upper = Inf, method="mle")
windows()
plot(nmegiexMLE)
summary(nmegiexMLE)
#####MLE nmegiex distribution data 2
x=c(3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11,4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43,2.95, 2.97, 3.39, 2.96, 2.53,2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85,2.56, 3.56, 3.15, 2.35, 2.55, 2.59,2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19,1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69,1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12,1.89, 2.88, 2.82, 2.05, 3.65)
length(x)
dnmegiex <- function(x,omega,psi,alpha,rho,sigma) 2*sigma*omega*psi*alpha*x**(-2)*rho*exp(-sigma*(alpha/x)) *(1-exp(-sigma*(alpha/x)))**(omega-1)*(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho-1)*(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))*(1-(1-(1-(1-exp(-sigma*(alpha/x)))**(omega))**(rho))**(2))**(psi-1)
pnmegiex <- function(q,omega,psi,alpha,rho,sigma) (1-(1-(1-(1-exp(-sigma*(alpha/q)))**(omega))**(rho))**(2))**(psi)
qnmegiex<- function(p,omega,psi,alpha,rho,sigma) (sigma*alpha)/(-log(1-(1-(1-(1-p**(1/psi))* *(1/2))**(1/rho))**(1/omega)))
nmegiexMLE<-
fitdist(x,"nmegiex",start=list(omega=0.005,psi=0.05,alpha=0.05,rho=0.005,sigma=0.005),lower = 0, upper = Inf, method="mle")
windows()
plot(nmegiexMLE)
summary(nmegiexMLE)
#####simulation of mle
rm(list=ls(all=TRUE))
####pdf
NMEGIExpdf<- function(x,sigma,psi,alpha,rho,omega){
((2*sigma*omega*rho*psi)*(alpha/x**2)*exp(-(sigma*(alpha/x)))*(1-exp(-(sigma*(alpha/x))))**(omega)*(1-(1-exp (-(sigma*(alpha/x))))**(omega))**(rho-1)*(1-(1-(1-exp(-(sigma*(alpha/x))))**(omega))**(rho))*(1-(1-(1-(1-exp (-(sigma *(alpha/x))))**(omega))**(rho))**(2))**(psi-1))}
###cdf
NMEGIExcdf<- function(x,sigma,psi,alpha,rho,omega){
((1-(1-(1-(1-exp(-(sigma*(alpha/x))))**(omega))**(rho))**(2))**(psi-1)) }
#likelihood
pdf<-function(par,x){
sigma<-par[1]; psi<-par[2]; alpha<-par[3]; rho<-par[4]; omega<-par[5] val<- -sum(log( NMEGIExpdf(x,sigma,psi, alpha,rho,omega))) val
}
```

```
#quantile function
q<-function(u,sigma,psi,alpha,rho,omega){
((sigma*alpha)/(-log(1-(1-(1-(1-(u**(1/psi)))**(1/2))**(1/rho))**(1/omega))) )}
#function to do the simulation
eqm<-function(n,sigma1,psi1,alpha1,rho1,omega1){
plot(0,0,ylim = c(0,1000))
sigma<-psi<-alpha<-rho<-omega<-c();
set.seed(123)
for (i in 1:1000){
u<-runif(n = n,min = 0,max = 1)
data<-q(u,sigma1,psi1,alpha1,rho1,omega1)
hat<-try(optim(c(sigma1,psi1,alpha1,rho1,omega1),pdf,x=data,control = list(maxit = 60)),silent=F)
sigma<-c(sigma,hat$par[1])
psi<-c(psi,hat$par[2])
alpha<-c(alpha,hat$par[3])
rho<-c(rho,hat$par[4])
omega<-c(omega,hat$par[5])
abline(h=i)
}
means<-c(mean(sigma),mean(psi),mean(alpha),mean(rho),mean(omega))
vars<-c(var(sigma),var(psi),var(alpha),var(rho),var(omega))
Bias<-means-c(sigma1,psi1,alpha1,rho1,omega1)
RMSE<-(vars+Bias^2)^(0.5)
result<-new.env()
result$means<-round(means,4)
result$Bias<-round(Bias,4)
result$RMSE<-round(RMSE,4)
return(as.list(result))
}
######################
eqm(20,0.5,1,0.5,0.8,0.7)
eqm(50,0.5,1,0.5,0.8,0.7)
eqm(100,0.5,1,0.5,0.8,0.7)
eqm(250,0.5,1,0.5,0.8,0.7)
eqm(500,0.5,1,0.5,0.8,0.7)
eqm(1000,0.5,1,0.5,0.8,0.7)
######################
eqm(20,2.5,0.5,3.1,3,4)
eqm(50,2.5,0.5,3.1,3,4)
eqm(100,2.5,0.5,3.1,3,4)
eqm(250,2.5,0.5,3.1,3,4)
eqm(500,2.5,0.5,3.1,3,4)
eqm(1000,2.5,0.5,3.1,3,4)
```