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Research Article

Existence Results for Fourth Order Non-Homogeneous Three-Point Boundary Value Problems

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Abstract: The present paper focuses on establishing the existence and uniqueness of solutions to the nonlinear differential equations of order four

$$y^{(4)}(t) + g(t, y(t)) = 0,\ \ t \in [a, b],$$

together with the non-homogeneous three-point boundary conditions

$$y(a) = 0,\ y'(a) = 0,\ y''(a) = 0,\ y(b) - \alpha y(\xi) = \lambda,$$

where $0 \le a < b$, $\xi \in (a, b)$, α, λ are real numbers and the function g: $[a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous with $g(t, 0) \ne 0$. With the aid of an estimate on the integral of kernel, the existence results to the problem are proved by employing fixed point theorem due to Banach.

***Keywords*:** nonlinear boundary value problem, non-homogeneous boundary conditions, kernel, existence and uniqueness of solution, fixed point theorem

MSC: 34B15, 34B10

1. Introduction

The theory of differential equations has served as a significant tool for describing and analyzing the problems in a broad range of scientific disciplines. The fact that laws governing such phenomena can be formulated as differential equations along with initial or boundary conditions. Boundary value problems are especially important as they model a wide range in applied mathematics, such as engineering design and manufacturing. Major industries like automotive, aerospace, chemical, pharmaceutical, as well as emerging technologies like biotechnology and nanotechnology, rely on boundary value problems to simulate complex phenomena on a variety of scales in the design and manufacture of high-

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technological products. For details, we refer to [1-4].

Two-point and multi-point problems associated with differential equations occur in various fields of mathematical sciences including three-layer beam, electromagnetic waves, etc. For instance, the vibrations of a wire with a uniform cross-section consisting of N components of various densities can be formulated as a multi-point problem [5] and several problems can also be expressed in the elastic stability theory as twopoint or multi-point problems [6]. Il'in and Moiseev [7, 8] addressed the multi-point problem involving differential equations of second order in 1987. Afterwards, several authors used various methods to focus on different multi-point problems.

In fact, the differential equations of fourth order are models for deformation or bending of elastic beams. In 1988, Gupta [9] established the existence and uniqueness results for the bending of an elastic beam with fully supported ends and is described by a non-homogeneous differential equation of fourth order

$$\frac{d^4u}{dx^4} = e(x), \quad 0 < x < 1,$$

satisfying

$$u(0) = 0, \ u(1) = 0, \ u''(0) = 0, \ u''(1) = 0.$$

In 2003, Ma [10] considered the differential equation of fourth order

$$u^{(4)}(x) = \lambda f(x, u(x), u'(x)), \quad x \in (0,1),$$

associated with two-point conditions

$$u(0) = 0, \ u'(0) = 0, \ u''(1) = 0, \ u'''(1) = 0,$$

and established the existence of multiple positive solutions. In 2016, Lakoud and Zenkoufi [11] proved the existence results for

$$u''''(t) + f(t, u(t), u'(t), u''(t)) = 0, \quad t \in (0,1),$$

$$u(0) = 0, \ u'(0) = 0, \ u''(0) = 0, \ u''(1) = \beta u''(\eta),$$

by applications of various fixed point theorems. In 2019, Smirnov [12] proved the existence and uniqueness of solutions for the boundary value problem of third order

$$x'''(t) + f(t, x(t)) = 0, \quad t \in [a, b],$$

$$x(a) = 0, \ x'(a) = 0, \ x(b) = kx(\eta),$$

based on Banach fixed point theorem. Later, these results are improved by Almuthaybiri and Tisdell [13]. In 2020, Bekri and Benaicha [14] dealt with the existence results to the problem

$$u^{(4)}(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$

$$u(0) = \alpha u(\eta), \quad u'(0) = u''(0) = 0, \quad u(1) = \beta u(\eta),$$

by utilizing Leray-Schauder nonlinear alternative. One can see the available literature on the study of the existence and positivity results for differential equations of second order, see [15-17]; third order, see [18-24]; fourth order, see [25-29]; and for difference equations, see [30-32] together with three-point conditions.

In applications of real-world problems and the analysis of differential equations with initial or boundary conditions, the problem must be well-posed. If there is only one solution to the problem and we also have specific acceptable conditions, then we can extend to a huge variety of possibilities to validate "well-posedness" of the problem. If there is no solution or several solutions to the problem, then it is not wellposed as per modelling perspective and should be ignored, and a new model needs to be developed [33]. In view of the importance in both theory and applications, we are interested in studying the results for the existence and uniqueness of solutions to the fourth order differential equations

$$y^{(4)}(t)+g(t,y(t))=0, \quad t\in[a,b], \tag{1}$$

satisfying

$$y(a)=0, \quad y'(a)=0, \quad y''(a)=0, \quad y(b)-\alpha y(\xi)=\lambda, \tag{2}$$

where $0 \le a < b$, $a < \xi < b$, α, λ are real numbers and $g \in C([a, b] \times \mathbb{R}, \mathbb{R})$ with $g(t, 0) \neq 0$, by employing fixed point theorem due to Banach. The problem in this paper generalizes the existing literature for the interval $[a, b]$.

For simplicity, the following notations are used.

$$\text{(A1)} \quad \gamma=\lambda+\frac{1}{3!}\int_a^b(b-s)^3h(s)ds-\frac{\alpha}{3!}\int_a^{\xi}(\xi-s)^3h(s)ds, \text{ and}$$

$$\text{(A2)} \quad \Delta=(b-a)^3-\alpha(\xi-a)^3.$$

The rest of the study is described below. In Section 2, the solution of the problem stated in (1)-(2) is expressed as a solution of an equivalent integral equation in terms of kernel and then the sharper estimate on the integral of kernel is determined. With the aid of the estimate on the integral of kernel, the results for the existence and uniqueness to the problem stated in (1)-(2) are proved and the result is then validated by an example in Section 3.

2. Preparatory results

We first express the solution of the problem (1)-(2) in terms of kernel as a solution of the analogous integral equation and then determine the bounds for the integral of kernel. These are useful in proving our main result.

Let $h(t)$ be a real-valued continuous function that defined on $[a, b]$. Then, we find the solution of the non-homogeneous problem

$$y^{(4)}(t)+h(t)=0, \quad t\in[a,b], \tag{3}$$

together with the conditions (2).

Theorem 2.1 If $\Delta \neq 0$, then the solution of the problem stated in (3), (2) is expressed uniquely and is given by

$$y(t)=\frac{\lambda(t-a)^3}{\Delta}+\int_a^b\mathcal{K}(t,s)h(s)ds,$$

where

$$\mathcal{K}(t,s)=\mathcal{H}(t,s)+\frac{\alpha(t-a)^3}{\Delta}\psi(s), \tag{4}$$

$$\mathcal{H}(t,s)=\begin{cases}\dfrac{(t-a)^3(b-s)^3}{3!(b-a)^3}-\dfrac{(t-s)^3}{3!}, & a\le s\le t\le b,\\ \dfrac{(t-a)^3(b-s)^3}{3!(b-a)^3}, & a\le t\le s\le b,\end{cases} \tag{5}$$

and $\psi(s)=\mathcal{H}(\xi,s)$.

Proof. An equivalent integral equation of (3) is expressed as

$$y(t)=B_0+B_1t+B_2t^2+B_3t^3-\frac{1}{3!}\int_a^t(t-s)^3h(s)ds, \tag{6}$$

where B_0, B_1, B_2 and B_3 are constants. Using the conditions (2), one can obtain

$$B_0+B_1a+B_2a^2+B_3a^3=0,$$

$$B_1+2B_2a+3B_3a^2=0,$$

$$B_2+3B_3a=0,$$

$$B_0(1-\alpha)+B_1(b-\alpha\xi)+B_2(b^2-\alpha\xi^2)+B_3(b^3-\alpha\xi^3)=\gamma,$$

where γ is given in (A1). On solving the above, we get

$$B_0=-\frac{a^3\gamma}{\Delta},\ B_1=\frac{3a^2\gamma}{\Delta},\ B_2=-\frac{3a\gamma}{\Delta}\ \text{ and }\ B_3=\frac{\gamma}{\Delta},$$

where Δ is given in (A2). Substituting these values in (6), we have

$$\begin{aligned}y(t)&=\frac{1}{\Delta}[-a^3+3a^2t-3at^2+t^3]\gamma-\frac{1}{3!}\int_a^t(t-s)^3h(s)ds\\
&=\frac{(t-a)^3}{(b-a)^3-\alpha(\xi-a)^3}\left[\lambda+\frac{1}{3!}\int_a^b(b-s)^3h(s)ds-\frac{\alpha}{3!}\int_a^\xi(\xi-s)^3h(s)ds\right]\\
&\quad-\frac{1}{3!}\int_a^t(t-s)^3h(s)ds\\
&=\frac{\lambda(t-a)^3}{(b-a)^3-\alpha(\xi-a)^3}+\frac{(t-a)^3[(b-a)^3-\alpha(\xi-a)^3+\alpha(\xi-a)^3]}{3!(b-a)^3[(b-a)^3-\alpha(\xi-a)^3]}\\
&\quad\int_a^b(b-s)^3h(s)ds-\frac{\alpha(t-a)^3}{3![(b-a)^3-\alpha(\xi-a)^3]}\int_a^\xi(\xi-s)^3h(s)ds\\
&\quad-\frac{1}{3!}\int_a^t(t-s)^3h(s)ds\\
&=\frac{\lambda(t-a)^3}{(b-a)^3-\alpha(\xi-a)^3}+\frac{(t-a)^3}{3!(b-a)^3}\int_a^b(b-s)^3h(s)ds\end{aligned}$$

$$+\frac{\alpha(t-a)^3(\xi-a)^3}{3!(b-a)^3[(b-a)^3-\alpha(\xi-a)^3]}\int_a^b(b-s)^3h(s)ds$$

$$-\frac{\alpha(t-a)^3}{3![(b-a)^3-\alpha(\xi-a)^3]}\int_a^\xi(\xi-s)^3h(s)ds-\frac{1}{3!}\int_a^t(t-s)^3h(s)ds$$

$$=\frac{\lambda(t-a)^3}{(b-a)^3-\alpha(\xi-a)^3}+\int_a^t\left[\frac{(t-a)^3(b-s)^3}{3!(b-a)^3}-\frac{(t-s)^3}{3!}\right]h(s)ds$$

$$+\int_t^b\left[\frac{(t-a)^3(b-s)^3}{3!(b-a)^3}\right]h(s)ds+\frac{\alpha(t-a)^3}{[(b-a)^3-\alpha(\xi-a)^3]}$$

$$\left\{\int_a^\xi\left[\frac{(\xi-a)^3(b-s)^3}{3!(b-a)^3}-\frac{(\xi-s)^3}{3!}\right]h(s)ds+\int_\xi^b\left[\frac{(\xi-a)^3(b-s)^3}{3!(b-a)^3}\right]h(s)ds\right\}$$

$$=\frac{\lambda(t-a)^3}{\Delta}+\int_a^b\mathcal{H}(t,s)h(s)ds+\frac{\alpha(t-a)^3}{\Delta}\int_a^b\psi(s)h(s)ds$$

$$=\frac{\lambda(t-a)^3}{\Delta}+\int_a^b\mathcal{K}(t,s)h(s)ds.$$

For the uniqueness of solution, assuming $x(t)$ is another solution of (3), (2). Let us take $u(t)=y(t)-x(t)$. Then, we have

$$u^{(4)}(t)=0,\quad t\in[a,b], \tag{7}$$

$$u(a)=0,\ u'(a)=0,\ u''(a)=0,\ u(b)-\alpha u(\xi)=0. \tag{8}$$

Therefore, the solution of (7) is

$$u(t)=C_0+C_1t+C_2t^2+C_3t^3,$$

where the constants C_0, C_1, C_2, and C_3 are to be determined. Using the conditions (8), we get the homogeneous system $AC=O$, where

$$A=\begin{bmatrix}1 & a & a^2 & a^3\\ 0 & 1 & 2a & 3a^2\\ 0 & 0 & 1 & 3a\\ 1-\alpha & b-\alpha\xi & b^2-\alpha\xi^2 & b^3-\alpha\xi^3\end{bmatrix}$$

and

$$C=\begin{bmatrix}C_0 & C_1 & C_2 & C_3\end{bmatrix}^T.$$

with the determinant $|A|=(b-a)^3-\alpha(\xi-a)^3\neq 0$. So, the homogeneous system $AC=O$ is consistent and has only a zero solution and, hence $u(t)=0$, for every $t\in[a,b]$, Therefore, the uniqueness is established.

Lemma 2.2 The function $\mathcal{H}(t,s)$ in (5) is non-negative for every $t,s\in[a,b]$.

Proof. The positivity of $\mathcal{H}(t, s)$ can be viewed by simple algebraic calculations.

Lemma 2.3 The function $\mathcal{H}(t, s)$ in (5) fulfills the below inequality

$$\int_a^b \mathcal{H}(t,s)ds \le \frac{9}{2048}(b-a)^4, \text{ for all } t \in [a,b]. \tag{9}$$

Proof. Consider, for every $t \in [a, b]$, then

$$\int_a^b \mathcal{H}(t,s)ds = \int_a^t \left[\frac{(t-a)^3(b-s)^3}{3!(b-a)^3} - \frac{(t-s)^3}{3!}\right]ds + \int_t^b \frac{(t-a)^3(b-s)^3}{3!(b-a)^3}ds$$

$$= \left[-\frac{(t-a)^3(b-s)^4}{24(b-a)^3} + \frac{(t-s)^4}{24}\right]_a^t + \left[-\frac{(t-a)^3(b-s)^4}{24(b-a)^3}\right]_t^b$$

$$= \frac{(t-a)^3(b-a)}{24} - \frac{(t-a)^4}{24}$$

$$= \frac{(t-a)^3(b-t)}{24}.$$

Let $\varphi(t) = \frac{(t-a)^3(b-t)}{24}$. Then $\varphi'(t) = \frac{(t-a)^2(-4t+a+3b)}{24}$. For stationary points, we have $\varphi'(t) = 0$, which implies that $t = \frac{a+3b}{4}$. Since $\varphi''(t) = -\frac{3(b-a)^2}{32} < 0$ at $t = \frac{a+3b}{4}$, the function $\varphi(t)$ has maximum at $t = \frac{a+3b}{4}$, that is

$$\max_{t\in[a,b]} \varphi(t) = \max_{t\in[a,b]} \frac{(t-a)^3(b-t)}{24} = \frac{9(b-a)^4}{2048}.$$

Hence, the inequality (9).

Lemma 2.4 The function $\psi(s)$ fulfills the following

$$\int_a^b |\psi(s)|\, ds \le \frac{1}{12}(b-a)^4.$$

Proof. Consider

$$\int_a^b |\psi(s)|\, ds = \int_a^b |\mathcal{H}(\xi,s)|\, ds$$

$$= \int_a^\xi |\mathcal{H}(\xi,s)|\, ds + \int_\xi^b |\mathcal{H}(\xi,s)|\, ds$$

$$= \int_a^\xi \left|\frac{(\xi-a)^3(b-s)^3}{3!(b-a)^3} - \frac{(\xi-s)^3}{3!}\right| ds + \int_\xi^b \left|\frac{(\xi-a)^3(b-s)^3}{3!(b-a)^3}\right| ds$$

$$\le \int_a^\xi \left[\frac{(\xi-a)^3(b-s)^3}{3!(b-a)^3} + \frac{(\xi-s)^3}{3!}\right] ds + \int_\xi^b \frac{(\xi-a)^3(b-s)^3}{3!(b-a)^3} ds$$

$$= \left[-\frac{(\xi-a)^3(b-s)^4}{24(b-a)^3} - \frac{(\xi-s)^4}{24} \right]_a^\xi + \left[-\frac{(\xi-a)^3(b-s)^4}{24(b-a)^3} \right]_\xi^b$$

$$= \frac{(\xi-a)^3(b-a)}{24} + \frac{(\xi-a)^4}{24}$$

$$\leq \frac{(b-a)^3(b-a)}{24} + \frac{(b-a)^4}{24}$$

$$= \frac{1}{12}(b-a)^4.$$

Lemma 2.5 If $\Delta \neq 0$, then the kernel $\mathcal{K}(t, s)$ in (4) fulfills the following

$$\int_a^b |\mathcal{K}(t,s)|\, ds \leq (b-a)^4 \left[\frac{9}{2048} + \frac{|\alpha|(b-a)^3}{12\left|(b-a)^3 - \alpha(\xi-a)^3\right|} \right].$$

Proof. Consider for every $t \in [a, b]$, we have

$$\int_a^b |\mathcal{K}(t,s)|\, ds = \int_a^b \left| \mathcal{H}(t,s) + \frac{\alpha(t-a)^3}{[(b-a)^3 - \alpha(\xi-a)^3]} \psi(s) \right| ds$$

$$\leq \int_a^b |\mathcal{H}(t,s)|\, ds + \left| \frac{\alpha(t-a)^3}{[(b-a)^3 - \alpha(\xi-a)^3]} \right| \int_a^b |\psi(s)|\, ds$$

$$\leq \frac{9}{2048}(b-a)^4 + \frac{|\alpha|(b-a)^3}{\left|(b-a)^3 - \alpha(\xi-a)^3\right|} \times \frac{(b-a)^4}{12}$$

$$= (b-a)^4 \left[\frac{9}{2048} + \frac{|\alpha|(b-a)^3}{12\left|(b-a)^3 - \alpha(\xi-a)^3\right|} \right].$$

The fixed point theorem described below due to Banach will be the fundamental tool to establish our result.

Theorem 2.6 [34] Let B be a any nonempty set and δ be a metric on B such that the ordered pair (B, δ) forms a complete metric space. If the mapping $\mathcal{G}: B \to B$ fulfills the condition that for some $0 < \theta < 1$,

$$\delta(\mathcal{G}y, \mathcal{G}z) \leq \theta\delta(y, z), \quad \text{for every } y, z \in B,$$

then the operator $\mathcal{G}$ has a unique fixed point $y^* \in B$.

3. Existence and uniqueness

This section establishes the existence and uniqueness of solutions to the problem stated in (1)-(2).

Let B be the set of real-valued functions belongs to $C^4([a, b])$. For functions $y(t)$, $z(t)$ in B, define a metric

$$\delta(y,z) = \max_{t\in[a,b]} | y(t) - z(t) |. \tag{10}$$

Then, the pair (B, δ) forms a complete metric space.

Theorem 3.1 Suppose that g: $[a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function with $g(t, 0) \neq 0$, for every $t \in [a, b]$. Let θ be a positive constant such that

$$\left|g(t,y) - g(t,z)\right| \leq \theta\left|y - z\right|, \text{ for every } (t,y),(t,z) \in [a,b]\times\mathbb{R}.$$

If $\Delta \neq 0$ and $(b - a)$ is small with the condition

$$(b-a)^4\left[\frac{9}{2048} + \frac{\left|\alpha\right|(b-a)^3}{12\left|(b-a)^3 - \alpha(\xi - a)^3\right|}\right] < \frac{1}{\theta} \tag{11}$$

then there is a unique solution of the problem stated in (1)-(2).

Proof. Let us consider the operator $\mathcal{G}$: $B \to B$ as

$$\mathcal{G}y(t) = \frac{\lambda(t-a)^3}{\Delta} + \int_a^b \mathcal{K}(t,s)g(s,y(s))ds, \text{ for all } t \in [a,b],$$

where $\mathcal{K}(t, s)$ is given in (4).

It is evident that, $y(t)$ is a solution of the problem stated in (1)-(2) if and only if $y(t)$ fulfills the following integral equation

$$y(t) = \frac{\lambda(t-a)^3}{\Delta} + \int_a^b \mathcal{K}(t,s)g(s,y(s))ds, \text{ for all } t \in [a,b].$$

The problem stated in (1)-(2) has only one solution if and only if the operator $\mathcal{G}$ has only one fixed point and we establish based on the Banach fixed point theorem.

Since (B, δ) is the complete metric space, then for any $y, z \in B$ and for $t \in [a, b]$, we conclude

$$\begin{aligned}
\left|\mathcal{G}y(t) - \mathcal{G}z(t)\right| &= \left|\frac{\lambda(t-a)^3}{\Delta} + \int_a^b \mathcal{K}(t,s)g(s,y(s))ds\right. \\
&\quad \left. - \frac{\lambda(t-a)^3}{\Delta} + \int_a^b \mathcal{K}(t,s)g(s,z(s))ds\right| \\
&\leq \int_a^b \left|\mathcal{K}(t,s)\right|\left|g(s,y(s)) - g(s,z(s))\right| ds \\
&\leq \theta\int_a^b \left|\mathcal{K}(t,s)\right|\left|y(s) - z(s)\right| ds \\
&\leq \theta\int_a^b \left|\mathcal{K}(t,s)\right| \delta(y,z)ds \\
&\leq \theta(b-a)^4\left[\frac{9}{2048} + \frac{\left|\alpha\right|(b-a)^3}{12\left|(b-a)^3 - \alpha(\xi - a)^3\right|}\right]\delta(y,z).
\end{aligned}$$

It is evident from the fact that

$$\delta(\mathcal{G}y, \mathcal{G}z) \leq \beta\delta(y, z),$$

where

$$\beta = \theta(b-a)^4 \left[\frac{9}{2048} + \frac{|\alpha|(b-a)^3}{12\left|(b-a)^3 - \alpha(\xi-a)^3\right|} \right].$$

In view of (11), $\beta < 1$ and the operator $\mathcal{G}$ fulfills the property of contraction mapping.
It follows that operator $\mathcal{G}$ has a unique fixed point and thus, the fixed point is the solution of (1)-(2).
As an application, the proved result is illustrated with an example.
Example 3.1 Let us take $a = \frac{1}{4}$, $b = 1$, $\xi = \frac{1}{2}$, $\alpha = \frac{1}{3}$ and $g(t, y) = 2 + t + \sin y$.
Now, consider differential equation of order four

$$y^{(4)} + 2 + t + \sin y = 0, \quad t \in \left[\frac{1}{4}, 1\right], \tag{12}$$

satisfying

$$y\left(\frac{1}{4}\right) = 0, \quad y'\left(\frac{1}{4}\right) = 0, \quad y''\left(\frac{1}{4}\right) = 0, \quad y(1) - \frac{1}{3}y\left(\frac{1}{2}\right) = \lambda. \tag{13}$$

It is evident that $g(t, 0) \neq 0$ and $\Delta = (b-a)^3 - \alpha(\xi - a)^3 \neq 0$. Then

$$\left|\frac{\partial g(t, y)}{\partial y}\right| = |\cos y| \leq 1 = \theta$$

and

$$(b-a)^4 \left[\frac{9}{2048} + \frac{|\alpha|(b-a)^3}{12\left|(b-a)^3 - \alpha(\xi-a)^3\right|} \right] = \frac{26973}{2621440} < \frac{1}{\theta} = 1.$$

So, all the claims in the Theorem 3.1 are fulfilled, and thus (12)-(13) has a unique solution.

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