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Research Article

Derivation of a New Differential Operator on Bi-Bounded Turning Functions

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Abstract: In this research, a novel generalized differential operator will be used to derive a new subclass of bi-univalent functions, specifically those that are subordinated to bounded turning functions with Gregory coefficients. This subclass is expected to provide precise estimates for various coefficient problems, such as coefficient estimates, the second Hankel determinant, and the Fekete-Szegő inequality. The identification of an extremal function will be crucial in establishing the validity and sharpness of the derived bounds. The introduction of this new subclass is expected to bridge the gap in the current literature on bi-univalent functions associated with Gregory coefficients and generalized bounded turning functions. These functions and the operator hold significant relevance across diverse technological and scientific disciplines, including, but not limited to, electromagnetic theory, plasma physics, mathematical biology, dynamical systems and optics.

***Keywords*:** Biunivalent functions, Gregory coefficients, differential operator, Hankel determinant, extremal functions, bounded turning functions, modelling operator

MSC: 30C80, 30C45, 30C50, 47B38, 11B65

Abbreviation

$\mathscr{U}$	Open unit disk
$\mathscr{DA}$	Class of analytic functions in $\mathscr{U}$
$\mathbb{C}$	Set of complex numbers
$\mathscr{A}$	Subclass of $\mathscr{DA}$ of normalised analytic functions in $\mathscr{U}$
$\star$	Hadamard product or convolution
$\prec$	Subordination

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$\mathscr{S}$	Univalent functions
$\mathscr{S}^*$	Starlike functions
$\mathscr{C}$	Convex functions
$\mathscr{K}$	Close-to-convex functions
$\mathscr{BT}$	Bounded turning functions
$T_{\tau,\varphi,k,x}^{c,\mu,n}$	New differential operator
$\mathscr{GFT}$	Geometric Function Theory
$\mathscr{BU}s$	Bi-univalent functions

1. Introduction and definitions

Let $\mathscr{DA}$ indicate the class of all analytic functions f defined in the unit disk $\mathscr{U}=\{z\in\mathscr{C}\colon |z|<1\}$, and let $\mathscr{A}$ be the subclass of $\mathscr{DA}$ consisting of all analytic functions f given by the series:

$$f(z)=z+\sum_{m=2}^{\infty}a_m z^m. \tag{1}$$

The following $\mathscr{S}$, $\mathscr{S}^*$, $\mathscr{C}$, $\mathscr{K}$, and $\mathscr{BT}$ are the different subclasses of $\mathscr{DA}$ known for containing univalent, starlike, convex, close-to-convex functions, and functions with bounded turning. Various methods and instruments have been used to thoroughly analyze these categories and their broader concepts in the last few decades. In Geometric Function Theory ($\mathscr{GFT}$), subordination plays a crucial role in defining new subclasses of analytic functions and proving various geometric properties of these subclasses.

We say that f is considered subordinate to g and represent this connection as $f\prec g$ if $|\omega(z)|\le|z|$ is satisfied, such that $f=g(\omega(z))$ where z in $\mathscr{U}$. Moreover, when g is a member of class S, the following statement holds. Equivalence is satisfied when $f\prec g$ in $(z\in\mathscr{U})$ iff $f(\mathscr{U})\subseteq g(\mathscr{U})$ and the function $f(0)=g(0)$.

The function $u(z)$, represented as

$$u(z)=\sum_{m=1}^{\infty}u_m z^m+1,$$

such that when z belongs to the unit disk and has $\Re(u(z))>0$, it is designated as $\mathscr{P}$ in $\mathscr{U}$, given that $|u_m|<2$ as stated in reference [1]. The well-known types of univalent functions, such as convex functions ($f\in\mathscr{C}$), starlike functions ($f\in\mathscr{S}^\star$) and bounded turning functions ($f\in\mathscr{BT}$), can be described in the following manner:

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>0,\;\; \Re\left(1+\frac{zf''(z)}{f'(z)}\right)>0 \text{ and } \Re(f'(z))>0. \tag{2}$$

Using the function $f(z)$ from equation (1), we can define the k^{th} Hankel determinant introduced in 1973 by Noonan and Thomas [2] when $k\ge1$, $\upsilon\ge1$, and $a_1=1$.

$$\mathscr{H}_k(\upsilon)=\begin{vmatrix} a_\upsilon & a_{\upsilon+1} & a_{\upsilon+2} & \cdots & a_{\upsilon+k-1} \\ a_{\upsilon+1} & a_{\upsilon+2} & a_{\upsilon+3} & \cdots & a_{\upsilon+k} \\ a_{\upsilon+2} & a_{\upsilon+3} & a_{\upsilon+4} & \cdots & a_{\upsilon+k+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{\upsilon+k-1} & a_{\upsilon+k} & a_{\upsilon+k+1} & \cdots & a_{\upsilon+2(k-1)} \end{vmatrix}. \tag{3}$$

Using the formula from equation (3) with $k=2$ and $\upsilon=1$, we obtain the well-known Fekete-Szego functional.

$$\mathscr{H}_2(1)=\begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix}=|a_3-a_2^2|. \tag{4}$$

This feature is further expanded in a broader manner as:

$$|a_3-\rho a_2^2|, \tag{5}$$

where ρ represents a number that can be either real or complex.

According to research in singularity theory and referenced in [3], the Hankel determinant plays a vital role, and its significance lies in its capacity to analyze power series with integer coefficients, as demonstrated in [4]. Many researchers have determined maximum limits for $\mathscr{H}_k(\upsilon)$ with varying values of k and υ in different categories of analytic functions (refer to [5–12], for more information).

Now utilizing the binomial theorem:

$$(1-\tau)^c=\sum_{l=0}^{c}\binom{c}{l}(-1)^l\tau^l.$$

For $f\in\mathscr{A}$, given any $\tau\in\mathbb{R}$, $k>0$, $\mu\geq 0$, $\varphi>0$ such that $\tau+\varphi>0$, and $n\in\mathbb{N}_0$ (where $\mathbb{N}_0$ is the set of natural numbers including 0), we introduce the new operator $T^{c,\,\mu,\,n}_{\tau,\,\varphi,\,k,\,x}\colon\mathscr{A}\longrightarrow\mathscr{A}$:

$$T^{c,\,\mu,\,0}_{\tau,\,\varphi,\,k,\,x}f(z)=f(z),$$

$$T^{c,\,\mu,\,1}_{\tau,\,\varphi,\,k,\,x}f(z)=\frac{(1-(1-\tau)^c)f(z)+\left(1-\left(1-k\Lambda_s^x(\varphi)\left[1-\frac{\mu}{\Lambda_s^x(\varphi)}\right]\right)^c\right)zf'(z)}{2-(1-\tau)^c-\left(1-k\Lambda_s^x(\varphi)\left[1-\frac{\mu}{\Lambda_s^x(\varphi)}\right]\right)^c}=T^{c,\,\mu}_{\tau,\,\varphi,\,k}f(z),$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$T^{c,\,\mu,\,n}_{\tau,\,\varphi,\,k,\,x}=T^{c,\,\mu}_{\tau,\,\varphi,\,k,\,x}(T^{c,\,\mu,\,n-1}_{\tau,\,\varphi,\,k,\,x}f(z)). \tag{6}$$

If function f is defined as shown in equation (1), it is apparent from equation (6) that

$$T_{\tau,\varphi,k,x}^{c,\mu,n}f(z)=z+\sum_{m=2}^{\infty}\left(\sum_{l=1}^{c}\binom{c}{l}(-1)^{l+1}\left(\frac{\tau^{l}+mk(\Lambda_s^x(\varphi))^{l}\left(1-\frac{\mu}{\Lambda_s^x(\varphi)}\right)^{l}}{\tau^{l}+k(\Lambda_s^x(\varphi))^{l}\left(1-\frac{\mu}{\Lambda_s^x(\varphi)}\right)^{l}}\right)\right)^{n}a_m z^m, \tag{7}$$

which can also be in the form

$$T_{\tau,\varphi,k,x}^{c,\mu,n}f(z)=z+\sum_{m=2}^{\infty}(\top(m))^{n}a_m z^m, \tag{8}$$

where

$$\top(m)=\sum_{l=1}^{c}\binom{c}{l}(-1)^{l+1}\left(\frac{\tau^{l}+mk(\Lambda_s^x(\varphi))^{l}\left(1-\frac{\mu}{\Lambda_s^x(\varphi)}\right)^{l}}{\tau^{l}+k(\Lambda_s^x(\varphi))^{l}\left(1-\frac{\mu}{\Lambda_s^x(\varphi)}\right)^{l}}\right) \tag{9}$$

and

$$\Lambda_s^x(\varphi)=\sum_{s=1}^{x}\binom{x}{s}(-1)^{s+1}\varphi^{s}.$$

Furthermore, the desired identity can be straightforwardly proven by employing equation (7):

$$z(T_{\tau,\varphi,k,x}^{c,\mu,n}f(z))'\cong\Theta(\mu,x,\varphi,l)T_{\tau,\varphi,k,x}^{c,\mu,n+1}f(z)-B(\mu,x,\varphi,l)T_{\tau,\varphi,k,x}^{c,\mu,n}f(z).$$

Here, the functions $\Theta(\mu, x, \varphi, l)$ and $B(\mu, x, \varphi, l)$ can be expressed using binomial series expansions:

$$\Theta(\mu,x,\varphi,l)=\sum_{l=1}^{c}\binom{c}{l}(-1)^{l+1}\left(\left(\frac{\tau}{k\Lambda_s^x(\varphi)\left(1-\frac{\mu}{\Lambda_s^x(\varphi)}\right)}\right)^{l}+1\right)$$

and

$$B(\mu,x,\varphi,l)=\sum_{l=1}^{c}\binom{c}{l}(-1)^{l+1}\left(\frac{\tau}{k\Lambda_s^x(\varphi)\left(1-\frac{\mu}{\Lambda_s^x(\varphi)}\right)}\right)^{l}.$$

Remark 1 The operator $T_{\tau,\varphi,k,x}^{c,\mu,n}$ is a generalization of previously known operators as listed below.
1. Setting $\mu=0$, $x=1$ and $k=1$, the operator $T_{\tau,\varphi,k,x}^{c,\mu,n}=W_{\tau,\varphi}^{c,n}$ was introduced and studied by Wanas [13].
2. Setting $\mu=0$, $x=1$, $c=1$ and $k=1$, the operator $T_{\tau,\varphi,k,x}^{c,\mu,n}=I_{\tau,\varphi}^{n}$ was introduced and studied by Swamy [14].

3. Setting $\mu = 0$, $x = 1$, $c = 1$, $\varphi = 1$, $\tau > -1$ and $k = 1$, the operator $T^{c,\,\mu,\,n}_{\tau,\,\varphi,\,k,\,x} = I^n_\tau$ $(\tau \geq 0)$ was examined and analyzed by Cho and Srivastava [15], along with Cho and Kim [16].

4. Setting $\mu = 0$, $x = 1$, $c = 1$, $\varphi = 1$, $\tau = L + 1 - \Lambda^x_s(\varphi)k + \mu k$ and $k = 1$, the operator $T^{c,\,\mu,\,n}_{\tau,\,\varphi,\,k,\,x} = I^n_{L,\,\varphi}$ $(L > -1,\, \varphi \geq 0)$ was introduced and studied by Catas [17].

5. Setting $\mu = 0$, $c = 1$, $\tau = 1 - \Lambda^x_s(\varphi)k + \mu k$ and $k = 1$, the operator $T^{c,\,\mu,\,n}_{\tau,\,\varphi,\,k,\,x} = D^n_{x,\,\varphi}$ was introduced and studied by Frasin [18].

6. Setting $\mu = 0$, $c = 1$, $x = 1$, $\tau = 1 - \Lambda^x_s(\varphi)k + \mu k$ and $k = 1$, the operator $T^{c,\,\mu,\,n}_{\tau,\,\varphi,\,k,\,x} = D^n_\varphi$ was introduced and studied by Al-Oboudi [19].

7. Setting $\mu = 0$, $c = 1$, $x = 1$, $\varphi = 1$, $\tau = 1 - \Lambda^x_s(\varphi)k + \mu k$ and $k = 1$, the operator $T^{c,\,\mu,\,n}_{\tau,\,\varphi,\,k,\,x} = D^n$ was introduced and studied by Salagean [20].

Every function f belonging to set $\mathscr{S}$ has a corresponding inverse presented as:

$$f^{-1}(f(z)) = z, \quad z \in \mathscr{U}$$

and

$$f(f^{-1}(w)) = w, \ \ |w| < x_0(f), \ \ x_0(f) \geq \frac{1}{4}.$$

The following series illustrates the inverse of $f(z)$, denoted as $f^{-1}(w)$.

$$f^{-1}(w) = h(w) = w + W_2w^2 + W_3w^3 + W_4w^4 + \cdots, \tag{10}$$

where

$$W_2 = -a_2, \quad W_3 = 2a_2^2 - a_3 \ \text{ and } \ W_4 = 5a_2a_3 - 5a_2^3 - a_4. \tag{11}$$

The bi-univalent functions ($\mathscr{BU}s$) in the field of $\mathscr{GFT}$, denoted by Ξ, was first researched by Lewin [21], who proved that $|a_2| \leq 4 \times 3^{-1}$. Later, following Lewin's research, Brannan and Clunie [22] expanded on the concept to establish $|a_2| \leq (2)^{1/2}$ and Netanyahu [23] demonstrated that $|a_2| < 4/3$. The bi-convex functions and bi-starlike functions were first presented by Brannan and Clunie [22] in 1985. Study on subclasses Ξ has been a major scholarly interest for the past decade. Finding the initial boundaries of coefficients for particular subclasses piqued the curiosity of Ξ. The importance of coefficient problems in the context of $\mathscr{BU}s$ has been emphasized by Srivastava et al. [24]. In 2010, Srivastava et al. [24] discovered two remarkable subclasses within the family of Ξ functions. He indicated that for functions in these subclasses, $|a_2|$ and $|a_3|$ act as boundaries. Frasin and Aouf [25] began calculating the precise values of $|a_2|$ and $|a_3|$ for functions within two new subclasses of the function category.

In 1965, Berezin's work [26], the function:

$$\Delta = z \times [\ln(1+z)]^{-1} = \Theta(z) = \sum_{m=0}^{\infty} J_m z^m \quad \text{for} \quad |z| < 1, \tag{12}$$

with Gregory coefficients exhibits starlike properties. Gregory coefficients, which are rational numbers that decrease in value, akin to Bernoulli numbers, are commonly found in numerical analysis and number theory applications. This concept has been rediscovered by prominent mathematicians, earning it a reputation as one of the most frequently rediscovered mathematical principles. References [26] and [27] provide additional information on this topic. It is evident that J_n takes on specific values for certain values of n, such as $J_0 = 1$, $J_1 = 1$, $J_2 = -1$, $J_3 = 1$, $J_4 = -19$, $J_5 = 3$, and $J_6 = -863$. For more recent work under Gregory coefficients see [28, 29]

Motivated by Tang et al. [30] and Shaba et al. [31–34], our goal is to create innovative bi-bounded turning functions defined by a new differential operator linked to the Gregory Coefficients. In our research, we are studying the coefficients $(|a_n|)$ $(n = 2, 3, 4)$ to analyze Hankel determinants related to the second order, Fekete-Szego inequality, and thoroughly investigating these mathematical entities.

2. Applications and importance of analytic and bi-univalent functions to physical problems

Friedlandi and Schiffer [35] addressed issues in control theory by applying the concept of coefficient problems found in the theory of analytic-univalent functions. Control theory involves the examination of specific controls related to particular physical processes and systems in relation to certain differential equations. The utilized approach involved comparing the coefficients of a slit function $f \in \mathscr{A}$ with those of the control function

$$\curlywedge(t, z) = e^t \left\{ z + \sum_{m=2}^{\infty} a_m(t) z^m \right\}.$$

Lehto [36] provided a comprehensive description of how univalent functions are used in Teichmüller spaces. The theory of Teichmuller spaces addresses the classes of quasiconformal mappings on a Riemann surface that are homotopic, allowing for conformal mappings. Vasilev [37] examined the time development of the free boundary of a viscous fluid in models with zero and non-zero surface tension for planar flows in Hele-Shaw cells, whether with an unbounded free boundary extending to infinity or with a confined free boundary. The author subsequently examined specific categories of univalent functions that allowed for a clear geometric interpretation to define the form of the free interface. Fadipe-Joseph et al. [38] began utilizing a modified sigmoid function

$$\varXi(z) = 2s(z) = \frac{2}{1+e^{-z}} = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \cdots,$$

within the realm of univalent functions. $\varXi(z)$ transforms the unit disk U into a domain $DS{:} = w \in \mathscr{C}{:}\ |\log \frac{w}{2-w}| < 1$, which exhibits symmetry around the real axis. The analysis of the logistic sigmoid function $s(z)$ as an activation mechanism pertains to the hardware realization of artificial neural networks. The activation function is an information processing mechanism that mimics how biological nervous systems (such as the human brain) handle information. It consists of a large quantity of interconnected processing units known as neurons that collaborate to accomplish particular tasks. Neural networks serve as a tool for complex learning tasks.

3. New class and lemmas

Definition 1 If the restrictions $z \in \mathscr{U}$ and $h(w)$ are satisfied according to (10), then $f \in \mathscr{T}\mathscr{S}_{x,s,\varphi}^{\mu,k}(n, \Xi)$ given the condition:

$$(T^{c,\mu,n}_{\tau,\varphi,k,x}f(z))' \prec \frac{z}{\ln(1+z)} \tag{13}$$

and

$$(T^{c,\mu,n}_{\tau,\varphi,k,x}h(w))' \prec \frac{w}{\ln(1+w)} \tag{14}$$

are satisfied.

Remark 2 Selecting specific parameters from the recently introduced differential operator in Definition 1 results in novel subclasses of $\mathscr{B}\mathscr{U}s$.

The Figure 1 below is the description of the Gregory Coefficients on how it looks like on the complex plane which actually falls at the positive right half plane.

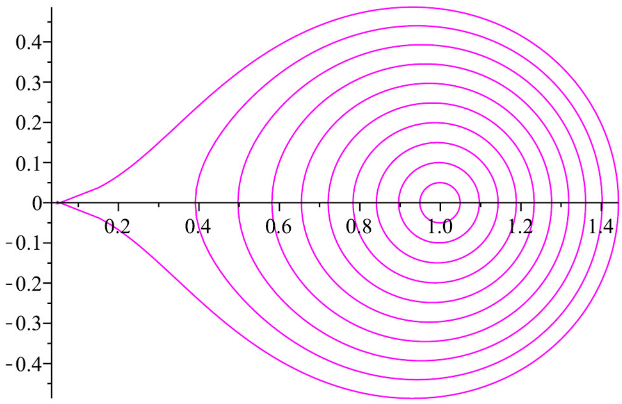

Figure 1. The diagram for the Gregory Coefficients $\frac{z}{\ln(1+z)}$

Combining the lemmas mentioned is expected to significantly help enhance and advance our main discoveries. Through the utilization of these lemmas, we are confident that our research will gain increased theoretical complexity and empirical reliability, ultimately leading to more robust and reliable findings.

Lemma 1 [31] Assume that $u \in \mathscr{P}$ has the form of a series

$$u(z) = 1 + \sum_{n=1}^{+\infty} u_n z^n \tag{15}$$

where $\Re(u(z)) > 0$ and z in the unit disk, then

$$|u_n| \leq 2, \quad n \in \mathbb{N}.$$

Lemma 2 [31] Assume that $u \in \mathscr{P}$ has the form of a series given in (15), and $\Re(u(z)) > 0$ where z in unit disk, then

$$u_2 = \frac{u_1^2 + (4 - u_1^2)e_1}{2},$$

$$u_3 = \frac{u_1^3 + 2(4 - u_1^2)u_1 e_1 - (4 - u_1^2)u_1 e_1^2 + 2(4 - u_1^2)(1 - |e_1|^2)z}{4}.$$

4. Coefficients bounds

Theorem 1 Suppose $f \in \mathscr{T}\mathscr{S}_{x,\, s,\, \varphi}^{\mu,\, k}(n,\, \Xi)$. Then,

$$|a_2| \leq \frac{1}{4(\top(2))^n},$$

$$|a_3| \leq \frac{1}{6(\top(3))^n},$$

$$|a_4| \leq \frac{1}{8(\top(4))^n}.$$

The distinctiveness of the mathematical equation was substantiated by the precisely specified upper bounds above the given function.

$$f_1(z) = z + \frac{1}{4(\top(2))^n} + \frac{1}{12(\top(3))^n} + \cdots,$$

$$f_2(z) = z + \frac{1}{6(\top(3))^n} z^3 + \frac{1}{16(\top(4))^n} z^4 + \cdots,$$

$$f_3(z) = z + \frac{1}{8(\top(4))^n} z^4 + \frac{1}{20(\top(5))^n} + \cdots.$$

Proof. If $f \in \mathscr{T}\mathscr{S}_{x,\, s,\, \varphi}^{\mu,\, k}(n,\, \Xi)$, $\tau \in \mathbb{R}$, $k > 0$, $\mu \geq 0$, $\varphi > 0$ with $\tau + \varphi > 0$, $c \in \mathbb{N} = \{1, 2, \cdots\}$, and $n, l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Following that, the designated criteria

$$(T_{\tau,\, \varphi,\, k,\, x}^{c,\, \mu,\, n} f(z))' = \frac{\upsilon_1(z)}{\ln(1 + \upsilon_1(z))} \tag{16}$$

and

$$(T_{\tau,\, \varphi,\, k,\, x}^{c,\, \mu,\, n} h(w))' = \frac{\upsilon_2(w)}{\ln(1 + \upsilon_2(w))} \tag{17}$$

are met by the analytic functions $\upsilon_1\colon \mathscr{U} \longrightarrow \mathscr{U}$ and $\upsilon_2\colon \mathscr{U}_{x_0} \longrightarrow \mathscr{U}_{x_0}$ together with the initial condition $\upsilon_1(0) = 0 = \upsilon_2(0)$, $|\upsilon_1(z)| < 1$, and $|\upsilon_2(w)| \leq 1$. By employing (16) and (17) and carrying out simple computations yields

$$(T^{c,\,\mu,\,n}_{\tau,\,\varphi,\,k,\,x}f(z))' = 1 + \frac{1}{4}u_1 z + \left(\frac{1}{4}u_2 - \frac{7}{48}u_1^2\right)z^2 + \left(\frac{17}{192}u_1^3 - \frac{7}{24}u_2u_1 + \frac{1}{4}u_3\right)z^3 + \cdots \tag{18}$$

and

$$(T^{c,\,\mu,\,n}_{\tau,\,\varphi,\,k,\,x}h(w))' = 1 + \frac{1}{4}b_1 w + \left(\frac{1}{4}b_2 - \frac{7}{48}b_1^2\right)w^2 + \left(\frac{17}{192}b_1^3 - \frac{7}{24}b_2b_1 + \frac{1}{4}b_3\right)w^3 + \cdots \tag{19}$$

where $u, b \in \mathscr{P}$ since

$$u(z) = \frac{1+\upsilon_1(z)}{1-\upsilon_1(z)} = 1 + \sum_{m=1}^{\infty} u_m z^m$$

and

$$b(z) = \frac{1+\upsilon_2(z)}{1-\upsilon_2(z)} = 1 + \sum_{m=1}^{\infty} b_m w^m.$$

To find the values of a_2, a_3, and a_4, it is necessary to compare and equate equations (18) and (19), which gives

$$2(\top(2))^n a_2 = \frac{1}{4}u_1, \tag{20}$$

$$3(\top(3))^n a_3 = \frac{1}{4}u_2 - \frac{7}{48}u_1^2, \tag{21}$$

$$4(\top(4))^n a_4 = \frac{17}{192}u_1^3 - \frac{7}{24}u_2u_1 + \frac{1}{4}u_3 \tag{22}$$

also,

$$-2(\top(2))^n a_2 = \frac{1}{4}b_1, \tag{23}$$

$$6(\top(3))^n a_2^2 - 3(\top(3))^n a_3 = \frac{1}{4}b_2 - \frac{7}{48}b_1^2, \tag{24}$$

$$-20(\top(4))^n a_2^3 + 20(\top(4))^n a_2a_3 - 4(\top(4))^n a_4 = \frac{17}{192}b_1^3 - \frac{7}{24}b_2b_1 + \frac{1}{4}b_3. \tag{25}$$

The following equation is obtained by using equations (20) and (23) in the manner described below

$$\frac{1}{8(\top(2))^n}u_1 = -\frac{1}{8(\top(2))^n}b_1 \Rightarrow u_1 = -b_1 \Rightarrow u_1^2 = b_1^2 \Rightarrow u_1^3 = -b_1^3. \tag{26}$$

By applying Lemma 1 and performing elementary calculations on the concluding equation, the theorem's conclusion becomes clear:

$$|a_2| \leq \frac{1}{4(\top(2))^n}. \tag{27}$$

By utilizing equations (24) and (21), we can calculate the boundary for a_3, with $u_1 = -b_1$ being established, which gives

$$a_3 = a_2^2 + \frac{(u_2 - b_2)}{24(\top(3))^n},$$

so that

$$a_3 = \frac{1}{64((\top(2))^n)^2}u_1^2 + \frac{(u_2 - b_2)}{24(\top(3))^n}. \tag{28}$$

In the same way, utilizing equations (22) and (25) with regard to equations (26) and (28) to determine a_4 leads to the subsequent equation:

$$a_4 = \frac{17}{768(\top(4))^n}u_1^3 + \frac{5(u_2 - b_2)u_1}{384(\top(2))^n(\top(3))^n} + \frac{(u_3 - b_3)}{32(\top(4))^n} - \frac{7u_1(u_2 + b_2)}{192(\top(4))^n}. \tag{29}$$

It can be seen from (26) and applying Lemma 2 with $|u_4| \leq 1$, $|b_4| \leq 1$, $|z| \leq 1$, and $|w| \leq 1$ results in:

$$u_2 - b_2 = \frac{4 - u_1^2}{2}(u_4 - b_4), \tag{30}$$

$$u_2 + b_2 = u_1^2 + \frac{4 - u_1^2}{2}(u_4 + b_4), \tag{31}$$

$$u_3 - b_3 = \frac{u_1^3}{2} + \frac{(4 - u_1^2)u_1}{2}(u_4 + b_4) - \frac{(4 - u_1^2)u_1}{4}(u_4^2 + b_4^2) + \frac{4 - u_1^2}{2}\left([1 - |u_4|^2]z - [1 - |b_4|^2]w\right). \tag{32}$$

Utilizing equations (28) and (30) leads to

$$a_3 = \frac{1}{64((\top(2))^n)^2}u_1^2 + \frac{(4 - u_1^2)(u_4 - b_4)}{48(\top(3))^n}. \tag{33}$$

Lemma 1 makes it clear that achieving $|u_1| = \gamma$ is a favorable result, resulting in the expression $4 - u_1^2 = 4 - \gamma^2$.

By choosing γ within the range of 0 to 2 and applying the triangular inequality, we find $|u_4| = u_5$ and $|b_4| = b_5$, leading to

$$|a_3| \leq \frac{1}{64((\top(2))^n)^2}\gamma^2 + \frac{(4-\gamma^2)(u_5+b_5)}{48(\top(3))^n}, \quad (u_5, b_5) \in [0, 1]^2.$$

Additionally, there is a requirement to define a function Q: $\mathbb{R} \longrightarrow \mathbb{R}$ that must be evaluated for its maximum value within the closed square $E = \{(u_5, b_5)\colon (u_5, b_5) \in [0, 1]^2\}$ as depicted:

$$Q(u_5, b_5) = \frac{1}{64((\top(2))^n)^2}\gamma^2 + \frac{(4-\gamma^2)(u_5+b_5)}{48(\top(3))^n}, \quad (u_5, b_5) \in [0, 1]^2.$$

The highest value of $Q(u_5, b_5)$ is found on the boundary of E, and by taking the derivative of $Q(u_5, b_5)$ with respect to u_5, it can be concluded:

$$Q_{u_5}(u_5, b_5) = \frac{(4-\gamma^2)}{48(\top(3))^n}.$$

Assuming $Q_{u_5}(u_5, b_5) \geq 0$, given that b_5 is within the range of $[0, 1]$ and γ falls within the range of $[0, 2]$. It can be inferred that as u_5 increases, the function $Q(u_5, b_5)$ also increases and peaks at $u_5 = 1$, suggesting that:

$$\max[Q(u_5, b_5)\colon u_5 \in [0, 1]] = Q(1, b_5) = \frac{1}{64((\top(2))^n)^2}\gamma^2 + \frac{(4-\gamma^2)(1+b_5)}{48(\top(3))^n}.$$

Taking further derivative of $Q(1, b_5)$ leads to

$$Q'(1, b_5) = \frac{(4-\gamma^2)}{48(\top(3))^n}.$$

Assuming $Q'(1, b_5) \geq 0$, given that $\gamma \in [0, 2]$. It is evident that the function $Q(1, b_5)$ rises and peaks at $b_5 = 1$, indicating that:

$$\max[Q(u_5, b_5)\colon u_5 \in [0, 1]] = Q(1, 1) = \frac{1}{64((\top(2))^n)^2}\gamma^2 + \frac{(4-\gamma^2)}{24(\top(3))^n}.$$

Furthermore,

$$Q(u_5, b_5) \leq \max[Q(u_5, b_5)\colon u_5 \in [0, 1]] = Q(1, 1) = \frac{1}{64((\top(2))^n)^2}\gamma^2 + \frac{(4-\gamma^2)}{24(\top(3))^n}.$$

If a_3 has an absolute value that is smaller or equal to $Q(u_5, b_5)$, it is easy to see that.

$$|a_3| \leq S(n)\gamma^2 + \frac{1}{6(\top(3))^n}, \quad \gamma \in [0, 2],$$

taking

$$S(n) = \frac{1}{4}\left[\frac{1}{16((\top(2))^n)^2} - \frac{1}{6(\top(3))^n}\right].$$

We present a function $Q_1 \colon \mathbb{R} \longrightarrow \mathbb{R}$ that is intended to find its highest possible value, given by:

$$Q_1(\gamma) = S(n)\gamma^2 + \frac{1}{6(\top(3))^n}, \quad \gamma \in [0, 2].$$

The equation $Q_1'(\gamma) = 2S(\zeta)\gamma$ is true when evaluating the derivative of $Q_1(\gamma)$ with γ values in the range of $[0, 2]$. When $S(\zeta) \leq 0$, $Q_1'(\gamma) \leq 0$, causing $Q_1(\gamma)$ to decrease and reach a peak at $\gamma = 0$. Thus,

$$\max[Q_1(\gamma) \colon \gamma \in [0, 2]] = Q_1(0) = \frac{1}{6(\top(3))^n},$$

If $S(\zeta) \geq 0$ holds, then $Q_1'(\gamma) \geq 0$, leading to $Q_1(\gamma)$ becoming a increasing function with its peak at $\gamma = 2$. Hence

$$\max[Q_1(\gamma) \colon \gamma \in [0, 2]] = Q_1(2) = \frac{1}{16((\top(2))^n)^2}.$$

Therefore, a successful calculation of the precise maximum value for $|a_3|$ is presented in the following manner:

$$|a_3| \leq \frac{1}{6(\top(3))^n}. \tag{34}$$

By using equations (29), (30), (31), and (32) in conjunction with the familiar triangular inequality, we can express the inequality for the magnitude of a_4 as follows:

$$|a_4| \leq s_1(\gamma) + s_2(\gamma)(u_5 + b_5) + s_3(\gamma)(u_5^2 + b_5^2) = Q_2(u_5, b_5),$$

so that

$$s_1(\gamma) = \frac{1}{768(\top(4))^n}\gamma^3 + \frac{(4-\gamma^2)}{32(\top(4))^n},$$

$$s_2(\gamma) = \frac{5(4-\gamma^2)}{768(\top(2))^n(\top(3))^n}\gamma + \frac{3(4-\gamma^2)}{832(\top(4))^n}\gamma,$$

$$s_3(\gamma) = \frac{(4-\gamma^2)(\gamma-2)}{128(\top(4))^n}.$$

The parameters $s_1(\gamma)$, $s_2(\gamma)$, and $s_3(\gamma)$ of $Q_2(u_5, b_5)$ change depending on the parameter γ, necessitating the maximization of $Q_2(u_5, b_5)$ on E for every $\gamma \in [0, 2]$. Following this, it is vital to identify the highest value that $Q_2(u_5, b_5)$ can take for diverse γ values. Due to the fact that gamma is equal to zero, as s_2 at 0 equals 0,

$$s_1(0) = \frac{1}{8(\top(4))^n}, \quad s_3(0) = -\frac{1}{16(\top(4))^n},$$

and gives the following:

$$Q_2(u_5, b_5) = \frac{1}{8(\top(4))^n} - \frac{1}{16(\top(4))^n}(u_5^2 + b_5^2), \quad (u_5, b_5) \in [0, 1]^2,$$

Consequently, we possess

$$Q_2(u_5, b_5) \leq \max[Q(u_5, b_5): (u_5, b_5) \in E] = Q(0, 0) = \frac{1}{8(\top(4))^n}.$$

Let's consider γ to be equal to 2. If $s_2(2) = s_3(2) = 0$, it follows that

$$s_1(2) = \frac{1}{96(\top(4))^n}.$$

As a result, we have a set value for the function $Q_2(u_5, b_5)$ as stated:

$$Q_2(u_5, b_5) = s_1(2) = \frac{1}{96(\top(4))^n}.$$

Noticing that $Q_2(u_5, b_5)$ does not achieve a highest value on E when γ is in the range of [0, 2], we can infer that

$$|a_4| \leq \frac{1}{8(\top(4))^n}. \tag{35}$$

Basically, we can verify that the outcomes achieved in equations (27), (34) and (35) are applicable to the functions mentioned here:

$$f_1(z) = z + \frac{1}{4(\top(2))^n} + \frac{1}{12(\top(3))^n} + \cdots,$$

$$f_2(z) = z + \frac{1}{6(\top(3))^n} z^3 + \frac{1}{16(\top(4))^n} z^4 + \cdots,$$

$$f_3(z) = z + \frac{1}{8(\top(4))^n} z^4 + \frac{1}{20(\top(5))^n} + \cdots.$$

□

The specified Corollary is valid if the condition $n = 1$ is met as stated in Theorem 1.

Corollary 1 If $f \in \mathscr{T}\mathscr{S}(\Xi)$. Then,

$$|a_2| \le \frac{1}{4}, \quad |a_3| \le \frac{1}{6} \quad |a_4| \le \frac{1}{8}.$$

5. Second Hankel determinant

Theorem 2 Suppose $f \in \mathscr{T}\mathscr{S}^{\mu,\,k}_{x,\,s,\,\varphi}(n,\,\Xi)$. Then,

$$|a_3^2 - a_2 a_3| \le \left(\frac{1}{6(\top(3))^n}\right)^2,$$

where

$$f_2(z) = z + \frac{1}{6(\top(3))^n} z^3 + \frac{1}{16(\top(4))^n} z^4 + \cdots.$$

Proof. If $f \in \mathscr{T}\mathscr{S}^{\mu,\,k}_{x,\,s,\,\varphi}(n,\,\Xi)$, $\tau \in \mathbb{R}$, $k > 0$, $\mu \ge 0$, $\varphi > 0$ with $\tau + \varphi > 0$, $c \in \mathbb{N} = \{1, 2, \cdots\}$, and $n,\, l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then let's consider the situation. By using equations (26), (28), and (29), we can show that $a_2 a_4 - a_3^2$ is equivalent, that is

$$\begin{aligned}
a_2 a_4 - a_3^2 =& \frac{17}{6144(\top(2))^n(\top(4))^n} u_1^4 - \frac{1}{4096((\top(2))^n)^4} u_1^4 + \frac{u_1^2(u_2 - b_2)}{3072((\top(2))^n)^2(\top(3))^n} \\
&+ \frac{1}{256(\top(2))^n(\top(4))^n} u_1(u_3 - b_3) - \frac{7}{1536(\top(2))^n(\top(4))^n} u_1^2(u_2 + b_2) \\
&- \frac{1}{576((\top(3))^n)^2}(u_2 - b_2)^2.
\end{aligned}$$

By utilizing equations (30), (31) and (32), and choosing γ values between 0 and 2 while employing the triangle inequality leads to $|u_4| = u_5$ and $|b_4| = b_5$, the outcome is:

$$|a_2a_4 - a_3^2| \leq S_1(\gamma) + S_2(\gamma)(u_5 + b_5) + S_3(\gamma)(u_5^2 + b_5^2) + S_4(\gamma)(u_5 + b_5)^2, \tag{36}$$

so that

$$S_1(\gamma) = \frac{1}{6144(\top(2))^n(\top(4))^n}\gamma^4 + \frac{1}{4096((\top(2))^n)^4}\gamma^4 + \frac{4-\gamma^2}{256(\top(2))^n(\top(4))^n}\gamma \geq 0,$$

$$S_2(\gamma) = \frac{(4-\gamma^2)}{6144((\top(2))^n)^2(\top(3))^n}\gamma^2 + \frac{(4-\gamma^2)}{3072(\top(2))^n(\top(4))^n}\gamma^2 \geq 0,$$

$$S_3(\gamma) = \frac{(4-\gamma^2)(\gamma-2)\gamma}{1024(\top(2))^n(\top(4))^n} \leq 0,$$

$$S_4(\gamma) = \frac{(4-\gamma^2)^2}{2304((\top(3))^n)^2} \geq 0.$$

Furthermore, the function Q_3: $\mathbb{R} \to \mathbb{R}$ aiming to identify its optimal value for γ within the specified range of [0, 2] as described by the provided expression:

$$Q_3(u_5, b_5) = S_1(\gamma) + S_2(\gamma)(u_5 + b_5) + S_3(\gamma)(u_5^2 + b_5^2) + S_4(\gamma)(u_5 + b_5)^2, \quad (u_5, b_5) \in [0, 1]^2.$$

The values of $Q_3(u_5, b_5)$ coefficients $S_1(\gamma)$, $S_2(\gamma)$, $S_3(\gamma)$, and $S_4(\gamma)$ change with the parameter γ, calling for optimization of $Q_3(u_5, b_5)$ over E for every γ in the range [0, 2]. Afterwards, we must find the maximum parameter of $Q_3(u_5, b_5)$ for different γ values.

(a) With γ equal to zero, since $S_1(0) = S_2(0) = S_3(0) = 0$,

$$S_4(0) = \frac{1}{144((\top(3))^n)^2}$$

which gives

$$Q_3(u_5, b_5) = \frac{1}{144((\top(3))^n)^2}(u_5 + b_5)^2, \quad (u_5, b_5) \in E.$$

One can conclude that the highest value of $Q_3(u_5, b_5)$ is situated at the boundary of E, and by taking the derivative of $Q_3(u_5, b_5)$ with respect to u_5, it can be deduced that:

$$(Q_3)_{u_5}(u_5, b_5) = \frac{1}{72((\top(3))^n)^2}(u_5 + b_5), \quad b_5 \in [0, 1].$$

Assuming that $(Q_3)_{u_5}(u_5, b_5) \geq 0$, with b_5 ranging from 0 to 1 and γ ranging from 0 to 2. The function $Q_3(u_5, b_5)$ is found to rise as u_5 increases and reaches its maximum at $u_5 = 1$, suggesting that:

$$\max[Q_3(u_5, b_5)\colon u_5 \in [0, 1]] = Q_3(1, b_5) = \frac{1}{144((\top(3))^n)^2}(1+b_5)^2, \ b_5 \in [0, 1].$$

Further derivation of $Q_3(1, b_5)$ leads to

$$Q_3'(1, b_5) = \frac{1}{72((\top(3))^n)^2}(1+b_5).$$

If $(Q_3)_{u_5}(1, b_5)$ is greater than or equal to 0, for γ in the range of $[0, 2]$. The function $Q_3(1, b_5)$ is observed to rise and reach its maximum at $b_5 = 1$, suggesting that:

$$\max[Q_3(1, b_5)\colon b_5 \in [0, 1]] = Q_3(1, 1) = \left(\frac{1}{6(\top(3))^n}\right)^2.$$

Therefore, when $\gamma = 0$, we get:

$$Q_3(u_5, b_5) \leq \max[Q_3(u_5, b_5); (u_5, b_5) \in [0, 1]^2] = Q_3(1, 1) = \left(\frac{1}{6(\top(3))^n}\right)^2.$$

Since $|a_3^2 - a_2a_4| \leq Q_3(u_5, b_5)$, yields

$$|a_3^2 - a_2a_3| \leq \left(\frac{1}{6(\top(3))^n}\right)^2.$$

(b) Let γ be equal to 2. If $S_2(2) = S_3(2) = S_4(2) = 0$, then

$$S_1(2) = \frac{1}{384(\top(2))^n(\top(4))^n} + \frac{1}{256((\top(2))^n)^4},$$

which offers the consistent function displayed underneath:

$$Q_3(u_5, b_5) = S_1(2) = \frac{1}{384(\top(2))^n(\top(4))^n} + \frac{1}{256((\top(2))^n)^4}.$$

Hence, yields

$$|a_3^2 - a_2a_3| \leq \frac{1}{384(\top(2))^n(\top(4))^n} + \frac{1}{256((\top(2))^n)^4},$$

for $\gamma = 2$.

(c) To examine the maximum point of $Q_3(u_5, b_5)$ in the interval $\gamma \in (0, 2)$, we will employ the function $\Xi(Q_3) = (Q_3)_{u_5u_5}(u_5, b_5)(Q_3)_{b_5b_5}(u_5, b_5) - ((Q_3)_{u_5b_5}(u_5, b_5))^2$.

Furthermore, we will analyze two situations to establish the preferred result for the expression $\Xi(Q_3) = 4S_3(\gamma)\{S_3(\gamma) + 2S_4(\gamma)\}$ in this case.

(i) If $s_3(\gamma) + 2S_4(\gamma) \leq 0$ for $\gamma \in (0, 2)$, then the function Q_3 will not possess a maximum on E since $(Q_3)_{u_5, b_5}(u_5, b_5) = (Q_3)_{b_5, u_5}(u_5, b_5) = 2S_4(\gamma) \geq 0$, and $\Xi(Q_3) \geq 0$.

(ii) In order for the maximum value of function Q_3 on E to occur, the condition $E(Q_3) \leq 0$ must hold when $S_3(\gamma) + 2S_4(\gamma) \geq 0$ for $\gamma \in (0, 2)$.

As a result of the results from the three cases, we develop

$$|a_3^2 - a_2a_3| \leq \left(\frac{1}{6(\top(3))^n}\right)^2. \tag{37}$$

In essence, we can establish that the outcome found in equation (37) is indeed valid for the function in question.

$$f_2(z) = z + \frac{1}{6(\top(3))^n}z^3 + \frac{1}{16(\top(4))^n}z^4 + \cdots.$$

□

According to Theorem 2, the Corollary holds true provided that the condition $\gamma = 1/2$ is satisfied.

Corollary 2 If $f \in \mathscr{T}\mathscr{S}(\Xi)$. Then,

$$|a_3^2 - a_2a_3| \leq \left(\frac{1}{6}\right)^2.$$

The function listed below is used to verify the accuracy of the result.

$$f_2(z) = z + \frac{1}{6}z^3 + \frac{1}{16}z^4 + \cdots.$$

6. Fekete-Szego inequality

Theorem 3 Suppose $f \in \mathscr{T}\mathscr{S}_{x,s,\varphi}^{\mu,k}(n, \Xi)$. Then,

$$\left|a_3 - \rho a_2^2\right| \leq \begin{cases} \frac{\rho(n)}{16((\top(2))^n)^2}, & |1-\rho| \leq \rho(n) \\ \frac{|1-\rho|}{16((\top(2))^n)^2}, & |1-\rho| \geq \rho(n) \end{cases},$$

where

$$\rho(n) = \frac{16((\top(2))^n)^2}{6(\top(3))^n}. \tag{38}$$

The precise maximum bounds on the specified function validated the accuracy of the equation:

$$f_2(z) = z + \frac{1}{6(\top(3))^n} z^3 + \frac{1}{16(\top(4))^n} z^4 + \cdots.$$

Proof. If $f \in \mathscr{T}\mathscr{S}^{\mu,k}_{x,s,\varphi}(n, \Xi)$, $\tau \in \mathbb{R}$, $k > 0$, $\mu \geq 0$, $\varphi > 0$ with $\tau + \varphi > 0$, $c \in \mathbb{N} = \{1, 2, \cdots\}$, and $n, l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then let's consider the situation. By using equations (26), (28), (30) and (31), we can show that $a_3 - \rho a_2^2$ is equivalent, that is

$$a_3 - \rho a_2^2 = \frac{1}{64((\top(2))^n)^2} \left[1 - \rho\right] u_1^2 + \frac{(4 - u_1^2)(u_4 + b_4)}{48(\top(3))^n}. \tag{39}$$

Using equation (39), we can choose γ between 0 and 2 and apply the triangle inequality to obtain $|u_4| = u_5$, and $|b_4| = b_5$, gives

$$\left|a_3 - \rho a_2^2\right| \leq \frac{|1 - \rho|}{64((\top(2))^n)^2} \gamma^2 + \frac{(4 - \gamma^2)(u_5 + b_5)}{48(\top(3))^n}.$$

Our goal is to find the highest value of the function Q_4: $\mathbb{R} \longrightarrow \mathbb{R}$ for $\gamma \in [0, 2]$. The function is defined as:

$$Q_4(u_5, b_5) = \frac{|1 - \rho|}{64((\top(2))^n)^2} \gamma^2 + \frac{(4 - \gamma^2)(u_5 + b_5)}{48(\top(3))^n}, \; (u_5, b_5) \in E, \; \gamma \in [0, 2].$$

The highest value of $Q_4(u_5, b_5)$ is found at the boundary of E, and by taking the derivative of $Q_4(u_5, b_5)$ with respect to u_5, we can ascertain that:

$$(Q_4)_{u_5}(u_5, b_5) = \frac{(4 - \gamma^2)}{48(\top(3))^n}, \quad \gamma \in [0, 2].$$

Assuming $(Q_4)_{u_5}(u_5, b_5) \geq 0$, with b_5 ranging from 0 to 1 and γ ranging from 0 to 2. One can ascertain that the function $Q_4(u_5, b_5)$ grows as u_5 increases and peaks at $u_5 = 1$, which suggests that:

$$\max[Q_4(u_5, b_5): u_5 \in [0, 1]] = Q_4(1, b_5)$$

$$= \frac{|1 - \rho|}{64((\top(2))^n)^2} \gamma^2 + \frac{(4 - \gamma^2)(u_5 + b_5)}{48(\top(3))^n}, \quad b_5 \in [0, 1], \quad \gamma \in [0, 2].$$

Further elaboration of $Q_4(1, b_5)$ leads to

$$Q_4'(1, b_5) = \frac{(4-\gamma^2)}{48(\top(3))^n}, \quad \gamma \in [0, 2].$$

Given that $(Q_4)_{u_5}(1, b_5) \geq 0$ for $\gamma \in [0, 2]$. It is evident that the function $Q_4(1, b_5)$ rises steadily and peaks at $b_5 = 1$, suggesting that:

$$\max[Q_4(1, b_5) \colon u_5 \in [0, 1]] = Q(1, 1) = \frac{|1-\rho|}{64((\top(2))^n)^2}\gamma^2 + \frac{(4-\gamma^2)}{48(\top(3))^n}, \quad \gamma \in [0, 2].$$

Futhermore, we have

$$Q_4(u_5, b_5) \leq \max[(u_5, b_5) \colon (u_5, b_5) \in E] = Q_4(1, 1)$$

$$= \frac{|1-\rho|}{64((\top(2))^n)^2}\gamma^2 + \frac{(4-\gamma^2)}{48(\top(3))^n}.$$

Since $\left|a_3 - \rho a_2^2\right| \leq Q_3(u_5, b_5)$, gives

$$\left|a_3 - \rho a_2^2\right| \leq \frac{1}{4}\left[\frac{|1-\rho| - \rho(n)}{16((\top(2))^n)^2}\right]\gamma^2 + \frac{\rho(n)}{16((\top(2))^n)^2},$$

where

$$\rho(\zeta) = \frac{16((\top(2))^n)^2}{6(\top(3))^n}.$$

Function $Q_5 \colon [0, 2] \longrightarrow \mathbb{R}$ is introduced to find the highest value it can take for γ in the interval $[0, 2]$, defined as:

$$Q_5(\gamma) = \frac{1}{4}\left[\frac{|1-\rho| - \rho(n)}{16((\top(2))^n)^2}\right]\gamma^2 + \frac{\rho(n)}{16((\top(2))^n)^2}.$$

Further differentiation of $Q_5(\gamma)$ leads to

$$Q_5'(\gamma) = \frac{1}{2}\left[\frac{|1-\rho| - \rho(n)}{16((\top(2))^n)^2}\right]\gamma.$$

If $Q_5'(\gamma) \leq 0$, $Q_5(\gamma)$ will decrease. The peak of the function occurs at $\gamma = 0$ if $|1-\rho| \leq \rho(\zeta)$. So

$$\max[Q_5(\gamma);\, \gamma \in [0,\, 2]] = Q_5(0) = \frac{\rho(n)}{16((\top(2))^n)^2}.$$

If $Q'_5(\gamma) \geq 0$, $Q_5(\gamma)$ will experience an increase. The function reaches its maximum at $\gamma = 2$ if $|1-\rho| \geq \rho(\zeta)$. Thus

$$\max[Q_5(\gamma);\, \gamma \in [0,\, 2]] = Q_5(2) = \frac{|1-\rho|}{16((\top(2))^n)^2}.$$

Hence, we get

$$\left|a_3 - \rho a_2^2\right| \leq \begin{cases} \frac{\rho(n)}{16((\top(2))^n)^2}, & |1-\rho| \leq \rho(n) \\ \\ \frac{|1-\rho|}{16((\top(2))^n)^2}, & |1-\rho| \geq \rho(n) \end{cases}. \tag{40}$$

In essence, we can verify that the outcome derived in equation (40) is indeed applicable to:

$$f_2(z) = z + \frac{1}{6(\top(3))^n} z^3 + \frac{1}{16(\top(4))^n} z^4 + \cdots.$$

□

According to Theorem 3, if the condition $n = 0$ holds true, the corresponding Corollary is valid.

Corollary 3 If $f \in \mathscr{G}\mathscr{B}_{\Xi}^{Sig}$. Then,

$$\left|a_3 - \rho a_2^2\right| \leq \begin{cases} \frac{\rho}{16}, & |1-\rho| \leq \rho \\ \\ \frac{|1-\rho|}{16}, & |1-\rho| \geq \rho \end{cases},$$

where

$$\rho = \frac{16}{6}.$$

The function listed below is used to verify the accuracy of the result.

$$f_2(z) = z + \frac{1}{6} z^3 + \frac{1}{16} z^4 + \cdots.$$

Theorem 4 Suppose $f \in \mathscr{T}\mathscr{S}_{x,\, s,\, \varphi}^{\mu,\, k}(n,\, \Xi)$. Then,

$$
\left|a_3-\rho a_2^2\right| \leq \begin{cases} \frac{(1-\rho)}{16((\top(2))^n)^2} & \text{if } \rho \leq 1-\rho(\zeta) \\ \\ \frac{\rho(\zeta)}{16((\top(2))^n)^2} & \text{if } 1-\rho(\zeta) \leq \rho \leq 1+\rho(\zeta) \\ \\ \frac{(\rho-1)}{16((\top(2))^n)^2} & \text{if } 1+\rho(\zeta) \leq \rho \end{cases} \tag{41}
$$

where

$$
\rho(\zeta)=\frac{16((\top(2))^n)^2}{6(\top(3))^n}.
$$

Proof. Assuming $f \in \mathscr{T}\mathscr{S}_{x,\,s,\,\varphi}^{\mu,\,k}(n,\,\Xi)$, $0<\zeta<1$. We have $|1-\rho| \geq \rho(\zeta)$ and $|1-\rho| \leq \rho(\zeta)$ when $\rho \in \mathbb{R}$. Yields:

$$
\rho \leq 1-\rho(\zeta) \text{ either } \rho \geq 1+\rho(\zeta)
$$

and

$$
1-\rho(\zeta) \leq \rho \leq 1+\rho(\zeta).
$$

□

Furthermore, the outcomes for $\rho=1$ can be obtained from Theorem 4.

Corollary 4 Suppose $f \in \mathscr{T}\mathscr{S}_{x,\,s,\,\varphi}^{\mu,\,k}(n,\,\Xi)$. Then,

$$
\left|a_3-a_2^2\right| \leq \frac{1}{6(\top(3))^n}.
$$

Corollary 5 If $f \in \mathscr{G}\mathscr{B}_{\Xi}^{Sig}$. Then,

$$
|a_3-a_2^2| \leq \frac{1}{6}.
$$

7. Conclusion

We explored a novel subclass of $\mathscr{B}\mathscr{U}s$ situated within the unit disk, leveraging subordination and the concept of Gregory coefficients. The article is structured into five distinct sections. The first section furnishes essential background information and definitions for this inquiry. The second section presents well-established lemmas alongside a novel subclass of $\mathscr{B}\mathscr{U}s$ generated through the integration of Gregory coefficients. The third section delves into the novel subclass, focusing on obtaining precise coefficient bounds, each of which is verified with utmost accuracy by applying the extremal function to the newfound category of $\mathscr{B}\mathscr{U}s$. In the fourth section, we extracted the exact limit for the Hankel determinant. The final section elucidates the precise limit for the Fekete-Szego Inequality, encompassing real and complex parameter values. It is crucial to note that the newly discovered subclass of $\mathscr{BU}s$ stands apart from existing methodologies, rendering direct comparison impossible. Thus, we accurately determined the extremal function for the novel category to corroborate the exact limits for the characteristics examined in this study. This operator can be used in creating a modelling part of dynamical systems which is suggested has a future research for researchers in the related fields. for more details see [39, 40].

Conflict of interest

The authors declare no competing financial interest.

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