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Research Article

An Multi-Attribute Decision Making Approach Based on Bipolar T-Spherical Fuzzy Hypersoft Set with Application to Industrial Air Filters

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Abstract: Introducing Bipolar T-Spherical Fuzzy Hypersoft Set as a hybrid extension of the bipolar fuzzy set and the T-spherical fuzzy set is a notable advancement in the fuzzy set theory. To address situations involving numerous sub-attributes, the Hypersoft set is presented as an additional generalization of the soft set. The theory of Bipolar T-Spherical Fuzzy Hypersoft Sets (BTSFHSS) with related score and accuracy functions are developed in this study to handle situations involving decision-making in a bipolar fuzzy environment with intricate attribute structures. In order to connect BTSFHSS to the fundamental algebraic structures of existing fuzzy set theories, fundamental algebraic properties and elementary theorems are established. There are many air filters that have some benefits and drawbacks compared to one another. The efficiency of air filters in lowering industrial air pollution is investigated as a real-world application. The best filter is then chosen from a range of options using a Multi-Attribute Decision-Making (MADM) technique.

***Keywords*:** Hypersoft, bipolar T-spherical fuzzy Hypersoft set, multi attribute decision making

MSC: 03B52, 03E72, 08A72

Abbreviation

FS	Fuzzy Set
IFS	Intuitionistic Fuzzy Set
PyFS	Pythagorean Fuzzy Set
PFS	Picture Fuzzy Set
qLDFS	q-Rung Linear Diophantine Fuzzy Set
SFS	Spherical Fuzzy Set
TSFS	T-spherical Fuzzy Set
MG	Membership Grade
NMG	Non Membership Grade
AG	Abstinence Grade
PMS	Positive Membership

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PNM	Positive Non-Membership
NM	Negative Membership
NNM	Negative Non-Membership
PAb	Positive Abstinence
NAb	Negative Abstinence
SS	Soft Set
FSS	Fuzzy Soft Set
IFSS	Intuitionistic Fuzzy Soft Set
PyFSS	Pythagorean Fuzzy Soft Set
PFSS	Pictue Fuzzy Soft Set
SFSS	Spherical Fuzzy Soft Set
TSFSS	T-Spherical Fuzzy Soft Set
BSFSS	Bipolar Spherical Fuzzy Soft Set
HSS	Hypersoft Set
BFS	Bipolar Fuzzy Set
BIFS	Bipolar Intuitionistic Fuzzy Set
BPyFS	Bipolar Pythagorean Fuzzy Set
BPFS	Bipolar Picture Fuzzy Set
BSFS	Bipolar Spherical Fuzzy Set
BTSFS	Bipolar T-Spherical Fuzzy Set
BFHSS	Bipolar Fuzzy Hypersoft Set
BLDFHSS	Bipolar Linear Diophantine Fuzzy Hypersoft Set
BTSFHSN	Bipolar T-Spherical Fuzzy Hypersoft Number
BTSFHSS	Bipolar T-Spherical Fuzzy Hypersoft Set

1. Introduction

Industries play a significant role in the economic progress of any nation. It opens up vast career opportunities for a large number of people. Industries are the foundation of modern society, producing everything from automobiles, technology, food, and pharmaceuticals. However, the importance of industries is overshadowed by the harmful influence they have on the environment. Industrial waste pollutes water, air, and land. Economic growth must be evaluated along with the health of the environment. To balance both concerns, pollution filters can be utilized to minimize pollutant particles effectively. This manuscript discusses a list of industrial air filters, and an optimal filter is chosen based on its effectiveness in all factors when compared to all other filters. To account for both negative grades and a multi-sub-attribute situation, this manuscript introduces the Bipolar T-spherical Fuzzy Hypersoft set theory.

1.1 *Outline of the study*

There are six separate sections in this manuscript. The introduction and the literature evaluation of existing theories are covered in Section 1. Also, the research gaps and the contribution in the manuscript have been discussed. Section 2 contains the fundamental ideas that serve as the framework for the suggested theory. The suggested theory, a few operators, and some basic theorems are defined in Section 3. Section 4 provides a detailed explanation of the case study. Furthermore, an algorithm is defined in Section 4 to choose an optimal solution from the given data. Section 5 describes a comparative analysis. In Section 6, a summary of the theoretical construction and possible advancement of the proposed theory is provided.

1.2 *Literature review*

To address real-world Multi-Attribute Decision Making (MADM) issues, a more effective framework is required than the classical set theory, which was developed by Zadeh [1] in 1965, who was the initiator of Fuzzy Sets (FS). The arrival of FS eased the insufficiencies to solve many real-world applications and led to the new branch of Mathematics. Despite having many advantages, FS faces some drawbacks. FS cannot handle the circumstances where both the truth and falsity values are to be considered. To eradicate it, Atanassov [2] commenced the theory of Intuitionistic Fuzzy Sets (IFS) where the range of the sum of membership and non-membership grades is restricted within the interval [0, 1]. However, it has never come to a conclusion. The new issue arises when the sum of the grades exceeds 1. To get rid of it, the theory was further developed to Picture Fuzzy Sets (PFS), Linear Diophantine Fuzzy Sets (LDFS) and q-Rung Linear Diophantine Fuzzy Sets (q-RLDFS) which was developed by Yager [3], Riaz and Hashmi [4] and Almagrabi [5] respectively. Later, many decision-making applications were discussed in [6, 7].

Even though having many innovations, there was a deficiency in dealing with neutral grades. Cuong [8] was the one who sorted it out through the introduction of PFS. In the PFS, when the Membership, non-membership, and abstinence grades are added, its total lies in the interval [0, 1]. When it comes to the situation where the sum of Membership Grade, Abstinence Grade and Non Membership Grade are greater than 1, PFS shows the inadequacy to handle them. Identifying the shortcomings of PFS, Muhmood et al. [9] devised Spherical Fuzzy Sets (SFS) and T-Spherical Fuzzy Sets (TSFS) to enhance the efficiency of the theory. After all these theoretical developments, questions arose about how to deal with all these grades in a negative sense, which motivates Zhang [10] to initiate Bipolar Fuzzy Sets (BFS). In a similar way, BFS was fused with IFS and PFS by Sankar et al. [11] and Riaz et al. [12]. To annihilate the inefficacy of constraints of grades that are evolved in PFS, Princy et al. [13] came up with a new notion, which were Bipolar Spherical Fuzzy Sets (BSFS) and Bipolar T-spherical Fuzzy Sets (BTSFS). Molodtsov [14] introduced Soft Sets (SS), which became an effective mathematical tool to deal with uncertainty. After that, SS was merged with the concept of FS by Roy and Maji [15]. Later, it was extended to Intuitionistic Fuzzy Soft Sets (IFSS), Pythagorean Fuzzy Soft Sets (PyFS), Picture Fuzzy Soft Sets (PFSS), Spherical Fuzzy Soft Sets (SFSS) and T-Spherical Fuzzy Soft Sets (TSFSS) by the researchers Maji et al. [16], Peng et al. [17], Yang et al. [18], Perveen et al. [19]. As a generalization, Hypersoft Sets (HSS) were devised by Smarandache [20] along with Fuzzy Hypersoft Sets (FHSS) and Intuitionistic Fuzzy Hypersoft Sets (IFHSS). Later, it is extended to Pythagorean Fuzzy Hypersoft Sets (PyFHSS) by Zulqarnain et al. [21]. To consider the reference parameter in a multi-sub attribute situation, q-Rung Linear Diophantine Fuzzy Hypersoft Sets (q-RLDFHSS) is introduced by Surya et al. [22]. Also, many hybrid versions of HSS were developed in picture, spherical and T-spherical fuzzy environments [23–25]. For utilizing some advanced structures on hypersoft, an algebraic structure is developed in [26]. To handle multiple sub-attributes under a bipolar fuzzy environment, Musa and Asaad introduced Bipolar Fuzzy Hypersoft Sets (BFHSS) [27]. Also, they discussed parameter reduction in BFHSS and presented a decision-making scenario in [28]. An MADM approach on selection of optimized air filtration technologies that are applied in petrochemical industries was discussed in [29]. To tackle the pollution control circumstances of the sewage system, the VlseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) method is used in [30]. In [31], an expanded the VIKOR approach is employed on a bipolar fuzzy environment to illustrate the effectiveness of solving real-life challenges. A Weighted Aggregated Sum Product Assessment (WASPAS) method is used to assess the service quality and to optimize the operational efficiency in [32]. In [33], a Fuzzy Analytical Hierarchy Process (AHP) is shown as an effective algorithm to handle classical AHP applications. Grey Measurement Alternatives and Ranking According to the Compromise Solution (MARCOS) method is used as a tool in [34] to choose the optimum supplier on the basis of various criteria. To evaluate the professor on the basis of teaching quality, ethical behaviour etc., a hybrid algorithm based on AHP and the Grey MARCOS method is discussed in [35].

Research gaps

• The existing research findings are capable of working with MADM problems but those are unable to do with MADM problems when negative grades, along with multi-sub attribute, are considered.

• It is already very complicated to work with BTSFS as it has parameters MG, NMG, and AG in both positive and negative grades. In addition to that, in a bipolar fuzzy environment, when the multiple sub-attributes have to be considered, it requires a better framework that leads to the path of combining both the BTSFS with a Hypersoft set.

• Another noteworthy feature of the BTSFHSS is its algebraic features. It is crucial to examine how the algebraic features suit the suggested theory of BTSFHSS and how the fundamental operations function with it.

Motivation

• A more thorough and organized decision-making framework that can successfully handle the duality of information and the intricacy of sub-attribute interactions in actual MADM issues is required to get beyond those restrictions. This encourages the proposal of the BTSFHSS model, which combines the adaptability of Hypersoft Sets with the advantages of BTSFS. A more comprehensive assessment framework that can fully capture the range of data present in intricate decision-making challenges is made possible by this integration.

• Furthermore, we explore the algebraic operations on BTSFHSS to ensure mathematical soundness and prepare the model for practical applications that comes under the scope of bipolar fuzzy environments.

Objectives

• To introduce the concept of BTSFHSS, which provides the gateway to deal with the BTSF environment along with several attributes.

• To define some algebraic structures and to discuss some elementary theorems for the theoretical construction of the framework of BTSFHSS.

• To present an efficient MADM strategy based on BTSFHSS for dealing with the order of ranking for the specified collection of items that are considered in real-world applications.

Contribution and novelty

A summary of the contribution and novelty of the proposed work is as follows:

• A novel concept is introduced by merging BTSFS with the Hypersoft set to handle the multiple sub-attributes situation along with both positive and negative aspects of MG, NMG, and AG. Also, its associated score function and accuracy function are analyzed.

• The basic algebraic properties like union, intersection, and complement for the proposed theory and its OR, AND operators are discussed.

•An algorithm is developed to find an appropriate filter for the industry using an MADM technique, and it is exhibited as a real-world application, demonstrating the effectiveness and efficiency of the proposed theory.

• To manage decision-making scenarios in which one must aggregate evidence that includes both positive and negative implications of uncertainty, as well as various sub-attributes, no existing framework can handle those situations which shows the novelty of the proposed BTSFHSS.

2. Preliminaries

In the context, the symbol $\acute{\mathfrak{V}}$ denotes the Universal Set.

A Abbreviation Table is provided with the abbreviated words that are frequently used in this manuscript for enhancing legibility.

Definition 2.1 [20] Let P($\acute{\mathfrak{V}}$) be the collection of possible subsets of $\acute{\mathfrak{V}}$. Let $\bar{\mathfrak{k}}_1,\ \bar{\mathfrak{k}}_2,\ \ldots,\ \bar{\mathfrak{k}}_n$, for $n \geq 1$ be some attributes that are distinct, where $\breve{\kappa}_1,\ \breve{\kappa}_2,\ \ldots,\ \breve{\kappa}_n$ are the corresponding mutually disjoint attribute values respectively. Then the tuple $(F,\ \breve{\kappa}_1 \times \breve{\kappa}_2 \times \ldots \times \breve{\kappa}_n)$, where F is a function from $\breve{\kappa}_1 \times \breve{\kappa}_2 \times \ldots \times \breve{\kappa}_n$ to $P(\acute{\mathfrak{V}})$ is called a HS over $\acute{\mathfrak{V}}$.

Definition 2.2 [20] Let FP($\acute{\mathfrak{V}}$) be the power set of FS over $\acute{\mathfrak{V}}$. For $n \geq 1$, let $\breve{\kappa}_1 \times \breve{\kappa}_2 \times \ldots \times \breve{\kappa}_n$ be the corresponding attribute values of the distinct attributes $\bar{\mathfrak{k}}_1,\ \bar{\mathfrak{k}}_2,\ \ldots,\ \bar{\mathfrak{k}}_n$ respectively such that $\breve{\kappa}_i \cap \breve{\kappa}_j = \emptyset$ for $i \neq j$. Let $\hat{\kappa}_i$ be the non-empty subset of $\breve{\kappa}_i$ for each $i \in \{1,\ 2,\ \cdots,\ n\}$. The tuple $(\theta,\ \breve{\kappa}_1 \times \breve{\kappa}_2 \times \ldots \times \breve{\kappa}_n\ )$ is called FHS, where θ assigns $\hat{\kappa}_1 \times \hat{\kappa}_2 \times \ldots \times \hat{\kappa}_n$ to an element belongs to $FP(\acute{\mathfrak{V}})$, i.e $\theta(\hat{\kappa}_1 \times \hat{\kappa}_2 \times \ldots \times \hat{\kappa}_n) = (\acute{v},\ < \dot{\varsigma}_{\theta(\hat{\mathfrak{o}})}(\acute{v}) >:\ \hat{\mathfrak{o}} \in \hat{\kappa}_1 \times \hat{\kappa}_2 \times \ldots \times \hat{\kappa}_n,\ \acute{v} \in \acute{\mathfrak{V}})$.

Definition 2.3 [36] The TSFS $\tilde{\mathfrak{R}}$ on $\acute{\mathfrak{V}}$ consists of the elements that are of the form

$$\tilde{\mathfrak{R}} = \{(\acute{v},\ < \dot{\varsigma}_{\tilde{\mathfrak{R}}}(\acute{v}),\ \dot{\rho}_{\tilde{\mathfrak{R}}}(\acute{v}),\ \dot{\vartheta}_{\tilde{\mathfrak{R}}}(\acute{v}) >) : \acute{v} \in \acute{\mathfrak{V}}\},$$

where $\dot{\varsigma}_{\tilde{\mathfrak{R}}}(\acute{v}),\ \dot{\rho}_{\tilde{\mathfrak{R}}}(\acute{v}),\ \dot{\vartheta}_{\tilde{\mathfrak{R}}}(\acute{v}) \in [0,\ 1]$ indicates the values of MG, AG and NMG respectively. These grades satisfy the condition $0 \leq \dot{\varsigma}^t_{\tilde{\mathfrak{R}}}(\acute{v}) + \dot{\rho}^t_{\tilde{\mathfrak{R}}}(\acute{v}) + \dot{\vartheta}^t_{\tilde{\mathfrak{R}}}(\acute{v}) \leq 1$, $\forall \acute{v} \in \acute{\mathfrak{V}}$, $t \geq 1$. The indeterminacy degree can be evaluated as

$$\pi_{\tilde{\mathfrak{R}}}(\acute{v}) = \sqrt[t]{1 - (\dot{\varsigma}^t_{\tilde{\mathfrak{R}}}(\acute{v}) + \dot{\rho}^t_{\tilde{\mathfrak{R}}}(\acute{v}) + \dot{\vartheta}^t_{\tilde{\mathfrak{R}}}(\acute{v}))},$$

where $\pi_{\tilde{\mathfrak{R}}}$ denotes degree of indeterminacy.

Definition 2.4 [37] The BTSFS $\tilde{\mathfrak{G}}$ on $\acute{\mathfrak{V}}$ is defined as

$$\tilde{\mathfrak{G}} = \ (\acute{v},\ \langle \dot{\varsigma}^+_{\tilde{\mathfrak{G}}}(\acute{v}),\ \dot{\rho}^+_{\tilde{\mathfrak{G}}}(\acute{v}),\ \dot{\vartheta}^+_{\tilde{\mathfrak{G}}}(\acute{v}),\ \dot{\varsigma}^-_{\tilde{\mathfrak{G}}}(\acute{v}),\ \dot{\rho}^-_{\tilde{\mathfrak{G}}}(\acute{v}),\ \dot{\vartheta}^-_{\tilde{\mathfrak{G}}}(\acute{v})\rangle : \acute{v} \in \acute{\mathfrak{V}}),$$

where $\dot{\varsigma}^+_{\tilde{\mathfrak{G}}}(\acute{v}),\ \dot{\rho}^+_{\tilde{\mathfrak{G}}}(\acute{v}),\ \dot{\vartheta}^+_{\tilde{\mathfrak{G}}}(\acute{v}) \in [0,\ 1]$ are the parameters that shows the value of MG, AG and NMG in the positive sense respectively and $\dot{\varsigma}^-_{\tilde{\mathfrak{G}}}(\acute{v}),\ \dot{\rho}^-_{\tilde{\mathfrak{G}}}(\acute{v}),\ \dot{\vartheta}^-_{\tilde{\mathfrak{G}}}(\acute{v}) \in [-1,\ 0]$ are the parameters that shows the value of MG, AG and NMG in the negative sense respectively, with conditions $0 \leq (\dot{\varsigma}^+_{\tilde{\mathfrak{G}}}(\acute{v}))^t + (\dot{\rho}^+_{\tilde{\mathfrak{G}}}(\acute{v}))^t + (\dot{\vartheta}^+_{\tilde{\mathfrak{G}}}(\acute{v}))^t \leq 1$ and $-1 \leq -[|\dot{\varsigma}^-_{\tilde{\mathfrak{G}}}(\acute{v})|^t + |\dot{\rho}^-_{\tilde{\mathfrak{G}}}(\acute{v})|^t + |\dot{\vartheta}^-_{\tilde{\mathfrak{G}}}(\acute{v})|^t] \leq 0$, $t \geq 1$.

3. Bipolar T-spherical fuzzy Hypersoft set

The concepts of BTSFHSS are formulated through some basic algebraic operations and the way to find out the comparison matrix on BTSFHSS was described in the following.

Definition 3.1 Let $BTSFP(\acute{\mathfrak{V}})$ be the collection of possible subsets of all bipolar $\ddot{\Omega}$: $\breve{\kappa}_1 \times \breve{\kappa}_2 \times ... \times \breve{\kappa}_n \to BTSFP(\acute{\mathfrak{V}})$ and

$$\ddot{\Omega}(\hat{\kappa}_1 \times \hat{\kappa}_2 \times ... \times \hat{\kappa}_n) = (\acute{v},\ < \dot{\varsigma}^+_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}),\ \dot{\rho}^+_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}),\ \dot{\vartheta}^+_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}),\ \dot{\varsigma}^-_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}),\ \dot{\rho}^-_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}),\ \dot{\vartheta}^-_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}) >$$

$$: \dot{\varepsilon} \in \hat{\kappa}_1 \times \hat{\kappa}_2 \times ... \times \hat{\kappa}_n,\ \acute{v} \in \acute{\mathfrak{V}}),$$

whose elements is viewed in the form of tuples

$$\{(\dot{\varepsilon},\ \ddot{\Omega}(\dot{\varepsilon})) : \dot{\varepsilon} \in \acute{\kappa}_1 \times \cdots \times \acute{\kappa}_n \text{ and } \ddot{\Omega}(\dot{\varepsilon}) \in BTSFP(\acute{\mathfrak{V}})\},$$

where $(\dot{\varsigma}^+_{\ddot{\Omega}_{\dot{\varepsilon}}}(\acute{v}),\ \dot{\rho}^+_{\ddot{\Omega}_{\dot{\varepsilon}}}(\acute{v}),\ \dot{\vartheta}^+_{\ddot{\Omega}_{\dot{\varepsilon}}}(\acute{v}),\ \dot{\varsigma}^-_{\ddot{\Omega}_{\dot{\varepsilon}}}(\acute{v}),\ \dot{\rho}^-_{\ddot{\Omega}_{\dot{\varepsilon}}}(\acute{v}),\ \dot{\varsigma}^+_{\ddot{\Omega}_{\dot{\varepsilon}}}(\acute{v}))$ is a BTSFHSN.

Definition 3.2 Let $(\ddot{\Omega}_1,\ \acute{\kappa}_1)$, $(\ddot{\Omega}_2,\ \acute{\kappa}_2)$ be BTSFHSS$(\acute{\mathfrak{V}})$, then $(\ddot{\Omega}_1,\ \acute{\kappa}_1)$ is said to be BTSFHS subset of $(\ddot{\Omega}_2,\ \acute{\kappa}_2)$ if

1. $\acute{\kappa}_1 \subseteq \acute{\kappa}_2$.
2. $\forall \dot{\varepsilon} \in \acute{\kappa}_1,\ \ddot{\Omega}_1(\dot{\varepsilon}) \subseteq \ddot{\Omega}_2(\dot{\varepsilon})$.

i.e. $\dot{\varsigma}^+_{\ddot{\Omega}_1(\dot{\varepsilon})}(\acute{v}) \leq \dot{\varsigma}^+_{\ddot{\Omega}_2(\dot{\varepsilon})}(\acute{v})$, $\dot{\rho}^+_{\ddot{\Omega}_1(\dot{\varepsilon})}(\acute{v}) \geq \dot{\rho}^+_{\ddot{\Omega}_2(\dot{\varepsilon})}(\acute{v})$, $\dot{\vartheta}^+_{\ddot{\Omega}_2(\dot{\varepsilon})}(\acute{v}) \leq \dot{\vartheta}^+_{\ddot{\Omega}_1(\dot{\varepsilon})}(\acute{v})$,
$\dot{\varsigma}^-_{\ddot{\Omega}_2(\dot{\varepsilon})}(\acute{v}) \leq \dot{\varsigma}^-_{\ddot{\Omega}_1(\dot{\varepsilon})}(\acute{v})$, $\dot{\rho}^-_{\ddot{\Omega}_2(\dot{\varepsilon})}(\acute{v}) \geq \dot{\rho}^-_{\ddot{\Omega}_1(\dot{\varepsilon})}(\acute{v})$, $\dot{\vartheta}^-_{\ddot{\Omega}_1(\dot{\varepsilon})}(\acute{v}) \leq \dot{\vartheta}^-_{\ddot{\Omega}_2(\dot{\varepsilon})}(\acute{v})$.

Definition 3.3 A BTSFHSS $(\ddot{\Omega},\ \acute{\kappa})$ over $\acute{\mathfrak{V}}$ is said to be null BTSFHSS if $\dot{\varsigma}^+_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}) = \dot{\rho}^+_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}) = \dot{\varsigma}^-_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}) = \dot{\rho}^-_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}) = 0$ and $\dot{\vartheta}^+_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}) = 1$, $\dot{\vartheta}^-_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}) = -1$, $\forall \acute{v} \in \acute{\mathfrak{V}}$, $\forall \dot{\varepsilon} \in \acute{\kappa}$ which is denoted by $(\ddot{\Omega},\ \acute{\kappa})_{\emptyset}$.

Definition 3.4 A BTSFHSS $(\ddot{\Omega},\ \acute{\kappa})$ over $\acute{\mathfrak{V}}$ is said to be absolute BTSFHSS if $\dot{\varsigma}^+_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}) = 1$, $\dot{\rho}^+_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}) = \dot{\rho}^-_{\ddot{\Omega}(\bar{\mathfrak{k}})}(\acute{v}) = \dot{\vartheta}^+_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}) = \dot{\vartheta}^-_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}) = 0$, $\dot{\varsigma}^-_{\ddot{\Omega}(\bar{\mathfrak{k}})}(\acute{v}) = -1 \forall \acute{v} \in \acute{\mathfrak{V}}$, $\forall \dot{\varepsilon} \in \acute{\kappa}$ which is denoted by $(\ddot{\Omega},\ \acute{\kappa})_{\acute{\mathfrak{V}}}$.

Definition 3.5 Let $(\ddot{\mathfrak{Q}}_1,\ \acute{\kappa}_1)$ and $(\ddot{\mathfrak{Q}}_2,\ \acute{\kappa}_2)$ be the two BTSFHSS. Then the extended union of $(\ddot{\mathfrak{Q}}_1,\ \acute{\kappa}_1)$ and $(\ddot{\mathfrak{Q}}_2,\ \acute{\kappa}_2)$ is defined as $(\ddot{\mathfrak{Q}}_1,\ \acute{\kappa}_1)\cup_e(\ddot{\mathfrak{Q}}_2,\ \acute{\kappa}_2)=(\ddot{\mathfrak{Q}}_3,\ \acute{\kappa}_3)$, where $\acute{\kappa}_3=\acute{\kappa}_1\cup\acute{\kappa}_2$ and

$$\dot{\varsigma}^{+}_{\ddot{\mathfrak{Q}}_3(\dot{\varepsilon})}(\acute{v})=\begin{cases}\dot{\varsigma}^{+}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon}\in\acute{\kappa}_1-\acute{\kappa}_2 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\\ \dot{\varsigma}^{+}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon}\in\acute{\kappa}_2-\acute{\kappa}_1 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\\ \max\left\{\dot{\varsigma}^{+}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}),\ \dot{\varsigma}^{+}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v})\right\} \text{ if } \dot{\varepsilon}\in\acute{\kappa}_1\cap\acute{\kappa}_2 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\end{cases}$$

$$\dot{\rho}^{+}_{\ddot{\mathfrak{Q}}_3(\dot{\varepsilon})}(\acute{v})=\begin{cases}\dot{\rho}^{+}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon}\in\acute{\kappa}_1-\acute{\kappa}_2 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\\ \dot{\rho}^{+}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon}\in\acute{\kappa}_2-\acute{\kappa}_1 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\\ \min\left\{\dot{\rho}^{+}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}),\ \dot{\rho}^{+}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v})\right\} \text{ if } \dot{\varepsilon}\in\acute{\kappa}_1\cap\acute{\kappa}_2 \quad \text{ and } \acute{v}\in\acute{\mathfrak{B}}\end{cases}$$

$$\dot{\vartheta}^{+}_{\ddot{\mathfrak{Q}}_3(\dot{\varepsilon})}(\acute{v})=\begin{cases}\dot{\vartheta}^{+}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon}\in\acute{\kappa}_1-\acute{\kappa}_2 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\\ \dot{\vartheta}^{+}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\dot{\varepsilon}) \text{ if } \dot{\varepsilon}\in\acute{\kappa}_2-\acute{\kappa}_1 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\\ \min\left\{\dot{\vartheta}^{+}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}),\ \dot{\vartheta}^{+}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v})\right\} \text{ if } \dot{\varepsilon}\in\acute{\kappa}_1\cap\acute{\kappa}_2 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\end{cases}$$

$$\dot{\varsigma}^{-}_{\ddot{\mathfrak{W}}_3(\dot{\varepsilon})}(\acute{v})=\begin{cases}\dot{\varsigma}^{-}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon}\in\acute{\kappa}_1-\acute{\kappa}_2 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\\ \dot{\varsigma}^{-}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon}\in\acute{\kappa}_2-\acute{\kappa}_1 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\\ \min\left\{\dot{\varsigma}^{-}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}),\ \dot{\varsigma}^{-}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v})\right\} \text{ if } \dot{\varepsilon}\in\acute{\kappa}_1\cap\acute{\kappa}_2 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\end{cases}$$

$$\dot{\rho}^{-}_{\ddot{\mathfrak{Q}}_3(\dot{\varepsilon})}(\acute{v})=\begin{cases}\dot{\rho}^{-}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon}\in\acute{\kappa}_1-\acute{\kappa}_2 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\\ \dot{\rho}^{-}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon}\in\acute{\kappa}_2-\acute{\kappa}_1 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\\ \max\left\{\dot{\rho}^{-}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}),\ \dot{\rho}^{-}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v})\right\} \text{ if } \dot{\varepsilon}\in\acute{\kappa}_1\cap\acute{\kappa}_2 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\end{cases}$$

$$\dot{\vartheta}^{-}_{\ddot{\mathfrak{Q}}_3(\dot{\varepsilon})}(\acute{v}) = \begin{cases} \dot{\vartheta}^{-}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon} \in \acute{\kappa}_1 - \acute{\kappa}_2 \text{ and } \acute{v} \in \acute{\mathfrak{B}} \\ \dot{\vartheta}^{-}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon} \in \acute{\kappa}_2 - \acute{\kappa}_1 \text{ and } \acute{v} \in \acute{\mathfrak{B}} \\ \max\left\{ \dot{\vartheta}^{-}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}),\ \dot{\vartheta}^{-}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v}) \right\} \text{ if } \dot{\varepsilon} \in \acute{\kappa}_1 \cap \acute{\kappa}_2 \text{ and } \acute{v} \in \acute{\mathfrak{B}}. \end{cases}$$

Illustrative example:

Let $(\ddot{\mathfrak{Q}}_1,\ \acute{\kappa}_1) = \{(\acute{v}_1, < 0.01,\ 0.57,\ 0.02,\ 0,\ -1,\ 0 >),\ (\acute{v}_2, < 0,\ 0.05,\ 0.059,\ -1,\ 0,\ 0 >)\}$ and $(\ddot{\mathfrak{Q}}_2,\ \acute{\kappa}_2) = \{(\acute{v}_1, < 0.01,\ 0.57,\ 0.02,\ 0,\ -1,\ 0 >)\}$.

Then $(\ddot{\mathfrak{Q}}_1,\ \acute{\kappa}_1) \cup_e (\ddot{\mathfrak{Q}}_2,\ \acute{\kappa}_2) = \{(\acute{v}_1, < 0.01,\ 0.57,\ 0.02,\ 0,\ -1,\ 0 >),\ (\acute{v}_2, < 0,\ 0.05,\ 0.059,\ -1,\ 0,\ 0 >)\}$.

Definition 3.6 Let $(\ddot{\mathfrak{Q}}_1,\ \acute{\kappa}_1)$ and $(\ddot{\mathfrak{Q}}_2,\ \acute{\kappa}_2)$ be the two BTSFHSS. Then the extended intersection of $(\ddot{\mathfrak{Q}}_1,\ \acute{\kappa}_1)$ and $(\ddot{\mathfrak{Q}}_2,\ \acute{\kappa}_2)$ is defined as $(\ddot{\mathfrak{Q}}_1,\ \acute{\kappa}_1) \cap_e (\ddot{\mathfrak{Q}}_2,\ \acute{\kappa}_2) = (\ddot{\mathfrak{Q}}_3,\ \acute{\kappa}_3)$, where $\acute{\kappa}_3 = \acute{\kappa}_1 \cup \acute{\kappa}_2$ and

$$\dot{\varsigma}^{+}_{\ddot{\mathfrak{Q}}_3(\dot{\varepsilon})}(\acute{v}) = \begin{cases} \dot{\varsigma}^{+}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon} \in \acute{\kappa}_1 - \acute{\kappa}_2 \text{ and } \acute{v} \in \acute{\mathfrak{B}} \\ \dot{\varsigma}^{+}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon} \in \acute{\kappa}_2 - \acute{\kappa}_1 \text{ and } \acute{v} \in \acute{\mathfrak{B}} \\ \min\left\{ \dot{\varsigma}^{+}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}),\ \dot{\varsigma}^{+}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v}) \right\} \text{ if } \dot{\varepsilon} \in \acute{\kappa}_1 \cap \acute{\kappa}_2 \text{ and } \acute{v} \in \acute{\mathfrak{B}} \end{cases}$$

$$\dot{\rho}^{+}_{\ddot{\mathfrak{Q}}_3(\dot{\varepsilon})}(\acute{v}) = \begin{cases} \dot{\rho}^{+}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon} \in \acute{\kappa}_1 - \acute{\kappa}_2 \text{ and } \acute{v} \in \acute{\mathfrak{B}} \\ \dot{\rho}^{+}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon} \in \acute{\kappa}_2 - \acute{\kappa}_1 \text{ and } \acute{v} \in \acute{\mathfrak{B}} \\ \min\left\{ \dot{\rho}^{+}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}),\ \dot{\rho}^{+}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v}) \right\} \text{ if } \dot{\varepsilon} \in \acute{\kappa}_1 \cap \acute{\kappa}_2 \text{ and } \acute{v} \in \acute{\mathfrak{B}} \end{cases}$$

$$\dot{\vartheta}^{+}_{\ddot{\mathfrak{Q}}_3(\dot{\varepsilon})}(\acute{v}) = \begin{cases} \dot{\vartheta}^{+}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon} \in \acute{\kappa}_1 - \acute{\kappa}_2 \text{ and } \acute{v} \in \acute{\mathfrak{B}} \\ \dot{\vartheta}^{+}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon} \in \acute{\kappa}_2 - \acute{\kappa}_1 \text{ and } \acute{v} \in \acute{\mathfrak{B}} \\ \max\left\{ \dot{\vartheta}^{+}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}),\ \dot{\vartheta}^{+}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v}) \right\} \text{ if } \dot{\varepsilon} \in \acute{\kappa}_1 \cap \acute{\kappa}_2 \text{ and } \acute{v} \in \acute{\mathfrak{B}} \end{cases}$$

$$\dot{\varsigma}^{-}_{\ddot{\mathfrak{Q}}_3(\dot{\varepsilon})}\left(v'\right)=\begin{cases}\dot{\varsigma}^{-}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon}\in\acute{\kappa}_1-\acute{\kappa}_2 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\\ \dot{\varsigma}^{-}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon}\in\acute{\kappa}_2-\acute{\kappa}_1 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\\ \max\left\{\dot{\varsigma}^{-}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}),\ \dot{\varsigma}^{-}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v})\right\} \text{ if } \dot{\varepsilon}\in\acute{\kappa}_1\cap\acute{\kappa}_2 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\end{cases}$$

$$\dot{\rho}^{-}_{\ddot{\mathfrak{Q}}_3(\dot{\varepsilon})}\left(v'\right)=\begin{cases}\dot{\rho}^{-}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}\left(v'\right) \text{ if } \dot{\varepsilon}\in\acute{\kappa}_1-\acute{\kappa}_2 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\\ \dot{\rho}^{-}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}\left(v'\right) \dot{\varepsilon}\in\acute{\kappa}_2-\acute{\kappa}_1 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\\ \max\left\{\dot{\rho}^{-}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}\left(v'\right),\ \dot{\rho}^{-}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}\left(v'\right)\right\} \text{ if } \dot{\varepsilon}\in\acute{\kappa}_1\cap\acute{\kappa}_2 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\end{cases}$$

$$\dot{\vartheta}^{-}_{\ddot{\mathfrak{Q}}_3(\dot{\varepsilon})}(\acute{v})=\begin{cases}\dot{\vartheta}^{-}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon}\in\acute{\kappa}_1-\acute{\kappa}_2 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\\ \dot{\vartheta}^{-}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v}) \text{ if } \dot{\varepsilon}\in\acute{\kappa}_1\cap\acute{\kappa}_2 \text{ and } \acute{v}\in\acute{\mathfrak{B}}\\ \min\left\{\dot{\vartheta}^{-}_{\ddot{\mathfrak{Q}}_1(\dot{\varepsilon})}(\acute{v}),\ \dot{\vartheta}^{-}_{\ddot{\mathfrak{Q}}_2(\dot{\varepsilon})}(\acute{v})\right\} \text{ if } \dot{\varepsilon}\in\acute{\kappa}_1\cap\acute{\kappa}_2 \text{ and } \acute{v}\in\acute{\mathfrak{B}}.\end{cases}$$

Illustrative example:

Let $(\ddot{\mathfrak{Q}}_1,\ \acute{\kappa}_1)=\{(\acute{v}_1,\ <0.01,\ 0.57,\ 0.02,\ 0,\ -1,\ 0>),\ (\acute{v}_2,\ <0,\ 0.05,\ 0.059,\ -1,\ 0,\ 0>)\}$ and $(\ddot{\mathfrak{Q}}_2,\ \acute{\kappa}_2)=\{(\acute{v}_1,\ <0.01,\ 0.57,\ 0.02,\ 0,\ -1,\ 0>)\}$.

Then $(\ddot{\mathfrak{Q}}_1,\ \acute{\kappa}_1)\cap_e(\ddot{\mathfrak{Q}}_2,\ \acute{\kappa}_2)=\{(\acute{v}_1,\ <0.01,\ 0.57,\ 0.02,\ 0,\ -1,\ 0>),\ (\acute{v}_2,\ <0,\ 0.05,\ 0.059,\ -1,\ 0,\ 0>)\}$.

Proposition 3.1 Let $(\ddot{\mathfrak{Q}},\ \acute{\kappa})$ be the BTSFHSS($\acute{\mathfrak{V}}$). Then

1. $(\ddot{\mathfrak{Q}},\ \acute{\kappa})_{\acute{\mathfrak{V}}}\cup(\ddot{\mathfrak{Q}},\ \acute{\kappa})=(\ddot{\mathfrak{Q}},\ \acute{\kappa})_{\acute{\mathfrak{V}}}$.
2. $(\ddot{\mathfrak{Q}},\ \acute{\kappa})_{\emptyset}\cup(\ddot{\mathfrak{Q}},\ \acute{\kappa})=(\ddot{\mathfrak{Q}},\ \acute{\kappa})$.
3. $(\ddot{\mathfrak{Q}},\ \acute{\kappa})_{\acute{\mathfrak{V}}}\cap(\ddot{\mathfrak{Q}},\ \acute{\kappa})=(\ddot{\mathfrak{Q}},\ \acute{\kappa})$.
4. $(\ddot{\mathfrak{Q}},\ \acute{\kappa})_{\emptyset}\cap(\ddot{\mathfrak{Q}},\ \acute{\kappa})=(\ddot{\mathfrak{Q}},\ \acute{\kappa})_{\emptyset}$.

Proof. 1. For $\dot{\varepsilon}\in\hat{\kappa}_1\times\hat{\kappa}_2\times\ldots\times\hat{\kappa}_n$ and $\acute{v}\in\acute{\mathfrak{V}}$, $(\ddot{\mathfrak{Q}},\ \acute{\kappa})_{\acute{\mathfrak{V}}}=\{\acute{v},\ <1,\ 0,\ 0,\ -1,\ 0,\ 0>:\acute{v}\in\acute{\mathfrak{V}}\}$ and $(\ddot{\mathfrak{Q}},\ \acute{\kappa})=(\acute{v},\ <\dot{\varsigma}^{+}_{\ddot{\mathfrak{Q}}(\dot{\varepsilon})}(\acute{v}),\ \dot{\rho}^{+}_{\ddot{\mathfrak{Q}}(\dot{\varepsilon})}(\acute{v}),\ \dot{\vartheta}^{+}_{\ddot{\mathfrak{Q}}(\dot{\varepsilon})}(\acute{v}),\ \dot{\varsigma}^{-}_{\ddot{\mathfrak{Q}}(\dot{\varepsilon})}(\acute{v}),\ \dot{\rho}^{-}_{\ddot{\mathfrak{Q}}(\dot{\varepsilon})}(\acute{v}),\ \dot{\vartheta}^{-}_{\ddot{\mathfrak{Q}}(\dot{\varepsilon})}(\acute{v})>)$.

The grades of $(\ddot{\mathfrak{Q}},\ \acute{\kappa})_{\acute{\mathfrak{V}}}\cup(\ddot{\mathfrak{Q}},\ \acute{\kappa})$ are determined as follows.

$\max\{1,\ \dot{\varsigma}^{+}_{\ddot{\mathfrak{Q}}(\dot{\varepsilon})}(\acute{v})\}=1$, $\min\{0,\ \dot{\rho}^{+}_{\ddot{\mathfrak{Q}}(\dot{\varepsilon})}(\acute{v})\}=0$, $\min\{0,\ \dot{\vartheta}^{+}_{\ddot{\mathfrak{Q}}(\dot{\varepsilon})}(\acute{v})\}=0$, $\min\{-1,\ \dot{\varsigma}^{-}_{\ddot{\mathfrak{Q}}(\dot{\varepsilon})}(\acute{v})\}=-1$, $\max\{0,\ \dot{\rho}^{-}_{\ddot{\mathfrak{Q}}(\dot{\varepsilon})}(\acute{v})\}=0$, $\max\{0,\ \dot{\vartheta}^{-}_{\ddot{\mathfrak{Q}}(\dot{\varepsilon})}(\acute{v})\}=0$.

Thus, $(\ddot{\mathfrak{Q}},\ \acute{\kappa})_{\acute{\mathfrak{V}}}\cup(\ddot{\mathfrak{Q}},\ \acute{\kappa})=(\ddot{\mathfrak{Q}},\ \acute{\kappa})_{\acute{\mathfrak{V}}}$.

2. As Similar as the proof of 1.

3. For $\dot{\varepsilon}\in\hat{\kappa}_1\times\hat{\kappa}_2\times\ldots\times\hat{\kappa}_n$ and $\acute{v}\in\acute{\mathfrak{V}}$, $(\ddot{\mathfrak{Q}},\ \acute{\kappa})_{\emptyset}=\{\acute{v},\ <0,\ 0,\ 1,\ 0,\ 0,\ -1>:\ \acute{v}\in\acute{\mathfrak{V}}\}$ and

$$(\ddot{\mathfrak{Q}},\ \acute{\kappa})=(\acute{v},\ <\dot{\varsigma}^{+}_{\ddot{\mathfrak{Q}}(\dot{\varepsilon})}(\acute{v}),\ \dot{\rho}^{+}_{\ddot{\mathfrak{Q}}(\dot{\varepsilon})}(\acute{v}),\ \dot{\vartheta}^{+}_{\ddot{\mathfrak{Q}}(\dot{\varepsilon})}(\acute{v}),\ \dot{\varsigma}^{-}_{\ddot{\mathfrak{Q}}(\dot{\varepsilon})}(\acute{v}),\ \dot{\rho}^{-}_{\ddot{\mathfrak{Q}}(\dot{\varepsilon})}(\acute{v}),\ \dot{\vartheta}^{-}_{\ddot{\mathfrak{Q}}(\dot{\varepsilon})}(\acute{v})>).$$

The grades of $(\ddot{\Omega},\ \acute{\kappa})_{\emptyset} \cup (\ddot{\Omega},\ \acute{\kappa})$ are determined as follows,

$$\min\{0,\ \dot{\varsigma}^{+}_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v})\} = 0, \min\{0,\ \dot{\rho}^{+}_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v})\} = 0,\ \max\{1,\ \dot{\vartheta}^{+}_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v})\} = 1,$$

$$\max\{0,\ \dot{\varsigma}^{-}_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v})\} = 0,\ \max\{0,\ \dot{\rho}^{-}_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v})\} = 0, \min\{-1,\ \dot{\vartheta}^{-}_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v})\} = -1.$$

Thus, $(\ddot{\Omega},\ \acute{\kappa})_{\emptyset} \cup (\ddot{\Omega},\ \acute{\kappa}) = (\ddot{\Omega},\ \acute{\kappa})$.

4. As similar as the proof of 4.

Proposition 3.2 Let $(\ddot{\Omega}_1,\ \acute{\kappa}_1)$, $(\ddot{\Omega}_2,\ \acute{\kappa}_2)$, $(\ddot{\Omega}_3,\ \acute{\kappa}_3)$ be the BTSFHSS($\acute{\mathfrak{V}}$). Then

(Associative laws)

1. $((\ddot{\Omega}_1,\ \acute{\kappa}_1) \cup (\ddot{\Omega}_2,\ \acute{\kappa}_2)) \cup (\ddot{\Omega}_3,\ \acute{\kappa}_3) = (\ddot{\Omega}_1,\ \acute{\kappa}_1) \cup ((\ddot{\Omega}_2,\ \acute{\kappa}_2) \cup (\ddot{\Omega}_3,\ \acute{\kappa}_3))$.
2. $((\ddot{\Omega}_1,\ \acute{\kappa}_1) \cap (\ddot{\Omega}_2,\ \acute{\kappa}_2)) \cap (\ddot{\Omega}_3,\ \acute{\kappa}_3) = (\ddot{\Omega}_1,\ \acute{\kappa}_1) \cap ((\ddot{\Omega}_2,\ \acute{\kappa}_2) \cap (\ddot{\Omega}_3,\ \acute{\kappa}_3))$.

(Distributive laws)

1. $((\ddot{\Omega}_1,\ \acute{\kappa}_1) \cup (\ddot{\Omega}_2,\ \acute{\kappa}_2)) \cap (\ddot{\Omega}_3,\ \acute{\kappa}_3) = ((\ddot{\Omega}_1,\ \acute{\kappa}_1) \cap (\ddot{\Omega}_3,\ \acute{\kappa}_3)) \cup ((\ddot{\Omega}_2,\ \acute{\kappa}_2) \cap (\ddot{\Omega}_3,\ \acute{\kappa}_3))$.
2. $((\ddot{\Omega}_1,\ \acute{\kappa}_1) \cap (\ddot{\Omega}_2,\ \acute{\kappa}_2)) \cup (\ddot{\Omega}_3,\ \acute{\kappa}_3) = ((\ddot{\Omega}_1,\ \acute{\kappa}_1) \cup (\ddot{\Omega}_3,\ \acute{\kappa}_3)) \cap ((\ddot{\Omega}_2,\ \acute{\kappa}_2) \cup (\ddot{\Omega}_3,\ \acute{\kappa}_3))$.

Proof. Since operator min and max are distributive and associative over each other. By using this property, the theorem can be proven obviously.

Definition 3.7 Let

$$(\ddot{\Omega},\ \acute{\kappa}) = \{(\dot{\varepsilon},\ < \dot{\varsigma}^{+}_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}),\ \dot{\rho}^{+}_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}),\ \dot{\vartheta}^{+}_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}),\ \dot{\varsigma}^{-}_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}),\ \dot{\rho}^{-}_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}),\ \dot{\vartheta}^{-}_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}) >)\ : \dot{\varepsilon} \in \acute{\kappa} \quad \text{and} \quad \acute{v} \in \acute{\mathfrak{V}}\},$$

be the BTSFHSS(U). Then the complement of $(\ddot{\Omega},\ \acute{\kappa})$ is defined by

$$(\ddot{\Omega},\ \acute{\kappa})^{C} = \{(\dot{\varepsilon},\ < \dot{\vartheta}^{+}_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}),\ \dot{\rho}^{+}_{\ddot{\Omega}(\bar{\mathfrak{k}})}(\acute{v}),\ \dot{\varsigma}^{+}_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}),\ \dot{\vartheta}^{-}_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}),\ \dot{\rho}^{-}_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}),\ \dot{\varsigma}^{-}_{\ddot{\Omega}(\dot{\varepsilon})}(\acute{v}) >)\ :\ \dot{\varepsilon} \in \acute{\kappa} \quad \text{and} \quad \acute{v} \in \acute{\mathfrak{V}}\}.$$

Proposition 3.3 Let $(\ddot{\Omega}_1,\ \acute{\kappa}_1)$, $(\ddot{\Omega}_2,\ \acute{\kappa}_2)$ and $(\ddot{\Omega}_3,\ \acute{\kappa}_3)$ be the BTSFHSS($\acute{\mathfrak{V}}$). Then

1. $((\ddot{\Omega}_1,\ \acute{\kappa}_1)^{C})^{C} = (\ddot{\Omega}_1,\ \acute{\kappa}_1)$.
2. $((\ddot{\Omega}_1,\ \acute{\kappa}_1)_{\emptyset})^{C} = (\ddot{\Omega}_1,\ \acute{\kappa}_1)_{\acute{\mathfrak{V}}}$.
3. $((\ddot{\Omega}_1,\ \acute{\kappa}_1)_{\acute{\mathfrak{V}}})^{C} = (\ddot{\Omega}_1,\ \acute{\kappa}_1)_{\emptyset}$.

Proof.

1. Taking the complement will shift positive membership to the place of positive non-membership and negative membership to the place of negative non-membership and operating it again will swap values to their original positions.

2. The proof can be done as follows.

$$(\ddot{\Omega}_1,\ \acute{\kappa}_1)_{\emptyset} = \{\acute{v},\ < 0,\ 0,\ 1,\ 0,\ 0,\ -1 >:\ \acute{v} \in \acute{\mathfrak{V}}\}.$$

$$\text{Then, } ((\ddot{\Omega}_1,\ \acute{\kappa}_1)_{\emptyset})^{C} = \{\acute{v},\ < 1,\ 0,\ 0,\ -1,\ 0,\ 0 >:\ \acute{v} \in \acute{\mathfrak{V}}\}.$$

$$\text{Thus, } ((\ddot{\Omega}_1,\ \acute{\kappa}_1)_{\emptyset})^{C} = (\ddot{\Omega}_1,\ \acute{\kappa}_1)_{\acute{\mathfrak{V}}}.$$

3. The proof can be done similarly as of 2.

Definition 3.8 $\daleth = \{\dot{\varsigma}^{+},\ \dot{\rho}^{+},\ \dot{\vartheta}^{+},\ \dot{\varsigma}^{-},\ \dot{\rho}^{-},\ \dot{\vartheta}^{-}\}$ be the BTSFHSN($\acute{\mathfrak{V}}$). Then the Score Function (SF) is the map $SF : BTSFHSN(\acute{\mathfrak{V}}) \to [-1,\ 1]$ and it is defined by

$$SF(\daleth) = \frac{(\dot{\varsigma}^{+})^{t} - (\dot{\rho}^{+})^{t} - (\dot{\vartheta}^{+})^{t} + (|\dot{\varsigma}^{-}|^{t} - |\dot{\rho}^{-}|^{t} - |\dot{\vartheta}^{-}|^{t})}{2};\ t \geq 1.$$

Definition 3.9 $\daleth = \{\dot{\varsigma}^{+},\ \dot{\rho}^{+},\ \dot{\vartheta}^{+},\ \dot{\varsigma}^{-},\ \dot{\rho}^{-},\ \dot{\vartheta}^{-}\}$ be the BTSFHSN($\acute{\mathfrak{V}}$). Then the Accuracy Function (AF) is the map $AF : BTSFHSN(\acute{\mathfrak{V}}) \to [0,\ 1]$ and it is defined by

$$AF(\daleth) = \frac{(\dot{\varsigma}^{+})^{t} + (\dot{\rho}^{+})^{t} + (\dot{\vartheta}^{+})^{t} + (|\dot{\varsigma}^{-}|^{t} + |\dot{\rho}^{-}|^{t} + |\dot{\vartheta}^{-}|^{t})}{2};\ t \geq 1.$$

Proposition 3.4 Let $\daleth_1 = \{\dot{\varsigma}_1^{+},\ \dot{\rho}_1^{+},\ \dot{\vartheta}_1^{+},\ \dot{\varsigma}_1^{-},\ \dot{\rho}_1^{-},\ \dot{\vartheta}_1^{-}\}$ and $\daleth_2 = \{\dot{\varsigma}_2^{+},\ \dot{\rho}_2^{+},\ \dot{\vartheta}_2^{+},\ \dot{\varsigma}_2^{-},\ \dot{\rho}_2^{-},\ \dot{\vartheta}_2^{-}\}$ be two BTSFHSN($\acute{\mathfrak{V}}$). Then we can compare them by SF and AF with the following procedure:

1. If $SF(\daleth_1) > SF(\daleth_2)$ *then* $\daleth_1 > \daleth_2$.
2. If $SF(\daleth_1) = SF(\daleth_2)$, then we have to use AF.

$AF(\daleth_1) > AF(\daleth_2)$ implies $\daleth_1 > \daleth_2$.

$AF(\daleth_1) = AF(\daleth_2)$ implies $\daleth_1 = \daleth_2$.

Proof. Obviously, the theorem is true.

Example: Consider the BTSFHSN $\daleth_1 = \{0.001,\ 0.5,\ 0.001,\ 0,\ 0 - 0.1\}$ and $\daleth_2 = \{0.02,\ 0.5,\ 0.02,\ 0,\ 0,\ -0.1\}$ and $t = 1$.

Then SF($\daleth_1$) $= -0.3$, SF($\daleth_2$) $= -0.3$. When the scores are equal then proceed to compute the data by using accuracy function. Thus, AF($\daleth_1$) $= 0.301$ and AF($\daleth_2$) $= 0.32$. Hence, $\daleth_1 < \daleth_2$.

Definition 3.10 Let $(\ddot{\Omega}_1,\ \acute{\kappa}_1)$ and $(\ddot{\Omega}_2,\ \acute{\kappa}_2)$ be BTSFHSS($\acute{\mathfrak{V}}$). Then OR operation is defined by $(\ddot{\Omega}_1,\ \acute{\kappa}_1) \vee (\ddot{\Omega}_2,\ \acute{\kappa}_2) = (\xi,\ \acute{\kappa}_1 \times \acute{\kappa}_2)$ where,

$$(\xi,\ \acute{\kappa}_1 \times \acute{\kappa}_2) = \left\{ \begin{array}{c} min\{\dot{\varsigma}^{+}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}),\ \dot{\varsigma}^{+}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v})\},\ min\{\dot{\rho}^{+}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}),\ \dot{\rho}^{+}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v})\} max\{\dot{\vartheta}^{+}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}),\ \dot{\vartheta}^{+}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v})\}, \\ max\{\dot{\varsigma}^{-}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}),\ \dot{\varsigma}^{-}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v})\}\ max\{\dot{\rho}^{-}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}),\ \dot{\rho}^{-}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v})\},\ min\{\dot{\vartheta}^{-}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}),\ \dot{\vartheta}^{-}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v})\} \end{array} \right\}.$$

Definition 3.11 Let $(\ddot{\Omega}_1,\ \acute{\kappa}_1)$ and $(\ddot{\Omega}_2,\ \acute{\kappa}_2)$ be BTSFHSS($\acute{\mathfrak{V}}$). Then AND operation is defined by $(\ddot{\Omega}_1,\ \acute{\kappa}_1) \wedge (\ddot{\Omega}_2,\ \acute{\kappa}_2) = (\xi,\ \acute{\kappa}_1 \times \acute{\kappa}_2)$ where,

$$(\xi,\ \acute{\kappa}_1 \times \acute{\kappa}_2) = \left\{ \begin{array}{c} min\{\dot{\varsigma}^{+}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}),\ \dot{\varsigma}^{+}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v})\},\ min\{\dot{\rho}^{+}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}),\ \dot{\rho}^{+}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v})\}\ , max\{\dot{\vartheta}^{+}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}),\ \dot{\vartheta}^{+}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v})\}, \\ max\{\dot{\varsigma}^{-}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}),\ \dot{\varsigma}^{-}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v})\} max\{\dot{\rho}^{-}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}),\ \dot{\rho}^{-}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v})\}, min\{\dot{\vartheta}^{-}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}),\ \dot{\vartheta}^{-}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v})\} \end{array} \right\}.$$

Proposition 3.5 Let $(\ddot{\Omega}_1,\ \acute{\kappa}_1)$ and $(\ddot{\Omega}_2,\ \acute{\kappa}_2)$ be BTSFHSS($\acute{\mathfrak{V}}$). Then $(\ddot{\Omega}_1,\ \acute{\kappa}_1) \vee (\ddot{\Omega}_2,\ \acute{\kappa}_2) \in$ BTSFHSS($\acute{\mathfrak{V}}$).

Proof. Let $(\ddot{\Omega}_1,\ \acute{\kappa}_1) = (\acute{v},\ <\dot{\varsigma}^{+}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}),\ \dot{\rho}^{+}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}),\ \dot{\vartheta}^{+}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}),\ \dot{\varsigma}^{-}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}),\ \dot{\rho}^{-}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}),\ \dot{\vartheta}^{-}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v})>)$ and $(\ddot{\Omega}_2,\ \acute{\kappa}_2) = (\acute{v},\ <\dot{\varsigma}^{+}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v}),\ \dot{\rho}^{+}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v}),\ \dot{\vartheta}^{+}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v}),\ \dot{\varsigma}^{-}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v}),\ \dot{\rho}^{-}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v}),\ \dot{\vartheta}^{-}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v})>)$ for $\acute{v} \in \acute{\mathfrak{V}}$, $\dot{\varepsilon}_i \in \acute{\kappa}_i$.

Thus for $t \geq 1$, $0 \leq (\dot{\varsigma}^{+}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}))^{t} + (\dot{\rho}^{+}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}))^{t} + (\dot{\vartheta}^{+}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v}))^{t} \leq 1$, $0 \leq (\dot{\varsigma}^{+}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v}))^{t} + (\dot{\rho}^{+}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v}))^{t} + (\dot{\vartheta}^{+}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v}))^{t} \leq 1$, $0 \leq |\dot{\varsigma}^{-}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v})|^{t} + |\dot{\rho}^{-}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v})|^{t} + |\dot{\vartheta}^{-}_{\acute{\kappa}_1(\dot{\varepsilon}_1)}(\acute{v})|^{t} \leq 1$ and $0 \leq |\dot{\varsigma}^{-}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v})|^{t} + |\dot{\rho}^{-}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v})|^{t} + |\dot{\vartheta}^{-}_{\acute{\kappa}_2(\dot{\varepsilon}_2)}(\acute{v})|^{t} \leq 1$.

Now, to prove $(\ddot{\Omega}_1,\ \acute{\kappa}_1) \vee (\ddot{\Omega}_2,\ \acute{\kappa}_2)$ is a BTSFHSS($\acute{\mathfrak{V}}$).

It is enough to show,

$$0 \leq (max\{\dot{\varsigma}^{+}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}),\ \dot{\varsigma}^{+}_{\acute{\kappa}_2(\dot{e}_2)}(\acute{v})\})^t + (min\{\dot{\rho}^{+}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}),\ \dot{\rho}^{+}_{\acute{\kappa}_2(\dot{e}_2)}(\acute{v})\})^t + (min\{\dot{\vartheta}^{+}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}),\ \dot{\varsigma}^{+}_{\acute{\kappa}_2(\dot{e}_2)}(\acute{v})\})^t \leq 1$$

and

$$0 \leq |min\{\dot{\varsigma}^{-}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}),\ \dot{\varsigma}^{-}_{\acute{\kappa}_2(\dot{e}_2)}(\acute{v})\}|^t + |max\{\dot{\rho}^{-}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}),\ \dot{\rho}^{-}_{\acute{\kappa}_2(\dot{e}_2)}(\acute{v})\}|^t + |max\{\dot{\vartheta}^{-}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}),\ \dot{\varsigma}^{-}_{\acute{\kappa}_2(\dot{e}_2)}(\acute{v})\}|^t \leq 1$$

let first prove the inequality,

i) $0 \leq (max\{\dot{\varsigma}^{+}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}),\ \dot{\varsigma}^{+}_{\acute{\kappa}_2(\dot{e}_2)}(\acute{v})\})^t + (min\{\dot{\rho}^{+}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}),\ \dot{\rho}^{+}_{\acute{\kappa}_2(\dot{e}_2)}(\acute{v})\})^t + (min\{\dot{\vartheta}^{+}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}),\ \dot{\varsigma}^{+}_{\acute{\kappa}_2(\dot{e}_2)}(\acute{v})\})^t \leq 1.$

Take $(\dot{\varsigma}^{+}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}))^t = a_1,\ (\dot{\varsigma}^{+}_{\acute{\kappa}_2(\dot{e}_2)}(\acute{v}))^t = a_2,\ (\dot{\rho}^{+}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}))^t = b_1,\ (\dot{\rho}^{+}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}))^t = b_2,\ (\dot{\vartheta}^{+}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}))^t = c_1,\ (\dot{\vartheta}^{+}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}))^t = c_2$ for simplicity. Obviously, $\max\{a_1,\ a_2\} + \min\{b_1,\ b_2\} + \min\{c_1,\ c_2\} \geq 0.$

Now to check, $\max\{a_1,\ a_2\} + \min\{b_1,\ b_2\} + \min\{c_1,\ c_2\} \leq 1.$ Notice that $a_1 \leq 1 - (b_1 + c_1)$ *and* $a_2 \leq 1 - (b_2 + c_2) \implies \max\{a_1,\ a_2\} \leq \max\{1 - (b_1 + c_1),\ 1 - (b_2 + c_2)\}$

$$\begin{aligned}\max\{a_1,\ a_2\} + \min\{b_1,\ b_2\} + \min\{c_1,\ c_2\} &\leq \max\{1 - (b_1 + c_1),\ 1 - (b_2 + c_2)\} + \min\{b_1,\ b_2\} + \min\{c_1,\ c_2\} \\ &= 1 - \min\{(b_1 + c_1),\ (b_2 + c_2)\} + \min\{b_1,\ b_2\} + \min\{c_1,\ c_2\} \\ &\leq 1 - \min\{b_1,\ b_2\} - \min\{c_1,\ c_2\} + \min\{b_1,\ b_2\} + \min\{c_1,\ c_2\}.\end{aligned}$$

Thus, $\max\{a_1,\ a_2\} + \min\{b_1,\ b_2\} + \min\{c_1,\ c_2\} \leq 1.$

Therefore, $0 \leq (max\{\dot{\varsigma}^{+}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}),\ \dot{\varsigma}^{+}_{\acute{\kappa}_2(\dot{e}_2)}(\acute{v})\})^t + (min\{\dot{\rho}^{+}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}),\ \dot{\rho}^{+}_{\acute{\kappa}_2(\dot{e}_2)}(\acute{v})\})^t + (min\{\dot{\vartheta}^{+}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}),\ \dot{\varsigma}^{+}_{\acute{\kappa}_2(\dot{e}_2)}(\acute{v})\})^t \leq 1.$

In a similar manner,

$$0 \leq |min\{\dot{\varsigma}^{-}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}),\ \dot{\varsigma}^{-}_{\acute{\kappa}_2(\dot{e}_2)}(\acute{v})\}|^t + |max\{\dot{\rho}^{-}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}),\ \dot{\rho}^{-}_{\acute{\kappa}_2(\dot{e}_2)}(\acute{v})\}|^t + |max\{\dot{\vartheta}^{-}_{\acute{\kappa}_1(\dot{e}_1)}(\acute{v}),\ \dot{\varsigma}^{-}_{\acute{\kappa}_2(\dot{e}_2)}(\acute{v})\}|^t \leq 1,$$

can be proven.

Thus, $(\ddot{\Omega}_1,\ \acute{\kappa}_1) \vee (\ddot{\Omega}_2,\ \acute{\kappa}_2) \in$ BTSFHSS($\acute{\mathfrak{V}}$).

Proposition 3.6 Let $(\ddot{\Omega}_1,\ \acute{\kappa}_1)$ and $(\ddot{\Omega}_2,\ \acute{\kappa}_2)$ be BTSFHSS($\acute{\mathfrak{V}}$). Then $(\ddot{\Omega}_1,\ \acute{\kappa}_1) \wedge (\ddot{\Omega}_2,\ \acute{\kappa}_2) \in$ BTSFHSS($\acute{\mathfrak{V}}$).

Proof. The proof of theorem requires the similar argument used in Theorem 3.5.

4. MADM technique based on BTSFHSS

In this section, comparison matrix is discussed with some scoring measures and presented a procedure to deal with it. So, using this technique, a problem is employed to MADM to identify the most acceptable and efficient air filters based on the parameters that are typically expected from industrial sides.

Definition 4.1 The Comparison matrix on BTSFHSS $(\omega,\ \Omega_1)$ is a $m \times m$ matrix in which the alternatives $\acute{v}_1,\ \acute{v}_2,\ \ldots,\ \acute{v}_m$ plays the role as row and column and the numerical value C_{ab} $(a,\ b$ are represented by the non negative integers that are upto m) taken by each entries is nothing but just evaluating how many parameters satisfying the condition $SF(\gimel_{\xi_{ac}}) \geq SF(\gimel_{\xi_{bc}})$, where $\gimel_{\xi_{ac}}$ represents the BTSFHSN $(\dot{\varsigma}^+_{\xi_{ac}}, \dot{\rho}^+_{\xi_{ac}}, \dot{\vartheta}^+_{\xi_{ac}}, \dot{\varsigma}^-_{\xi_{ac}}, \dot{\rho}^-_{\xi_{ac}}, \dot{\vartheta}^-_{\xi_{ac}})$, $\gimel_{\xi_{bc}}$ represents the BTSFHSN $(\dot{\varsigma}^+_{\xi_{bc}}, \dot{\rho}^+_{\xi_{bc}}, \dot{\vartheta}^+_{\xi_{bc}}, \dot{\varsigma}^-_{\xi_{bc}}, \dot{\rho}^-_{\xi_{bc}}, \dot{\vartheta}^-_{\xi_{bc}})$ and $0 \leq C_{ab}$ is any integer that can takes values upto n, where n is the total number of parameters in Ω_1.

Key elements for evaluating the values acquired in the matrix

• The rowwise total $\Re\Sigma_a$ of the $\acute{v}_j$ is given by the formula

$$\Re\Sigma_a = C_{a1} + C_{a2} + \cdot + C_{am}.$$

• The columnwise total $\mathfrak{C}\Sigma_b$ of the $\acute{v}_j$ is given by the formula

$$\mathfrak{C}\Sigma_b = C_{1a} + C_{2a} + \cdot + C_{ma}.$$

Definition 4.2 The score total Υ_j for each alternative $\acute{v}_j$ is calculated by

$$\Upsilon_j = \Re\Sigma_j - \mathfrak{C}\Sigma_j.$$

Algorithm The best alternative can be picked up by the algorithm depicted in the following steps. Also, it is illustrated as a flow chart in Figure 1.

Step 1: Input the BTSFHSSs $(\ddot{\Omega}_1, \acute{\kappa}_1)$ and$(\ddot{\Omega}_2, \acute{\kappa}_2)$.

Step 2: compute the resultant BTSFHSS $(\ddot{\Omega}_1, \acute{\kappa}_1) \wedge (\ddot{\Omega}_2, \acute{\kappa}_2)$ =$(\xi, \acute{\kappa}_1 \times \acute{\kappa}_2)$ by above mentioned Defintion 3.11.

Step 3: using the Definition 3.8, estimate the score of BTSFHSN in $(\xi, \acute{\kappa}_1 \times \acute{\kappa}_2)$.

Step 4: The comparison table $(\xi, \acute{\kappa}_1 \times \acute{\kappa}_2)$ is constructed using the Definition 4.1.

Step 5: Estimate the score total of u_j $\forall j$ using Definition 4.2.

Step 6: Find $\Upsilon_l = max_a \Upsilon_a$ and select it as ideal alternative.

Step 7: If best alternative was attained more than one, choose any one of them.

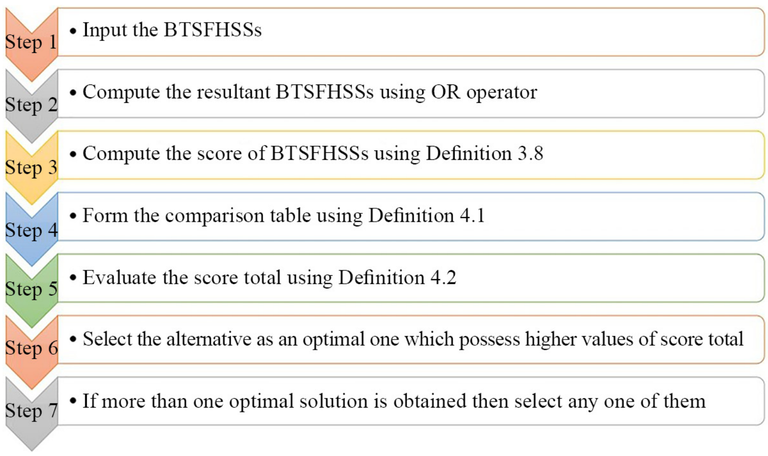

Figure 1. Proposed algorithm's flowchart

Case study: In day-to-day life, clean air is the most indispensable thing and is the basic need that is essential for living. The major air pollution was caused by the industries, vehicles, houses etc., in this study, the main interest is to select a suitable filter for reducing the air pollutants that are emitted from industries. The four appropriate filters are Cylcone Seperators (CYS), Electrostatic Precipitators (EP), Wet Scrubbers (WS) and Fabric Filters (FF). From these four the most suitable one will be selected. A brief outlook on these four filters is discussed in this study. EP is a filterless device that removes fine particles, such as dust and smoke, from the dirty air using the electrostatic charge before it enters into the chimney.

A wet scrubber refers to a group of air pollution control systems designed to remove contaminants from furnace flue gas or other industrial gas streams. In these systems, the polluted gas is brought into contact with a scrubbing medium by spraying, passing the gas through a liquid pool, or using another contact method, so that the pollutants can be captured. Dust particles are trapped within the liquid droplets, which are then collected. If the pollutants are gaseous, they may be dissolved or absorbed by the liquid. To prevent liquid droplets from being carried into the cleaned gas outlet, a mist eliminator is installed downstream. The scrubbing liquid, once used, requires treatment before it can be discharged or reused within the facility.

A FF operates by allowing flue gas to pass through a porous medium that blocks particles larger than its pore size. Over time, collected ash particles form a dust layer, which itself becomes an additional filtering surface and can also capture metallic compounds present in the gas. Cleaning of the filter is typically triggered either by a set pressure drop across the filter or after a fixed operating period. During cleaning, a pressure pulse or other method dislodges the dust layer from the gas-facing side. Various FF designs exist, differing mainly in their dust removal techniques, but all follow the same principle. As polluted air enters, dust is trapped on the filter surfaces, and accumulated material is periodically removed to maintain efficiency.

Similar to a centrifuge, cyclone separators need a constant supply of contaminated air. Unclean flue gas is pumped into a chamber in a cyclone separator. The chamber's inside produces a tornado-like spiral vortex. It is easier for the lighter parts of this gas to be impacted by the vortex and move up it since they have less inertia. On the other hand, larger particulate matter components are more resistant to the vortex's action since they have greater inertia. These larger particles strike the container's interior walls and fall into a collection hopper because they find it impossible to follow the gas's fast spiral motion and the vortex. To encourage the collection of these particles at the bottom of the container, these chambers are fashioned like an upside-down cone. The top of the chamber is where the cleaned flue gas exits.

By trapping dust particles in liquid droplets, wet scrubbers eliminate them. Following the collection of the droplets, the liquid absorbs or dissolves the noxious gases. An additional device known as a mist eliminator or entrainment separator is required to separate any droplets present in the scrubber's incoming gas from the exit gas stream. Additionally, the resulting scouring liquid needs to be treated before it is finally released or used again in the plant. By the use of suggested notions, the algorithm of MADM Problem is demonstrated in the following.

Problem Let $\mathfrak{D} = \{\acute{v}_1 = \{\text{Electrostatic precipitator}\}, \acute{v}_2 = \{\text{Wet Scrubbers}\}, \acute{v}_3 = \{\text{Cyclone Seperators}\}, \acute{v}_4 = \{\text{Fabric Filters}\}\}$ be the set of filters that are provided. Two expertise teams $\{\mathfrak{E}_1, \mathfrak{E}_2\}$ were set to find an optimal filter by analyzing the alternatives.

Let $f_1 = \{\text{Cost}\}$, $f_2 = \{\text{Efficiency}\}$, $f_3 = \{\text{Maintenance}\}$, $f_4 = \{\text{life span}\}$ be the attributes and $\varrho_1 = \{$Purchase cost (p_{11}), Maintenance cost $(p_{12})\}$, $\varrho_2 = \{$Efficiency (p_{21}), Pore size $(p_{22})\}$, $\varrho_3 = \{$Maintenance period $(p_{31})\}$, $\varrho_4 = \{$span rate $(p_{41})\}$ be their corresponding attribute values respectively. The possible alternatives are

$$\ddot{\mathfrak{Q}} = \{\tau_1 = (p_{11}, p_{21}, p_{31}, p_{41}), \tau_2 = (p_{11}, p_{22}, p_{31}, p_{41}), \tau_3 = (p_{12}, p_{21}, p_{31}, p_{41}), \tau_4 = (p_{12}, p_{22}, p_{31}, p_{41})\}.$$

Step 1:

The data that are given by the expertise team $\mathfrak{E}_1$ by analyzing the alternatives are constructed as BTSFHSS. Then, BTSFHSS $(\ddot{\mathfrak{Q}}_1, \acute{\kappa}_1)$ may be expressed as

$$(\ddot{\Omega}_1,\ \acute{\kappa}_1) = \begin{cases} \left\langle \tau_1,\ \left(\dfrac{\acute{v}_1}{[(0.512,\ 0.017,\ 0.0156),\ (-0.06,\ -0.125,\ -0.343)]}, \right.\right. \\ \dfrac{\acute{v}_2}{[(0.064,\ 0.125,\ 0.027),\ (-0.125,\ -0.216,\ -0.357)]}, \\ \dfrac{\acute{v}_3}{[(0.024,\ 0.074,\ 0.064),\ (-0.343,\ -0.064,\ -0.421)]}, \\ \left.\left. \dfrac{\acute{v}_4}{[(0.274,\ 0.008,\ 0.042),\ (-0.091,\ -0.1851,\ -0.456)]} \right) \right\rangle \\ \left\langle \tau_2,\ \left(\dfrac{\acute{v}_1}{[(0.3579,\ 0.003,\ 0.024),\ (-0.027,\ -0.008,\ -0.064)]}, \right.\right. \\ \dfrac{\acute{v}_2}{[((0.027,\ 0.0506,\ 0.0593),\ (-0.125,\ -0.008,\ -0.027)]}, \\ \dfrac{\acute{v}_3}{[(0.125,\ 00156,\ 0.0506),\ (-0.064,\ -0.008,\ -0.027)]}, \\ \left.\left. \dfrac{\acute{v}_4}{[(0.042,\ 0.008,\ 0.125),\ (-0.343,\ -0.001,\ -0.008)]} \right) \right\rangle \\ \left\langle \tau_3,\ \left(\dfrac{\acute{v}_1}{[(0.592,\ 0.001,\ 0.009),\ (-0.05,\ -0.0092,\ -0.064)]}, \right.\right. \\ \dfrac{\acute{v}_2}{[(0.064,\ 0.027,\ 0.006),\ (-0.008,\ -0.091,\ -0.027)]}, \\ \dfrac{\acute{v}_3}{[(0.012,\ 0.064,\ 0.035),\ (-0.027,\ -0.064,\ -0.012)]}, \\ \left.\left. \dfrac{\acute{v}_4}{[(0.079,\ 0.05,\ 0.024),\ (-0.003,\ -0.019,\ -0.216)]} \right) \right\rangle \end{cases}$$

$$\left\langle \tau_4, \left(\frac{\acute{v}_1}{[(0.421,\ 0.009,\ 0.027),\ (-0.05,\ -0.024,\ -0.068)]}, \frac{\acute{v}_2}{[(0.042,\ 0.079,\ 0.019),\ (-0.059,\ -0.091,\ -0.008)]}, \frac{\acute{v}_3}{[(0.013,\ 0.012,\ 0.216),\ (-0.216,\ -0.027,\ -0.006)]}, \frac{\acute{v}_4}{[(0.05,\ 0.024,\ 0.125),\ (-0.068,\ -0.019,\ -0.029)]} \right) \right\rangle. \Bigg\}$$

Here the value of t is taken as 3.

The data that are given by the expertise team $\mathfrak{E}_2$ by analyzing the alternatives are constructed as BTSFHSS. Then, BTSFHSS $(\ddot{\Omega}_2,\ \acute{\kappa}_2)$ may be expressed as

$$(\ddot{\Omega}_2,\ \acute{\kappa}_2) = \left\{ \begin{array}{l} \left\langle \tau_1, \left(\frac{\acute{v}_1}{[(0.61,\ 0.006,\ 0.029),\ (-0.054,\ -0.166,\ -0.314)]}, \frac{\acute{v}_2}{[(0.054,\ 0.140,\ 0.027),\ (-0.148,\ -0.262,\ -0.389)]}, \frac{\acute{v}_3}{[(0.029,\ 0.091,\ 0.054),\ (-0.373,\ -0.074,\ -0.314)]}, \frac{\acute{v}_4}{[(0.238,\ 0.0156,\ 0.035),\ (-0.074,\ -0.148,\ -0.373)]} \right) \right\rangle \\ \left\langle \tau_2, \left(\frac{\acute{v}_1}{[(0.30,\ 0.002,\ 0.019),\ (-0.019,\ -0.010,\ -0.05)]}, \frac{\acute{v}_2}{[(0.032,\ 0.0548,\ 0.0506),\ (-0.068,\ -0.012,\ -0.024)]}, \frac{\acute{v}_3}{[(0.1038,\ 0.0121,\ 0.042),\ (-0.059,\ -0.012,\ -0.0297)]}, \frac{\acute{v}_4}{[(0.0506,\ 0.009,\ 0.103),\ (-0.328,\ -0.003,\ -0.027)]} \right) \right\rangle \end{array} \right.$$

$$
\left\langle \tau_3, \left(\frac{\acute{v}_1}{[(0.551,\ 0.004,\ 0.027),\ (-0.068,\ -0.015,\ -0.012)]}, \frac{\acute{v}_2}{[(0.074,\ 0.042,\ 0.196),\ (-0.006,\ -0.074,\ -0.05)]}, \frac{\acute{v}_3}{[(0.024,\ 0.079,\ 0.029),\ (-0.0506,\ -0.059,\ -0.006)]}, \frac{\acute{v}_4}{[(0.068,\ 0.05,\ 0.009),\ (-0.004,\ -0.0029,\ -0.166)]} \right) \right\rangle
$$

$$
\left.\left\langle \tau_4, \left(\frac{\acute{v}_1}{[(0.493,\ 0.006,\ 0.0196),\ (-0.068,\ -0.059,\ -0.05)]}, \frac{\acute{v}_2}{[(0.059,\ 0.103,\ 0.035),\ (-0.05,\ -0.074,\ -0.024)]}, \frac{\acute{v}_3}{[(0.039,\ 0.006,\ 0.274),\ (-0.22,\ -0.024,\ -0.0156)]}, \frac{\acute{v}_4}{[(0.059,\ 0.042,\ 0.14),\ (-0.079,\ -0.019,\ -0.042)]} \right) \right\rangle. \right\}
$$

Here the value of t is taken as 3.

Step 2:

The opinions of both expertise teams are presented in the form of BTSFHSS $(\ddot{\Omega}_1,\ \acute{\kappa}_1)$ and $(\ddot{\Omega}_2,\ \acute{\kappa}_2)$ are combined by AND operation and the resultant BTSFHSS $(\xi,\ \acute{\kappa}_1 \times \acute{\kappa}_2)$ is

$$(\xi,\ \acute{\kappa}_1\times\acute{\kappa}_2)=\left\{\begin{array}{l}
\left\langle(\tau_1,\ \tau_1)\left(\dfrac{\acute{v}_1}{\langle(0.512,\ 0.017,\ 0.029,\ -0.05,\ -0.166,\ -0.343)\rangle},\ \dfrac{\acute{v}_2}{\langle(0.054,\ 0.140,\ 0.027,\ -0.125,\ -0.262,\ -0.389)\rangle},\right.\right.\\
\left.\left.\dfrac{\acute{v}_3}{\langle(0.024,\ 0.091,\ 0.064,\ -0.343,\ -0.074,\ -0.421)\rangle},\ \dfrac{\acute{v}_4}{\langle(0.238,\ 0.0156,\ 0.042,\ -0.074,\ -0.185,\ -0.456)\rangle}\right)\right\rangle,\\
\left\langle(\tau_1,\ \tau_2)\left(\dfrac{\acute{v}_1}{\langle(0.30,\ 0.017,\ 0.019,\ -0.019,\ -0.125,\ -0.343)\rangle},\ \dfrac{\acute{v}_2}{\langle(0.032,\ 0.125,\ 0.05,\ -0.125,\ -0.012,\ -0.024)\rangle},\right.\right.\\
\left.\left.\dfrac{\acute{v}_3}{\langle(0.024,\ 0.074,\ 0.064,\ -0.059,\ -0.064,\ -0.421)\rangle},\ \dfrac{\acute{v}_4}{\langle(0.0506,\ 0.009,\ 0.103,\ -0.091,\ -0.185,\ -0.456)\rangle}\right)\right\rangle,\\
\left\langle(\tau_1,\ \tau_3)\left(\dfrac{\acute{v}_1}{\langle(0.512,\ 0.017,\ 0.027,\ -0.06,\ -0.125,\ -0.343)\rangle},\ \dfrac{\acute{v}_2}{\langle(0.064,\ 0.125,\ 0.027,\ -0.006,\ -0.216,\ -0.357)\rangle},\right.\right.\\
\left.\left.\dfrac{\acute{v}_3}{\langle(0.024,\ 0.079,\ 0.064,\ -0.343,\ -0.064,\ -0.421)\rangle},\ \dfrac{\acute{v}_4}{\langle(0.068,\ 0.05,\ 0.042,\ -0.004,\ -0.148,\ -0.373)\rangle}\right)\right\rangle,\\
\left\langle(\tau_1,\ \tau_4)\left(\dfrac{\acute{v}_1}{\langle(0.493,\ 0.017,\ 0.0196,\ -0.05,\ -0.024,\ -0.343)\rangle},\ \dfrac{\acute{v}_2}{\langle(0.421,\ 0.017,\ 0.027,\ -0.05,\ -0.125,\ -0.343)\rangle},\right.\right.\\
\left.\left.\dfrac{\acute{v}_3}{\langle(0.013,\ 0.074,\ 0.216,\ -0.216,\ -0.064,\ -0.421)\rangle},\ \dfrac{\acute{v}_4}{\langle(0.05,\ 0.0.024,\ 0.125,\ -0.068,\ -0.1851,\ -0.456)\rangle}\right)\right\rangle,
\end{array}\right\}$$

$$(\xi,\ \acute{\kappa}_1 \times \acute{\kappa}_2) = \left\{ \begin{array}{l} \left\langle (\tau_2,\ \tau_1) \left(\dfrac{\acute{v}_1}{\begin{array}{l}\langle (0.3579,\ 0.006,\ 0.029, \\ \quad -0.027,\ -0.166,\ -0.314)\rangle\end{array}},\ \dfrac{\acute{v}_2}{\begin{array}{l}\langle (0.027,\ 0.140,\ 0.059, \\ \quad -0.125,\ -0.008,\ -0.389)\rangle\end{array}}, \right. \right. \\ \left. \left. \dfrac{\acute{v}_3}{\begin{array}{l}\langle (0.029,\ 0.091,\ 0.054, \\ \quad -0.064,\ -0.074,\ -0.314)\rangle\end{array}},\ \dfrac{\acute{v}_4}{\begin{array}{l}\langle (0.042,\ 0.0156,\ 0.125, \\ \quad -0.074,\ -0.148,\ -0.373)\rangle\end{array}} \right) \right\rangle, \\ \left\langle (\tau_2,\ \tau_2) \left(\dfrac{\acute{v}_1}{\begin{array}{l}\langle (0.30,\ 0.003,\ 0.024, \\ \quad -0.019,\ -0.01,\ -0.064)\rangle\end{array}},\ \dfrac{\acute{v}_2}{\begin{array}{l}\langle (0.027,\ 0.0548,\ 0.0593, \\ \quad -0.068,\ -0.012,\ -0.027)\rangle\end{array}}, \right. \right. \\ \left. \left. \dfrac{\acute{v}_3}{\begin{array}{l}\langle (0.1038,\ 0.0156,\ 0.0506, \\ \quad -0.059,\ -0.012,\ -0.029)\rangle\end{array}},\ \dfrac{\acute{v}_4}{\begin{array}{l}\langle (0.042,\ 0.009,\ 0.125, \\ \quad -0.328,\ -0.003,\ -0.027)\rangle\end{array}} \right) \right\rangle, \\ \left\langle (\tau_2,\ \tau_3) \left(\dfrac{\acute{v}_1}{\begin{array}{l}\langle (0.3579,\ 0.004,\ 0.027, \\ \quad -0.027,\ -0.015,\ -0.064)\rangle\end{array}},\ \dfrac{\acute{v}_2}{\begin{array}{l}\langle (0.027,\ 0.05,\ 0.059, \\ \quad -0.006,\ -0.074,\ -0.05)\rangle\end{array}}, \right. \right. \\ \left. \left. \dfrac{\acute{v}_3}{\begin{array}{l}\langle (0.024,\ 0.079,\ 0.050, \\ \quad -0.05,\ -0.059,\ -0.006)\rangle\end{array}},\ \dfrac{\acute{v}_4}{\begin{array}{l}\langle (0.042,\ 0.08,\ 0.125, \\ \quad -0.004,\ -0.029,\ -0.166)\rangle\end{array}} \right) \right\rangle, \\ \left\langle (\tau_2,\ \tau_4) \left(\dfrac{\acute{v}_1}{\begin{array}{l}\langle (0.3579,\ 0.006,\ 0.024, \\ \quad -0.027,\ -0.059,\ -0.064)\rangle\end{array}},\ \dfrac{\acute{v}_2}{\begin{array}{l}\langle (0.027,\ 0.103,\ 0.059, \\ \quad -0.05,\ -0.074,\ -0.027)\rangle\end{array}}, \right. \right. \\ \left. \left. \dfrac{\acute{v}_3}{\begin{array}{l}\langle (0.125,\ 0.015,\ 0.274, \\ \quad -0.064,\ -0.024,\ -0.015)\rangle\end{array}},\ \dfrac{\acute{v}_4}{\begin{array}{l}\langle (0.042,\ 0.042,\ 0.125, \\ \quad -0.079,\ -0.0019,\ -0.042)\rangle\end{array}} \right) \right\rangle, \end{array} \right\}$$

$$
(\xi, \acute{\kappa}_1 \times \acute{\kappa}_2) = \left\{ \begin{array}{l}
\left\langle (\tau_3,\ \tau_1) \left(\frac{\acute{v}_1}{\langle (0.592,\ 0.006,\ 0.029,\ -0.05,\ -0.166,\ -0.314) \rangle},\ \frac{\acute{v}_2}{\langle (0.054,\ 0.14,\ 0.027,\ -0.008,\ -0.262,\ -0.389) \rangle}, \right.\right. \\
\left.\left. \frac{\acute{v}_3}{\langle (0.012,\ 0.091,\ 0.054,\ -0.027,\ -0.074,\ -0.314) \rangle},\ \frac{\acute{v}_4}{\langle (0.079,\ 0.05,\ 0.035,\ -0.003,\ -0.148,\ -0.373) \rangle} \right) \right\rangle, \\
\left\langle (\tau_3,\ \tau_2) \left(\frac{\acute{v}_1}{\langle (0.3,\ 0.002,\ 0.019,\ -0.019,\ -0.1,\ -0.0064) \rangle},\ \frac{\acute{v}_2}{\langle (0.032,\ 0.05,\ 0.05,\ -0.008,\ -0.091,\ -0.027) \rangle}, \right.\right. \\
\left.\left. \frac{\acute{v}_3}{\langle (0.012,\ 0.064,\ 0.042,\ -0.027,\ -0.064,\ -0.029) \rangle},\ \frac{\acute{v}_4}{\langle (0.05,\ 0.05,\ 0.103,\ -0.003,\ -0.019,\ -0.216) \rangle} \right) \right\rangle, \\
\left\langle (\tau_3,\ \tau_3) \left(\frac{\acute{v}_1}{\langle (0.551,\ 0.004,\ 0.027,\ -0.05,\ -0.015,\ -0.064) \rangle},\ \frac{\acute{v}_2}{\langle (0.064,\ 0.042,\ 0.019,\ -0.006,\ -0.091,\ -0.05) \rangle}, \right.\right. \\
\left.\left. \frac{\acute{v}_3}{\langle (0.012,\ 0.079,\ 0.035,\ -0.027,\ -0.064,\ -0.012) \rangle},\ \frac{\acute{v}_4}{\langle (0.068,\ 0.05,\ 0.024,\ -0.003,\ -0.029,\ -0.216) \rangle} \right) \right\rangle, \\
\left\langle (\tau_3,\ \tau_4) \left(\frac{\acute{v}_1}{\langle (0.493,\ 0.006,\ 0.0196,\ -0.05,\ -0.059,\ -0.064) \rangle},\ \frac{\acute{v}_2}{\langle (0.059,\ 0.103,\ 0.035,\ -0.008,\ -0.091,\ -0.027) \rangle}, \right.\right. \\
\left.\left. \frac{\acute{v}_3}{\langle (0.012,\ 0.064,\ 0.035,\ -0.027,\ -0.064,\ -0.015) \rangle},\ \frac{\acute{v}_4}{\langle (0.059,\ 0.05,\ 0.14,\ -0.003,\ -0.019,\ -0.216) \rangle} \right) \right\rangle,
\end{array} \right\}
$$

$$
(\xi,\ \acute{\kappa}_1 \times \acute{\kappa}_2) = \left\{ \begin{array}{l}
\left\langle (\tau_4,\ \tau_1) \left(\dfrac{\acute{v}_1}{\langle(0.421,\ 0.009,\ 0.029,\ -0.05,\ -0.166,\ -0.314)\rangle},\ \dfrac{\acute{v}_2}{\langle(0.042,\ 0.14,\ 0.027,\ -0.059,\ -0.262,\ -0.389)\rangle}, \right.\right. \\
\left.\left. \dfrac{\acute{v}_3}{\langle(0.013,\ 0.091,\ 0.054,\ -0.216,\ -0.074,\ -0.314)\rangle},\ \dfrac{\acute{v}_4}{\langle(0.05,\ 0.024,\ 0.125,\ -0.068,\ -0.148,\ -0.373)\rangle} \right) \right\rangle, \\
\left\langle (\tau_4,\ \tau_2) \left(\dfrac{\acute{v}_1}{\langle(0.3,\ 0.009,\ 0.027,\ -0.019,\ -0.024,\ -0.068)\rangle},\ \dfrac{\acute{v}_2}{\langle(0.032,\ 0.079,\ 0.05,\ -0.059,\ -0.091,\ -0.024)\rangle}, \right.\right. \\
\left.\left. \dfrac{\acute{v}_3}{\langle(0.103,\ 0.012,\ 0.216,\ -0.059,\ -0.027,\ -0.029)\rangle},\ \dfrac{\acute{v}_4}{\langle(0.05,\ 0.024,\ 0.125,\ -0.068,\ -0.019,\ -0.029)\rangle} \right) \right\rangle, \\
\left\langle (\tau_4,\ \tau_3) \left(\dfrac{\acute{v}_1}{\langle(0.421,\ 0.009,\ 0.027,\ -0.05,\ -0.024,\ -0.068)\rangle},\ \dfrac{\acute{v}_2}{\langle(0.042,\ 0.079,\ 0.019,\ -0.006,\ -0.091,\ -0.05)\rangle}, \right.\right. \\
\left.\left. \dfrac{\acute{v}_3}{\langle(0.013,\ 0.079,\ 0.216,\ -0.05,\ -0.059,\ -0.006)\rangle},\ \dfrac{\acute{v}_4}{\langle(0.05,\ 0.05,\ 0.125,\ -0.004,\ -0.059,\ -0.166)\rangle} \right) \right\rangle, \\
\left\langle (\tau_4,\ \tau_4) \left(\dfrac{\acute{v}_1}{\langle(0.421,\ 0.009,\ 0.027,\ -0.068,\ -0.059,\ -0.068)\rangle},\ \dfrac{\acute{v}_2}{\langle(0.042,\ 0.103,\ 0.035,\ -0.05,\ -0.091,\ -0.024)\rangle}, \right.\right. \\
\left.\left. \dfrac{\acute{v}_3}{\langle(0.013,\ 0.012,\ 0.274,\ -0.22,\ -0.027,\ -0.0156)\rangle},\ \dfrac{\acute{v}_4}{\langle(0.05,\ 0.042,\ 0.125,\ -0.068,\ -0.019,\ -0.042)\rangle} \right) \right\rangle.
\end{array} \right\}
$$

Step 3: The opinions of both expertise teams which are in the form of BTSFHSS $(\ddot{\Omega}_1,\ \acute{\kappa}_1)$ and $(\ddot{\Omega}_2,\ \acute{\kappa}_2)$ are combined by AND operation and the resultant Table 1 $(\xi,\ \acute{\kappa}_1 \times \acute{\kappa}_2)$ is

Table 1. The resultant table obtained by AND operator in $(\xi,\ \acute{\kappa}_1 \times \acute{\kappa}_2)$

$(\xi,\ \acute{\kappa}_1 \times \acute{\kappa}_2)$	$\acute{v}_1$	$\acute{v}_2$	$\acute{v}_3$	$\acute{v}_4$
$(\tau_1,\ \tau_1)$	0.0035	-0.3195	-0.1415	-0.1933
$(\tau_1,\ \tau_2)$	-0.0925	-0.027	-0.27	-0.3057
$(\tau_1,\ \tau_3)$	0.03	-0.3275	-0.1305	-0.2705
$(\tau_1,\ \tau_4)$	0.0697	-0.0205	-0.273	-0.2728
$(\tau_2,\ \tau_1)$	-0.065	-0.222	-0.22	-0.2728
$(\tau_2,\ \tau_2)$	0.109	-0.029	0.0278	0.103
$(\tau_2,\ \tau_3)$	0.1374	-0.1	-0.06	-0.177
$(\tau_2,\ \tau_4)$	0.1159	-0.093	-0.0695	-0.0449
$(\tau_3,\ \tau_1)$	0.0635	-0.342	-0.247	-0.262
$(\tau_3,\ \tau_2)$	0.112	-0.089	-0.08	-0.1675
$(\tau_3,\ \tau_3)$	0.2455	-0.066	-0.0755	-0.124
$(\tau_3,\ \tau_4)$	0.1975	-0.0945	-0.0695	-0.3115
$(\tau_4,\ \tau_1)$	-0.0235	-0.358	-0.152	-0.126
$(\tau_4,\ \tau_2)$	0.095	-0.0765	-0.106	-0.0395
$(\tau_4,\ \tau_3)$	0.1715	-0.095	-0.1485	-0.173
$(\tau_4,\ \tau_4)$	0.163	-0.08	-0.0475	-0.055

Step 4: From the acquired score total of BTSFHSN in $(\xi,\ \acute{\kappa}_1 \times \acute{\kappa}_2)$, the alternatives are compared and the value is given in the Table 2 of comparison $(\xi,\ \acute{\kappa}_1 \times \acute{\kappa}_2)$.

Table 2. Table of comparison of BTSFHSNs in $(\xi,\ \acute{\kappa}_1 \times \acute{\kappa}_2)$

$(\xi,\ \acute{\kappa}_1 \times \acute{\kappa}_2)$	$\acute{v}_1$	$\acute{v}_2$	$\acute{v}_3$	$\acute{v}_4$
$\acute{v}_1$	16	15	16	16
$\acute{v}_2$	1	16	5	8
$\acute{v}_3$	0	11	16	12
$\acute{v}_4$	0	8	12	16

Step 5: From the numerical values of the Table of comparison 3 $(\xi,\ \acute{\kappa}_1 \times \acute{\kappa}_2)$, the rowwise total, columnwise total and score total are obtained.

Table 3. score total of BTSFHSNs in $(\xi,\ \acute{\kappa}_1 \times \acute{\kappa}_2)$

$(\xi,\ \acute{\kappa}_1 \times \acute{\kappa}_2)$	$\mathfrak{R}\Sigma$	$\mathfrak{C}\Sigma$	Υ_j
$\acute{v}_1$	63	17	46
$\acute{v}_2$	30	50	-20
$\acute{v}_3$	39	49	-10
$\acute{v}_4$	36	52	-16

From the above Table 3, clearly $\acute{v}_1$ is the best filter by analyzing the alternatives corresponding score total and the rank of the filters is in the order $\acute{v}_2 < \acute{v}_4 < \acute{v}_3 < \acute{v}_1$.

5. Comparative analysis

The Proposed theory is compared to the existing theories under many circumstances, various factors and its key advancements.

Advantages:

• The theory like FS [1], IFS [2], PyFS [3], PFS [4], SFS [5], TSFS [5], parameters of the grades are represented only in positive sense. These theories fails to address the situation where negative grades have to be considered for the parameters of MG, NMG and AG along with multi sub-attributes. But, adopting BTSFHSS paves the way for handling such scenarios which reveals that the proposed theory is superior than those theories that are in the existence.

• Soft sets such as FSS [15], IFSS [16], PyFSS [17], PFSS [18], SFSS [19], and TSFSS [38] only address the issue through sub-attributes. However, a problem with multi sub-attributed data cannot be addressed in these theories.

• Eventhough the theories such as FHSS [20], PyFHSS [21], PFHSS [23], SFHSS [24], TSFHSS [25] can handle multi sub-attributed decision making issues, those are not capable of addressing the issues in BTSF environment. The suggested BTSFHSS outperforms those theories and exhibits the superiority in such circumstances.

• In comparison to BPFHSS [39], the BTSFHSS can accommodate a large collection of data than the BPF environment, since the constraints employed in BPFS shorten the range of data.

The superiority of the BTSFHSS theory is depicted in the Table 4.

Table 4. Outperforming factors of the proposed theory with the existing theory on basis of grades and attributes

Sets	PMS grade	PNMG grade	NMG grade	NNMG grade	PAb grade	NAb grade	Attributes	Multi-sub attributes
BFS [10]	✓	×	✓	×	×	×	×	×
BIFS [11]	✓	✓	✓	✓	×	×	×	×
BPFS [12]	✓	✓	✓	✓	✓	✓	×	×
BTSFS [13]	✓	✓	✓	✓	✓	✓	×	×
BFSS [40]	✓	×	✓	×	×	×	✓	×
BIFSS [41]	✓	✓	✓	✓	×	×	✓	×
BPyFSS [42]	✓	✓	✓	✓	×	×	✓	×
BSFSS [43]	✓	✓	✓	✓	×	×	✓	×
BFHSS [44, 45]	✓	×	✓	×	×	×	✓	✓
BLDFHSS [46]	✓	✓	✓	✓	×	×	✓	✓
BTSFSS [47]	✓	✓	✓	✓	✓	✓	✓	×
BTSFHSS (Proposed)	✓	✓	✓	✓	✓	✓	✓	✓

As the defined algorithm is used in the existing theories such as FSS [48], IFSS [49] and q-RLDFHSS [50]. But FSS and IFSS reflects the inability to handle the numerous sub-attributes that present with real life circumstances, while q-RLDFHSS does not possess the ability to handle the negative grades. So, the numerical data which is used for ranking the provided industrial air filters, cannot be directly applied and compared to the existing studies. To compare the ranking results by using the given data, it should be reduced to those structures.

Table 5. Comparative analysis with existing methodologies

Methodologies	Ranking
FSS [48]	Not applicable
IFSS [49]	Not applicable
q-RLDFHSS [50]	Not applicable
BTSFHSS (Proposed)	$\acute{v}_2 < \acute{v}_4 < \acute{v}_3 < \acute{v}_1$

Limitation: Eventhough possessing many advancement than the existing works, still BTSFHSS has some fewer drawbacks. They are,

1. The proposed BTSFHSS model lacks the ability to handle the data that are not restricted to the constraint of BTSFHSS.
2. As algorithms such as TOPSIS, VIKOR, and PROMETHEE have not yet been developed in the BTSFHSS framework. So, it is not possible to directly compare the defined MADM algorithm with them.

6. Conclusion

The BTSFHSS theory is proposed as a hybrid extension of BTSFS and Hypersoft sets, with the advantages of taking into account grades in both positive and negative aspects, as well as dealing with decision-making issues in the real world with multiple sub-attributed situations. The theory of BTSFHSS is introduced along with the elementary theorems, algebraic operators like extended union, intersection, OR operator and AND operator. For country's well-being, economic growth must be upgraded to higher level without degrading the quality of the natural resources and peace of environment. Because many sectors that support economic growth are also responsible for causing environmental pollution. So, the country's economic development and nature should be equally concerned. A country's progress requires a balance of both variables. Industries contribute significantly to economic growth while also deteriorating environmental health by emitting hazardous wastes. In this manuscript, a list of industrial air filters is discussed in a case study and an ideal filter is selected as an optimal filter by considering the attributes that are required by industries. An algorithm is constructed to formulate the optimal solution from the data that is provided. To evaluate the data that are given as an opinion of an expert team, a comparison matrix is defined along with row-wise total, column-wise total and score total. The obtained numerical values that are in the form of matrix are finally converted into score values of the corresponding alternatives by using the defined score function. A comparative analysis is included in this article for showcasing the new advancement and superiority of the proposed theory than the stuides that are in the existence and the limitation of the proposed theory. To describe the pathway of the research and future findings, the future direction is provided as follows.

7. Future direction

To overcome the restriction of the BTSFHSS constraints, many hybrid fuzzy notions will be developed in future work. The data in the framework of BTSFHSS cannot be directly compared to any algorithms as there are no Algorithms like TOPSIS, VIKOR, PROMTHEE etc., were developed in the framework of BTSFHSS yet. So it is aimed to develop those methodologies in the BTSFHSS theory. Many aggregation operators and information measures have to be developed as an effective tool in computing fuzzy values. Also, it is aimed to develop some new software to apply the defined algorithms.

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Conflict of interest

The authors declare no competing financial interest.

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