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Research Article

On Certain Normalized Classes Involving Higher-Order q-Derivatives

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Abstract: The current research addresses the study of novel subclasses of analytic functions constructed within the open unit disk by employing a generalized operator based on q-calculus and higher-order q-derivatives. In particular, we introduce and investigate subclasses of multivalent analytic functions associated with the generalized operator $\mathscr{V}_{q,\rho}^{\xi}$. We establish several inclusion relations among these classes, highlighting the structural hierarchy induced by the parameterization of the q-operators. Additionally, we derive sharp coefficient inequalities of Fekete–Szegő type, providing precise bounds for the initial coefficients in the power series expansion of functions within these subclasses. The results presented extend and unify several existing findings in the theory of q-analytic functions and reduce to known results in the classical setting when the deformation parameter tends to one. The paper also presents sufficient conditions for class membership, discusses notable special cases, and offers new subordination results that connect these classes to established families of univalent functions. These contributions emphasize the importance of the generalized higher-order q-differential operator as a versatile tool in geometric function theory and underline the role of q-calculus in generating and analyzing rich families of analytic functions with potential applications beyond the classical framework.

***Keywords*:** multivalent functions, q-calculus, q-differential operator, q-starlike functions, coefficient estimates, q-differential subordination

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1. Introduction

Over the past few years, q-calculus has garnered significant interest and intensive research attention due to its wide-ranging applications across mathematics and physics. The q-derivative operator has proven highly effective in the study of diverse subclasses of analytic functions. Moreover, the q-difference operator is extensively used in the context of hyper-geometric series and quantum physics (see [1, 2]).

In 1990, Ismail and collaborators [3] pioneered the introduction of q-starlike functions in the literature. However, a solid foundation for the integration of q-calculus into the framework of Geometric Function Theory was effectively established through the pioneering work of Srivastava, particularly via the application of generalized basic (or q-) hypergeometric functions in this area (see [4]). Following this foundational work, many researchers have contributed substantially to the development of the theory, producing a rich body of results. Notably, Srivastava et al. (see [5]) extended the study of q-starlike functions to those associated with conic regions, while other works explored q-starlike

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classes connected to Janowski functions (see [6, 7]), analyzing them from multiple perspectives. Furthermore, a recent and comprehensive survey-cum-expository article (see [8]) by Srivastava offers valuable insights into the theoretical framework and practical implementation of extended q-analogues in the fractional setting along with their associated q-differential operators, particularly in the context of function-theoretic analysis from a geometric perspective. This review serves as a useful reference for researchers working in this domain. For details and examples of q-convex functions, see [9]. For additional recent developments involving q-calculus, readers may consult sources such as [9–20].

Motivated by the foundational contributions presented in the works of Khan et al. [21], Rehman et al. [22], Wongsaijai et al. [23] which investigate q-analogues and their implications in the context of geometric function theory, we aim to extend this line of research by introducing and analyzing new subclasses of analytic functions defined via a generalized higher-order q-differential operator. In this paper, we establish several inclusion results among these newly defined classes, derive sharp coefficient inequalities of Fekete-Szegő type, and provide sufficient conditions for class membership.

We additionally contribute uniquely by introducing generalized higher-order q-differential operators, expanding upon existing categories and yielding new findings. These extensions underline the significance of q-calculus as a versatile framework, which not only recovers the classical setting in the limit $q \to 1^-$, but also yields genuinely new phenomena for $0 < q < 1$. Finally, the overall method follows a step-by-step strategy: we define the operator, investigate the associated subclasses, derive their analytic properties, and validate the findings through examples and limiting cases.

2. Basic concepts and preliminary results

$\mathscr{H}(U)$ denotes holomorphic mappings in $U = \{\mu \in \mathbb{C}\colon |\mu| < 1\}$.

Within this framework, we focus on the subclass $\mathscr{H}(e, \rho)$, consisting of all functions in $\mathscr{H}(U)$ that admit an analytic expansion expressed as $\mathfrak{f}(\mu) = e + e_\rho \mu^\rho + e_{\rho+1}\mu^{\rho+1} + \ldots$, $\mu \in U$, with $e \in \mathbb{C}$ and $\rho \in \mathbb{N} = \{1, 2, 3....\}$.

The subclass $\mathscr{A}_\rho \in \mathscr{H}(U)$ comprises functions that comply with the following normalization criteria:

$$\mathscr{A}_\rho = \{\mathfrak{f} \in \mathscr{H}(U)\colon \mathfrak{f}(\mu) = \mu^\rho + e_{\rho+1}\mu^{\rho+1} + \ldots,\ \mu \in U\}. \tag{1}$$

In the special case $\rho = 1$, the subclass $\mathscr{A}_\rho$ becomes the classical normalized analytic function class $\mathscr{A}$, commonly written as $\mathscr{A} = \mathscr{A}_1$.

Let us consider two analytic functions $\mathfrak{f}, \mathfrak{g} \in \mathscr{A}_\rho$, defined as

$$\mathfrak{f}(\mu) = \mu^\rho + \sum_{\iota=1}^{\infty} e_{\iota+\rho}\mu^{\iota+\rho} \text{ and } \mathfrak{g}(\mu) = \mu^\rho + \sum_{\iota=1}^{\infty} l_{\iota+\rho}\mu^{\iota+\rho}. \tag{2}$$

The Hadamard product (or convolution product) of $\mathfrak{f}$ and $\mathfrak{g}$, denoted $(\mathfrak{f} * \mathfrak{g})(\mu)$, is given by

$$\mathfrak{f}(\mu) * \mathfrak{g}(\mu) = (\mathfrak{f} * \mathfrak{g})(\mu) = \mu^\rho + \sum_{\iota=1}^{\infty} e_{\iota+\rho} l_{\iota+\rho}\mu^{\iota+\rho}. \tag{3}$$

We define $\mathfrak{P}$ as the family of holomorphic functions $\theta(\mu)$ defined on U, whose range lies entirely within the region $\{\mu \in \mathbb{C}\colon \mathrm{Re}(\mu) > 0\}$. Functions in this family satisfy the condition

$$Re(\theta(\mu)) > 0, \text{ for all } \mu \in U, \tag{4}$$

and are normalized such that $\theta(0) = 1$. These mappings serve as a cornerstone in geometric function theory, as they frequently arise in the investigation of univalent and starlike function classes. Numerous foundational results and deep structural properties have been developed with respect to this class. Each function $\theta \in \mathfrak{P}$ admits a development in series expressed as

$$\theta(\mu) = 1 + \sum_{\iota=1}^{\infty} w_\iota \mu^\iota, \ \mu \in U, \tag{5}$$

where the coefficients w_ι are complex-valued and the series converges for all $\mu \in U$. These functions are commonly referred to as Carathéodory functions.

Definition 1 [24] Given a pair of holomorphic mappings $\mathfrak{f}$, $\mathfrak{g}$ defined in U, subordination of $\mathfrak{f}$ to $\mathfrak{g}$ is denoted by

$$\mathfrak{f}(\mu) \prec \mathfrak{g}(\mu), \quad \mu \in U, \tag{6}$$

whenever there exists a Schwarz mapping $w(\mu)$, analytic in U, which meets the conditions $w(0) = 0$ and $|w(\mu)| < 1$, for which

$$\mathfrak{f}(\mu) = \mathfrak{g}(w(\mu)), \quad \mu \in U. \tag{7}$$

We briefly revisit several concepts and notational conventions from q-calculus that will be used in the subsequent analysis.

The foundation of q-generalizations of well-known results, is built upon the insight that

$$\lim_{q\to 1} \frac{1-q^\iota}{1-q} = \iota, \ q \in (0, 1), \ \iota \in \mathbb{N} = \{1, 2, 3....\}. \tag{8}$$

Hence, the value $\frac{1-q^\iota}{1-q}$ is typically denoted by the symbol $[\iota]_q$, known as the q-integer, while the related factorial in the q-setting, referred to as $[\iota]_q!$ takes the form:

$$[\iota]_q! = \begin{cases} [\iota]_q \cdot [\iota-1]_q \cdots [1]_q, \ \text{for } \iota = 1, 2, ...; \\ \\ 1, \ \text{for } n = 0. \end{cases} \tag{9}$$

The defining characteristic of a q-analogue is that, as $q \to 1^-$, it reduces to its classical counterpart, such as $[\iota]_q \to \iota$. Jackson, in [25] and [26], defined the q-derivative D_q applied to functions $\mathfrak{f}(\mu) \in \mathscr{A}$ in the following manner:

$$(D_q\mathfrak{f})(\mu) = \begin{cases} \frac{\mathfrak{f}(\mu)-\mathfrak{f}(q\mu)}{\mu(1-q)}, & \mu \neq 0, \ 0 < q < 1, \\ \\ \mathfrak{f}'(0), & \mu = 0. \end{cases} \tag{10}$$

One observes that as q tends toward 1 from the left, the q-difference operator yields the usual derivative $\mathfrak{f}'(\mu)$. Accordingly, applying the q-differentiation to $\mathfrak{f}(\mu) = \mu^{\iota}$ yields

$$(D_q\mathfrak{f})(\mu) = D_q(\mu^{\iota}) = \frac{1-q^{\iota}}{1-q} \cdot \mu^{\iota-1} = [\iota]_q \mu^{\iota-1}. \tag{11}$$

Taking the limit as $q \to 1^-$, we obtain

$$\lim_{q\to 1}(D_q\mathfrak{f})(\mu) = \lim_{q\to 1}[\iota]_q \mu^{\iota-1} = \iota\mu^{\iota-1} = \mathfrak{f}'(\mu), \tag{12}$$

which corresponds to the standard (classical) derivative. Using (10), one finds that

$$(D_q\mathfrak{f})(\mu) = 1 + \sum_{\iota=2}^{\infty} \frac{1-q^{\iota}}{1-q} e_{\iota}\mu^{\iota}, \ \ \mu \neq 0. \tag{13}$$

Based on the formulation of the q-derivate, the subsequent properties arise for functions $\mathfrak{f}$ and $\mathfrak{g}$ in $\mathscr{A}$:

$$D_q((a\mathfrak{f}(\mu)) \pm b\mathfrak{g}(\mu)) = aD_q\mathfrak{f}(\mu) \pm bD_q\mathfrak{g}(\mu), \ \ a, b \in \mathbb{C}, \tag{14}$$

$$D_q(\mathfrak{f}(\mu)\mathfrak{g}(\mu)) = \mathfrak{g}(\mu)D_q\mathfrak{f}(\mu) + \mathfrak{f}(q\mu)D_q\mathfrak{g}(\mu), \tag{15}$$

$$D_q\left(\frac{\mathfrak{f}(\mu)}{\mathfrak{g}(\mu)}\right) = \frac{\mathfrak{g}(\mu)D_q\mathfrak{f}(\mu) - \mathfrak{f}(\mu)D_q\mathfrak{g}(\mu)}{\mathfrak{g}(\mu)\mathfrak{g}(q\mu)}, \ \ \mathfrak{g}(\mu)\mathfrak{g}(q\mu) \neq 0, \tag{16}$$

$$D_q(\log \mathfrak{f}(\mu)) = \frac{\log q}{q-1}\frac{D_q\mathfrak{f}(\mu)}{\mathfrak{f}(\mu)}. \tag{17}$$

Let $\mathfrak{S}_q(\theta)$ represent the subclass of $\mathscr{A}$ defined by means of analytic subordination, $\theta \in \mathfrak{P}$, as outlined in [27]:

$$\mathfrak{S}_q(\theta) = \left\{\mathfrak{f} \in \mathscr{A}\colon \frac{\mu D_q\mathfrak{f}(\mu)}{\mathfrak{f}(\mu)} \prec \theta(\mu), \mu \in U\right\}. \tag{18}$$

Let $\mathfrak{S}_q(\beta, \theta)$ denote the subclass of $\mathscr{A}$, defined via analytic subordination for $0 \leq \beta < 1$ and a function $\theta \in \mathfrak{P}$, as described in [28]:

$$\mathfrak{S}_q(\beta, \theta) = \left\{\mathfrak{f} \in \mathscr{A}\colon \frac{1}{1-\beta}\left(\frac{\mu D_q\mathfrak{f}(\mu)}{\mathfrak{f}(\mu)} - \beta\right) \prec \theta(\mu), \mu \in U\right\}. \tag{19}$$

Remark 1 As q approaches 1 from the left, $\mathfrak{S}_q(\theta)$ recovers the family $\mathfrak{S}(\theta)$, as considered by [29].

Remark 2 As $q \to 1^-$, the q-analog class $\mathfrak{S}_q(\beta, \theta)$ converges to the classical class $\mathfrak{S}(\beta, \theta)$, which is defined through standard analytic subordination, as detailed in [30].

Remark 3 Furthermore, when $q \to 1^-$ and the function $\theta(\mu)$ is chosen as $\frac{1+\mu}{1-\mu}$, $\mathfrak{S}_q(\theta)$ recovers the conventional starlike function class.

Remark 4 When the function $\theta(\mu)$ is taken to be $\frac{1+L\mu}{1+M\mu}$, where the parameters satisfy $-1 \leq M < L \leq 1$, the corresponding class $\mathfrak{S}_q(\theta)$ becomes equivalent to the class $\mathfrak{S}_q(L, M)$, which was studied in detail by Noor and collaborators, as documented in [31]. Furthermore, in the limiting case as q approaches 1^-, $\mathfrak{S}_q(L, M)$ converges to $\mathfrak{S}(L, M)$, previously analyzed in the work of Janowski, referenced [32].

Remark 5 If the function $\theta(\mu)$ is defined as $\frac{1}{1-q\mu}$, then the associated class $\mathfrak{S}_q(\theta)$ is equivalent to the class considered by Noor in [33].

Remark 6 When $\theta(\mu) = \frac{1+\mu}{1-q\mu}$, the corresponding class $\mathfrak{S}_q(\theta)$ coincides with the q-starlike function class $\mathfrak{S}_q(1, -q)$, discussed in [34] and [35].

Equivalently, belonging to the family $\mathfrak{S}_q(1, -q)$ of starlike mappings defined via q-calculus in U (see [3]), an analytic mapping $\mathfrak{f} \in \mathscr{A}$ must meet the condition stated below:

$$\left| \frac{\mu D_q \mathfrak{f}(\mu)}{\mathfrak{f}(\mu)} - \frac{1}{1-q} \right| < \frac{1}{1-q}. \tag{20}$$

Let $\mathfrak{f} \in \mathscr{A}_\rho$, given by $\mathfrak{f}(\mu) = \mu^\rho + \sum_{\iota=1}^{\infty} e_{\iota+\rho}\mu^{\iota+\rho}$. We consider the following q-differential operator $\mathscr{V}_{q,\rho}^{\xi}: \mathscr{A}_\rho \to \mathscr{A}_\rho$, defined as follows:

$$\mathscr{V}_{q,\rho}^{\xi}\mathfrak{f}(\mu) = \mu^\rho + \sum_{\iota=1}^{\infty} [\iota+1]_q^{\xi} e_{\iota+\rho}\mu^{\iota+\rho}. \tag{21}$$

This q-differential operator was introduced by convolution in [36].

Remark 7 The operator $\mathscr{V}_{q,\rho}^{\xi}\mathfrak{f}(\mu)$ corresponds to a generalization that simplifies to the Salagean type q-operator for $\rho = 1$, originally formulated by Govindaraj and Sivasubramanian [37]. In the limit as $q \to 1^-$ with $\rho = 1$, the operator specified in (21) recovers the well-known Salagean derivative [38]. This shows that the generalized operator naturally extends classical results.

Differentiating $\mathscr{V}_{q,\rho}^{\xi}$ with respect to the q-derivative, we arrive at:

$$\left(D_q \mathscr{V}_{q,\rho}^{\xi}\right)\mathfrak{f}(\mu) = \frac{\mathscr{V}_{q,\rho}^{\xi}\mathfrak{f}(q\mu) - \mathscr{V}_{q,\rho}^{\xi}\mathfrak{f}(\mu)}{\mu(q-1)} = [\rho]_q\mu^{\rho-1} + \sum_{\iota=1}^{\infty}\left([\iota+1]_q\right)^{\xi}[\iota+\rho]_q e_{\iota+\rho}\mu^{\iota+\rho-1}. \tag{22}$$

Applying the identities given in (22) and (10), we obtain :

$$\left(D_q^{(1)}\mathscr{V}_{q,\rho}^{\xi}\right)\mathfrak{f}(\mu) = \left(D_q \mathscr{V}_{q,\rho}^{\xi}\right)\mathfrak{f}(\mu), \tag{23}$$

$$\left(D_q^{(2)}\mathscr{V}_{q,\rho}^{\xi}\mathfrak{f}\right)(\mu) = [\rho]_q[\rho-1]_q\mu^{\rho-2} + \sum_{\iota=1}^{\infty}\left([\iota+1]_q\right)^{\xi}[\iota+\rho]_q[\iota+\rho-1]_q e_{\iota+\rho}\mu^{\iota+\rho-2}, \tag{24}$$

$$\left(D_q^{(\rho)}\mathscr{V}_{q,\rho}^{\xi}\mathfrak{f}\right)(\mu) = [\rho]_q! + \sum_{\iota=1}^{\infty}\frac{[\iota+\rho]_q!}{[\iota]_q!}\left([\iota+1]_q\right)^{\xi} e_{\iota+\rho}\mu^{\iota}. \tag{25}$$

Here, $\left(D_q^{(\rho)}\mathscr{V}_{q,\rho}^{\xi}\mathfrak{f}\right)(\mu)$ denotes the ρ-th order q-derivative of $\mathscr{V}_{q,\rho}^{\xi}f$, where $\rho\in\mathbb{N}$.

In a more general setting, the k-th q-derivative of $\mathscr{V}_{q,\rho}^{\xi}f$, with $0\leq k\leq\rho$, takes the form:

$$\left(D_q^{(\kappa)}\mathscr{V}_{q,\rho}^{\xi}\mathfrak{f}\right)(\mu)=\frac{[\rho]_q!}{[\rho-k]_q!}\mu^{\rho-k}+\sum_{\iota=1}^{\infty}\frac{[\iota+\rho]_q!}{[\iota+\rho-k]_q!}\left([\iota+1]_q\right)^{\xi}e_{\iota+\rho}\mu^{\iota+\rho-k}. \tag{26}$$

Proposition 1 The identity given below is valid for the involved operators and holds for all functions $\mathfrak{f}\in\mathscr{A}_\rho$:

$$\mu\left(D_q^{(\rho)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)=\left(D_q^{(\rho-1)}\mathscr{V}_{q,\rho}^{\xi}\mathfrak{f}\right)(\mu). \tag{27}$$

Proof. Indeed,

$$\begin{aligned}\mu\left(D_q^{(\rho)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu) &= [\rho]_q!\mu+\sum_{\iota=1}^{\infty}\frac{[\iota+\rho]_q!}{[\iota]_q!}\left([\iota+1]_q\right)^{\xi-1}e_{\iota+\rho}\mu^{\iota+1}\\ &= [\rho]_q!\mu+\sum_{\iota=1}^{\infty}\frac{[\iota+\rho]_q!}{[\iota+1]_q!}\left([\iota+1]_q\right)^{\xi}e_{\iota+\rho}\mu^{\iota+1}\\ &= \left(D_q^{(\rho-1)}\mathscr{V}_{q,\rho}^{\xi}\mathfrak{f}\right)(\mu).\end{aligned}$$

The proof is complete. □

Employing the operator $D_q^{(\rho)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}$, we introduce the ensuing families of holomorphic functions parameterized by $0\leq\beta<1$, along with a function $\theta\in\mathfrak{P}$,

$$\mathfrak{S}_{q,\rho-1}^{\xi-1}(\theta)=\left\{\mathfrak{f}\in\mathscr{A}:\frac{1}{[\rho]_q!}D_q^{(\rho-1)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\in\mathfrak{S}_q(\theta)\right\}, \tag{28}$$

$$\mathfrak{S}_{q,\rho-1}^{\xi-1}(\beta,\theta)=\left\{\mathfrak{f}\in\mathscr{A}:\frac{1}{[\rho]_q!}D_q^{(\rho-1)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\in\mathfrak{S}_q(\beta,\theta)\right\}. \tag{29}$$

Specifically, we define

$$\mathfrak{S}_{q,\rho-1}^{\xi-1}\left(\frac{1+L\mu}{1+M\mu}\right)=\mathfrak{S}_{q,\rho-1}^{\xi-1}(L,M),\ \ -1<M<L\leq1, \tag{30}$$

and

$$\mathfrak{S}_{q,\rho-1}^{\xi-1}\left(\beta,\frac{1+L\mu}{1+M\mu}\right)=\mathfrak{S}_{q,\rho-1}^{\xi-1}(\beta,L,M),\ \ -1<M<L\leq1. \tag{31}$$

3. Inclusion results involving higher order q-derivatives of the operator $\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}$

To lay the groundwork for our principal results from the current part, we first state the lemma presented hereafter.

Lemma 1 [28] Let $\theta \in \mathfrak{P}$ and $Re\{\varsigma\theta(\mu)+\tau\}>0$, ς, $\tau \in \mathbb{C}$. If χ is holomorphic in U and satisfies the normalization $\chi(0)=1$, it follows that

$$\chi(\mu)+\frac{\mu D_q\chi(\mu)}{\varsigma\chi(\mu)+\tau} \prec \theta(\mu),\ \ \mu \in U, \tag{32}$$

implies $\chi(\mu) \prec \theta(\mu)$, $\mu \in U$.

This part of the paper focuses on establishing inclusion criteria among the classes $\mathfrak{S}_{q,\rho-1}^{\xi}(\theta)$ and $\mathfrak{S}_{q,\rho-1}^{\xi}(\beta,\theta)$. The analysis is based on applying higher-order q-derivatives of the operator $\mathscr{V}_{q,\rho}^{\xi}$ and making use of Lemma 2.1, showing that class membership is preserved when transitioning from the parameter ξ to $\xi-1$. The results are accompanied by corollaries involving specific choices of the comparison function $\theta(\mu)$, such as $\frac{1+\mu}{1-q\mu}$, which play key roles in geometric function theory.

Theorem 1 Let $\mathfrak{f} \in \mathscr{A}_\rho$, and $\theta \in \mathfrak{P}$. Then the following inclusion property is satisfied:

$$\mathfrak{S}_{q,\rho-1}^{\xi}(\theta) \subseteq \mathfrak{S}_{q,\rho-1}^{\xi-1}(\theta). \tag{33}$$

Proof. Suppose that $\mathfrak{f}$ belongs to the class $\mathfrak{S}_{q,\rho}^{\xi}(\theta)$.

We define

$$\frac{\mu D_q\left(\frac{1}{[\rho]_q!}D_q^{(\rho-1)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}{\left(\frac{1}{[\rho]_q!}D_q^{(\rho-1)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}=\chi(\mu), \tag{34}$$

which is equivalent to

$$\frac{\mu D_q^{(\rho)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}(\mu)}{\left(D_q^{(\rho-1)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}=\chi(\mu), \tag{35}$$

under the conditions that $\chi \in \mathscr{H}(U)$, satisfies $\chi(0)=1$ and $\theta(\mu)$ does not vanish. Substituting equation (27) into (35), we obtain:

$$\frac{\left(D_q^{(\rho-1)}\mathscr{V}_{q,\rho}^{\xi}\mathfrak{f}\right)(\mu)}{\left(D_q^{(\rho-1)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}=\chi(\mu). \tag{36}$$

Logarithmic q-differentiation of (36) yields

$$\frac{\mu\left(D_q^{(\rho)}\mathscr{V}_{q,\rho}^{\xi}\mathfrak{f}\right)(\mu)}{\left(D_q^{(\rho-1)}\mathscr{V}_{q,\rho}^{\xi}\mathfrak{f}\right)(\mu)}=\chi(\mu)+\frac{\mu D_q\chi(\mu)}{\chi(\mu)}. \tag{37}$$

Given that $\mathfrak{f}\in\mathfrak{S}_{q,\rho-1}^{\xi}(\theta)$, we infer from (37) that

$$\chi(\mu)+\frac{\mu D_q\chi(\mu)}{\chi(\mu)}\prec\theta(\mu). \tag{38}$$

Through the use of Lemma 1, it follows that

$$\frac{\mu D_q^{(\rho)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}(\mu)}{\left(D_q^{(\rho-1)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}\prec\theta(\mu). \tag{39}$$

Consequently, $\mathfrak{f}\in\mathfrak{S}_{q,\rho-1}^{\xi-1}(\theta)$, completing the proof. □

Corollary 1 For $\mathfrak{f}\in\mathscr{A}_\rho$, the inclusion below is satisfied:

$$\mathfrak{S}_{q,\rho-1}^{\xi}(L,M)\subseteq\mathfrak{S}_{q,\rho-1}^{\xi-1}(L,M), \tag{40}$$

where $-1\leq M<L\leq 1$

Proof. Let $\theta(\mu)=\frac{1+L\mu}{1+M\mu}$, where $-1\leq M<L\leq 1$. By applying Theorem 1, the desired inclusion follows directly. □

Corollary 2 Given $\mathfrak{f}\in\mathscr{A}_\rho$, the inclusion below follows:

$$\mathfrak{S}_{q,\rho-1}^{\xi}(1,-q)\subseteq\mathfrak{S}_{q,\rho-1}^{\xi-1}(1,-q).$$

Proof. Taking $\theta(\mu)=\frac{1+\mu}{1-q\mu}$ and applying Theorem 1, yields the stated outcome. □

Corollary 3 Suppose $\mathfrak{f}\in\mathscr{A}_\rho$. Then the following inclusion relation holds:

$$\mathfrak{S}_{q,\rho-1}^{\xi}(0,-q)\subseteq\mathfrak{S}_{q,\rho-1}^{\xi-1}(0,-q).$$

Proof. Let $\theta(\mu)=\frac{1}{1-q\mu}$. By applying Theorem 1, the conclusion follows immediately. □

Theorem 2 Let $\mathfrak{f}\in\mathscr{A}_\rho$ and $\theta\in\mathfrak{S}$ such that

$$Re\{\theta(\mu)\}<\frac{\beta}{1-\beta}. \tag{41}$$

Then the inclusion

$$\mathfrak{S}_{q,\,\rho-1}^{\xi}(\beta,\,\theta) \subseteq \mathfrak{S}_{q,\,\rho-1}^{\xi-1}(\beta,\,\theta) \tag{42}$$

holds.

Proof. Assume $\mathfrak{f}(\mu) \in \mathfrak{S}_{q,\,\rho-1}^{\xi}(\beta,\,\theta)$ and set

$$\chi(\mu) = \frac{1}{1-\beta}\left(\frac{\mu D_q\left(\frac{1}{[\rho]_q!}D_q^{(\rho-1)}\mathscr{V}_{q,\,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}{\frac{1}{[\rho]_q!}\left(D_q^{(\rho-1)}\mathscr{V}_{q,\,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)} - \beta\right), \tag{43}$$

provided χ is holomorphic, normalized so that $\chi(0) = 1$. Expression (43) is equivalent to

$$\chi(\mu) = \frac{1}{1-\beta}\left(\frac{\mu\left(D_q^{(\rho)}\mathscr{V}_{q,\,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}{\left(D_q^{(\rho-1)}\mathscr{V}_{q,\,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)} - \beta\right), \tag{44}$$

Using identity (27), we find

$$\frac{\mu\left(D_q^{(\rho)}\mathscr{V}_{q,\,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}{\left(D_q^{(\rho-1)}\mathscr{V}_{q,\,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)} = \frac{D_q^{(\rho-1)}\mathscr{V}_{q,\,\rho}^{\xi}\mathfrak{f}(\mu)}{\left(D_q^{(\rho-1)}\mathscr{V}_{q,\,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}. \tag{45}$$

Substituting this into (43) we obtain

$$\frac{D_q^{(\rho-1)}\mathscr{V}_{q,\,\rho}^{\xi}\mathfrak{f}(\mu)}{\left(D_q^{(\rho-1)}\mathscr{V}_{q,\,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)} = (1-\beta)\chi(\mu) + \beta, \tag{46}$$

Now, logarithmically q-differentiating equation (46), we derive

$$\frac{1}{1-\beta}\left[\frac{\mu\left(D_q^{(\rho)}\mathscr{V}_{q,\,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}{\left(D_q^{(\rho-1)}\mathscr{V}_{q,\,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)} - \beta\right] = \chi(\mu) + \frac{\mu D_q\chi(\mu)}{(1-\beta)\chi(\mu)+\beta}. \tag{47}$$

Since the condition $Re\{\theta(\mu)\} < \frac{\beta}{1-\beta}$ implies $Re\{(1-\beta)\chi(\mu)+\beta\} > 0$, we can apply Lemma 1 to (47) thereby obtaining

$$\mathfrak{f}(\mu) \in \mathfrak{S}_{q,\,\rho-1}^{\xi-1}(\beta,\,\theta), \tag{48}$$

as required. □

Corollary 4 For $\mathfrak{f} \in \mathscr{A}_{\rho}$, the inclusion below is satisfied:

$$\mathfrak{S}_{q,\,\rho-1}^{\xi}(\beta,\,L,\,M) \subseteq \mathfrak{S}_{q,\,\rho-1}^{\xi-1}(\beta,\,L,\,M)\,, \tag{49}$$

where $-1 \leq M < L \leq 1$.

Proof. Let $\theta(\mu) = \frac{1+L\mu}{1+M\mu}$, where $-1 \leq M < L \leq 1$. Applying Theorem 2, and considering the notation (31), the desired inclusion follows directly. □

Corollary 5 Given $\mathfrak{f} \in \mathscr{A}_\rho$, the inclusion below follows:

$$\mathfrak{S}_{q,\,\rho-1}^{\xi}(\beta,\,1,\,-q) \subseteq \mathfrak{S}_{q,\,\rho-1}^{\xi-1}(\beta,\,1,\,-q)\,. \tag{50}$$

Proof. Taking $\theta(\mu) = \frac{1+\mu}{1-q\mu}$ and applying Theorem 2, along with the notation from (31), we obtain the result. □

Corollary 6 Suppose $\mathfrak{f} \in \mathscr{A}_\rho$. Then the following inclusion relation holds:

$$\mathfrak{S}_{q,\,\rho-1}^{\xi}(\beta,\,0,\,-q) \subseteq \mathfrak{S}_{q,\,\rho-1}^{\xi-1}(\beta,\,0,\,-q)\,. \tag{51}$$

Proof. Let $\theta(\mu) = \frac{1}{1-q\mu}$. By applying Theorem 2, and taking into account notation 31 conclusion follows immediately. □

Observation 1 The inclusion theorems established in this section illustrate the central role played by subordination in the analysis of the operator-based classes. In particular, the subordination results obtained above highlight how the generalized higher-order q-differential operator creates structural connections between the newly introduced subclasses and well-studied families of analytic functions. In particular, these relations show that the q-calculus framework preserves many of the hierarchical properties known from the classical theory while also providing genuinely new phenomena for $0 < q < 1$. This underlines the applicability of subordination as a powerful tool in extending operator-based results into the setting of q-calculus.

Observation 2 We note that the method used here relies essentially on subordination. Although duality methods may also be applicable, their investigation is left for future research.

4. Coefficient inequalities for q-analytic function classes $\mathscr{L}\mathfrak{S}_{q,\rho-1}^{\xi-1}$ and $\mathscr{L}\mathfrak{S}_{q,\rho-1}^{\xi-1}(\beta)$

For simplicity of notation, we will denote

$$\mathfrak{S}_{q,\,\rho-1}^{\xi-1}\left(\frac{1+\mu}{1-q\mu}\right) = \mathfrak{S}_{q,\,\rho-1}^{\xi-1}(1,\,-q) = \mathscr{L}\mathfrak{S}_{q,\,\rho-1}^{\xi-1} \tag{52}$$

and

$$\mathfrak{S}_{q,\,\rho-1}^{\xi-1}\left(\beta,\,\frac{1+\mu}{1-q\mu}\right) = \mathfrak{S}_{q,\,\rho-1}^{\xi-1}(\beta,\,1,\,-q) = \mathscr{L}\mathfrak{S}_{q,\,\rho-1}^{\xi-1}(\beta) \tag{53}$$

In this section, we investigate sharp coefficient inequalities in $\mathscr{L}\mathfrak{S}_{q,\,\rho-1}^{\xi-1}$ and its generalized form $\mathscr{L}\mathfrak{S}_{q,\,\rho-1}^{\xi-1}(\beta)$. By employing the higher order q-derivative of $\mathscr{V}_{q,\,\rho}^{\xi-1}$, together with subordination techniques and expansions of associated

Carathéodory functions, we derive Fekete–Szegő type estimates for the coefficients $e_{\rho+1}$ and $e_{\rho+2}$. Additionally, we provide adequate analytic constraints under which $\mathfrak{f} \in \mathscr{A}_\rho$ is contained in these function subclasses, and we analyze special cases obtained by choosing specific values of the parameter ξ. As a preliminary step toward the main results from this section, we present the Lemma below.

Lemma 2 [29] Let the function $w(\mu)$ given by $w(\mu) = 1 + w_1\mu + w_2\mu^2 + \cdots$ belonging to $\mathfrak{P}$, the set of mappings in U whose range lies in the $\{\mu \in \mathbb{C}\colon \operatorname{Re}(\mu) > 0\}$. Thus, for every complex constant K, the relation below is satisfied:

$$\left|w_2 - Kw_1^2\right| \le \begin{cases} -4K+2, & K < 0; \\ 2, & 0 \le K \le 1; \\ 4K-2, & K > 1. \end{cases} \tag{54}$$

For K outside the interval $[0, 1]$, the identity in (54) is valid exclusively when $w(\mu) = \frac{1+\mu}{1-\mu}$, or a rotated instance of it. For K in the interval $(0, 1)$, the identity in (54) holds precisely if $w(\mu)$ assumes the form $\frac{1+\mu^2}{1-\mu^2}$ or a rotated counterpart. The relation stated in (54) is valid under the condition $K = 0$, if

$$w(\mu) = \left(\frac{1+\sigma}{2}\right)\frac{1+\mu}{1-\mu} + \left(\frac{1-\sigma}{2}\right)\frac{1-\mu}{1+\mu}, \quad 0 \le \sigma \le 1, \tag{55}$$

or a rotated analogue. In the case $K = 1$, the identity in (54) is fulfilled whenever $w(\mu)$ is the inverse (in the compositional sense) of a function for which the relation holds in the $K = 0$ scenario.

Theorem 3 Consider $\mathfrak{f} \in \mathscr{A}_\rho$ belonging to the class $\mathscr{L}\mathfrak{S}_{q,\rho-1}^{\xi-1}$. Then

$$\left|e_{\rho+2} - \lambda e_{\rho+1}^2\right| \le \begin{cases} \frac{[2]_q}{q^2[1+\rho]_q[2+\rho]_q\left([3]_q\right)^{\xi-2}} K(q, \rho, \xi) & \text{if } \lambda < \gamma_1; \\ \frac{[2]_q}{q[1+\rho]_q[2+\rho]_q\left([3]_q\right)^{\xi-2}} & \text{if } \gamma_1 \le \lambda \le \gamma_2; \\ -\frac{[2]_q}{q^2[1+\rho]_q[2+\rho]_q\left([3]_q\right)^{\xi-2}} K(q, \rho, \xi) & \text{if } \gamma_2 < \lambda, \end{cases} \tag{56}$$

where

$$K(q, \rho, \xi) = \frac{q[1+\rho]_q\left([2]_q\right)^{2(\xi-2)} + (q^2+1)[1+\rho]_q\left([2]_q\right)^{2(\xi-2)} - \lambda[2]_q[2+\rho]_q\left([3]_q\right)^{\xi-2}}{[1+\rho]_q\left([2]_q\right)^{2(\xi-2)}}, \tag{57}$$

$$\gamma_1 = \frac{(q^2+1)[1+\rho]_q\left([2]_q\right)^{2(\xi-2)}}{[2]_q[2+\rho]_q\left([3]_q\right)^{\xi-2}}, \tag{58}$$

and

$$\gamma_2 = \frac{[1+\rho]_q \left([2]_q\right)^{2(\xi-1)}}{[2]_q [2+\rho]_q \left([3]_q\right)^{\xi-2}}. \tag{59}$$

Each of these results is sharp.

Proof. Assume that $\mathfrak{f} \in \mathscr{L}\mathfrak{S}^{\xi-1}_{q,\rho-1}$. By (28) and (52), it follows that

$$\frac{\mu D_q^{(\rho)} \mathscr{V}_{q,\rho}^{\xi-1} \mathfrak{f}(\mu)}{\left(D_q^{(\rho-1)} \mathscr{V}_{q,\rho}^{\xi-1} \mathfrak{f}\right)(\mu)} \prec \frac{1+\mu}{1-q\mu}. \tag{60}$$

We proceed to define the function $w \in \mathfrak{P}$,

$$w(\mu) = \frac{1+v(\mu)}{1-v(\mu)} = 1 + w_1\mu + w_2\mu^2 + w_3\mu^3 + \ldots. \tag{61}$$

From (61) we deduce that

$$v(\mu) = \frac{w(\mu)-1}{w(\mu)+1}. \tag{62}$$

Hence, from (60), it follows that

$$\frac{\mu D_q^{(\rho)} \mathscr{V}_{q,\rho}^{\xi-1} \mathfrak{f}(\mu)}{\left(D_q^{(\rho-1)} \mathscr{V}_{q,\rho}^{\xi-1} \mathfrak{f}\right)(\mu)} = \frac{1+v(\mu)}{1-qv(\mu)}, \tag{63}$$

where

$$\frac{1+v(\mu)}{1-qv(\mu)} = \frac{2w(\mu)}{(1-q)w(\mu)+(1+q)}. \tag{64}$$

So, from (63) and (64), we obtain

$$\frac{\mu D_q^{(\rho)} \mathscr{V}_{q,\rho}^{\xi-1} \mathfrak{f}(\mu)}{\left(D_q^{(\rho-1)} \mathscr{V}_{q,\rho}^{\xi-1} \mathfrak{f}\right)(\mu)} = \frac{2w(\mu)}{(1-q)w(\mu)+(1+q)}. \tag{65}$$

Using (63), it follows, upon simplification, that

$$\frac{2w(\mu)}{(1-q)w(\mu)+(1+q)}=1+\frac{1}{2}(q+1)w_1\mu+\frac{1}{4}(q+1)\left[w_1^2(q-1)+2w_2\right]\mu^2+\ldots. \tag{66}$$

Proceeding similarly, we obtain that

$$\begin{aligned}&\frac{\mu D_q^{(\rho)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}(\mu)}{\left(D_q^{(\rho-1)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}\\ =&1+[1+\rho]_q\left([2]_q\right)^{\xi-2}qe_{\rho+1}\mu+[1+\rho]_q\left\{[2+\rho]_q\left([3]_q\right)^{\xi-2}qe_{\rho+2}-[1+\rho]_q\left([2]_q\right)^{2(\xi-2)}qe_{\rho+1}^2\right\}\mu^2+\ldots.\end{aligned} \tag{67}$$

Thus, we obtain

$$e_{\rho+1}=\frac{(q+1)}{2q[1+\rho]_q\left([2]_q\right)^{\xi-2}}w_1, \tag{68}$$

and

$$e_{\rho+2}=\frac{[2]_q\left[w_1^2\left(q^2+1\right)+2qw_2\right]}{4q^2[1+\rho]_q[2+\rho]_q\left([3]_q\right)^{\xi-2}}. \tag{69}$$

It is clear, therefore, that

$$\left|e_{\rho+2}-\lambda e_{\rho+1}^2\right|=\frac{[2]_q}{2q[1+\rho]_q[2+\rho]_q\left([3]_q\right)^{\xi-2}}\left|w_2-kw_1^2\right|, \tag{70}$$

where

$$k=\frac{\lambda[2]_q[2+\rho]_q\left([3]_q\right)^{\xi-2}-\left(q^2+1\right)[1+\rho]_q\left([2]_q\right)^{2(\xi-2)}}{2q[1+\rho]_q\left([2]_q\right)^{2(\xi-2)}}. \tag{71}$$

The desired result is obtained through the application of the Lemma 2 in (70). □

As a direct consequence of Theorem 3 with $\xi=1$, the subsequent outcome emerges.

Corollary 7 Assume $\mathfrak{f}\in\mathscr{A}_\rho$ belongs to the family $\mathscr{L}\mathfrak{S}_{q,\rho-1}^0$. Then

$$\left|e_{\rho+2}-\lambda e_{\rho+1}^{2}\right| \leq \begin{cases} \frac{[2]_q[3]_q}{q^2[1+\rho]_q[2+\rho]_q} K(q,\rho) & \text{if } \lambda<\gamma_1\ ; \\ \frac{[2]_q[3]_q}{q[1+\rho]_q[2+\rho]_q} & \text{if } \gamma_1 \leq \lambda \leq \gamma_2\ ; \\ -\frac{[2]_q[3]_q}{q^2[1+\rho]_q[2+\rho]_q} K(q,\rho) & \text{if } \gamma_2<\lambda, \end{cases} \tag{72}$$

where

$$K(q,\rho)=q^2+q+1-\frac{\lambda\,[2+\rho]_q\left([2]_q\right)^3}{[3]_q\,[1+\rho]_q}, \tag{73}$$

$$\gamma_1=\frac{\left(q^2+1\right)[1+\rho]_q\,[3]_q}{\left([2]_q\right)^3[2+\rho]_q}, \tag{74}$$

and

$$\gamma_2=\frac{[1+\rho]_q\,[3]_q}{[2]_q\,[2+\rho]_q}. \tag{75}$$

All of these results represent best-possible cases..

Theorem 4 Assume that $\mathfrak{f}\in\mathscr{A}_\rho$ is contained in the family $\mathscr{L}\mathfrak{S}_{q,\rho-1}^{\xi-1}(\beta)$. Then

$$\left|e_{\rho+2}-\lambda e_{\rho+1}^{2}\right| \leq \begin{cases} \frac{[2]_q(1-\beta)}{q^2[1+\rho]_q[2+\rho]_q\left([3]_q\right)^{\xi-2}} K(q,\rho,\xi,\beta) & \text{if } \lambda<\gamma_3\ ; \\ \frac{[2]_q(1-\beta)}{q[1+\rho]_q[2+\rho]_q\left([3]_q\right)^{\xi-2}} & \text{if } \gamma_3 \leq \lambda \leq \gamma_4\ ; \\ -\frac{[2]_q(1-\beta)}{q^2[1+\rho]_q[2+\rho]_q\left([3]_q\right)^{\xi-2}} K(q,\rho,\xi,\beta) & \text{if } \gamma_4<\lambda, \end{cases} \tag{76}$$

where

$$K(q,\rho,\xi,\beta)=(1-\beta)\,[2]_q-q^2+2q-\frac{\lambda\,(1-\beta)\,[2+\rho]_q\left([3]_q\right)^{\xi-2}}{[1+\rho]_q\left([2]_q\right)^{2\xi-5}}, \tag{77}$$

$$\gamma_3 = \frac{[1+\rho]_q \left([2]_q\right)^{2\xi-5} \left\{(1-\beta)[2]_q - q(q-1)\right\}}{(1-\beta)[2+\rho]_q \left([3]_q\right)^{\xi-2}}, \tag{78}$$

and

$$\gamma_4 = \frac{[1+\rho]_q \left([2]_q\right)^{2\xi-5} \left\{(1-\beta)[2]_q - q^2 + 3q\right\}}{(1-\beta)[2+\rho]_q \left([3]_q\right)^{\xi-2}}. \tag{79}$$

Each result is the best possible one.
Proof. Let $\mathfrak{f} \in \mathscr{L}\mathfrak{S}_{q,\rho-1}^{\xi-1}(\beta)$. Using (28) and (53), we obtain that

$$\frac{1}{1-\beta}\left(\frac{\mu\left(D_q^{(\rho)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}{\left(D_q^{(\rho-1)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)} - \beta\right) \prec \frac{1+\mu}{1-q\mu}. \tag{80}$$

Define the function $w \in \mathfrak{P}$ as follows.

$$w(\mu) = \frac{1+v(\mu)}{1-v(\mu)} = 1 + w_1\mu + w_2\mu^2 + w_3\mu^3 + \ldots. \tag{81}$$

It can be deduced from (81) that

$$v(\mu) = \frac{w(\mu)-1}{w(\mu)+1}.$$

Consequently, by (80), we obtain

$$\frac{1}{1-\beta}\left(\frac{\mu\left(D_q^{(\rho)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}{\left(D_q^{(\rho-1)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)} - \beta\right) = \frac{1+v(\mu)}{1-qv(\mu)}, \tag{82}$$

where

$$\frac{1+v(\mu)}{1-qv(\mu)} = \frac{2w(\mu)}{(1-q)w(\mu)+(1+q)}. \tag{83}$$

It follows from (82) and (83) that

$$\frac{1}{1-\beta}\left(\frac{\mu\left(D_q^{(\rho)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}{\left(D_q^{(\rho-1)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}-\beta\right)=\frac{2w(\mu)}{(1-q)\,w(\mu)+(1+q)}.$$

By using (82), it can be deduced, after simplification, that

$$\frac{2w(\mu)}{(1-q)\,w(\mu)+(1+q)}=1+\frac{1}{2}(q+1)\,w_1\mu+\frac{1}{4}(q+1)\left[w_1^2(q-1)+2w_2\right]\mu^2+\ldots.$$

By proceeding in a similar manner, we obtain

$$\frac{1}{1-\beta}\left(\frac{\mu\left(D_q^{(\rho)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}{\left(D_q^{(\rho-1)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}-\beta\right)=\frac{\mu\left(D_q^{(\rho)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)-\beta\left(D_q^{(\rho-1)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}{(1-\beta)\left(D_q^{(\rho-1)}\mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}$$

$$=1+\frac{[1+\rho]_q\left([2]_q\right)^{\xi-2}q}{1-\beta}e_{\rho+1}\mu+\frac{[1+\rho]_q}{1-\beta}\left\{[2+\rho]_q\left([3]_q\right)^{\xi-2}qe_{\rho+2}-[1+\rho]_q\left([2]_q\right)^{2(\xi-2)}qe_{\rho+1}^2\right\}\mu^2+\ldots.$$

Hence, we find

$$e_{\rho+1}=\frac{(1-\beta)(q+1)}{2q\,[1+\rho]_q\left([2]_q\right)^{\xi-2}}w_1,$$

and

$$e_{\rho+2}=\frac{[2]_q\left\{(1-\beta)\left[(1-\beta)[2]_q+q(q-1)\right]w_1^2+2qw_2\right\}}{4q^2\,[1+\rho]_q\,[2+\rho]_q\left([3]_q\right)^{\xi-2}}.$$

Clearly, we have that

$$\left|e_{\rho+2}-\lambda e_{\rho+1}^2\right|=\frac{[2]_q(1-\beta)}{2q\,[1+\rho]_q\,[2+\rho]_q\left([3]_q\right)^{\xi-2}}\left|w_2-lw_1^2\right|, \tag{84}$$

where

$$l=\frac{\lambda\left(1-\beta\right)\left[2\right]_{q}\left[2+\rho\right]_{q}\left(\left[3\right]_{q}\right)^{\xi-2}-\left[1+\rho\right]_{q}\left(\left[2\right]_{q}\right)^{2(\xi-2)}\left\{\left[2\right]_{q}\left(1-\beta\right)-q\left(q-1\right)\right\}}{2q\left[1+\rho\right]_{q}\left(\left[2\right]_{q}\right)^{2(\xi-2)}}.$$

Applying Lemma 2 in (84) yields the desired result. □

Choosing $\xi=1$ in Theorem 4 leads directly to the subsequent result.

Corollary 8 Suppose that the function $\mathfrak{f}\in\mathscr{A}_{\rho}$ lies within the subclass $\mathscr{L}\mathfrak{S}_{q,\,\rho-1}^{0}\left(\beta\right)$. Then

$$\left|e_{\rho+2}-\lambda e_{\rho+1}^{2}\right|\leq\begin{cases}\frac{[2]_{q}[3]_{q}(1-\beta)}{q^{2}[1+\rho]_{q}[2+\rho]_{q}}K\left(q,\,\rho,\,\xi,\,\beta\right) & \text{if } \lambda<\gamma_{3}\,;\\ \\ \frac{[2]_{q}[3]_{q}(1-\beta)}{q[1+\rho]_{q}[2+\rho]_{q}} & \text{if } \gamma_{3}\leq\lambda\leq\gamma_{4}\,;\\ \\ -\frac{[2]_{q}[3]_{q}(1-\beta)}{q^{2}[1+\rho]_{q}[2+\rho]_{q}}K\left(q,\,\rho,\,\xi,\,\beta\right) & \text{if } \gamma_{4}<\lambda,\end{cases}\tag{85}$$

where

$$K\left(q,\,\rho,\,1,\,\beta\right)=\left(1-\beta\right)\left[2\right]_{q}-q^{2}+2q-\frac{\lambda\left(1-\beta\right)\left(\left[2\right]_{q}\right)^{3}\left[2+\rho\right]_{q}}{\left[1+\rho\right]_{q}\left[3\right]_{q}},\tag{86}$$

$$\gamma_{3}=\frac{\left[1+\rho\right]_{q}\left[3\right]_{q}\left\{\left[2\right]_{q}\left(1-\beta\right)-q^{2}+q\right\}}{\left(1-\beta\right)\left(\left[2\right]_{q}\right)^{3}\left[2+\rho\right]_{q}},\tag{87}$$

and

$$\gamma_{4}=\frac{\left[1+\rho\right]_{q}\left[3\right]_{q}\left\{\left[2\right]_{q}\left(1-\beta\right)-q\left(q-1\right)+2q\right\}}{\left(1-\beta\right)\left(\left[2\right]_{q}\right)^{3}\left[2+\rho\right]_{q}}.\tag{88}$$

Each of these results is sharp.

The following equivalent formulations of the definitions of the classes $\mathscr{L}\mathfrak{S}_{q,\,\rho-1}^{\xi-1}$ and $\mathfrak{S}_{q,\,\rho-1}^{\xi-1}\left(\beta\right)$ will be useful in our subsequent analysis.

$$\mathfrak{f}\in\mathscr{L}\mathfrak{S}_{q,\,\rho-1}^{\xi-1}\Leftrightarrow\left|\frac{\mu\left(D_{q}^{(\rho)}\mathscr{V}_{q,\,\rho}^{\xi-1}\mathfrak{f}\right)\left(\mu\right)}{\left(D_{q}^{(\rho-1)}\mathscr{V}_{q,\,\rho}^{\xi-1}\mathfrak{f}\right)\left(\mu\right)}-\frac{1}{1-q}\right|<\frac{1}{1-q}.\tag{89}$$

$$\mathfrak{f} \in \mathfrak{S}_{q,\rho-1}^{\xi-1}(\beta) \Leftrightarrow \left| \frac{\mu\left(D_q^{(\rho)} \mathscr{V}_{q,\rho}^{\xi-1} \mathfrak{f}\right)(\mu)}{\left(D_q^{(\rho-1)} \mathscr{V}_{q,\rho}^{\xi-1} \mathfrak{f}\right)(\mu)} - \left(\beta + \frac{1-\beta}{1-q}\right) \right| < \frac{1-\beta}{1-q}. \tag{90}$$

Theorem 5 We say that $\mathfrak{f} \in \mathscr{A}_\rho$ is a member of the family $\mathscr{L}\mathfrak{S}_{q,\rho-1}^{\xi-1}$ when the following condition holds:

$$\sum_{\iota=1}^{\infty} \frac{[\iota+\rho]_q!}{[\iota]_q![\rho]_q!} \left([\iota+1]_q\right)^{\xi-1} |e_{\iota+\rho}| < 1. \tag{91}$$

Proof. Given that (91) is satisfied, it is enough to demonstrate that

$$\left| \frac{\mu\left(D_q^{(\rho)} \mathscr{V}_{q,\rho}^{\xi-1} \mathfrak{f}\right)(\mu)}{\left(D_q^{(\rho-1)} \mathscr{V}_{q,\rho}^{\xi-1} \mathfrak{f}\right)(\mu)} - \frac{1}{1-q} \right| < \frac{1}{1-q}. \tag{92}$$

Thus,

$$\begin{aligned}
&\left| \frac{\mu\left(D_q^{(\rho)} \mathscr{V}_{q,\rho}^{\xi-1} \mathfrak{f}\right)(\mu)}{\left(D_q^{(\rho-1)} \mathscr{V}_{q,\rho}^{\xi-1} \mathfrak{f}\right)(\mu)} - \frac{1}{1-q} \right| < \left| \frac{\mu\left(D_q^{(\rho)} \mathscr{V}_{q,\rho}^{\xi-1} \mathfrak{f}\right)(\mu)}{\left(D_q^{(\rho-1)} \mathscr{V}_{q,\rho}^{\xi-1} \mathfrak{f}\right)(\mu)} - 1 \right| + \frac{q}{1-q} \\
&= \left| \frac{\left(D_q^{(\rho-1)} \mathscr{V}_{q,\rho}^{\xi} \mathfrak{f}\right)(\mu) - \left(D_q^{(\rho-1)} \mathscr{V}_{q,\rho}^{\xi-1} \mathfrak{f}\right)(\mu)}{\left(D_q^{(\rho-1)} \mathscr{V}_{q,\rho}^{\xi-1} \mathfrak{f}\right)(\mu)} \right| + \frac{q}{1-q} \\
&= \left| \frac{\sum_{\iota=1}^{\infty} \frac{[\iota+\rho]_q!}{[\iota+1]_q!} \left([\iota+1]_q\right)^{\xi-1} \left([\iota+1]_q - 1\right) e_{\iota+\rho} \mu^{\iota}}{[\rho]_q! + \sum_{\iota=1}^{\infty} \frac{[\iota+\rho]_q!}{[\iota+1]_q!} \left([\iota+1]_q\right)^{\xi-1} e_{\iota+\rho} \mu^{\iota}} \right| + \frac{q}{1-q} \\
&< \frac{\sum_{\iota=1}^{\infty} \frac{[\iota+\rho]_q!}{[\iota]_q!} \left([\iota+1]_q\right)^{\xi-2} \left([\iota+1]_q - 1\right) |e_{\iota+\rho}|}{[\rho]_q! - \sum_{\iota=1}^{\infty} \frac{[\iota+\rho]_q!}{[\iota]_q!} \left([\iota+1]_q\right)^{\xi-2} |e_{\iota+\rho}|} + \frac{q}{1-q}.
\end{aligned} \tag{93}$$

Under the assumption that (91) is satisfied, the last expression does not exceed $\frac{1}{1-q}$, which concludes the proof. □

Example 1 Consider $\mathfrak{f} \in \mathscr{A}_\rho$, defined by $\mathfrak{f}(\mu) = \mu^\rho + \mu^{\rho+1}$ which implies that $e_{\rho+1} = 1$ and $e_{\iota+\rho} = 0$, for all $\iota \geq 2$. Therefore, using the criterion established in Theorem 5, $\mathfrak{f} \in \mathscr{L}\mathfrak{S}_{q,\rho-1}^{\xi-1}$ provided that the condition below is satisfied:

$$[1+\rho]_q \left([2]_q\right)^{\xi-1} < 1. \tag{94}$$

The following result is obtained by substituting $\xi = 1$ in Theorem 5.

Corollary 9 A function $\mathfrak{f} \in \mathscr{A}_\rho$ belongs to $\mathscr{L}\mathfrak{S}^{0}_{q,\rho-1}$ under condition:

$$\sum_{\iota=1}^{\infty} \frac{[\iota+\rho]_q!}{[\iota]_q!\,[\rho]_q!} \left|e_{\iota+\rho}\right| < 1. \tag{95}$$

Theorem 6 The analytic mapping $\mathfrak{f} \in \mathscr{A}_\rho$ lies within the subclass $\mathscr{L}\mathfrak{S}^{\xi-1}_{q,\rho-1}(\beta)$ if the following condition holds:

$$\sum_{\iota=1}^{\infty} \frac{[\iota+\rho]_q!}{[\iota]_q!\,[\rho]_q!} \left([\iota+1]_q\right)^{\xi-2} \left([\iota+1]_q - \beta\right) \left|e_{\iota+\rho}\right| < 1-\beta. \tag{96}$$

Proof. Once condition (96) is assumed to be valid, the goal reduces to proving that:

$$\left| \frac{\mu\left(D_q^{(\rho)} \mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}{\left(D_q^{(\rho-1)} \mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)} - \left(\beta + \frac{1-\beta}{1-q}\right) \right| < \frac{1-\beta}{1-q}. \tag{97}$$

□

Hence,

$$\begin{aligned}
\left| \frac{\mu\left(D_q^{(\rho)} \mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}{\left(D_q^{(\rho-1)} \mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)} - \left(\beta + \frac{1-\beta}{1-q}\right) \right| &< \left| \frac{\mu\left(D_q^{(\rho)} \mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}{\left(D_q^{(\rho-1)} \mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)} - 1 \right| + \frac{(1-\beta)q}{1-q} \\
&= \left| \frac{\left(D_q^{(\rho-1)} \mathscr{V}_{q,\rho}^{\xi}\mathfrak{f}\right)(\mu) - \left(D_q^{(\rho-1)} \mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)}{\left(D_q^{(\rho-1)} \mathscr{V}_{q,\rho}^{\xi-1}\mathfrak{f}\right)(\mu)} \right| + \frac{(1-\beta)q}{1-q} \\
&= \left| \frac{\sum_{\iota=1}^{\infty} \frac{[\iota+\rho]_q!}{[\iota+1]_q!} \left([\iota+1]_q\right)^{\xi-1} \left([\iota+1]_q - 1\right) e_{\iota+\rho}\mu^{\iota}}{[\rho]_q! + \sum_{\iota=1}^{\infty} \frac{[\iota+\rho]_q!}{[\iota+1]_q!} \left([\iota+1]_q\right)^{\xi-1} e_{\iota+\rho}\mu^{\iota}} \right| + \frac{(1-\beta)q}{1-q} \\
&< \frac{\sum_{\iota=1}^{\infty} \frac{[\iota+\rho]_q!}{[\iota]_q!} \left([\iota+1]_q\right)^{\xi-2} \left([\iota+1]_q - 1\right) \left|e_{\iota+\rho}\right|}{[\rho]_q! - \sum_{\iota=1}^{\infty} \frac{[\iota+\rho]_q!}{[\iota]_q!} \left([\iota+1]_q\right)^{\xi-2} \left|e_{\iota+\rho}\right|} + \frac{(1-\beta)q}{1-q}.
\end{aligned} \tag{98}$$

Under the assumption that (96) holds, the final expression remains less than $\frac{1-\beta}{1-q}$, which completes the proof.

Example 2 Consider $\mathfrak{f} \in \mathscr{A}_\rho$, defined by $\mathfrak{f}(\mu) = \mu^{\rho} + \mu^{\rho+1}$ which implies that $e_{\rho+1} = 1$ and $e_{\iota+\rho} = 0$, for all $\iota \geq 2$. We now verify whether $\mathfrak{f}$ belongs to the class $\mathscr{L}\mathfrak{S}^{\xi-1}_{q,\rho-1}(\beta)$ based on the sufficient condition stated in Theorem 6. Since all terms vanish except for $\iota = 1$, the inequality (96) simplifies to

$$[1+\rho]_q \left([2]_q\right)^{\xi-2} \left([2]_q - \beta\right) < 1-\beta.$$

In particular, for $\xi = 1$, the inequality reduces to

$$\frac{[1+\rho]_q \left([2]_q - \beta\right)}{[2]_q} < 1-\beta,$$

which coincides with the required condition, thereby confirming the validity of the example.

Corollary 10 Let $\mathfrak{f} \in \mathscr{A}_\rho$. Then $\mathfrak{f}$ is included in the subclass $\mathscr{L}\mathfrak{S}^0_{q,\rho-1}$ whenever the ensuing inequality is satisfied::

$$\sum_{\iota=1}^{\infty} \frac{[\iota+\rho]_q!}{[\iota+1]_q! [\rho]_q!} \left([\iota+1]_q - \beta\right) |e_{\iota+\rho}| < 1. \tag{99}$$

Proof. Substituting $\xi = 1$ into Theorem 6 yields the result. □

5. Conclusions

This study is devoted to the introduction and analysis of families of normalized transformations arising from the application of a generalized higher-order q-differential operator, contributing to the broader understanding of their behavior under subordination and coefficient bounds. By employing subordination principles involving univalent functions, we established several inclusion relationships between these subclasses and derived sharp coefficient inequalities of Fekete–Szegő type. Additionally, sufficient conditions for function membership within these classes were provided. The results obtained generalize and extend various known results in geometric function theory and demonstrate the versatility of q-calculus in constructing and analyzing function classes.The present investigation opens several avenues for further research. One natural extension would be to study subclasses of multivalent analytic functions associated with fractional q-differential operators, thereby bridging the gap between fractional calculus and q-calculus in the setting of geometric function theory. Another promising direction is the analysis of bi-univalent function classes generated by the generalized operator $\mathscr{V}^{\xi}_{q,\rho}$, with a particular focus on obtaining sharp coefficient bounds and Fekete–Szegő inequalities. Furthermore, the operator-based approach adopted here could be adapted to explore subclasses connected with q-close-to-convex and q-convex functions, which would broaden the applicability of the results. It would also be of interest to investigate inequality versions under different fractional operators, such as the Hilfer and Prabhakar types, and to examine numerical aspects of the associated inequalities.

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Conflict of interest

The author declares no competing financial interest.

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