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A Systematic Comparison of Ten Odd-G Family-Based Probability Models

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Abstract: Generalized probability distributions play a crucial role in statistical modeling across various disciplines. However, selecting an appropriate model remains a challenge for researchers. This study provides a comprehensive comparative analysis of ten odd-related families of distributions (five single and five double parameters) to aid researchers in choosing suitable models. To ensure a robust comparison, we selected the Weibull distribution as the baseline and applied each distribution to eight datasets exhibiting diverse characteristics, including left-skewed, right-skewed, bi-modal, tri-modal, and symmetrical patterns. The results indicate that among three-parameter models, the Odd Lindley Weibull Distribution (OLWD) demonstrated superior flexibility, while the Odd Burr-III Weibull Distribution (OBIIIWD) was the most adaptable among four-parameter models. Consequently, the Odd Lindley-G (OL-G) and Odd Burr-III-G (OBIII-G) families emerged as the most effective among the examined distributions. Moreover, the Odd-G family exhibited strong modeling capabilities for moderately skewed (left or right), symmetrical, bi-modal, and tri-modal data. However, it was found to be less suitable for highly skewed data. Based on these findings, we recommend the OL-G family (single-parameter) and the OBIII-G family (two-parameter) as promising candidates for modeling real-world data with diverse distributional characteristics.

Keywords: Odd G-family, odd function, comparison, probability model, application

MSC: 35A01, 65L10, 65L12, 65L20, 65L70

1. Introduction

Probability distributions form the foundation of statistical inference. From classical statistical methods to modern machine learning techniques, many tools and models are inherently built upon probability distributions. With advancements in technology, data collection and generation have become more accessible, leading to the availability of diverse and complex datasets across a wide range of fields. However, these datasets often present unique challenges where traditional distributions may fail to provide an adequate fit [1]. Among the many types of measurements researchers encounter in applied research, positive data situations frequently arise across a wide range of disciplines. Fields such as biomedical sciences (e.g., survival times, blood concentrations), engineering and reliability (e.g., failure times, lifetimes

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of components), economics and finance (e.g., income, expenditures, durations of unemployment), environmental science (e.g., pollutant levels, rainfall amounts), and social sciences (e.g., response times, counts of events) often involve strictly positive data. Some popular distributions that have been widely applied for modeling positive data include the Gamma, Exponential and Weibull distributions. Despite their popularity, these classical models have limitations in addressing the full range of positive data characteristics and reliability contexts. For instance, the exponential distribution assumes a constant hazard rate and thus cannot capture situations where the hazard changes over time. The gamma distribution, while flexible, does not have closed-form expressions for its distribution and survival functions when the shape parameter is non-integer, complicating analysis. Similarly, the Weibull distribution struggles with data exhibiting hazard rates that vary non-monotonically. These shortcomings have been discussed in several studies [2–4].

To address the limitations of classical distributions in modeling complex and diverse data structures, statisticians have developed new families of distributions by introducing additional parameters to existing ones. These generalizations enhance the flexibility and adaptability of probability models in various applied contexts. Prominent examples include the Marshall-Olkin-G (MO-G) family [5], the Odd-G family [6], the T-X family [7], the Beta-G family [8], the Kumaraswamy-G (Kw-G) family [9], the Exponentiated-G family [10], Marshall-Olkin-Weibull-H family [11], weighted Lindley-G family [12], the Sine π-power Odd-G family [13], and the Odd inverse Chen-G family [14], among others. Among them, the Odd-G family of distributions has gained considerable attention in recent years due to its ability to model diverse data behaviors across applied disciplines. The Odd-G family of distributions offers remarkable flexibility, particularly in modeling various shapes of hazard functions, including increasing, decreasing, J-shaped, and reverse J-shaped, bathtub-shaped, and inverted bathtub-shaped forms [6]. The main idea behind generating the Odd-G family of distributions is to increase the flexibility of a baseline distribution $G(x)$ by applying an odds-based transformation to its Cumulative Distribution Function (CDF) [6, 15]. This transformation introduces additional shape parameters that can capture a wide variety of distributional forms. Several distributions have been developed under the Odd-G framework, each extending a different baseline distribution. Examples include the Exponentiated Weibull distribution family [10], the Odd Exponential-G (OE-G) family [15], the Odd Rayleigh-G (OR-G) family [16], the Odd Fréchet-G family [17], the Odd Half-Logistic-G family [18], the Odd Lindley-G family [19], the Odd Weibull-G family [15], the Odd Lomax-G (OLx-G) family [20], the Odd Burr III-G (OBIII-G) family [21], and the Odd Chen-G (OC-G) family [22], power exponential Odd-G (no parameter in this family) [23], among others.

There are several Odd family distributions constructed using different baseline distributions. In applied fields, datasets may exhibit a wide range of distributional shapes and measurement characteristics. For instance, the survival time of leukemia patients, as presented by [24], appears approximately symmetric; in contrast, the survival times in a blood cancer dataset studied by [25] are left-skewed; and measurements of petroleum rock samples, as analyzed by [26], are right-skewed. Moreover, data may exhibit unimodal, bimodal, trimodal, or even multimodal behavior.

A natural question arises: do all Odd family distributions provide comparably good fits across different data shapes, or are certain Odd family distributions consistently more suitable for specific types of data? This creates a dilemma for practitioners when selecting an appropriate model for their datasets.

To address this issue, the current study compares five two-parameter Odd family distributions and five one-parameter Odd family distributions across eight diverse positive data scenarios, including right-skewed, left-skewed, symmetric, bimodal, and trimodal distributions.

The detailed manuscript is structured as follows: Section 2 introduces specific models of the Odd-G family of distributions; Section 3 discusses parameter estimation; Section 4 presents simulations of all the proposed distributions; Section 5 covers model comparison and applications; and Section 6 provides the conclusion.

2. Particular models of Odd-G family of distributions

In this study, we selected the Odd-G family based on parameter complexity, restricting our focus to one- and two-parameter G families. Families with more than two parameters were excluded because their associated distributions often involve five or more parameters, adding unnecessary complexity to our comparative analysis.

To compare the performance of the Odd-G family of distributions, we have developed ten particular models based on these families. Among these, five families are single-parameter, while the other five are two-parameter G-families presented in Table 1. Most authors have utilized the Weibull distribution as a base distribution for odd-related families. Therefore, in this section, we have used the Weibull distribution as a base, with its CDF and Probability Density Function (PDF) defined as follows:

$$G(x;\, \gamma,\, \lambda) = 1 - e^{-\gamma x^{\lambda}};\ (\gamma,\, \lambda) > 0,\ x > 0, \tag{1}$$

and

$$g(x;\, \gamma,\, \lambda) = \gamma\lambda x^{\lambda-1} e^{-\gamma x^{\lambda}};\ (\gamma,\, \lambda) > 0,\ x > 0. \tag{2}$$

Table 1. Family of distributions under study

SN	G-family	CDF	Source
1	OE-G	$F(x;\, \xi) = 1 - \exp\left(-\alpha\left(\frac{G(x;\, \xi)}{1-G(x;\, \xi)}\right)\right),\ \alpha > 0$	[15]
2	OR-G	$F(x;\, \xi) = 1 - \exp\left(-\frac{1}{2\alpha^2}\left(\frac{G(x;\, \xi)}{1-G(x;\, \xi)}\right)^2\right),\ \alpha > 0$	[16]
3	OF-G	$F(x;\, \xi) = \exp\left(-\left(\frac{G(x;\, \xi)}{1-G(x;\, \xi)}\right)^{-\alpha}\right),\ \alpha > 0$	[17]
4	OHL-G	$F(x;\, \xi) = \frac{1-\exp\left(-\alpha\frac{G(x;\, \xi)}{1-G(x;\, \xi)}\right)}{1+\exp\left(-\alpha\frac{G(x;\, \xi)}{1-G(x;\, \xi)}\right)},\ \alpha > 0$	[18]
5	OL-G	$F(x;\, \xi) = 1 - \frac{\alpha+\overline{G}(x;\, \xi)}{(1+\alpha)\overline{G}(x;\, \xi)}\exp\left(-\alpha\frac{G(x;\, \xi)}{\overline{G}(x;\, \xi)}\right),\ \alpha > 0$	[19]
6	OW-G	$F(x;\, \xi) = 1 - \exp\left(-\alpha\left(\frac{G(x;\, \xi)}{1-G(x;\, \xi)}\right)^{\beta}\right),\ \alpha,\ \beta > 0$	[15]
7	OLx-G	$F(x;\, \xi) = 1 - \alpha^{\beta}\left(\alpha + \frac{G(x;\, \xi)}{1-G(x;\, \xi)}\right)^{-\beta},\ \alpha,\ \beta > 0$	[20]
8	OBIII-G	$F(x;\, \xi) = \left(1+\left(\frac{G(x;\, \xi)}{1-G(x;\, \xi)}\right)^{-\alpha}\right)^{-\beta},\ \alpha,\ \beta > 0$	[21]
9	OG-G	$F(x;\, \xi) = 1 - \exp\left(-\frac{\alpha}{\beta}\left(\exp\left(\beta\frac{G(x;\, \xi)}{1-G(x;\, \xi)}\right)-1\right)\right),\ \alpha,\ \beta > 0$	[27]
10	OC-G	$F(x;\, \xi) = 1 - \exp\left(\alpha\left(1-\exp\left(\left(\frac{G(x;\, \xi)}{1-G(x;\, \xi)}\right)^{\beta}\right)\right)\right),\ \alpha,\ \beta > 0$	[22]

2.1 *The Odd Exponential Weibull Distribution (OEWD)*

Using the CDF of the OE-G family of distributions presented in Table 1 and the CDF of the base distribution 1, we get the following CDF and PDF of OEWD

$$F(x;\ \alpha,\ \gamma,\ \lambda) = 1 - e^{-\alpha\left(e^{\gamma x^{\lambda}}-1\right)};\ (\alpha,\ \gamma,\ \lambda) > 0,\ x > 0.$$

$$f(x;\ \alpha,\ \gamma,\ \lambda) = \alpha\gamma\lambda x^{\lambda-1} e^{\gamma x^{\lambda}-\alpha\left(e^{\gamma x^{\lambda}}-1\right)};\ (\alpha,\ \gamma,\ \lambda) > 0,\ x > 0. \tag{3}$$

To know more about this OE-G family of distributions, like shapes of PDF, Hazard Rate Function (HRF), and properties, readers can go through [15].

The quantile function of the OEWD was used to generate the random numbers for the simulation study. The expression of the quantile function is

$$Q(q) = \left[\frac{1}{\gamma}\log\left\{1 - \frac{1}{\alpha}\log(1-q)\right\}\right]^{1/\lambda};\ 0 < q < 1,$$

where Q follows uniform distribution $U(0,\ 1)$.

2.2 *The Odd Rayleigh Weibull Distribution (ORWD)*

As discussed in subsection 2.1, we have created ORWD using the OR-G family of distributions defined in Table 1 and CDF, PDF, and quantile functions are expressed as

$$F(x;\ \alpha,\ \gamma,\ \lambda) = 1 - e^{-\frac{1}{2\alpha^2}\left(e^{\gamma x^{\lambda}}-1\right)^2};\ (\alpha,\ \gamma,\ \lambda) > 0,\ x > 0.$$

$$f(x;\ \alpha,\ \gamma,\ \lambda) = \frac{\gamma\lambda}{\alpha^2}\left\{e^{\gamma x^{\lambda}}-1\right\} x^{\lambda-1} e^{\gamma x^{\lambda}-\frac{1}{2\alpha^2}\left\{e^{\gamma x^{\lambda}}-1\right\}^2};\ (\alpha,\ \gamma,\ \lambda) > 0,\ x > 0$$

$$Q(q) = \left[\frac{1}{\gamma}\log\left\{\left\{-2\alpha^2\log(1-q)\right\}^{1/2}+1\right\}\right]^{1/\lambda};\ 0 < q < 1.$$

2.3 *The Odd Frenchet Weibull Distribution (OFWD)*

Also, as discussed in subsection 2.1, we have created ORWD using the OF-G family of distribution defined in Table 1 and CDF, PDF, and quantile functions are expressed as

$$F(x;\ \alpha,\ \gamma,\ \lambda) = e^{-\left\{e^{\gamma x^{\lambda}}-1\right\}^{-\alpha}};\ (\alpha,\ \gamma,\ \lambda) > 0,\ x > 0.$$

$$f(x;\, \alpha,\, \gamma,\, \lambda) = \alpha\gamma\lambda e^{\gamma x^{\lambda}} x^{\lambda-1} e^{-\left\{e^{\gamma x^{\lambda}}-1\right\}^{-\alpha}} \left\{e^{\gamma x^{\lambda}}-1\right\}^{-(\alpha+1)};\ (\alpha,\, \gamma,\, \lambda) > 0,\ x > 0.$$

$$Q(q) = \left[\frac{1}{\gamma}\log\left\{1+\{-\log(q)\}^{-1/\alpha}\right\}\right]^{1/\lambda};\ 0 < q < 1.$$

2.4 *The Odd Half-Logistic Weibull Distribution (OHLWD)*

Also, OHLWD was developed as in subsection 2.1, using the OHL-G family of distribution defined in Table 1 and CDF, PDF, and quantile functions are expressed as

$$F(x;\, \alpha,\, \gamma,\, \lambda) = \frac{1-e^{-\alpha\left\{e^{\gamma x^{\lambda}}-1\right\}}}{1+e^{-\alpha\left\{e^{\gamma x^{\lambda}}-1\right\}}};\ (\alpha,\, \gamma,\, \lambda) > 0,\ x > 0.$$

$$f(x;\, \alpha,\, \gamma,\, \lambda) = \alpha\gamma\lambda x^{\lambda-1} e^{\gamma x^{\lambda}} e^{-\alpha\left\{e^{\gamma x^{\lambda}}-1\right\}} \left[1+e^{-\alpha\left\{e^{\gamma x^{\lambda}}-1\right\}}\right]^{-1};\ (\alpha,\, \gamma,\, \lambda) > 0,\ x > 0.$$

$$Q(q) = \left[\frac{1}{\gamma}\log\left\{1-\frac{1}{\alpha}\log\left(\frac{1-q}{1+q}\right)\right\}\right]^{1/\lambda};\ 0 < q < 1.$$

2.5 *The Odd Lindley Weibull Distribution (OLWD)*

Using the OL-G family of distributions, we have defined a new model, OLWD. The CDF, PDF, and quantile functions are expressed as follows

$$F(x;\, \alpha,\, \gamma,\, \lambda) = 1-\frac{\alpha+e^{-\gamma x^{\lambda}}}{(1+\alpha)e^{-\gamma x^{\lambda}}} e^{-\alpha\left(e^{\gamma x^{\lambda}}-1\right)};\ (\alpha,\, \gamma,\, \lambda) > 0,\ x > 0.$$

$$f(x;\, \alpha,\, \gamma,\, \lambda) = \frac{\alpha^{2}\gamma\lambda}{(1+\alpha)} x^{\lambda-1} e^{2\gamma x^{\lambda}} e^{-\alpha\left(e^{\gamma x^{\lambda}}-1\right)};\ (\alpha,\, \gamma,\, \lambda) > 0,\ x > 0.$$

$$(1-q)(1+\alpha)-\left\{\alpha+e^{-\gamma x^{\lambda}}\right\} e^{\gamma x^{\lambda}-\alpha\left(e^{\gamma x^{\lambda}}-1\right)} = 0;\ 0 < q < 1.$$

The quantile function of the OLWD does not have a closed-form expression. However, we can numerically generate random numbers for simulation studies using the R programming software. The results of such a simulation are presented in Table 2.

Table 2. Results of the simulation study of the OWWD

n	B_α	B_β	B_γ	B_λ	MSE_α	MSE_β	MSE_γ	MSE_λ
50	0.2529	-0.0528	0.0415	0.3086	0.3294	0.2904	1.0319	0.3428
80	0.1552	-0.0532	0.0362	0.1856	0.1588	0.0285	0.7375	0.1598
110	0.0989	-0.0342	0.0441	0.1188	0.1034	0.0216	0.4467	0.0904
140	0.0931	-0.0352	-0.0066	0.1085	0.0829	0.0184	0.2878	0.0722
170	0.0664	-0.0146	0.0156	0.0687	0.0627	0.0159	0.2278	0.0464
200	0.0687	-0.0205	-0.0097	0.0703	0.0542	0.0129	0.2102	0.0412
230	0.0579	-0.0201	4.0E-04	0.0589	0.0466	0.0116	0.1744	0.0316
260	0.0407	-0.0114	0.0104	0.0435	0.0369	0.0092	0.1466	0.0249
290	0.0403	-0.0128	0.0119	0.0423	0.0361	0.0087	0.1433	0.0240
320	0.0344	-0.0105	0.0022	0.0382	0.0305	0.0082	0.1122	0.0201
350	0.0341	-0.0068	0.0059	0.0330	0.0289	0.0074	0.1205	0.0179
380	0.0244	-0.0063	0.0189	0.0267	0.0254	0.0069	0.1101	0.0164
410	0.0268	-0.0093	0.0078	0.0301	0.0248	0.0065	0.0974	0.0160
440	0.0262	-0.0057	0.0054	0.0256	0.0228	0.0059	0.0889	0.0137
470	0.0268	-0.0086	-0.0022	0.0257	0.0203	0.0050	0.0771	0.0122
500	0.0254	-0.0083	2.0E-04	0.0270	0.0204	0.0053	0.0805	0.0132

2.6 *The Odd Weibull Weibull of Distribution (OWWD)*

A four-parameter OWWD was defined by using the CDF of the OW-G family of distributions presented in Table 1. The CDF, PDF, and quantile functions of this model are respectively expressed as follows;

$$F(x;\ \alpha,\ \beta,\ \gamma,\ \lambda) = 1 - e^{-\alpha\left(e^{\gamma x^{\lambda}}-1\right)^{\beta}};\ (\alpha,\ \beta,\ \gamma,\ \lambda) > 0,\ x > 0.$$

$$f(x;\ \alpha,\ \beta,\ \gamma,\ \lambda) = \alpha\beta\gamma\lambda x^{\lambda-1}(z)^{\beta-1}e^{\gamma x^{\lambda}-\alpha(z)^{\beta}};\ (\alpha,\ \beta,\ \gamma,\ \lambda) > 0,\ x > 0,$$

where $z = e^{\gamma x^{\lambda}} - 1$.

$$Q(q) = \left[\frac{1}{\gamma}\log\left\{1+\left\{-\frac{1}{\alpha}\log(1-q)\right\}^{1/\beta}\right\}\right]^{1/\lambda};\ 0 < q < 1.$$

2.7 *The Odd Lomax Weibull Distribution (OLxWD)*

Another four-parameter OLxWD was defined by using the CDF of the OLx-G family of distributions presented in Table 1. The CDF, PDF, and quantile functions of this model are respectively expressed as follows;

$$F(x;\ \alpha,\ \beta,\ \gamma,\ \lambda) = 1 - \alpha^{\beta}\left[\alpha + e^{\gamma x^{\lambda}} - 1\right]^{-\beta};\ (\alpha,\ \beta,\ \gamma,\ \lambda) > 0,\ x > 0.$$

$$f(x;\ \alpha,\ \beta,\ \gamma,\ \lambda) = \alpha^{\beta}\cdot\beta\gamma\lambda x^{\lambda-1}e^{\gamma x^{\lambda}}\left[\alpha + e^{\gamma x^{\lambda}} - 1\right]^{-\beta-1};\ (\alpha,\ \beta,\ \gamma,\ \lambda) > 0,\ x > 0.$$

$$Q(q) = \left[\frac{1}{\gamma}\log\left\{\left\{\frac{1}{\alpha^{\beta}}(1-q)\right\}^{-1/\beta} + 1 - \alpha\right\}\right]^{1/\lambda};\ 0 < q < 1.$$

2.8 *The Odd Burr-III Weibull Distribution (OBIIIWD)*

Similarly, OBIIIWD was defined by using the CDF of the OBIII-G family of distributions presented in Table 1. The CDF, PDF, and quantile functions of this model are respectively expressed as follows;

$$F(x;\ \alpha,\ \beta,\ \gamma,\ \lambda) = \left[1 + \left(e^{\gamma x^{\lambda}} - 1\right)^{-\alpha}\right]^{-\beta};\ (\alpha,\ \beta,\ \gamma,\ \lambda) > 0,\ x > 0.$$

$$f(x;\ \alpha,\ \beta,\ \gamma,\ \lambda) = \alpha\beta\gamma\lambda x^{\lambda-1} e^{\gamma x^{\lambda}} \left(e^{\gamma x^{\lambda}} - 1\right)^{-\alpha-1} \left[1 + \left(e^{\gamma x^{\lambda}} - 1\right)^{-\alpha}\right]^{-\beta-1};\ (\alpha,\ \beta,\ \gamma,\ \lambda) > 0,\ x > 0.$$

$$Q(q) = \left[\frac{1}{\gamma}\log\left\{\left(q^{-1/\beta} - 1\right)^{-1/\alpha} + 1\right\}\right]^{1/\lambda};\ 0 < q < 1.$$

2.9 *The Odd Gompertz Weibull Distribution (OGWD)*

Using the CDF of the OG-G family of distributions presented in Table 1, we have developed a four-parameter OGWD. The CDF, PDF, and quantile functions of this model are respectively expressed as follows;

$$F(x;\ \alpha,\ \beta,\ \gamma,\ \lambda) = 1 - e^{-\frac{\alpha}{\beta}\left\{e^{\beta\left\{e^{\gamma x^{\lambda}} - 1\right\}} - 1\right\}};\ (\alpha,\ \beta,\ \gamma,\ \lambda) > 0,\ x > 0.$$

$$f(x;\ \alpha,\ \beta,\ \gamma,\ \lambda) = \alpha\gamma\lambda x^{\lambda-1} e^{\gamma x^{\lambda}} e^{\beta\left\{e^{\gamma x^{\lambda}} - 1\right\}} e^{-\frac{\alpha}{\beta}\left\{e^{\beta\left\{e^{\gamma x^{\lambda}} - 1\right\}} - 1\right\}};\ (\alpha,\ \beta,\ \gamma,\ \lambda) > 0,\ x > 0.$$

$$Q(q) = \left[\frac{1}{\gamma}\log\left\{1 + \frac{1}{\beta}\log\left\{1 - \frac{\beta}{\alpha}\log(1-q)\right\}\right\}\right]^{1/\lambda};\ 0 < q < 1.$$

2.10 *The Odd Chen Weibull Distribution (OCWD)*

Similarly, using the CDF of the OC-G family of distributions presented in Table 1, we have developed a four-parameter OCWD. The CDF, PDF, and quantile functions of this model are respectively expressed as follows;

$$F(x;\ \alpha,\ \beta,\ \gamma,\ \lambda) = 1 - e^{\alpha\left(1 - e^{\left(e^{\gamma x^{\lambda}} - 1\right)^{\beta}}\right)};\ (\alpha,\ \beta,\ \gamma,\ \lambda) > 0,\ x > 0.$$

$$f(x;\ \alpha,\ \beta,\ \gamma,\ \lambda) = \alpha\beta\gamma\lambda x^{\lambda-1} e^{\gamma x^{\lambda}} \left(e^{\gamma x^{\lambda}} - 1\right)^{\beta-1} e^{\left(e^{\gamma x^{\lambda}} - 1\right)^{\beta}} e^{\alpha\left(1 - e^{\left(e^{\gamma x^{\lambda}} - 1\right)^{\beta}}\right)};\ (\alpha,\ \beta,\ \gamma,\ \lambda) > 0,\ x > 0.$$

$$Q(q) = \left[\frac{1}{\gamma}\log\left\{\left\{\log\left\{1-\frac{1}{\alpha}\log(1-q)\right\}\right\}^{1/\beta}+1\right\}\right]^{1/\lambda}; \ 0<q<1.$$

3. Parameter estimation

In statistical modeling, parameter estimation is a crucial step in defining the behavior of a probability distribution based on observed data. One of the most widely used methods for parameter estimation is Maximum Likelihood Estimation (MLE), which finds the parameter values that maximize the likelihood function. Using this MLE method, we have estimated the parameters of the following ten models under study

3.1 *MLE for OEWD*

Assume that $\underline{x} = (x_1, \ldots, x_n)$ represents a random sample of independent and identically distributed observations from the OEWD. Using the PDF defined in 3 we can express the log-likelihood function for a single observation as:

$$\log f(x) = \log(\alpha\gamma\lambda) + (\lambda-1)\log x + \gamma x^{\lambda} - \alpha\left(e^{\gamma x^{\lambda}}-1\right).$$

For a given sample $\underline{x} = (x_1, \ldots, x_n)$ of size n, the full log-likelihood function is obtained by summing the individual log-likelihood contributions:

$$\log \ell(\underline{x}) = n\log(\alpha\gamma\lambda) + (\lambda-1)\sum_{i=1}^{n}\log x_i + \gamma\sum_{i=1}^{n}x_i^{\lambda} - \alpha\sum_{i=1}^{n}\left(e^{\gamma x_i^{\lambda}}-1\right).$$

The MLE method involves finding the estimates $(\hat{\alpha}, \hat{\gamma}, \hat{\lambda})$ that maximize $\log \ell(\underline{x})$. This is done by differentiating $\log \ell(\underline{x})$ with respect to α, γ, and λ, and solving the resulting system of nonlinear equations:

$$\frac{\partial \log \ell}{\partial \alpha} = 0, \quad \frac{\partial \log \ell}{\partial \gamma} = 0, \quad \frac{\partial \log \ell}{\partial \lambda} = 0.$$

As closed-form solutions may not be available for these equations, numerical optimization techniques-such as the Newton-Raphson algorithm or other suitable methods can be employed to estimate the parameters. By maximizing the likelihood function, we obtain parameter estimates that best explain the observed data, making MLE a powerful tool for statistical inference in the OEWD framework.

Using the same parameter estimation technique as described in Subsection 3.1, we have estimated the parameters of the models ORWD, OFWD, OHLWD, OLWD, OWWD, OLxWD, OBIIIWD, OGWD, and OCWD.

4. Simulation study

To evaluate the consistency and efficiency of the MLE method for parameter estimation, we conducted a comprehensive Monte Carlo simulation study. The primary objective was to assess the asymptotic behavior of the estimators by examining the Bias (B) and Mean Squared Error (MSE) across varying sample sizes. The simulation experiment was executed with 1,000 replications for each considered sample size, ranging from 50 to 500 in increments of 30. The parameter configurations were chosen to reflect realistic scenarios for both three-parameter and four-parameter models. The values of parameters used in the simulation scenarios were selected to reflect a broad range of distributional shapes observed in real-world applications, including right-skewed and heavy-tailed behaviors. Specifically, for the three-parameter model, the true parameter values were set as $\alpha = 0.25$, $\gamma = 0.75$, and $\lambda = 0.5$. Similarly, for the four-parameter model, the initial values were fixed at $\alpha = 0.45$, $\beta = 0.5$, $\gamma = 0.75$, and $\lambda = 0.5$.

The estimation of parameters was performed using the `MaxLik` package [28] in the *R* statistical computing environment [29]. The bias and MSE were computed for each parameter at different sample sizes to assess the performance of the MLE method.

The numerical results, presented in Tables 2 to 11, demonstrate a clear trend: as the sample size increases, both the bias and MSE exhibit a decreasing pattern, ultimately converging towards zero. This observation strongly supports the consistency property of the MLE method, reinforcing its reliability in parameter estimation for the studied distributions. The findings underscore the importance of sufficiently large sample sizes to achieve accurate and stable estimates in practical applications.

Table 3. Results of the simulation study of the OEWD

n	B_α	B_γ	B_λ	MSE_α	MSE_γ	MSE_λ
50	1.1880	0.1154	0.0724	23.3330	0.8313	0.0487
80	0.4217	0.1181	0.0485	2.4583	0.6988	0.0331
110	0.2591	0.1837	0.0262	0.7664	0.7519	0.0266
140	0.1682	0.1601	0.0184	0.2412	0.6749	0.0202
170	0.1876	0.1361	0.0200	0.5539	0.5832	0.0197
200	0.1457	0.1212	0.0179	0.175	0.5248	0.0176
230	0.1115	0.1458	0.0091	0.1238	0.5013	0.0158
260	0.0907	0.1307	0.0093	0.0878	0.4724	0.0139
290	0.0829	0.1350	0.0056	0.0817	0.4249	0.0131
320	0.0837	0.0861	0.0111	0.0789	0.3167	0.0115
350	0.0778	0.1097	0.0083	0.0638	0.4167	0.0114
380	0.0598	0.1070	0.0032	0.0543	0.3255	0.0102
410	0.0547	0.0918	0.0038	0.0465	0.2648	0.0092
440	0.0591	0.0989	0.0045	0.0463	0.3172	0.0097
470	0.0517	0.0709	0.0040	0.0391	0.2069	0.0080
500	0.0434	0.0847	0.0026	0.0343	0.2321	0.0081

Table 4. Results of the simulation study of the ORWD

n	B_α	B_γ	B_λ	MSE_α	MSE_γ	MSE_λ
50	1.1008	0.4078	-0.0199	12.4642	1.6144	0.0174
80	0.9177	0.3250	-0.0196	13.4485	1.2767	0.0138
110	0.8115	0.3110	-0.0201	9.9162	1.1506	0.0118
140	0.6461	0.2383	-0.0181	8.9944	0.9476	0.0097
170	0.4256	0.1396	-0.0085	4.7145	0.7495	0.0079
200	0.4852	0.1525	-0.0109	8.9365	0.7005	0.0076
230	0.3700	0.1709	-0.0136	3.6766	0.6342	0.0070
260	0.4037	0.1344	-0.0086	5.5072	0.6211	0.0066
290	0.2568	0.1146	-0.0080	3.0938	0.4672	0.0052
320	0.2741	0.1016	-0.0062	4.5496	0.4615	0.0050
350	0.1947	0.0591	-0.0026	2.2765	0.3951	0.0043
380	0.1175	0.0681	-0.0050	0.7116	0.2992	0.0036
410	0.1763	0.0778	-0.0053	2.5046	0.3379	0.0039
440	0.1057	0.0431	-0.0021	0.3361	0.3074	0.0037
470	0.1106	0.0450	-0.0041	1.2937	0.2709	0.0032
500	0.1095	0.0673	-0.0046	0.5475	0.2656	0.0032

Table 5. Results of the simulation study of the OFWD

n	B_α	B_γ	B_λ	MSE_α	MSE_γ	MSE_λ
50	0.0088	-0.0204	0.0275	0.0053	0.0617	0.0141
80	0.0032	-0.0256	0.0189	0.0027	0.0384	0.0072
110	8.0E-04	-0.0201	0.0163	0.0020	0.0302	0.0055
140	6.0E-04	-0.0133	0.0112	0.0015	0.0228	0.0040
170	0.0036	-0.0018	0.0056	0.0012	0.0192	0.0032
200	0.0019	-0.0099	0.0065	0.0011	0.0158	0.0027
230	0.0010	-0.0020	0.0059	0.0010	0.0133	0.0023
260	9.0E-04	-0.0066	0.0057	7.0E-04	0.0120	0.0018
290	8.0E-04	-0.0032	0.0048	7.0E-04	0.0103	0.0016
320	2.0E-04	-0.0103	0.0063	7.0E-04	0.0097	0.0017
350	0.0016	-0.0051	0.0036	6.0E-04	0.0093	0.0014
380	8.0E-04	-0.0018	0.0031	5.0E-04	0.0086	0.0013
410	-1.0E-04	-0.0056	0.0046	5.0E-04	0.0077	0.0012
440	0.0011	-0.0045	0.0030	5.0E-04	0.0068	0.0011
470	0.0010	-0.0024	0.0015	4.0E-04	0.0066	0.0010
500	4.0E-04	-0.0044	0.0033	4.0E-04	0.0063	0.0010

Table 6. Results of the simulation study of the OHLWD

n	B_α	B_γ	B_λ	MSE_α	MSE_γ	MSE_λ
50	1.0321	0.1724	0.1337	17.5479	1.4427	0.0877
80	0.4089	0.1499	0.1039	1.9621	0.9551	0.0631
110	0.2541	0.2521	0.0682	1.4520	1.0329	0.0491
140	0.1145	0.2356	0.0525	0.2795	0.8846	0.0364
170	0.1413	0.2737	0.0453	1.3190	0.9559	0.0372
200	0.0859	0.2356	0.0443	0.2261	0.7724	0.0334
230	0.0424	0.2481	0.0325	0.1381	0.6903	0.0280
260	0.0143	0.2865	0.024	0.0982	0.7527	0.0262
290	0.0104	0.2132	0.0285	0.0842	0.5457	0.0220
320	0.0047	0.2253	0.0264	0.0795	0.5728	0.0219
350	-0.0049	0.2574	0.0201	0.0651	0.6321	0.0214
380	-0.0307	0.2979	0.0075	0.0562	0.6611	0.0192
410	-0.0351	0.2782	0.0081	0.0472	0.5797	0.0178
440	-0.0315	0.2486	0.0117	0.0450	0.5040	0.0172
470	-0.0409	0.2517	0.0071	0.0391	0.4803	0.0156
500	-0.0483	0.2587	0.0063	0.0346	0.4897	0.0154

Table 7. Results of the simulation study of the OLWD

n	B_α	B_γ	B_λ	MSE_α	MSE_γ	MSE_λ
50	0.2693	0.6504	0.0906	0.4096	4.1363	0.1044
80	0.2135	0.3566	0.0805	0.2735	1.8316	0.0813
110	0.1872	0.2073	0.0794	0.2162	1.1099	0.0663
140	0.2120	0.1095	0.0921	0.2272	0.7153	0.0659
170	0.1545	0.1564	0.0626	0.1684	0.6857	0.0529
200	0.1806	0.0878	0.0778	0.1910	0.5557	0.0554
230	0.1645	0.0460	0.0746	0.1583	0.4215	0.0468
260	0.1464	0.0507	0.0671	0.1456	0.3529	0.0455
290	0.1533	0.0054	0.0739	0.1368	0.3027	0.0419
320	0.1500	0.0042	0.0736	0.1322	0.2984	0.0413
350	0.1371	0.0123	0.0656	0.1216	0.2753	0.0378
380	0.1450	-0.0179	0.0727	0.1155	0.2599	0.0376
410	0.1509	-0.0381	0.0775	0.1203	0.2308	0.0379
440	0.1385	-0.0139	0.0684	0.1162	0.2252	0.0369
470	0.1419	-0.0317	0.0707	0.1140	0.2128	0.0356
500	0.1416	-0.0554	0.0746	0.1075	0.1840	0.0337

Table 8. Results of the simulation study of the OLmxWD

n	B_α	B_β	B_γ	B_λ	MSE_α	MSE_β	MSE_γ	MSE_λ
50	1.2042	0.6177	0.2878	0.1168	26.8994	10.5607	2.9489	0.0609
80	0.4830	0.3153	0.2719	0.0734	9.7496	3.6353	1.8592	0.0274
110	0.2993	0.3026	0.2461	0.0496	3.2962	2.0316	1.6806	0.0169
140	0.1478	0.2978	0.1621	0.0443	1.5700	2.1395	1.1677	0.0128
170	0.0678	0.2324	0.1902	0.0352	0.7401	1.2433	1.1979	0.0096
200	0.0305	0.1570	0.1558	0.0347	0.5362	0.9397	0.8177	0.0085
230	0.0535	0.1440	0.2427	0.0270	0.5050	0.8904	2.3138	0.0068
260	0.0339	0.2051	0.1406	0.0235	0.3394	1.2032	0.7269	0.0057
290	0.0164	0.1755	0.1243	0.0221	0.2203	0.9330	0.5669	0.0052
320	0.0089	0.1537	0.1535	0.0218	0.2137	0.7382	0.7269	0.0048
350	-0.0113	0.1440	0.0903	0.0211	0.1399	0.8655	0.4796	0.0041
380	0.0101	0.1297	0.1063	0.0157	0.1708	0.7908	0.4394	0.0039
410	-0.0087	0.1399	0.1013	0.0166	0.0999	0.4886	0.4898	0.0035
440	-1.0E-04	0.1618	0.0662	0.0156	0.1071	0.8189	0.4423	0.0034
470	-0.0171	0.1356	0.0582	0.0141	0.0691	0.6515	0.3055	0.0028
500	-0.0095	0.1036	0.0438	0.0154	0.0711	0.2992	0.2448	0.0032

Table 9. Results of the simulation study of the OBIIIWD

n	B_α	B_β	B_γ	B_λ	MSE_α	MSE_β	MSE_γ	MSE_λ
50	0.7618	0.3163	0.3860	0.0754	14.5412	0.8627	0.8634	0.1471
80	0.4465	0.2149	0.3185	0.0239	6.3178	0.3343	0.6784	0.0764
110	0.4031	0.1788	0.2861	0.0039	5.1797	0.2668	0.5628	0.0468
140	0.3877	0.1641	0.2754	0.0024	12.5820	0.2035	0.5035	0.0368
170	0.2299	0.1626	0.2752	-0.0170	3.6735	0.2098	0.5103	0.0267
200	0.1767	0.1728	0.2929	-0.0178	3.0965	0.2069	0.5223	0.0205
230	0.1275	0.1385	0.2475	-0.0119	0.8794	0.1609	0.4274	0.0196
260	0.0965	0.1403	0.2437	-0.0176	1.0936	0.1564	0.4093	0.0137
290	0.0655	0.1295	0.2317	-0.0157	0.3849	0.1395	0.3817	0.0123
320	0.0961	0.1258	0.2232	-0.0205	0.6157	0.1414	0.3825	0.0114
350	0.0534	0.1535	0.2686	-0.0233	0.5146	0.1616	0.4493	0.0110
380	0.0784	0.1101	0.2016	-0.0217	0.4129	0.1191	0.3350	0.0092
410	0.0424	0.1228	0.2215	-0.0189	0.1985	0.1358	0.3810	0.0087
440	0.0342	0.1226	0.2173	-0.0220	0.1404	0.1254	0.3446	0.0080
470	0.0171	0.1134	0.1970	-0.0170	0.1219	0.1161	0.3206	0.0074
500	0.0189	0.0966	0.1741	-0.0120	0.1268	0.0939	0.2664	0.0067

Table 10. Results of the simulation study of the OGWD

n	B_α	B_β	B_γ	B_λ	MSE_α	MSE_β	MSE_γ	MSE_λ
50	0.1773	0.2570	0.1552	0.0224	0.4619	2.0708	0.4426	0.0375
80	0.1151	0.2705	0.1313	0.0120	0.3046	2.1188	0.3314	0.0276
110	0.0458	0.3266	0.1724	-0.0059	0.2004	2.4368	0.4122	0.0229
140	0.0288	0.3166	0.1750	-0.0105	0.1420	2.3165	0.4314	0.0173
170	0.0529	0.5343	0.1058	-0.0048	0.1324	4.0888	0.2727	0.0141
200	0.0290	0.3470	0.1205	-0.0053	0.1053	2.6318	0.2765	0.0133
230	0.0048	0.3699	0.1133	-0.0128	0.0744	2.4810	0.2021	0.0119
260	0.0145	0.4165	0.1327	-0.0120	0.0857	3.1158	0.3052	0.0107
290	0.0091	0.4611	0.0898	-0.0131	0.0669	2.9620	0.2087	0.0096
320	0.0126	0.4128	0.0869	-0.0091	0.0617	2.5292	0.1813	0.0082
350	0.0183	0.4326	0.0805	-0.0081	0.0670	2.9847	0.1764	0.0079
380	0.0086	0.4451	0.0760	-0.0139	0.0544	2.3540	0.1933	0.0074
410	0.0131	0.4486	0.0730	-0.0120	0.0534	2.6086	0.1741	0.0063
440	0.0118	0.4213	0.0673	-0.0103	0.0510	2.3299	0.1398	0.0065
470	0.0204	0.4178	0.0563	-0.0103	0.0590	2.4302	0.1101	0.0052
500	3.0E-04	0.3995	0.0779	-0.0123	0.0463	2.4907	0.1463	0.0056

Table 11. Results of the simulation study of the OCWD

n	B_α	B_β	B_γ	B_λ	MSE_α	MSE_β	MSE_γ	MSE_λ
50	0.1940	0.0439	-0.0970	0.4826	0.3084	1.0220	0.2758	0.6992
80	0.1189	0.0037	-0.0723	0.3110	0.1368	0.3062	0.2041	0.3734
110	0.0737	0.0171	-0.0399	0.2045	0.0905	0.2496	0.1465	0.2200
140	0.0668	-0.0097	-0.0538	0.1961	0.0677	0.1062	0.1171	0.1995
170	0.0571	0.0225	-0.0412	0.1278	0.0601	0.1471	0.0875	0.1276
200	0.0585	-0.0039	-0.0457	0.1289	0.0519	0.0635	0.0831	0.1137
230	0.0466	-0.0071	-0.0304	0.1114	0.0446	0.0480	0.0730	0.0947
260	0.0348	0.0030	-0.0235	0.0833	0.0373	0.0381	0.0634	0.0728
290	0.0334	-0.0019	-0.0224	0.0797	0.0345	0.0336	0.0580	0.0657
320	0.0297	0.0032	-0.0228	0.0704	0.0299	0.0337	0.0510	0.0578
350	0.0320	0.0062	-0.0274	0.0654	0.0302	0.0330	0.0487	0.0530
380	0.0198	0.0074	-0.0133	0.0521	0.0254	0.0294	0.0430	0.0461
410	0.0210	8.0E-04	-0.0155	0.0561	0.0241	0.0238	0.0410	0.0457
440	0.0239	0.0039	-0.0182	0.0502	0.0238	0.0221	0.0402	0.0410
470	0.0238	-0.0049	-0.0204	0.0512	0.0207	0.0172	0.0348	0.0359
500	0.0207	-0.0029	-0.0172	0.0518	0.0202	0.0189	0.0363	0.0381

5. Model comparison and application

To conduct a robust comparison of models, we selected ten distinct families of distributions associated with odds ratios. Among these, five families comprise a single parameter, while the remaining five incorporate two parameters. Utilizing these families, we developed ten different models by employing the Weibull distribution as the baseline. Specifically, the first five models feature three parameters, whereas the latter five extend to four parameters.

For a meaningful and structured comparison, we evaluated the three-parameter and four-parameter models separately before performing an overall assessment across eight distinct data sets, as detailed in Table 12. The models are categorized as follows:

- Three-parameter models: OEWD, ORWD, OFWD, OHLWD, and OLWD.
- Four-parameter models: OWWD, OLmxWD, OBIIIWD, OGWD, and OCWD.

Table 12. Data sets used under study

Data set	Nature of the data	Description of the data	Source of data
I	Left-skewed	Strength of 1.5 cm glass fibers	[30]
II	Left-skewed	Survival time of blood cancer patients (in years)	[25]
III	Symmetric	Survival time of 20 acute myeloid leukemia patients	[24]
IV	Symmetric	Milk production of 107 cows	[26]
V	Bi-modal	Life-time of 20 electric components	[31]
VI	Tri-modal	Life-time of 30 items (censored)	[31]
VII	Right-skewed	Phosphorus concentration in leaves of 128 plants	[32]
VIII	Right-skewed	Measurement of 48 petroleum rock samples	[26]

Parameter estimation for all models was conducted using the MLE method via the `MaxLik` package [28] in R [29]. The estimated Parameters (Par) and their corresponding Standard Errors (SE) are systematically presented in Tables 13 to 20 for each data set.

To evaluate the comparative performance of the models, we employed multiple statistical criteria, including the log-likelihood function (-2log(L)), Akaike Information Criterion (AIC), Hannan-Quinn Information Criterion (HQIC), Kolmogorov-Smirnov (KS) statistic, Cramér-von Mises (CVM) statistic, and Anderson-Darling (AD) statistic, along with their respective p-values. The comprehensive model comparison results are provided in Tables 21 to 28 for all data sets under study.

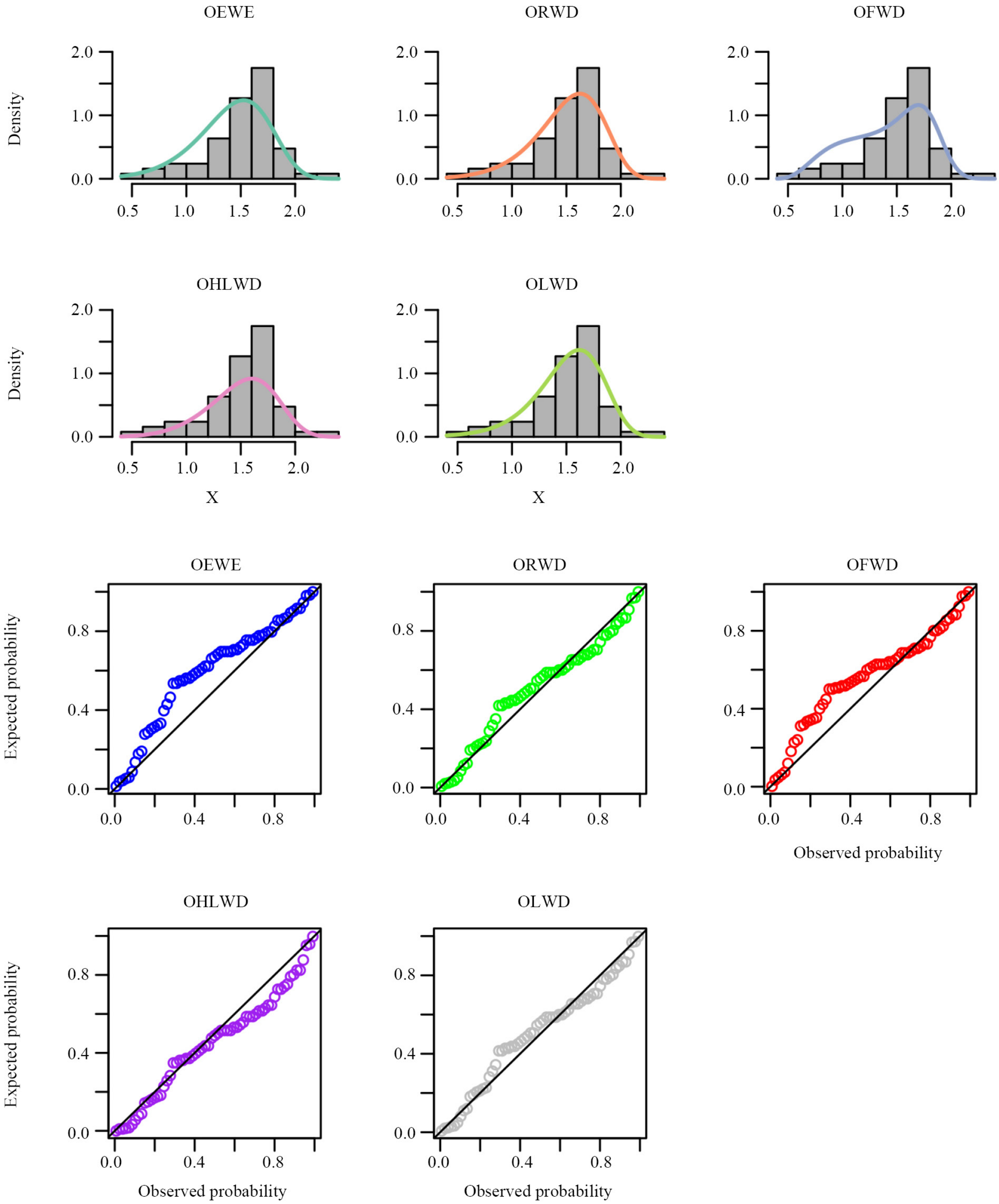

Figure 1. Fitted PDF and PP plots of the models having three parameters for data set I

Furthermore, we visually assessed the model fit using PDF plots and Probability-Probability (P-P) plots. The graphical representations for the three-parameter models are illustrated in Figures 1 to 8, while those for the four-parameter models are displayed in Figures 9 to 16. Finally, the best-performing models among the three-parameter and four-parameter categories for each data set are summarized in Table 29.

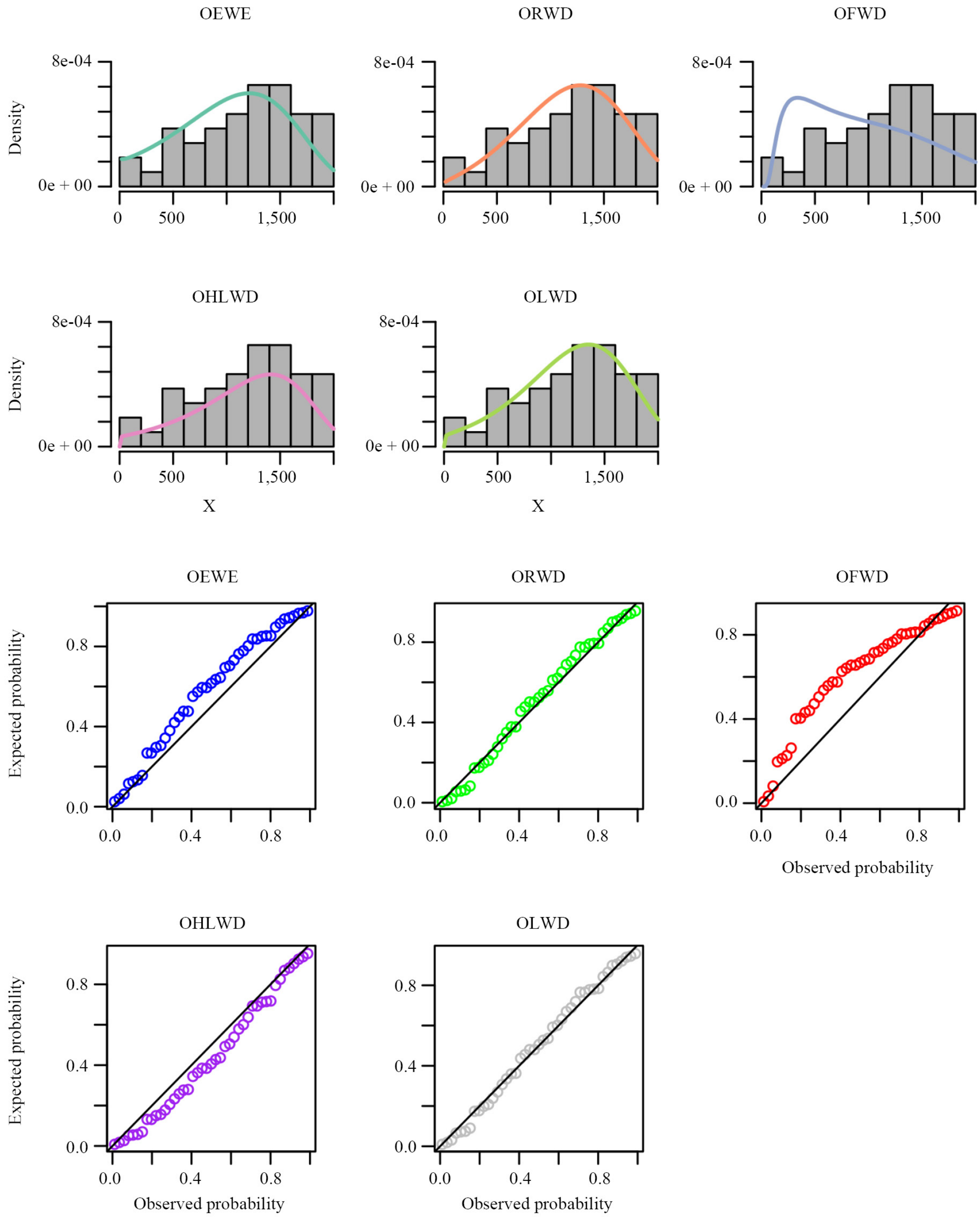

Figure 2. Fitted PDF and PP plots of the models having three parameters for data set II

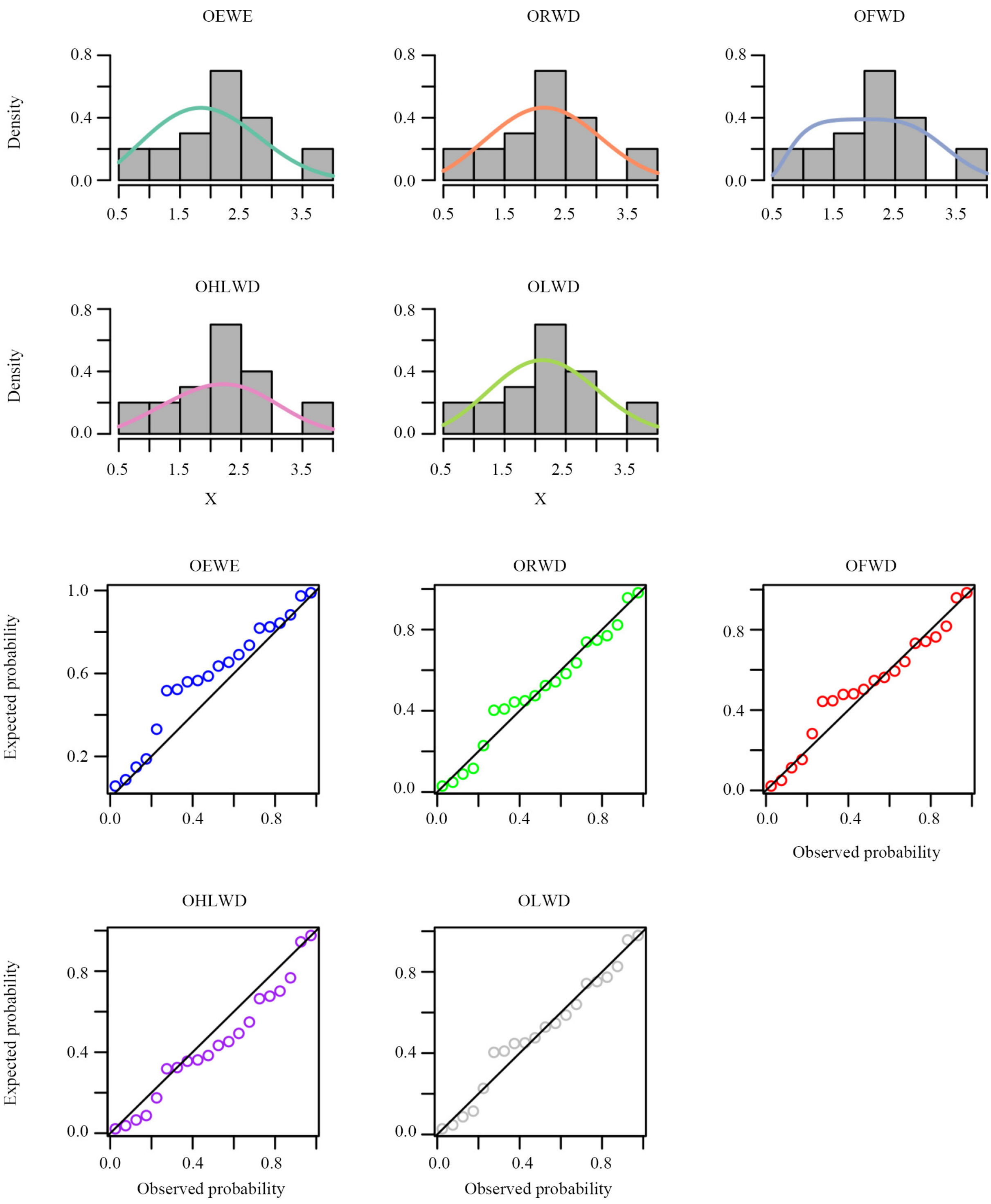

Figure 3. Fitted PDF and PP plots of the models having three parameters for data set III

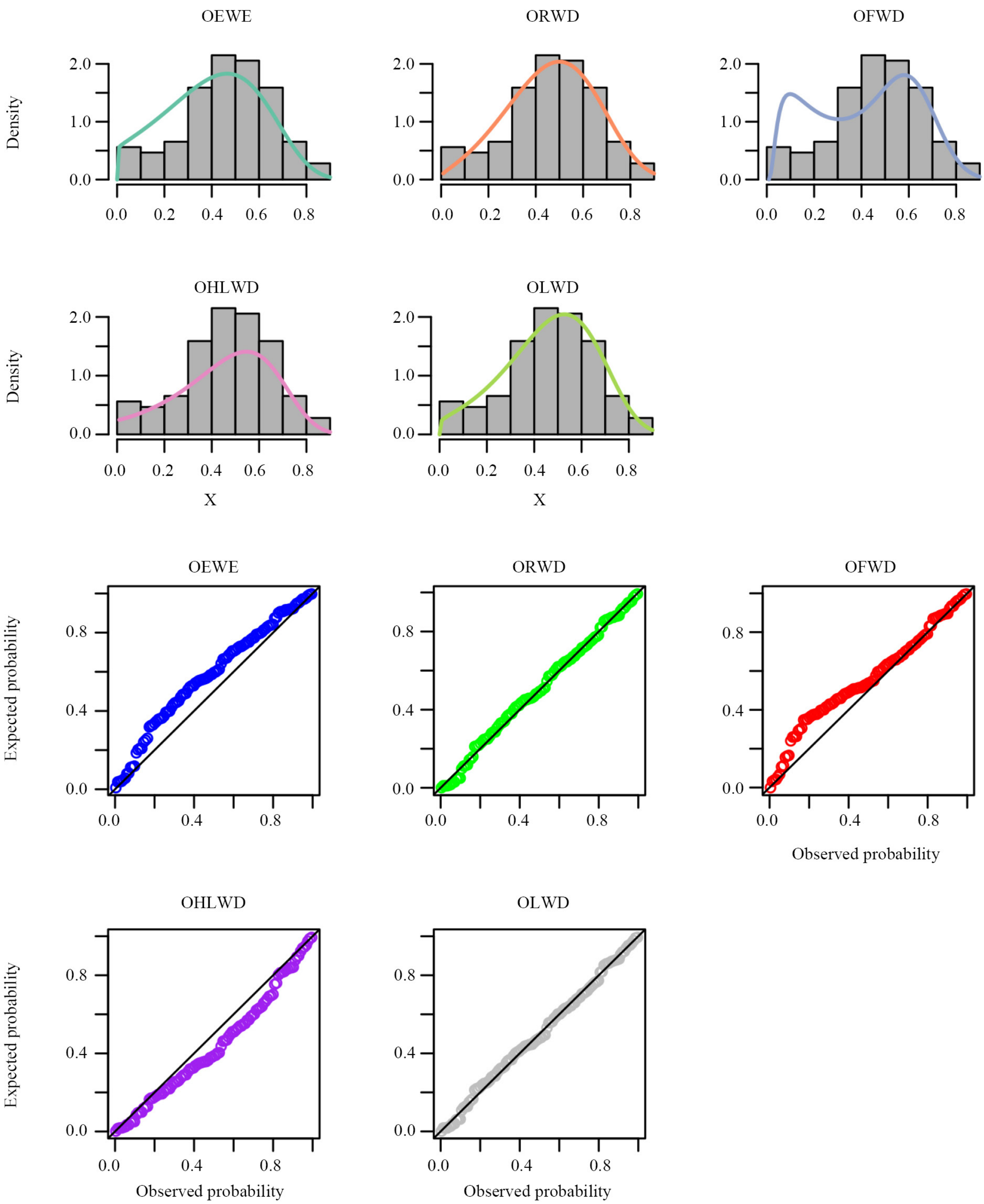

Figure 4. Fitted PDF and PP plots of the models having three parameters for data set IV

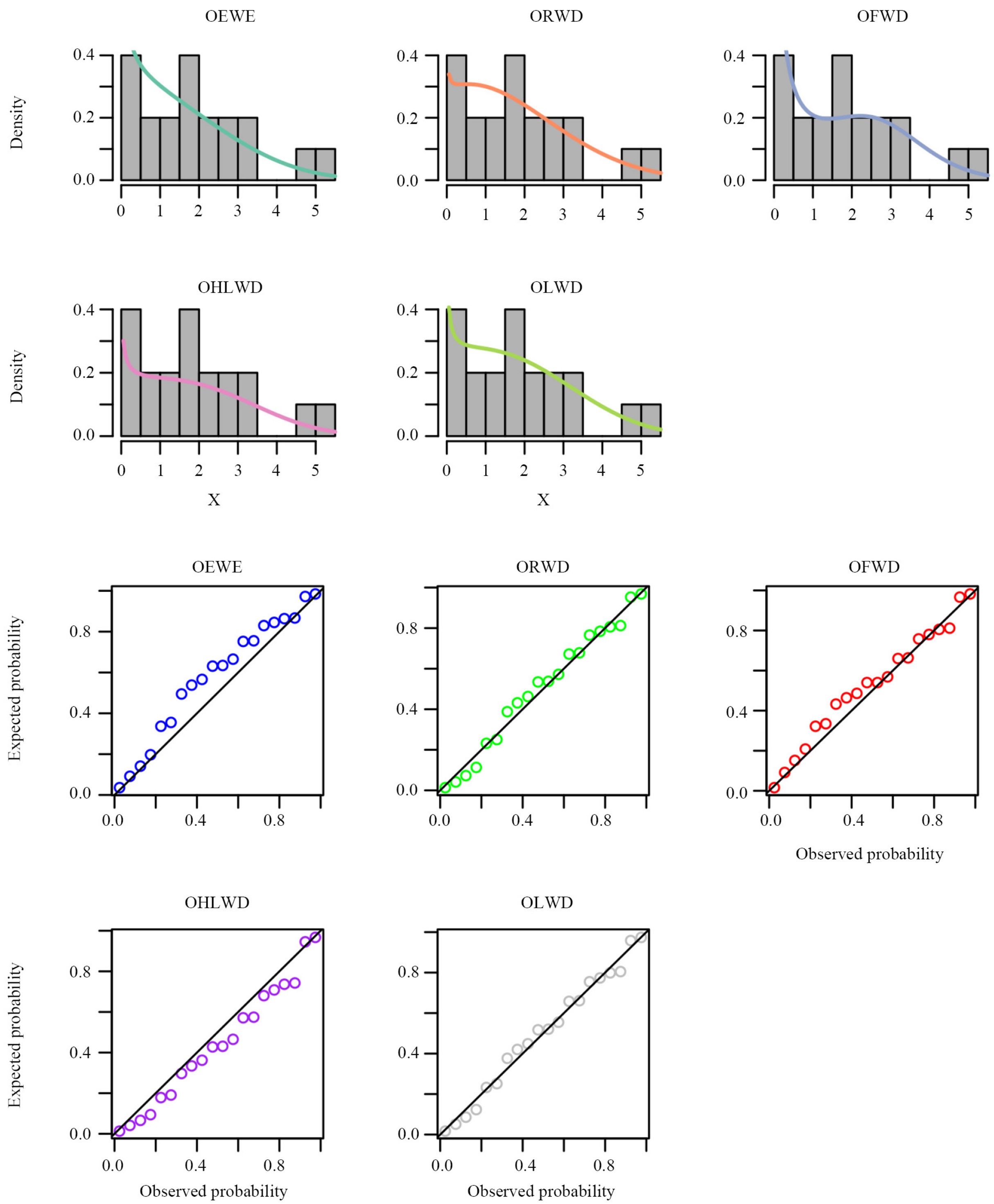

Figure 5. Fitted PDF and PP plots of the models having three parameters for data set V

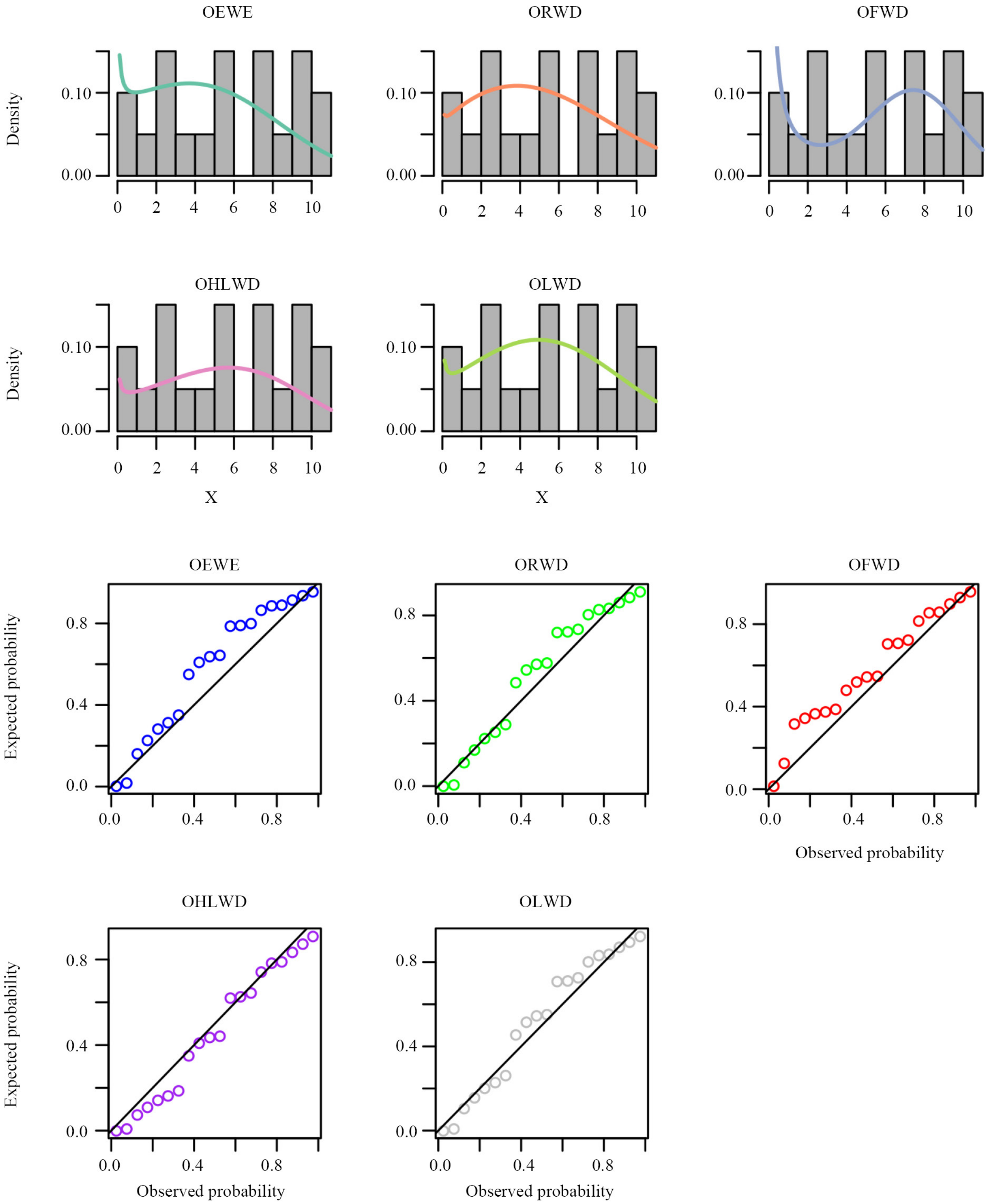

Figure 6. Fitted PDF and PP plots of the models having three parameters for data set VI

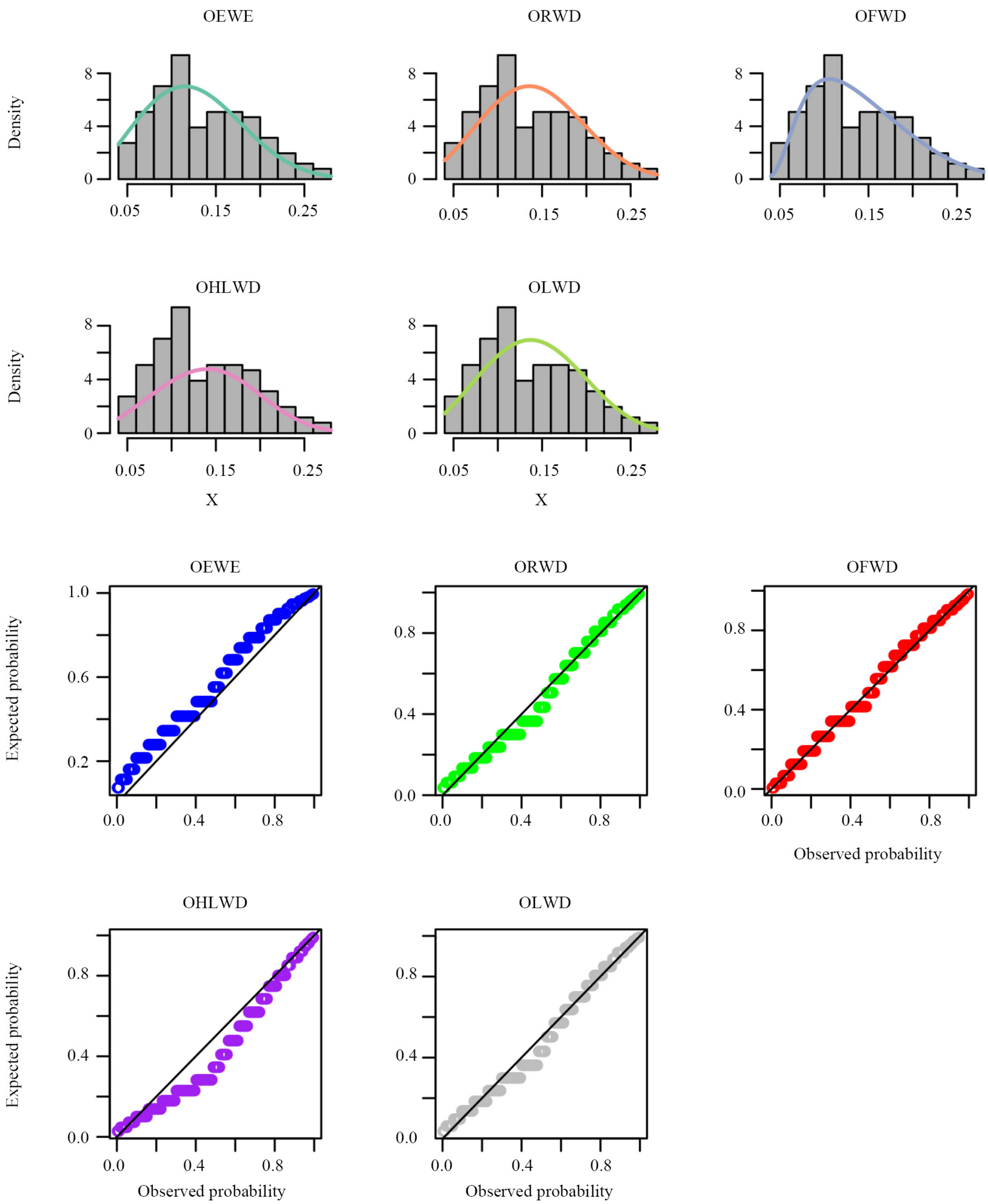

Figure 7. Fitted PDF and PP plots of the models having three parameters for data set VII

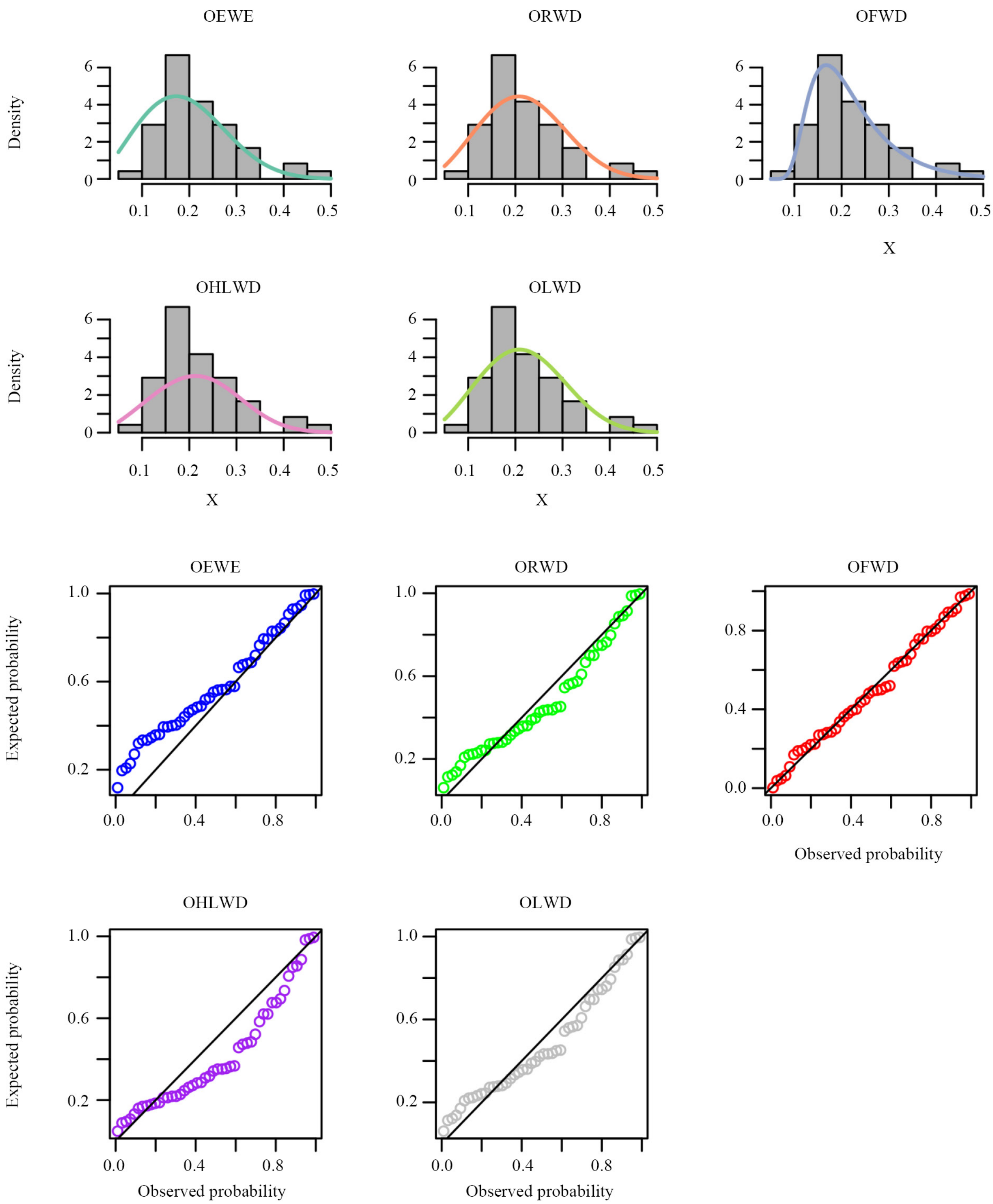

Figure 8. Fitted PDF and PP plots of the models having three parameters for data set VIII

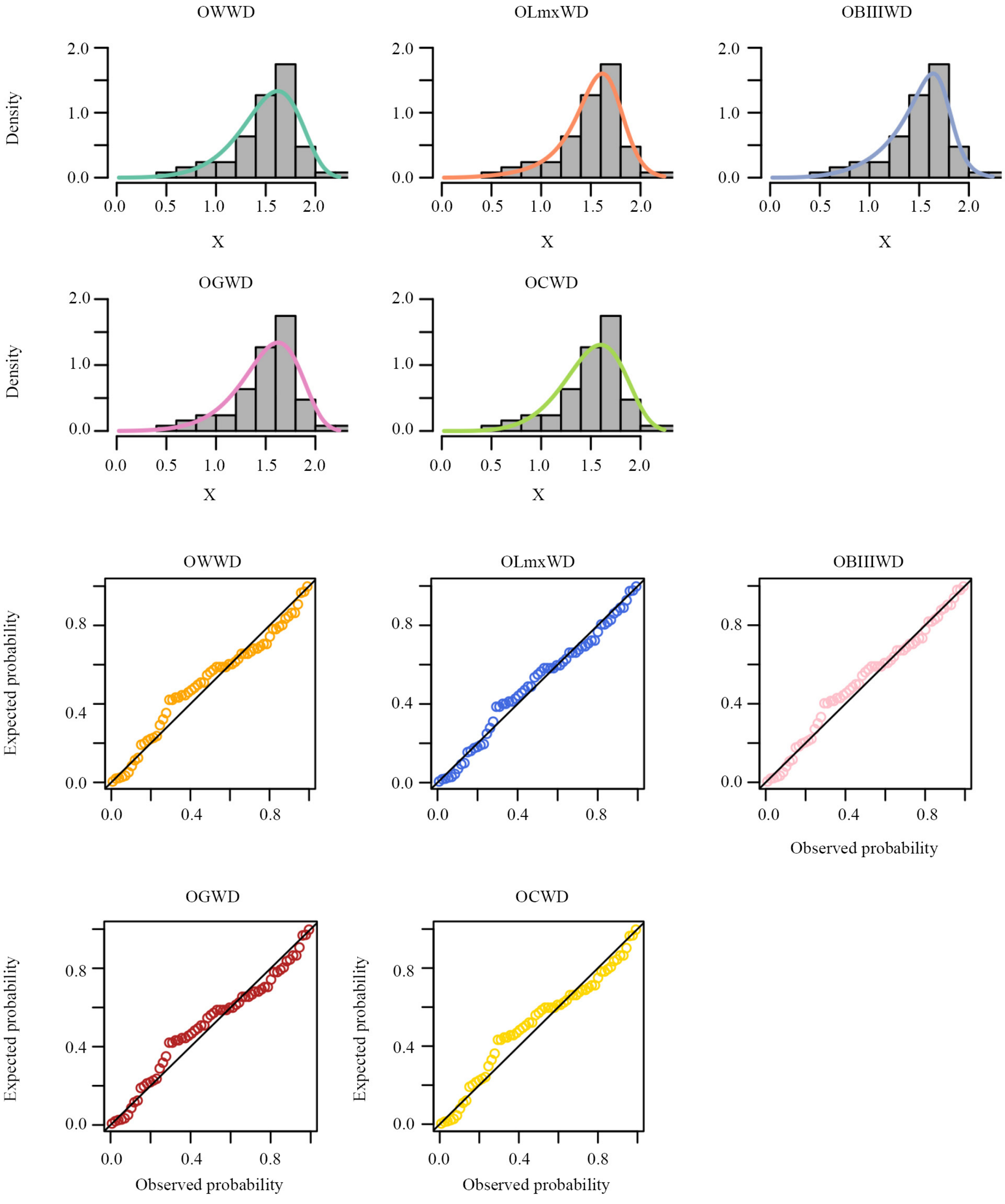

Figure 9. Fitted PDF and PP plots of the models having four parameters for data set I

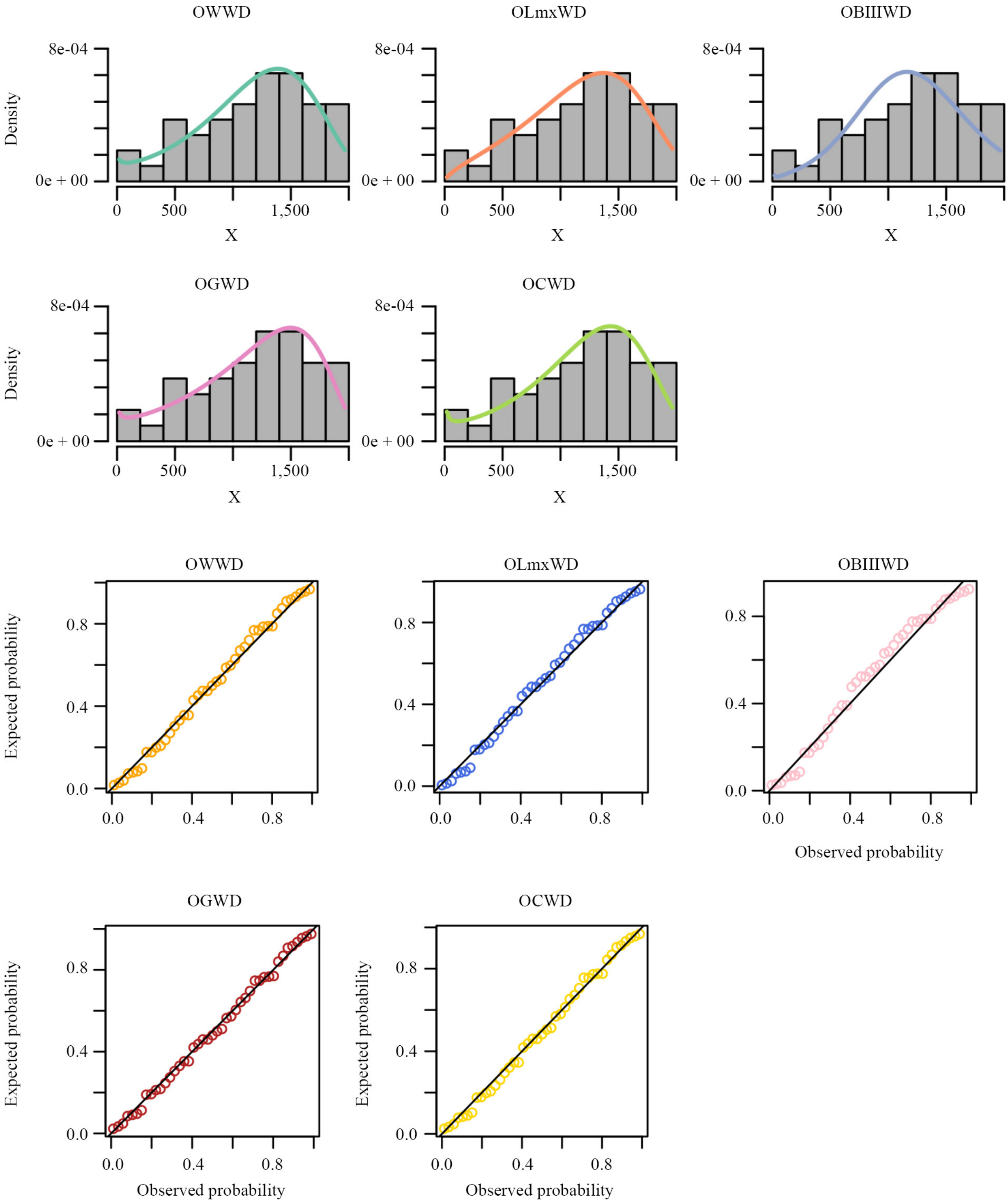

Figure 10. Fitted PDF and PP plots of the models having four parameters for data set II

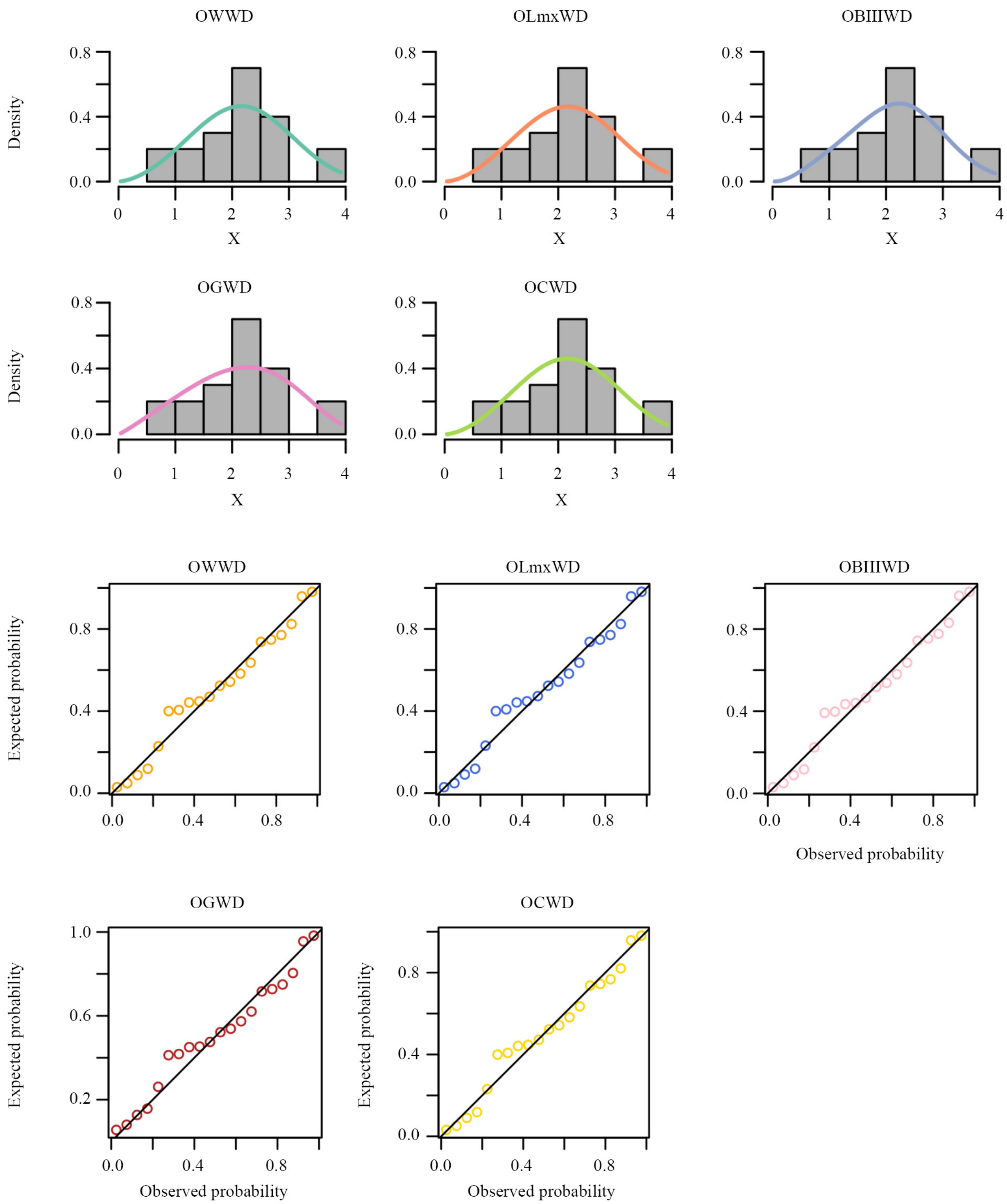

Figure 11. Fitted PDF and PP plots of the models having four parameters for data set III

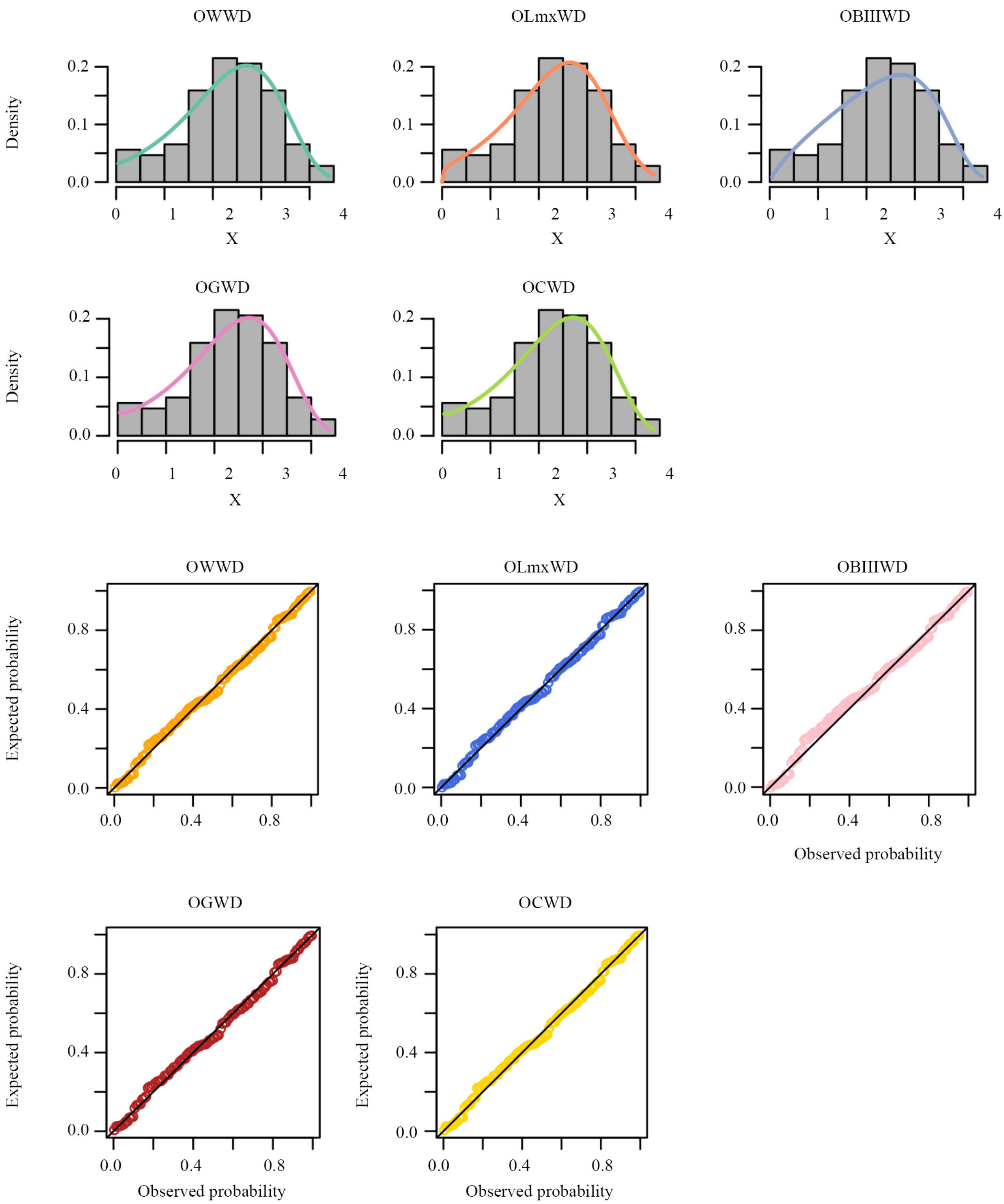

Figure 12. Fitted PDF and PP plots of the models having four parameters for data set IV

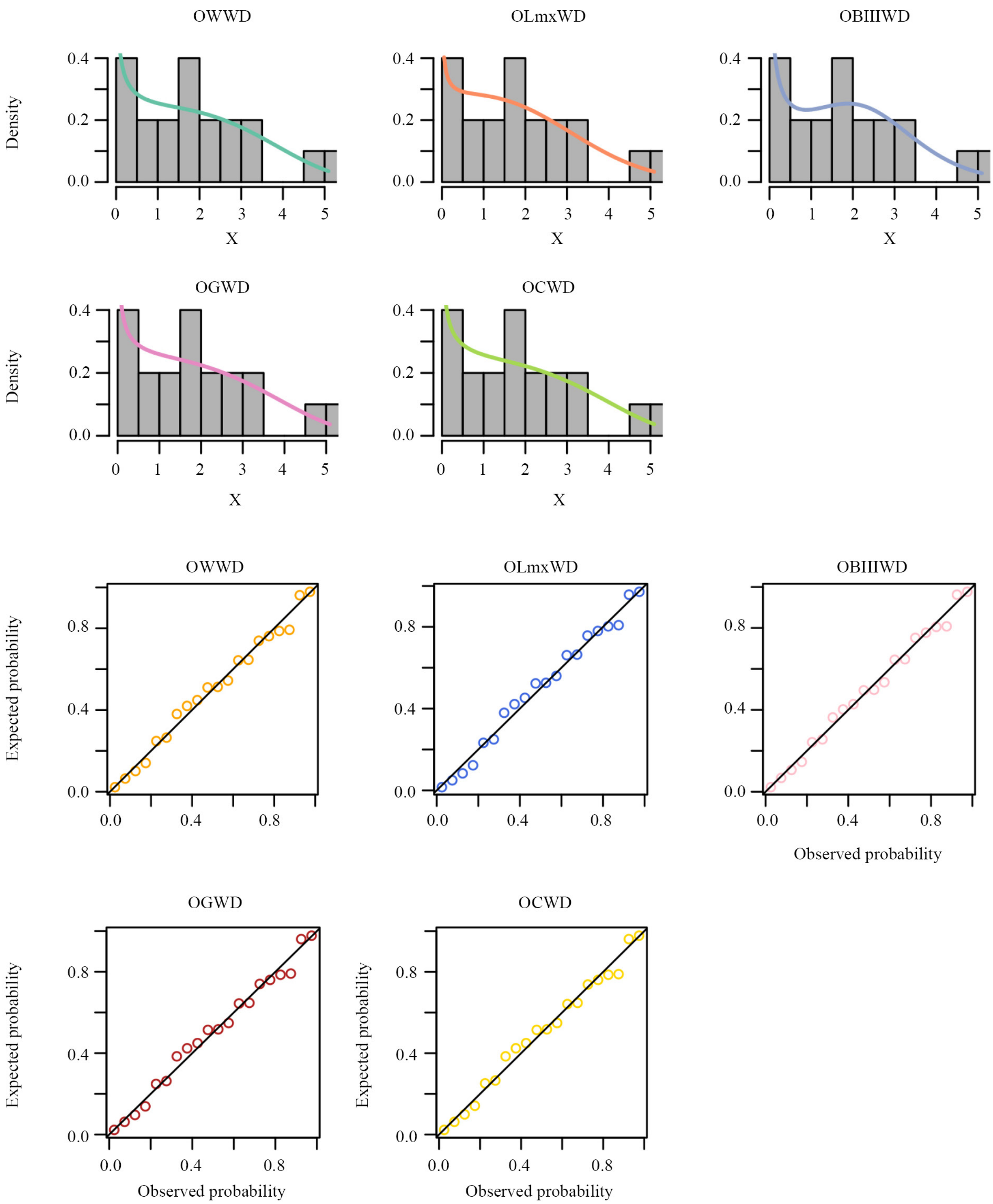

Figure 13. Fitted PDF and PP plots of the models having four parameters for data set V

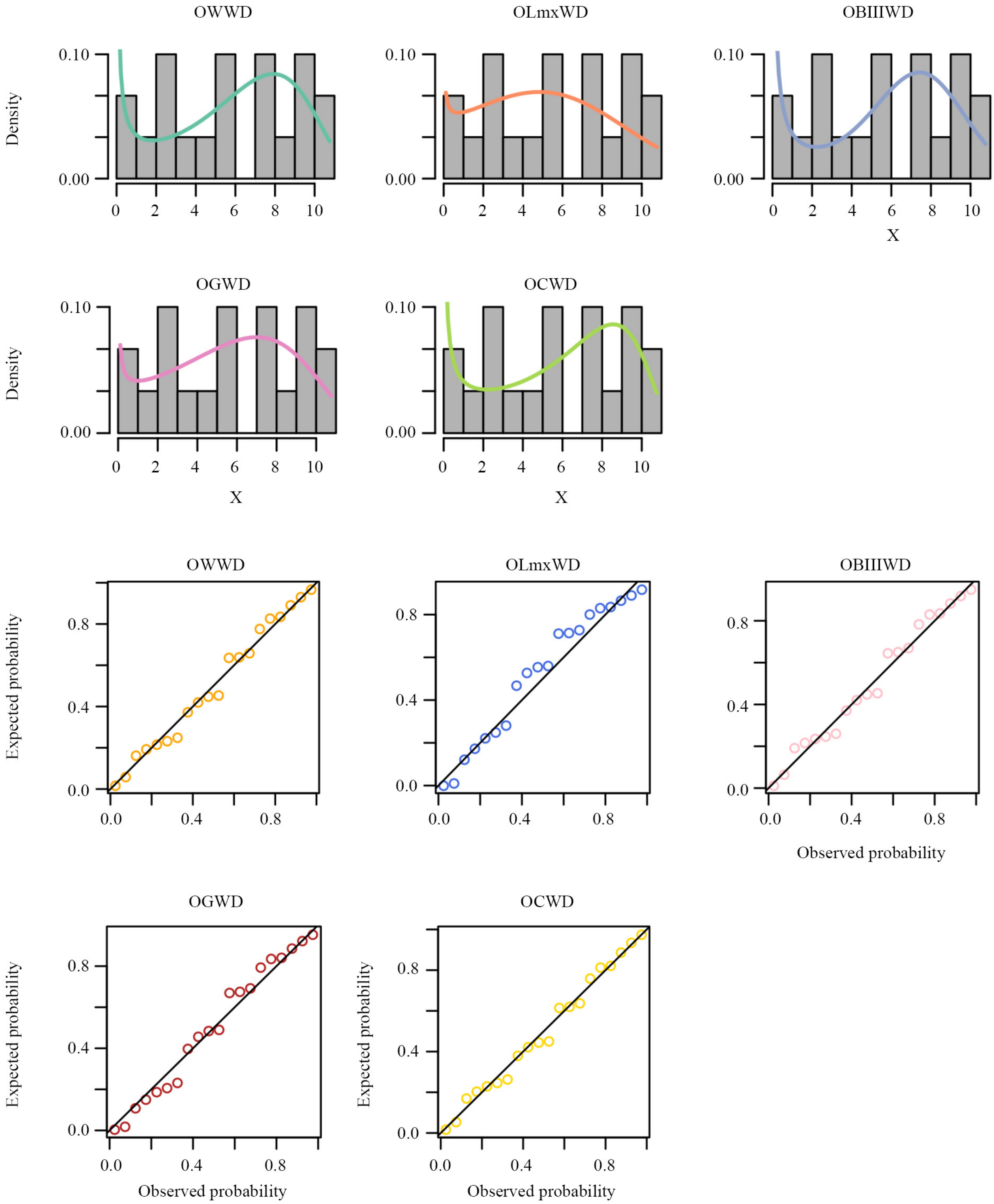

Figure 14. Fitted PDF and PP plots of the models having four parameters for data set VI

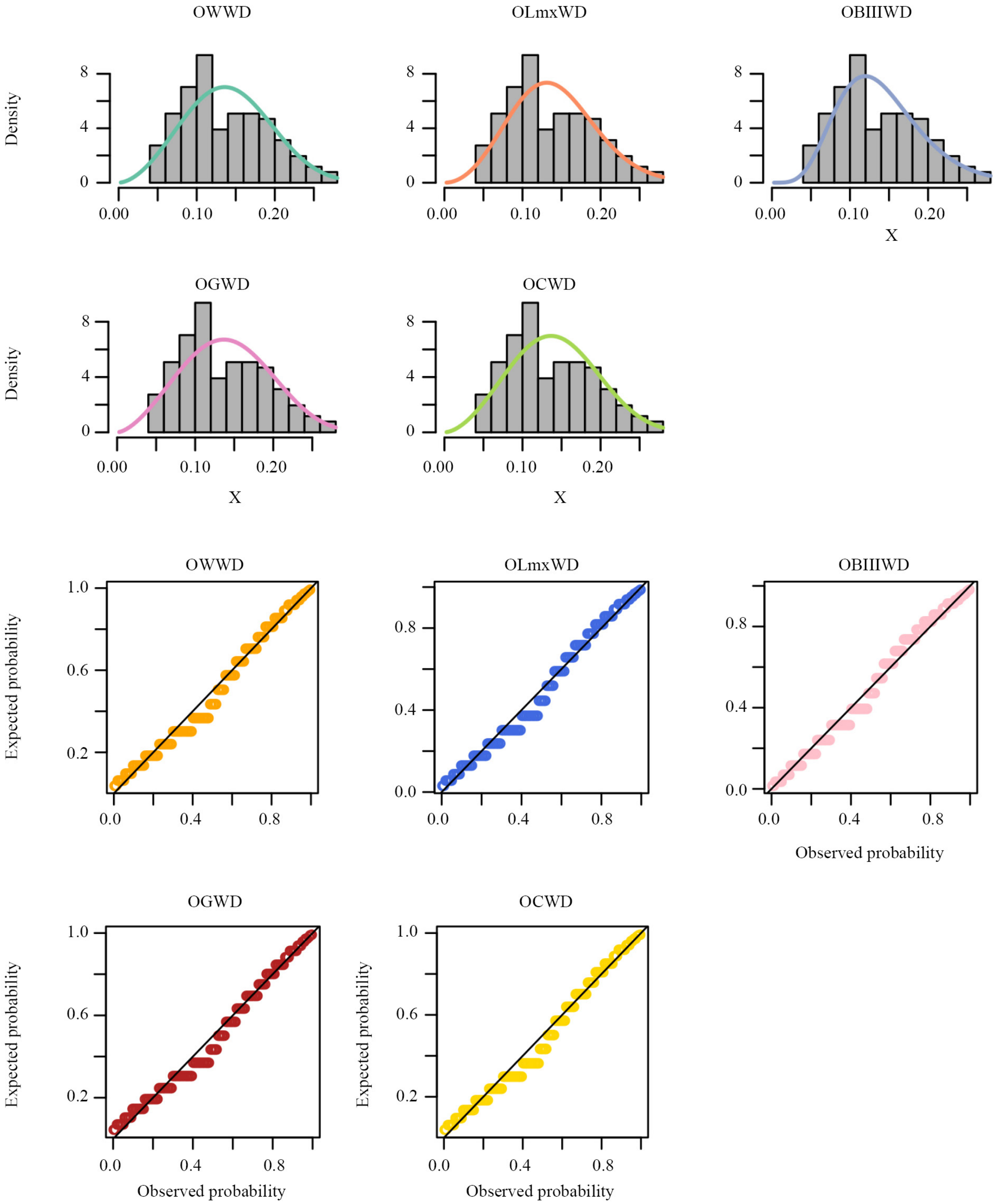

Figure 15. Fitted PDF and PP plots of the models having four parameters for data set VII

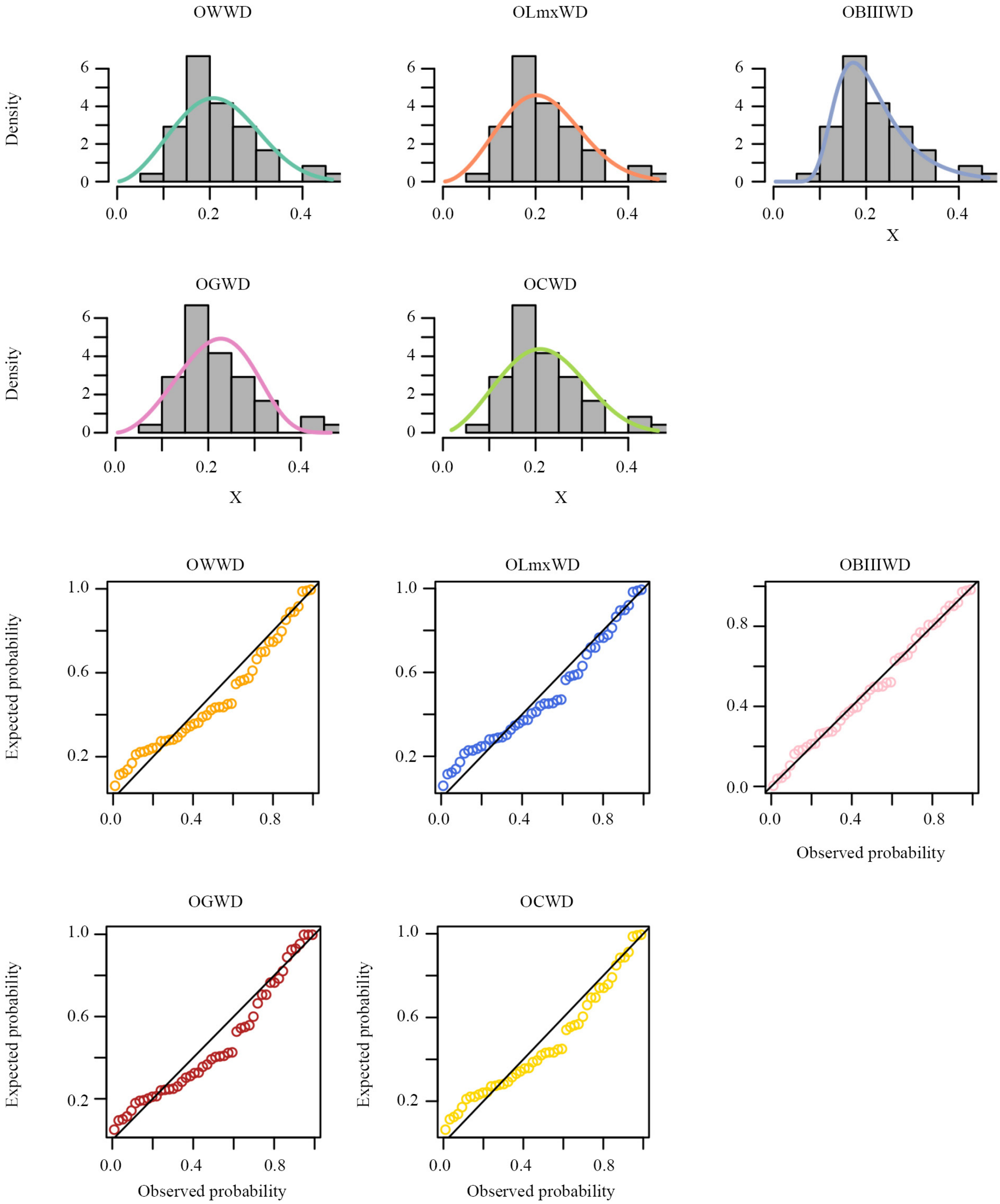

Figure 16. Fitted PDF and PP plots of the models having four parameters for data set VIII

Table 13. Estimated parameters with SE of the data set I

Model	Par	SE	Par	SE	Par	SE	Par	SE
OEWD (α, γ, λ)	3.11E-05	1.39E-05	8.3098	0.3472	0.5022	0.0476	-	-
ORWD (α, γ, λ)	74.2083	7.0899	3.4020	0.1646	0.6347	0.0654	-	-
OFWD (α, γ, λ)	0.2598	0.0487	0.0929	0.041	7.3202	0.873	-	-
OHLWD (α, γ, λ)	19.3795	7.1812	0.0049	0.0025	5.1905	0.5203	-	-
OLWD (α, γ, λ)	0.0456	0.0853	2.2545	1.7000	1.0791	0.5359	-	-
OWWD (α, β, γ, λ)	0.0028	0.0136	3.0785	2.5134	1.3646	1.0121	0.8196	0.1802
OLmxWD (α, β, γ, λ)	22.0552	11.0886	1.1917	0.5873	0.7773	0.2983	2.9814	0.5181
OBIIIWD (α, β, γ, λ)	6.4475	12.9271	0.2745	0.0586	0.2003	0.4455	2.1841	4.004
OGWD (α, β, γ, λ)	1.96E-05	8.05E-06	7.1182	11.2118	0.8906	0.8408	0.2914	0.0816
OCWD (α, β, γ, λ)	23.8507	25.5276	4.2377	2.5564	0.2239	0.1753	1.1059	0.7497

Table 14. Estimated parameters with SE of the data set II

Model	Par	SE	Par	SE	Par	SE	Par	SE
OEWD (α, γ, λ)	0.0911	0.0448	0.0028	0.0046	0.9474	0.1901	-	-
ORWD (α, γ, λ)	7.7906	4.6793	0.0464	0.0331	0.5506	0.0554	-	-
OFWD (α, γ, λ)	0.3897	0.067	1.01E-06	1.70E-07	2.061	0.0184	-	-
OHLWD (α, γ, λ)	0.0906	0.0591	0.0016	0.0019	1.0254	0.1459	-	-
OLWD (α, γ, λ)	0.3366	0.1159	7.00E-04	6.00E-04	1.0817	0.102	-	-
OWWD (α, β, γ, λ)	0.0279	0.036	0.8226	15.535	0.0068	0.0036	0.8922	2.4467
OLmxWD (α, β, γ, λ)	29.1202	2.1692	8.0000	9.1895	9.51e-06	1.27e-06	1.6591	0.0536
OBIIIWD (α, β, γ, λ)	0.007	0.0261	5.4149	1.0931	0.0062	0.0233	1.5142	0.2717
OGWD (α, β, γ, λ)	0.0311	0.0473	0.0465	0.1124	0.0227	0.0549	0.6727	0.2856
OCWD (α, β, γ, λ)	0.0116	0.0068	0.6557	0.2776	0.1078	0.0514	0.426	0.0784

Table 15. Estimated parameters with SE of the data set III

Model	Par	SE	Par	SE	Par	SE	Par	SE
OEWD (α, γ, λ)	20.0865	55.8432	0.0066	0.0185	2.5125	0.4023	-	-
ORWD (α, γ, λ)	0.0669	18.9165	0.0257	7.2876	1.3993	16.4408	-	-
OFWD (α, γ, λ)	0.3944	0.1471	0.1000	0.0809	3.4098	0.9302	-	-
OHLWD (α, γ, λ)	19.9081	19.8774	0.0056	0.0049	2.6326	0.4620	-	-
OLWD (α, γ, λ)	0.0011	0.0037	6.0068	3.3920	0.2637	0.1147	-	-
OWWD (α, β, γ, λ)	1.0374	12.1890	8.4586	12.8396	0.5511	0.6693	0.2512	0.4129
OLmxWD (α, β, γ, λ)	1.3472	3.0108	2.2812	12.2650	0.0453	0.2336	2.7799	1.1004
OBIIIWD (α, β, γ, λ)	0.5311	0.3958	3.1299	3.2916	0.4835	0.9677	2.1992	1.0939
OGWD (α, β, γ, λ)	1.4508	0.6779	3.31E-07	0.0322	0.0763	0.0210	2.1075	0.0435
OCWD (α, β, γ, λ)	19.3508	31.7510	18.2613	39.8376	0.5533	0.3193	0.1165	0.2702

Table 16. Estimated parameters with SE of the data set IV

Model	Par	SE	Par	SE	Par	SE	Par	SE
OEWD (α, γ, λ)	0.1521	0.3181	4.2365	1.9991	1.0434	0.6002	-	-
ORWD (α, γ, λ)	6.2983	5.8461	3.2644	0.8887	0.5752	0.1589	-	-
OFWD (α, γ, λ)	0.1717	0.0125	56.2615	0.3008	3.9842	0.1991	-	-
OHLWD (α, γ, λ)	0.0962	0.2392	4.7189	2.3405	0.9871	0.6552	-	-
OLWD (α, γ, λ)	0.4092	0.3329	3.3904	0.7436	1.1487	0.3952	-	-
OWWD (α, β, γ, λ)	0.0572	0.1294	1.0213	0.3888	5.1146	2.9924	0.9763	0.5674
OLmxWD (α, β, γ, λ)	46.9253	1.7722	4.9043	1.3099	5.2596	0.4951	1.2660	0.1277
OBIIIWD (α, β, γ, λ)	1.0333	0.1535	0.2561	0.0435	10.4940	3.00E-04	6.6527	0.0837
OGWD (α, β, γ, λ)	0.9743	2.4582	19.7814	6.8618	0.2421	0.0653	0.8797	0.6083
OCWD (α, β, γ, λ)	0.0538	0.1410	9.2718	4.6751	0.7872	0.0426	0.0733	0.0373

Table 17. Estimated parameters with SE of the data set V

Model	Par	SE	Par	SE	Par	SE	Par	SE
OEWD (α, γ, λ)	0.4372	1.0403	0.8144	1.2967	0.6532	0.4200	-	-
ORWD (α, γ, λ)	3.4634	6.6546	1.3881	1.4348	0.3139	0.1830	-	-
OFWD (α, γ, λ)	0.1955	0.0696	0.8030	0.4295	1.9947	0.5162	-	-
OHLWD (α, γ, λ)	0.3977	0.9372	0.8048	1.2678	0.6797	0.4284	-	-
OLWD (α, γ, λ)	0.8119	1.1595	0.6201	0.8226	0.7110	0.4128	-	-
OWWD (α, β, γ, λ)	0.4415	0.9826	0.7421	0.6364	0.6071	1.3235	0.9796	1.1661
OLmxWD (α, β, γ, λ)	18.9709	3.7676	5.5285	5.0269	0.8436	0.6120	0.7614	0.2161
OBIIIWD (α, β, γ, λ)	0.3109	0.1597	2.2124	1.2602	1.3865	1.5708	1.4533	0.5863
OGWD (α, β, γ, λ)	0.7977	13.7467	0.8032	34.9606	0.3432	6.1755	0.7121	0.5135
OCWD (α, β, γ, λ)	0.9327	1.4916	0.8078	1.7891	0.2407	0.8051	0.9051	2.3184

Table 18. Estimated parameters with SE of the data set VI

Model	Par	SE	Par	SE	Par	SE	Par	SE
OEWD (α, γ, λ)	0.0466	0.0746	1.3375	1.0672	0.4828	0.1973	-	-
ORWD (α, γ, λ)	10.4388	10.5436	1.6518	0.8056	0.2750	0.0930	-	-
OFWD (α, γ, λ)	0.0629	0.0162	0.0410	0.0164	2.9864	0.0163	-	-
OHLWD (α, γ, λ)	0.0393	0.0622	1.3424	1.0469	0.4961	0.1970	-	-
OLWD (α, γ, λ)	0.2697	0.2225	0.7085	0.5056	0.5720	0.1995	-	-
OWWD (α, β, γ, λ)	0.1824	0.1138	0.2191	0.1260	0.3996	0.4504	1.4780	0.4142
OLmxWD (α, β, γ, λ)	63.7165	3.0284	5.2349	2.6183	0.7668	0.3395	0.6611	0.1568
OBIIIWD (α, β, γ, λ)	0.0997	0.0394	2.1572	0.4241	0.0965	0.0062	2.5084	0.2289
OGWD (α, β, γ, λ)	0.0275	0.0528	0.1671	0.4157	1.2764	1.3253	0.3509	0.2115
OCWD (α, β, γ, λ)	0.1488	0.0891	0.2040	0.1291	0.1950	0.2509	1.4279	0.5833

Table 19. Estimated parameters with SE of the data set VII

Model	Par	SE	Par	SE	Par	SE	Par	SE
OEWD (α, γ, λ)	53.5756	2.3386	2.1675	0.6894	2.4395	0.1785	-	-
ORWD (α, γ, λ)	3.0E-04	1.0E-04	0.0051	0.0024	1.4087	0.0971	-	-
OFWD (α, γ, λ)	0.7079	0.0464	116.765	3.2142	2.355	0.0367	-	-
OHLWD (α, γ, λ)	44.9641	3.1571	3.0230	0.9021	2.5713	0.1749	-	-
OLWD (α, γ, λ)	25.8396	2.765	6.2275	2.7021	2.7552	0.2332	-	-
OWWD (α, β, γ, λ)	11.4019	3.7652	54.1807	3.1575	0.7195	0.0040	0.0379	0.0034
OLmxWD (α, β, γ, λ)	0.2143	0.6985	4.3507	7.7954	16.2114	3.5921	3.0728	0.6923
OBIIIWD (α, β, γ, λ)	1.0015	0.1961	3.7177	2.3397	35.7005	3.8154	1.4928	0.2498
OGWD (α, β, γ, λ)	10.7676	0.9207	1.7E-06	0.0111	10.7761	0.0028	2.6150	0.0114
OCWD (α, β, γ, λ)	52.4703	2.3505	59.2399	1.4584	0.7034	0.0029	0.0345	0.0022

Table 20. Estimated parameters with SE of the data set VIII

Model	Par	SE	Par	SE	Par	SE	Par	SE
OEWD (α, γ, λ)	49.3387	7.6150	0.7255	0.2311	2.3458	0.2487	-	-
ORWD (α, γ, λ)	2.0E-04	1.0E-04	0.0017	0.0010	1.3710	0.1476	-	-
OFWD (α, γ, λ)	1.2751	0.4622	11.9843	10.3647	1.6493	0.5158	-	-
OHLWD (α, γ, λ)	63.5042	12.2610	0.6305	0.1825	2.4878	0.2643	-	-
OLWD (α, γ, λ)	67.8443	5.1411	0.6679	0.2587	2.7130	0.2856	-	-
OWWD (α, β, γ, λ)	7.5960	6.7569	68.0849	3.6151	0.7068	0.0059	0.0293	0.0034
OLmxWD (α, β, γ, λ)	0.0016	0.0066	11.1546	1.8150	0.0089	0.0364	2.8604	0.2230
OBIIIWD (α, β, γ, λ)	1.8217	0.8993	10.9509	4.5683	7.1772	5.2821	0.8964	0.3034
OGWD (α, β, γ, λ)	3.3810	0.5045	4.65E-07	0.0250	13.7075	0.0107	2.8345	0.0071
OCWD (α, β, γ, λ)	28.5000	47.7069	0.2979	0.1408	3.7272	0.0077	9.0146	0.0072

Table 21. Evaluation of model selection criteria and goodness-of-fit statistics for data set I

Model	-2logL	AIC	HQIC	KS	K_p	CVM	C_p	AD	A_p
OEWD	85.1230	91.1230	93.6517	0.9997	0.0000	0.8927	0.0042	3.9678	0.0091
ORWD	28.6527	34.6527	37.1814	0.1347	0.2034	0.1669	0.3427	0.9256	0.3984
OFWD	35.5564	41.5564	44.0851	0.2153	0.0058	0.6033	0.0217	2.8192	0.0340
OHLWD	75.3221	81.3221	83.8508	0.1483	0.1254	0.2643	0.1710	1.7137	0.1329
OLWD	28.3851	34.3851	36.9138	0.1301	0.2366	0.1526	0.3824	0.8606	0.4389
OWWD	28.7343	36.7343	40.106	0.1368	0.1888	0.1725	0.3283	0.9592	0.3790
OLmxWD	23.9811	31.9811	35.3527	0.1013	0.5379	0.0815	0.6847	0.5338	0.7119
OBIIIWD	23.0705	31.0705	34.4421	0.1183	0.3416	0.0985	0.5940	0.5498	0.6960
OGWD	28.6767	36.6767	40.0483	0.1348	0.2026	0.1670	0.3424	0.9281	0.3969
OCWD	29.7362	37.7362	41.1079	0.1469	0.1318	0.2006	0.2666	1.1263	0.2971

Table 22. Evaluation of model selection criteria and goodness-of-fit statistics for data set II

Model	-2logL	AIC	HQIC	KS	K_p	CVM	C_p	AD	A_p
OEWD	690.1085	696.1085	698.0569	0.9776	0.0000	0.3521	0.0967	1.8614	0.1099
ORWD	653.0246	659.0246	660.9730	0.0800	0.9462	0.0499	0.8791	0.4449	0.8025
OFWD	667.6017	673.6017	675.5502	0.2399	0.0142	0.8606	0.0050	4.0142	0.0087
OHLWD	681.7668	687.7668	689.7152	0.1195	0.5712	0.1769	0.3180	1.0786	0.3181
OLWD	651.2878	657.2878	659.2362	0.0699	0.9845	0.0284	0.9821	0.2617	0.9637
OWWD	650.7744	658.7744	661.3723	0.0721	0.9788	0.0280	0.9835	0.2325	0.9788
OLmxWD	650.9407	658.9407	661.5387	0.0717	0.9800	0.0316	0.9721	0.3034	0.9352
OBIIIWD	661.0032	669.0032	671.6011	0.0823	0.9325	0.0730	0.7361	0.5407	0.7047
OGWD	649.2485	657.2485	659.8464	0.0502	0.9999	0.0167	0.9993	0.1414	0.9992
OCWD	650.8493	658.8493	661.4472	0.0590	0.9983	0.0220	0.9952	0.1847	0.9941

Table 23. Evaluation of model selection criteria and goodness-of-fit statistics for data set III

Model	-2logL	AIC	HQIC	KS	K_p	CVM	C_p	AD	A_p
OEWD	66.5670	72.5670	73.1502	0.9894	0.0000	0.2260	0.2229	1.0857	0.3144
ORWD	48.2434	54.2434	54.8265	0.1529	0.6826	0.0496	0.8843	0.3043	0.9342
OFWD	48.3576	54.3576	54.9408	0.1933	0.3931	0.0792	0.7018	0.4135	0.8335
OHLWD	62.9119	68.9119	69.4951	0.1560	0.6586	0.1279	0.4678	0.7105	0.5482
OLWD	48.3558	54.3558	54.9389	0.1542	0.6730	0.0495	0.8846	0.3044	0.9341
OWWD	48.2682	56.2682	57.0457	0.1512	0.6952	0.0487	0.8892	0.3003	0.9373
OLmxWD	48.2508	56.2508	57.0283	0.1524	0.6866	0.0493	0.8860	0.3017	0.9362
OBIIIWD	48.0149	56.0149	56.7924	0.1433	0.7542	0.0435	0.9189	0.2788	0.9528
OGWD	48.8456	56.8456	57.6231	0.1647	0.5930	0.0627	0.8026	0.3597	0.8863
OCWD	48.2539	56.2539	57.0314	0.1519	0.6904	0.0496	0.8841	0.3041	0.9344

Table 24. Evaluation of model selection criteria and goodness-of-fit statistics for data set IV

Model	-2logL	AIC	HQIC	KS	K_p	CVM	C_p	AD	A_p
OEWD	39.6903	45.6903	48.9409	0.9975	0.0000	0.9566	0.0030	4.7018	0.0040
ORWD	-55.6593	-49.6593	-46.4087	0.0512	0.9421	0.0401	0.9337	0.4209	0.8275
OFWD	-42.5230	-36.5230	-33.2724	0.1805	0.0019	0.6825	0.0139	3.4127	0.0170
OHLWD	20.2446	26.2446	29.4952	0.1269	0.0637	0.4893	0.0422	2.3841	0.0571
OLWD	-58.2888	-52.2888	-49.0382	0.0459	0.9778	0.0292	0.9792	0.2332	0.9785
OWWD	-58.2558	-50.2558	-45.9217	0.0508	0.9452	0.0351	0.9572	0.2367	0.9769
OLmxWD	-58.1956	-50.1956	-45.8615	0.0453	0.9805	0.0277	0.9834	0.2334	0.9785
OBIIIWD	-55.4483	-47.4483	-43.1142	0.0732	0.6144	0.0830	0.6755	0.5296	0.7164
OGWD	-58.0723	-50.0723	-45.7381	0.0527	0.9278	0.0387	0.9404	0.2464	0.9722
OCWD	-58.1430	-50.1430	-45.8089	0.0516	0.9382	0.0365	0.9508	0.2377	0.9765

Table 25. Evaluation of model selection criteria and goodness-of-fit statistics for data set V

Model	-2logL	AIC	HQIC	KS	K_p	CVM	C_p	AD	A_p
OEWD	81.3283	87.3283	87.9114	0.9847	0.0000	0.1900	0.2892	0.9528	0.3818
ORWD	63.6664	69.6664	70.2495	0.088 0	0.9939	0.0340	0.9646	0.2543	0.9679
OFWD	62.9089	68.9089	69.4920	0.1335	0.8230	0.0573	0.8367	0.3407	0.9037
OHLWD	77.5987	83.5987	84.1819	0.1555	0.6626	0.0966	0.6066	0.5872	0.6580
OLWD	63.0342	69.0342	69.6174	0.0941	0.9870	0.0259	0.9894	0.1994	0.9907
OWWD	62.8297	70.8297	71.6073	0.1070	0.9575	0.0261	0.9891	0.1933	0.9923
OLmxWD	63.1432	71.1432	71.9207	0.0912	0.9908	0.0270	0.9870	0.2059	0.9889
OBIIIWD	62.4409	70.4409	71.2184	0.0897	0.9924	0.0191	0.9984	0.1558	0.9983
OGWD	62.8639	70.8639	71.6415	0.1072	0.9568	0.0269	0.9874	0.1976	0.9912
OCWD	62.8553	70.8553	71.6328	0.1092	0.9501	0.0276	0.9855	0.2008	0.9903

Table 26. Evaluation of model selection criteria and goodness-of-fit statistics for data set VI

Model	-2logL	AIC	HQIC	KS	K_p	CVM	C_p	AD	A_p
OEWD	114.1795	120.1795	120.7627	0.9973	0.0000	0.2515	0.1866	1.4664	0.1848
ORWD	100.6068	106.6068	107.1899	0.1705	0.5498	0.0998	0.5904	0.9103	0.4066
OFWD	94.9636	100.9636	101.5467	0.2178	0.2588	0.1771	0.3184	0.9281	0.3960
OHLWD	110.3733	116.3733	116.9564	0.1612	0.6196	0.0755	0.7235	0.8312	0.4575
OLWD	97.4502	103.4502	104.0333	0.1599	0.6295	0.0787	0.7047	0.7179	0.5421
OWWD	90.3738	98.3738	99.1513	0.1008	0.9745	0.0301	0.9785	0.1857	0.9940
OLmxWD	97.9024	105.9024	106.6799	0.1624	0.6101	0.0821	0.6852	0.7228	0.5381
OBIIIWD	92.6753	100.6753	101.4528	0.0964	0.9834	0.0344	0.9630	0.2274	0.9811
OGWD	92.5696	100.5696	101.3472	0.1219	0.8931	0.0479	0.8944	0.4205	0.8264
OCWD	89.7441	97.7441	98.5216	0.0990	0.9783	0.0244	0.9923	0.1566	0.9982

Table 27. Evaluation of model selection criteria and goodness-of-fit statistics for data set VII

Model	-2logL	AIC	HQIC	KS	K_p	CVM	C_p	AD	A_p
OEWD	-269.1480	-263.1480	-259.6720	0.9958	0.0000	0.7651	0.0088	5.0387	0.0028
ORWD	-389.6270	-383.6270	-380.1510	0.1178	0.0574	0.1855	0.2977	1.0572	0.3285
OFWD	-398.8600	-392.8600	-389.3840	0.0674	0.6057	0.0809	0.6876	0.4800	0.7670
OHLWD	-293.6740	-287.6740	-284.1980	0.1992	1.0E-04	1.0023	0.0024	4.8518	0.0034
OLWD	-389.3680	-383.3680	-379.8920	0.1194	0.0521	0.1916	0.2845	1.0923	0.3121
OWWD	-389.4210	-381.4210	-376.7860	0.1164	0.0622	0.1816	0.3064	1.0542	0.3299
OLmxWD	-390.1260	-382.1260	-377.4910	0.1110	0.0851	0.1687	0.3376	0.9592	0.3792
OBIIIWD	-393.4340	-385.4340	-380.7990	0.0897	0.2542	0.1384	0.4274	0.7860	0.4910
OGWD	-388.7790	-380.7790	-376.1440	0.1136	0.0735	0.1717	0.3301	1.0460	0.3338
OCWD	-389.3320	-381.3320	-376.6970	0.1187	0.0544	0.1891	0.2898	1.0819	0.3169

Table 28. Evaluation of model selection criteria and goodness-of-fit statistics for data set VIII

Model	-2logL	AIC	HQIC	KS	K_p	CVM	C_p	AD	A_p
OEWD	-59.8117	-53.8117	-51.6903	0.9981	0.0000	0.4663	0.0481	2.9070	0.0307
ORWD	-105.4820	-99.4815	-97.3601	0.1493	0.2350	0.1938	0.2804	1.2279	0.2570
OFWD	-116.6910	-110.6910	-108.5700	0.0845	0.8834	0.0303	0.9762	0.2066	0.9885
OHLWD	-68.5470	-62.5470	-60.4256	0.2349	0.0100	0.6207	0.0195	2.9740	0.0284
OLWD	-105.1620	-99.1622	-97.0408	0.1510	0.2235	0.2015	0.2650	1.2684	0.2427
OWWD	-105.3310	-97.3306	-94.5021	0.1500	0.2306	0.1962	0.2754	1.2418	0.2520
OLmxWD	-107.1520	-99.1523	-96.3238	0.1327	0.3661	0.1500	0.3906	1.0181	0.3475
OBIIIWD	-116.7310	-108.7310	-105.9020	0.0825	0.8994	0.0280	0.9833	0.1839	0.9942
OGWD	-87.1371	-79.1371	-76.3086	0.1781	0.0951	0.2529	0.1848	2.1033	0.0809
OCWD	-104.7450	-96.7453	-93.9168	0.1537	0.2070	0.2112	0.2472	1.3186	0.2263

5.1 *Result of model comparison*

In this subsection, we present the comparison results separately for the three-parameter and four-parameter models, along with an overall comparison. The final results are detailed in Table 29 and can be interpreted as follows

• The **4-parameter models** demonstrate greater flexibility compared to the **3-parameter models** in the overall comparison.

• The **best overall model** is **OGWD (4-parameter)** for **data set II** (p-value = 0.9999), which represents **moderately left-skewed data**. The next best model is **OBIIIWD (4-parameter)** for the same data set (p-value = 0.9983).

• Among **3-parameter models**, **OLWD** is identified as the best model for **data sets I, II, IV, V, and VI**.

• All models appear to perform poorly for **right-skewed data**. However, **OFWD (3-parameter)** performs better compared to the 4-parameter models for this type of data.

• For **left-skewed and symmetrical data**, **OBIIIWD** outperforms all competing models.

• Hence, the best model among all ten models is **OBIIIWD**, making the **OBIII-G family** the best-performing family in this study.

Table 29. Best models in overall comparison

Data Set	Model (3-par)	AIC	KS (p)	Model (4-par)	AIC	KS (p)
I	OLWD	34.3851	0.1301 (0.2366)	OlmxWD	31.9811	0.1013 (0.5379)
II	OLWD	657.2878	0.0699 (0.9845)	OGWD	657.2485	0.0502 (0.9999)
III	ORWD	54.2434	0.1529 (0.6826)	OBIIIWD	56.0149	0.1433 (0.7542)
IV	OLWD	-52.2888	0.0459 (0.9778)	OlmxWD	-50.1956	0.0453 (0.9805)
V	OLWD	69.0342	0.0941 (0.9870)	OBIIIWD	70.4409	0.0897 (0.9924)
VI	OLWD	103.4502	0.1599 (0.6295)	OCWD	97.7441	0.0990 (0.9783)
VII	OFWD	-392.8600	0.0674 (0.6057)	OBIIIWD	-385.4340	0.0897 (0.2542)
VIII	OFWD	-110.6910	0.0845 (0.8834)	OBIIIWD	-108.7310	0.0825 (0.8994)

6. Conclusion

This study conducted a comprehensive comparison of ten distributions derived from ten distinct Odd-G families across eight different types of data sets, including left-skewed, right-skewed, bi-modal, tri-modal, and symmetrical. The findings revealed that among the three-parameter models, the Odd Lindley Weibull Distribution (OLWD) exhibited superior flexibility, whereas the Odd Burr-III Weibull Distribution (OBIIIWD) emerged as the most adaptable among the four-parameter models. Consequently, the Odd Lindley-G (OL-G) and Odd Burr-III-G (OBIII-G) families of distributions were identified as the most effective among the families examined in this study.

Furthermore, the Odd-G family of distributions demonstrated strong modeling capabilities for moderately skewed (left or right), symmetrical, bi-modal, and tri-modal data. However, the study also found that this family is not particularly suitable for highly left-skewed or right-skewed data. Based on these findings, we recommend the OL-G family (single-parameter) and the OBIII-G family (two-parameter) as viable choices for modeling real-world data.

Future Directions: Upcoming studies can further investigate the robustness of the Odd-G family by utilizing alternative baseline distributions, such as the Rayleigh, Lindley, and generalized exponential distributions, to evaluate how these choices affect model performance. Moreover, future research could incorporate and compare other flexible distribution families, including the Marshall-Olkin-Weibull-H family [11], the New Lomax-G family of distributions [33], the weighted Lindley-G family [12], A new family using the delta power transformation [34], and the modified Kies-Kavya-Manoharan Family (MKKM-F) [35]. Applying these models to real-world datasets from various domains, such as finance, healthcare, and environmental science, could further validate their practical utility. Ultimately, adopting Bayesian estimation methods and machine learning-based model selection techniques may enhance both the accuracy and interpretability of these models in practical applications.

Limitations: One limitation of this study is the selection of the Weibull distribution as the base distribution for constructing and evaluating the Odd-G family of distributions. While Weibull is a widely used and flexible choice, alternative base distributions could yield different comparative results. Therefore, the generalizability of our findings may be constrained by the choice of the base distribution.

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Availability of data and materials

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Conflict of interest

The authors declare no competing financial interest.

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