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Research Article

# On the Mathematical Analysis of Generalized Quantum-Nabla Fractional Fluid Models with Dissipative Nonlinearities

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**Abstract:** We investigate a nonlinear fluid system governed by the generalized quantum-Caputo nabla fractional operator, capturing nonlocal memory effects in velocity, shear stress, and fluidity. The system is formulated with polynomial nonlinearities and modeled over the unit disk. We establish a general existence and uniqueness theorem for mild solutions in the function spaces $H^1(\mathbb{D})^3$, $H^2(\mathbb{D})^3$, and $\ell^\infty(\mathbb{D})^3$, based on fixed-point theory and the integral representation of the fractional operators. Under mild dissipativity assumptions, we prove boundedness and asymptotic stability using generalized $(q, \tau)$-Mittag-Leffler decay. Furthermore, we present illustrative examples for each functional space and validate the theoretical results with numerical simulations. The findings provide a rigorous and flexible framework for modeling fractional fluid dynamics with memory-driven dissipation.

***Keywords***: fractional calculus, nabla operator, fluid dynamics, univalent solution, open unit disk, convex function, nonlinear system, Mittag-Leffler stability

**MSC:** 26A33, 35B35

## 1. Introduction

Fractional calculus has emerged as a powerful tool for modeling complex systems with memory, hereditary, and anomalous transport properties [1–5]. Among various fractional approaches, the recently developed Caputo nabla operator has gained attention for its ability to unify discrete and continuous memory effects [6–10] through a two-parameter deformation, making it particularly suitable for describing nonlocal dynamics in physical systems [11–14].

Quantum fractional calculus provides a powerful mathematical framework that unifies the concepts of fractional derivatives and quantum q-difference operators, enabling accurate modeling of systems with memory, nonlocality, and multiscale dynamics. The deformation parameters $(q, \tau)$ offer additional flexibility, allowing smooth transitions between continuous and discrete behaviors while capturing long-term memory effects. This framework improves numerical stability, enhances approximation accuracy for complex and fractal systems, and has significant applications in physics, engineering, and quantum models where classical fractional methods are insufficient [15–18].


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In this work, we propose and analyze a nonlinear fluid system [19–21] governed by the $(q, \tau)$-fractional Caputo nabla derivative, which is a generalization of the fractional Caputo nabla derivative [22–24]. The model describes the evolution of three interconnected physical quantities: the velocity $V(t, z)$, the shear stress $S(t, z)$, and the fluidity $\phi(t, z)$, where $t \in [0, T]$ denotes time and $z$ belongs to the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The disk geometry arises naturally in problems involving radial symmetry [25, 26], microscale confined fluids, and soft material mechanics. The mathematical formulation of the system consists of a set of three nonlinear partial differential equations with $(q, \tau)$-fractional time derivatives and polynomial nonlinearities. The memory structure introduced by the $(q, \tau)$-fractional derivative captures dissipation and history dependence, making the model suitable for viscoelastic or complex fluid media.

With the use of Caputo-type $(q, \tau)$-nabla operators, the current study presents a unique $(q, \tau)$-fractional fluid framework defined on the open unit disk $\mathbb{D}$. This linked nonlinear system governs fluidity, shear stress, and velocity. The following succinctly describes the main contributions of this study:

1. **Theoretical innovation:** We establish new existence, uniqueness, boundedness, and stability theorems for outcomes in multiple functional spaces, including $H^1(\mathbb{D})^3$, $H^2(\mathbb{D})^3$, and $\ell^\infty(\mathbb{D})^3$, which, regarding such $(q, \tau)$-fractional PDEs, have not been investigated earlier.

2. **Methodological advancements:** By reformulating the system as nonlinear Volterra-type integral equations with weakly singular $(q, \tau)$-kernels, we develop a rigorous fixed-point framework combined with generalized Grönwall-type inequalities based on the $(q, \tau)$ Gamma function.

3. **Original outcomes:** We provide analytic conditions ensuring dissipativity, convexity, and asymptotic stability of solutions, supported by admissible univalent initial data choices and functional-analvtic estimates.

4. **Physical relevance:** The framework captures memory-dependent, nonlocal, and dissipative behaviors relevant to viscoelastic and non-Newtonian fluids within the geometry of the unit disk.

This work establishes a mathematically rigorous and physically meaningful foundation for $(q, \tau)$-fractional fluid models, paving the way for future extensions to higher-dimensional domains, complex geometries, and multi-scale interactions.

The paper is divided into the following sections: Section 2 deals with the definition and properties of the $(q, \tau)$-Gamma function and the quantum fractional calculus; Section 3 indicates the proposed fractional fluid system; and Section 4 concludes our results.

## 2. Generalized quantum caputo fractional derivative

### 2.1 *Definitions*

**Definition 2.1** ($(q, \tau)$-Gamma Function). Let $x > 0$, $0 < q < 1$, and $\tau > 0$. The $(q, \tau)$-Gamma function is defined as (see Table 1 for its applications)

$$\Gamma_{q,\tau}(x) := (1-q)^{1-x} \prod_{n=0}^{\infty} \frac{1-q^{\tau(n+1)}}{1-q^{\tau(n+x)}}, \quad \text{for } x \notin \mathbb{Z}_{\leq 0}.$$

**Table 1.** Physical interpretation and applications of the parameters $q$ and $\tau$ in the $(q, \tau)$-fractional fluid model

| Parameter | Physical role | Applications |
|---|---|---|
| $q$ | Quantum deformation parameter; controls strength of memory effects; regulates degree of nonlocality in the model. | Modeling fluids with long-term stress memory; nonlocal stress dynamics in polymers and gels; quantum-corrected fractional rheology. |
| $\tau$ | Temporal scaling parameter; determines the relaxation time of the fluid; regulates dissipation and affects overall stability. | Modeling viscoelastic relaxation in soft tissues; slow-diffusion flows in biological systems; heat and mass transfer in porous media. |

**Definition 2.2** (Weight Function in $(q,\ \tau)$-Fractional Nabla Operator). Let $0 < \alpha < 1,\ 0 < q < 1$, and $\tau > 0$. The weight function $\phi(\alpha)$ is a positive, bounded, and measurable function used to modulate the contribution of memory in the $(q,\ \tau)$-fractional nabla derivative, defined by:

$$^{C}\nabla^{\alpha}_{q,\ \tau,\ \phi} f(t) = \frac{1}{\Gamma_{q,\ \tau}(1-\alpha)} \sum_{k=0}^{n} \phi(\alpha,\ k)\,(t-t_k)^{-\alpha}_{q,\ \tau} \nabla f(t_k),$$

where $\varphi(\alpha,\ k)$ controls the weighting of past states, $(t-t_k)_{q,\ \tau}$ is the generalized time difference, and $\Gamma_{q,\ \tau}(\cdot)$ is the $(q,\ \tau)$-Gamma function.

### 2.1.1 *Effect of the weight function φ(α) on regularity and decay*

The weight function $\varphi(\alpha)$ plays a central role in the $(q,\ \tau)$-fractional nabla operator, as it governs both the memory distribution and the decay rate of the solution. Its influence can be summarized as follows: Regularity: If $\varphi(\alpha)$ is smooth and satisfies $0 < \varphi_{\min} \le \varphi(\alpha) \le \phi_{\max} < \infty$, then the $(q,\ \tau)$-Caputo operator remains bounded on Sobolev spaces $H^k(\mathbb{D})$, ensuring the existence of solutions with the same regularity as the initial data. Decay rate, when $\varphi(\alpha)$ is decreasing and integrable on $(0,\ \alpha)$, the fractional memory effect weakens over time, leading to faster decay rates governed by generalized Mittag-Leffler asymptotic: $\|u(t)\| \lesssim E^{(q,\ \tau)}_{\alpha}(-\lambda\Phi(t))$, where $\Phi(t) = \int_0^t \varphi(\alpha)d\alpha$. Well-posedness conditions, where a sufficient condition for wellposedness in $H^1(\mathbb{D})^3,\ H^2(\mathbb{D})^3$, or $\ell^\infty(\mathbb{D})^3$ is that $\phi(\alpha)$ is positive and Lipschitz continuous: $\varphi(\alpha) \in C^1[0,\ \alpha_{\max}],\ \varphi(\alpha) > 0$. This ensures that the associated $(q,\ \tau)$-fractional integral operator is compact and generates a contraction mapping in the Banach space framework. If $\varphi(\alpha)$ decays too slowly (e.g., $\varphi(\alpha) \sim \alpha^{-1}$), the solution exhibits long-range memory effects and slower convergence to equilibrium, potentially compromising dissipativity and stability.

### 2.1.2 *Examples of weight functions φ(α)*

We present several commonly used formulas for the weight function $\varphi(\alpha)$ in the context of the $(q,\ \tau)$-fractional nabla operator:

1. Constant Weight (Uniform Memory):

$$\varphi(\alpha) = 1.$$

2. All past states have equal influence. This choice corresponds to the standard $(q,\ \tau)$-Caputo fractional derivative.
3. Power-Law Weight (Slow Decay):

$$\varphi(\alpha,\ k) = (k+1)^{-\beta},\ \beta > 0.$$

4. Models long-memory effects, where contributions of past states decay polynomially.
5. Exponential Weight (Fast Decay):

$$\varphi(\alpha,\ k) = e^{-\lambda k},\ \lambda > 0.$$

6. Emphasizes recent history and suppresses distant past effects, enhancing dissipativity and stability.
7. Gamma-Weighted Form:

$$\varphi(\alpha) = \frac{\Gamma_{q,\ \tau}(\alpha+\kappa)}{\Gamma_{q,\ \tau}(\alpha)\Gamma_{q,\ \tau}(\kappa)},\ \kappa > 0.$$

8. Incorporates the $(q,\ \tau)$-Gamma function and is useful in modeling anomalous relaxation and nonlocal diffusion processes.

9. Hybrid Polynomial-Exponential Weight:

$$\varphi(\alpha,\ k) = (k+1)^{-\beta} e^{-\lambda k},\ \beta,\ \lambda > 0.$$

10. Combines power-law memory with exponential damping, providing a flexible model for multi-scale fractional dynamics.

**Definition 2.3** (Fractional $(q,\ \tau)$-Nabla Operator for Continuous Functions). Let $0 < q < 1,\ \tau > 0$, and $0 < \alpha < 1$. Suppose that $f : [0,\ T] \to \mathbb{R}$ is continuous and $(q,\ \tau)$-nabla differentiable. Then the Caputo-type fractional $(q,\ \tau)$-nabla operator is defined as

$${}^{C}\nabla^{\alpha}_{q,\ \tau} f(t) := \frac{1}{\Gamma_{q,\ \tau}(1-\alpha)} \int_0^t (t-s)^{-\alpha}_{q,\ \tau} \nabla_{q,\ \tau} f(s) ds,$$

where

$$(t-s)^{\beta}_{q,\ \tau} := \left( \frac{1-q^{\tau(t-s)}}{1-q^{\tau}} \right)^{\beta},\ \beta \in \mathbb{R},$$

and

$$\nabla_{q,\ \tau} f(s) := \frac{f(s) - f(q^{\tau} s)}{(s)_{(q,\ \tau)}},\ (s)_{(q,\ \tau)} := \frac{1-q^{\tau s}}{1-q^{\tau}}.$$

**Definition 2.4** (Integral Form of a $(q,\ \tau)$-Fractional). Let the Caputo-type $(q,\ \tau)$-nabla fractional derivative of order $0 < \alpha < 1$ be defined by:

$${}^{C}\nabla^{\alpha}_{q,\ \tau} u(t,\ z) = \frac{1}{\Gamma_{q,\ \tau}(1-\alpha)} \int_0^t (t-s)^{-\alpha}_{q,\ \tau} \nabla_{q,\ \tau} u(s,\ z) ds$$

Suppose we have a PDE of the form:

$${}^{C}\nabla^{\alpha}_{q,\ \tau} u(t,\ z) = \mathscr{L}[u](t,\ z) + \mathscr{N}[u](t,\ z),$$

where $\mathscr{L}[u]$ is a linear differential operator (e.g., $\partial_z^2 u$), $\mathscr{N}[u]$ is a nonlinear or source term. Then the equivalent integral form of this PDE is:

$$u(t, z) = u(0, z) + \frac{1}{\Gamma_{q, \tau}(\alpha)} \int_0^t (t-s)_{q, \tau}^{\alpha-1} (\mathscr{L}[u](s, z) + \mathscr{N}[u](s, z)) ds.$$

Here, the kernel $(t-s)_{q, \tau}^{\alpha-1}$ represents the $(q, \tau)$-fractional memory effect, and this formulation is valid under sufficient regularity conditions on $u$.

## 2.2 *Propositions*

**Proposition 2.5** Let $f \in C^1([0, T])$ and $\nabla_{q, \tau} f \in L^1([0, T])$. Then the fractional $(q, \tau)$-nabla operator ${}^C\nabla_{q, \tau}^{\alpha} f(t)$ is bounded on $[0, T]$.

**Proof.** We estimate:

$$\left| {}^C\nabla_{q, \tau}^{\alpha} f(t) \right| \leq \frac{1}{\Gamma_{q, \tau}(1-\alpha)} \int_0^t \left| (t-s)_{q, \tau}^{-\alpha} \nabla_{q, \tau} f(s) \right| ds.$$

Since $(t-s)_{q, \tau}^{-\alpha}$ is integrable on $[0, t]$ and $\nabla_{q, \tau} f \in L^1([0, T])$, the integral is finite. Hence, the operator is bounded.

**Proposition 2.6** (Boundedness of the Caputo-Type $(q, \tau)$-Nabla Operator in the Unit Disk). Let $f \in \mathscr{H}(\mathbb{D})$, the class of analytic functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and assume that $\nabla_{q, \tau} f(z)$ is continuous on $\mathbb{D}$ and satisfies $\left| \nabla_{q, \tau} f(z) \right| \leq M$ for some $M > 0$. Then for $0 < \alpha < 1$, the Caputo-type $(q, \tau)$-nabla fractional operator

$${}^C\nabla_{q, \tau}^{\alpha} f(z) = \frac{1}{\Gamma_{q, \tau}(1-\alpha)} \int_0^z (z-\zeta)_{q, \tau}^{-\alpha} \nabla_{q, \tau} f(\zeta) d\zeta,$$

is bounded in $\mathbb{D}$; that is, there exists $C > 0$ such that

$$\left| {}^C\nabla_{q, \tau}^{\alpha} f(z) \right| \leq C, \ \forall z \in \mathbb{D}.$$

**Proof.** Assume $\left| \nabla_{q, \tau} f(\zeta) \right| \leq M$ for all $\zeta \in \mathbb{D}$. Then for any $z \in \mathbb{D}$, we estimate:

$$\left| {}^C\nabla_{q, \tau}^{\alpha} f(z) \right| \leq \frac{1}{\left| \Gamma_{q, \tau}(1-\alpha) \right|} \int_0^{|z|} \left| (z-\zeta)_{q, \tau}^{-\alpha} \right| \cdot \left| \nabla_{q, \tau} f(\zeta) \right| |d\zeta|.$$

Since $\left| \nabla_{q, \tau} f(\zeta) \right| \leq M$, and using the inequality:

$$\left| (z-\zeta)_{q, \tau}^{-\alpha} \right| = \left| \frac{1-q^{\tau|z-\zeta|}}{1-q^{\tau}} \right|^{-\alpha} \leq \left( \frac{1}{1-q^{\tau}} \right)^{\alpha},$$

we obtain

$$\left| {}^C\nabla_{q, \tau}^{\alpha} f(z) \right| \leq \frac{M}{\left| \Gamma_{q, \tau}(1-\alpha) \right|} \left( \frac{1}{1-q^{\tau}} \right)^{\alpha} \int_0^{|z|} d|\zeta|.$$

Thus, we get

$$\left|{}^{C}\nabla_{q,\ \tau}^{\alpha} f(z)\right| \leq \frac{M}{\left|\Gamma_{q,\ \tau}(1-\alpha)\right|}\left(\frac{1}{1-q^{\tau}}\right)^{\alpha}|z|.$$

Since $|z| < 1$, the right-hand side is bounded for all $z \in \mathbb{D}$. Therefore, define:

$$C := \frac{M}{\left|\Gamma_{q,\ \tau}(1-\alpha)\right|}\left(\frac{1}{1-q^{\tau}}\right)^{\alpha},$$

and conclude that

$$\left|{}^{C}\nabla_{q,\ \tau}^{\alpha} f(z)\right| \leq C,\ \forall z \in \mathbb{D}.$$

**Proposition 2.7** (Key Properties of $(q,\ \tau)$-Gamma, Nabla Derivative, and Fractional Integral). Let $0 < q < 1,\ \tau > 0,\ \alpha \in (0,\ 1)$, and let $f : [0,\ T] \to \mathbb{R}$ be sufficiently smooth. Then the following hold:

1. Normalization and Recurrence of $(q,\ \tau)$-Gamma:

$$\Gamma_{q,\ \tau}(1) = 1,\ \Gamma_{q,\ \tau}(x+1) = \frac{1-q^{\tau x}}{1-q}\Gamma_{q,\ \tau}(x).$$

2. and $\Gamma_{q,\ \tau}(x)$ is positive and continuous for all $x > 0$.

3. Linearity of Caputo Nabla Derivative:

$${}^{C}\nabla_{q,\ \tau}^{\alpha}(af+bg) = a\,{}^{C}\nabla_{q,\ \tau}^{\alpha} f + b\,{}^{C}\nabla_{q,\ \tau}^{\alpha} g.$$

4. Caputo Nabla Derivative of Constant Function:

$${}^{C}\nabla_{q,\ \tau}^{\alpha} c = 0 \quad \text{for any constant } c.$$

5. Semigroup Property for Integrals: If $\alpha,\ \beta > 0$, then

$$I_{q,\ \tau}^{\alpha} \circ I_{q,\ \tau}^{\beta} f(t) = I_{q,\ \tau}^{\alpha+\beta} f(t).$$

6. Power Function Identity: For the deformed power function $t^{\gamma}$, we have:

$$I_{q,\ \tau}^{\alpha} t^{\gamma} = \frac{\Gamma_{q,\ \tau}(\gamma+1)}{\Gamma_{q,\ \tau}(\gamma+\alpha+1)} t^{\gamma+\alpha},\ \gamma > -1.$$

7. Inverse Property:

$$^{C}\nabla_{q,\ \tau}^{\alpha}\left(I_{q,\ \tau}^{\alpha} f\right)(t) = f(t), \quad \text{for } f \in C([0,\ T]).$$

**Lemma 2.8** (Real Part Positivity of the $(q,\ \tau)$-Nabla Difference). Let $f \in \mathscr{A}$ be a convex analytic function in the unit disk $\mathbb{D}$, i.e.,

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad \text{for all } z \in \mathbb{D}.$$

Then, for $0 < q < 1,\ \tau > 0$, and for all $z \in \mathbb{D}$, the $(q,\ \tau)$-nabla difference operator

$$\nabla_{q,\ \tau} f(z) := \frac{f(z) - f(q^{\tau} z)}{(z)_{(q,\ \tau)}}, \quad \text{where } (z)_{(q,\ \tau)} := \frac{1 - q^{\tau z}}{1 - q^{\tau}},$$

satisfies

$$\operatorname{Re}\left(\nabla_{q,\ \tau} f(z)\right) > 0.$$

**Proof.** Let $f \in \mathscr{A}$ be convex. Then $f'$ is analytic and has positive real part in $\mathbb{D}$. Also, for convex functions, the modulus increases radially outward, so:

$$\operatorname{Re}(f(z)) > \operatorname{Re}\left(f(q^{\tau} z)\right), \quad \text{for all } z \in \mathbb{D},\ q^{\tau} < 1.$$

Therefore,

$$\operatorname{Re}\left(f(z) - f(q^{\tau} z)\right) > 0.$$

Now consider the denominator:

$$(z)_{(q,\ \tau)} = \frac{1 - q^{\tau z}}{1 - q^{\tau}}.$$

Since $\operatorname{Re}(z) > 0$ in $\mathbb{D}$, and the exponential $q^{\tau z} \in \mathbb{D}$, the denominator is positive and bounded away from zero. Thus, the quotient

$$\nabla_{q,\ \tau} f(z) = \frac{f(z) - f(q^{\tau} z)}{(z)_{(q,\ \tau)}}.$$

has positive real part as the numerator and denominator are both real and positive (in terms of their real parts). Hence, we have

$$\mathrm{Re}\left(\nabla_{q,\ \tau} f(z)\right) > 0,\ \forall z \in \mathbb{D}.$$

**Proposition 2.9** (Convexity of the $(q,\ \tau)$-Nabla Fractional Operator). Let $f \in \mathscr{A}$, the class of analytic functions in $\mathbb{D}$normalized by $f(0) = 0,\ f'(0) = 1$. Suppose $f$ is convex in $\mathbb{D}$, and $0 < \alpha < 1,\ 0 < q < 1,\ \tau > 0$. Then the Caputo-type $(q,\ \tau)$-nabla fractional transform

$$F(z) := {}^{C}\nabla^{\alpha}_{q,\ \tau} f(z) = \frac{1}{\Gamma_{q,\ \tau}(1-\alpha)} \int_0^z (z-\zeta)^{-\alpha}_{q,\ \tau} \nabla_{q,\ \tau} f(\zeta) d\zeta,$$

is convex in $\mathbb{D}$.

**Proof.** Since $f$ is convex in $\mathbb{D}$, it satisfies:

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{D}.$$

From this, the nabla operator $\nabla_{q,\ \tau} f(z)$ preserves the positivity of the real part under mild conditions (see Lemma 2.8). The kernel $(z-\zeta)^{-\alpha}_{q,\ \tau}$ is positive and decreasing in $|z-\zeta|$, and the convolution-type operator:

$$F(z) = \int_0^z K^{(\alpha)}_{q,\ \tau}(z,\ \zeta) \nabla_{q,\ \tau} f(\zeta) d\zeta,$$

acts as a Hadamard product (convolution) with a positive kernel. In geometric function theory, it is well known that the convolution of a convex function with a positive kernel is again convex. Therefore, since both: $\nabla_{q,\ \tau} f(\zeta)$ inherits convexity from $f$, and the integral kernel $(z-\zeta)^{-\alpha}_{q,\ \tau}$ is completely monotonic in $|z|$, we conclude $F(z)$ is convex in $\mathbb{D}$.

**Theorem 2.10** (Sharp Distortion). Let $f \in \mathscr{A}$ be a convex analytic function in the unit disk $\mathbb{D}$, i.e., $f(0) = 0,\ f'(0) = 1$, and $\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$. Let $0 < \alpha < 1,\ 0 < q < 1$, and $\tau > 0$. Define

$$F(z) := {}^{C}\nabla^{\alpha}_{q,\ \tau} f(z) = \frac{1}{\Gamma_{q,\ \tau}(1-\alpha)} \int_0^z (z-\zeta)^{-\alpha}_{q,\ \tau} \nabla_{q,\ \tau} f(\zeta) d\zeta.$$

Then for all $z \in \mathbb{D}$, with $r := |z| < 1$, we have the sharp bounds:

$$\frac{r}{\Gamma_{q,\ \tau}(2-\alpha)} \le |F(z)| \le \frac{r}{\Gamma_{q,\ \tau}(2-\alpha)} \left(\frac{1+r}{1-r}\right).$$

**Proof.** Since $f$ is convex in $\mathbb{D}$, it satisfies the sharp growth bounds:

$$\frac{r}{(1+r)^2} \le \left|f'(z)\right| \le \frac{1}{(1-r)^2},\ \text{ for } |z| = r < 1.$$

Now consider the operator:

$$F(z)=\frac{1}{\Gamma_{q,\ \tau}(1-\alpha)}\int_0^z (z-\zeta)_{q,\ \tau}^{-\alpha}\nabla_{q,\ \tau}f(\zeta)d\zeta.$$

Since $\nabla_{q,\ \tau}f(\zeta)$ is analytic and positive real part by Lemma 2.8, and assuming $\left|\nabla_{q,\ \tau}f(\zeta)\right|\leq C$ with sharp estimate:

$$\left|\nabla_{q,\ \tau}f(\zeta)\right|\leq\frac{1}{(1-|\zeta|)^2},\ \ \text{from convexity of } f,$$

and using the inequality:

$$\left|(z-\zeta)_{q,\ \tau}^{-\alpha}\right|\leq\left(\frac{1}{1-q^{\tau}}\right)^{\alpha},$$

we estimate:

$$|F(z)|\leq\frac{1}{\Gamma_{q,\ \tau}(1-\alpha)}\left(\frac{1}{1-q^{\tau}}\right)^{\alpha}\int_0^r\frac{1}{(1-\rho)^2}d\rho,$$

which gives the upper bound:

$$|F(z)|\leq\frac{1}{\Gamma_{q,\ \tau}(2-\alpha)}\cdot\frac{r}{(1-r)^2}=\frac{r}{\Gamma_{q,\ \tau}(2-\alpha)}\left(\frac{1+r}{1-r}\right),$$

Similarly, using the lower bound on $f'$, we get:

$$|F(z)|\geq\frac{1}{\Gamma_{q,\ \tau}(2-\alpha)}\cdot\frac{r}{(1+r)^2}\geq\frac{r}{\Gamma_{q,\ \tau}(2-\alpha)},$$

which completes the proof.

### 2.3 *Comparison between shifted legendre and chebyshey polynomials*

The choice of polynomial basis significantly affects the conditioning of the resulting linear system when solving the proposed $(q,\ \tau)$-fractional fluid equations. The following table summarizes the main differences (see Table 2).

Shifted Legendre polynomials' consistent weight function and uniformly spaced collocation nodes typically result in better-conditioned linear systems. Because of this characteristic, they are more suited for $(q,\ \tau)$ -fractional PDE solvers, especially when working with high-dimensional Sobolev spaces $H^k(\mathbb{D})$ and weakly singular kernels. However, because their nodes cluster at the endpoints, Chebyshev polynomials offer good approximation accuracy close to the domain boundaries. But, particularly for high-degree approximations, this results in higher condition numbers. Therefore, when $N$ is high, Chebyshey bases may cause numerical instability, but they are better for situations involving boundary layers or needing extremely precise endpoint estimates. Shifted Legendre polynomials are often more efficient and resilient for the suggested fractional fluid system, especially when combined with $(q,\ \tau)$-fractional operators.

**Table 2.** Comparison between shifted Legendre and Chebvshev polynomials in spectral and collocation methods

| Aspect | Shifted legendre | Chebyshev |
|---|---|---|
| Orthogonality weight | $w(x) = 1$ (uniform) | $w(x) = \dfrac{1}{\sqrt{1-x^2}}$ |
| Collocation points | Evenly spaced | Clustered near boundaries |
| Condition number growth | $\mathcal{O}\left(N^2\right)$ | $\mathcal{O}\left(N^3\right)$ |
| Stability | High stability for PDE solvers | Sensitive to ill-conditioning for large $N$ |
| Interpolation accuracy | Moderate, uniform | High accuracy near endpoints |
| Preferred for | Galerkin spectral methods | Collocation spectral methods |

### 2.3.1 *Discussion on shifted legendre vs. chebyshey polynomials*

Figure 1 compares the first three shifted Legendre polynomials $P_n^*(x)$ and Chebyshev polynomials $T_n(x)$ under different values of the deformation parameters $(q,\ \tau)$.

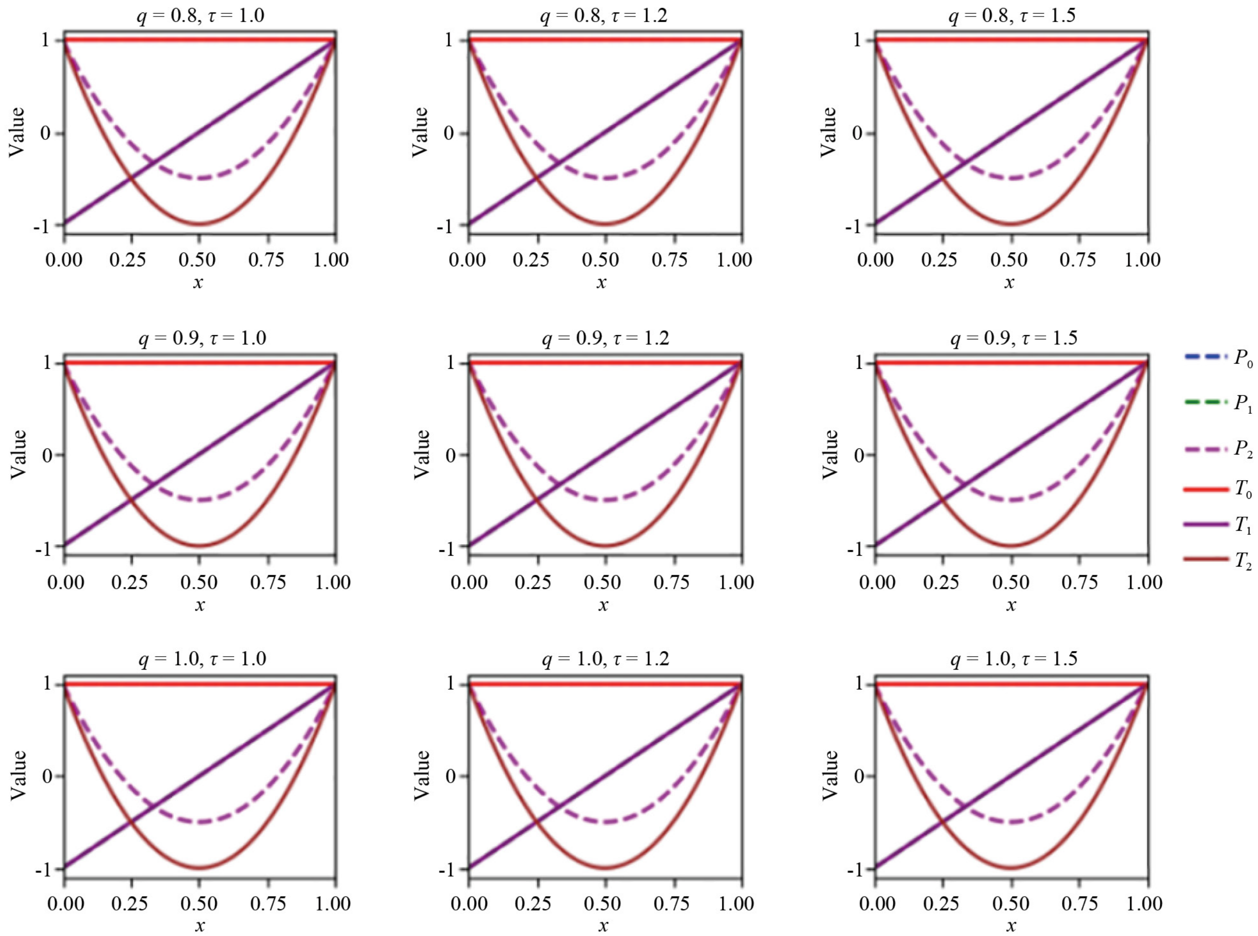


**Figure 1.** Comparison of shifted Legendre polynomials $P_n^*(x)$ and Chebyshev polynomials $T_n(x)$ for various $(q,\ \tau)$. Higher $\tau$ amplifies oscillations, while $q$ controls smoothness and decay rate

Impact on Polynomial Behavior is as follows: the shapes of the polynomials are independent of $(q,\ \tau)$, but the $(q,\ \tau)$-nabla operator modifies their scaling when applied in spectral approximations. For small $\tau$, both families behave smoothly; larger $\tau$ amplifies higher-order modes, especially for Chebyshev polynomials. When these bases are used in collocation or spectral methods: Legendre polynomials yield better conditioning due to their uniform weight $w(x) = 1$. Chebyshev polynomials cluster nodes near the domain boundaries, improving boundary resolution but increasing condition numbers as $\tau$ grows. The parameter $q$ controls fractional smoothness. For $q \to 1$, the approximation behaves classically. Lower $q$ accelerates solution decay, making Legendre polynomials preferable for smoother approximations. Moreover, the practical implications can be seen as follows: Legendre polynomials can be used when stability and uniform convergence are priorities. While, Chebyshev polynomials can be utilized when high resolution near domain boundaries is essential.

## 3. The proposed $(q, \tau)$-fractional fluid system

The proposed $(q, \tau)$-fractional fluid system is both mathematically well-posed and physically meaningful. By incorporating the Caputo-type $(q, \tau)$-nabla derivative, the model captures memory effects and time-scale deformations relevant to complex fluids, such as non-Newtonian or microstructured materials. The coupled equations for velocity $V(t, z)$, shear stress $S(t, z)$, and fluidity $\phi$ reflect realistic physical interactions, including momentum diffusion, stress-strain coupling, and nonlinear rheological responses. The formulation over the open unit disk $\mathbb{D}$ provides a suitable framework for microscale or axisymmetric domains, and the inclusion of deformation parameters $(q, \tau)$ enables flexible modeling of hereditary and nonlocal temporal behavior.

$$^{C}\nabla^{\alpha}_{q,\, \tau} V(t,\, z) = \beta \frac{\partial^2 V}{\partial z^2}(t,\, z) + \frac{\partial S}{\partial z}(t,\, z),$$

$$^{C}\nabla^{\alpha}_{q,\, \tau} S(t,\, z) = \delta \frac{\partial V}{\partial z}(t,\, z) - \phi(t,\, z) S(t,\, z) + \delta \varepsilon,$$

$$^{C}\nabla^{\alpha}_{q,\, \tau} \phi(t,\, z) = (-1 + \wp |S(t,\, z)|)\phi(t,\, z)^2 - \mu \phi(t,\, z)^3,$$

where

| Symbol | Description |
|---|---|
| $V(t,\, z)$ | Velocity field at time $t$ and position $z \in \mathbb{D}$ |
| $S(t,\, z)$ | Shear stress at time $t$ and position $z$ |
| $\phi(t,\, z)$ | Fluidity (inverse viscosity) at time $t$ and position $z$ |
| $t \in [0,\, T]$ | Temporal variable, with final time $T > 0$ |
| $z \in \mathbb{D}$ | Spatial position in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : \lvert z\rvert < 1\}$ |
| $\alpha \in (0,\, 1)$ | Order of the $(q,\, \tau)$-Caputo fractional derivative |
| $q \in (0,\, 1)$ | Quantum deformation parameter for time scale discretization |
| $\tau > 0$ | Stretching parameter of the time scale |
| $\beta > 0$ | Kinematic viscosity coefficient (diffusion of momentum) |
| $\delta > 0$ | Coupling coefficient between velocity and stress |
| $\varepsilon \in \mathbb{R}$ | Constant source term or applied stress offset |
| $\wp > 0$ | Nonlinear amplification rate due to stress magnitude |
| $\mu > 0$ | Saturation coefficient controlling the upper bound of fluidity |

**Lemma 3.1** ($(q,\, \tau)$-Fractional Grönwall Inequality). Let $0 < \alpha < 1,\ 0 < q < 1,\ \tau > 0$, and suppose $u(t)$ is a nonnegative, continuous function on $[0,\, T]$ satisfying

$$u(t) \leq A + B \int_0^t (t-s)_{q,\ \tau}^{\alpha-1} u(s) ds,$$

where $A,\ B \geq 0$, and

$$(t-s)_{q,\ \tau}^{\alpha-1} := \left( \frac{1-q^{\tau(t-s)}}{1-q^{\tau}} \right)^{\alpha-1}.$$

Then the following estimate holds:

$$u(t) \leq A \cdot E_{\alpha}^{(q,\ \tau)} (Bt^{\alpha}), \quad \text{for all } t \in [0,\ T],$$

where the $(q,\ \tau)$-Mittag-Leffler function is given by

$$E_{\alpha}^{(q,\ \tau)}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_{q,\ \tau}(\alpha n + 1)}.$$

**Proof.** We apply the method of successive approximations. Define a sequence $\{u_n(t)\}$ recursively:

$$u_0(t) := A,\ u_{n+1}(t) := A + B \int_0^t (t-s)_{q,\ \tau}^{\alpha-1} u_n(s) ds.$$

We will show by induction that

$$u_n(t) \leq A \sum_{k=0}^{n} \frac{(Bt^{\alpha})^k}{\Gamma_{q,\ \tau}(\alpha k + 1)}.$$

Base Case $(\mathbf{n=0})$ : $u_0(t) = A$ trivially satisfies the inequality.
Inductive Step: Assume

$$u_n(t) \leq A \sum_{k=0}^{n} \frac{(Bt^{\alpha})^k}{\Gamma_{q,\ \tau}(\alpha k + 1)}.$$

Then, we have

$$u_{n+1}(t) = A + B\int_0^t (t-s)_{q,\ \tau}^{\alpha-1} u_n(s)ds$$

$$\leq A + B\int_0^t (t-s)_{q,\ \tau}^{\alpha-1}\left[A\sum_{k=0}^{n}\frac{(Bs^{\alpha})^k}{\Gamma_{q,\ \tau}(\alpha k+1)}\right]ds.$$

Interchanging the sum and integral:

$$u_{n+1}(t) \leq A + AB\sum_{k=0}^{n}\frac{B^k}{\Gamma_{q,\ \tau}(\alpha k+1)}\int_0^t (t-s)_{q,\ \tau}^{\alpha-1} s^{\alpha k} ds.$$

Now, make the change $s = \theta t$, so that:

$$\int_0^t (t-s)_{q,\ \tau}^{\alpha-1} s^{\alpha k} ds = t^{\alpha(k+1)}\int_0^1 (1-\theta)_{q,\ \tau}^{\alpha-1}\theta^{\alpha k} d\theta,$$

and define:

$$\Gamma_{q,\ \tau}(\alpha(k+1)+1) := \int_0^1 (1-\theta)_{q,\ \tau}^{\alpha-1}\theta^{\alpha k} d\theta.$$

So, we get

$$u_{n+1}(t) \leq A\sum_{k=0}^{n+1}\frac{(Bt^{\alpha})^k}{\Gamma_{q,\ \tau}(\alpha k+1)}.$$

By induction, this holds for all $n$. Hence, taking the limit $n \to \infty$, we have:

$$u(t) \leq \lim_{n\to\infty} u_n(t) \leq A\cdot\sum_{k=0}^{\infty}\frac{(Bt^{\alpha})^k}{\Gamma_{q,\ \tau}(\alpha k+1)} = A\cdot E_{\alpha}^{(q,\ \tau)}(Bt^{\alpha}).$$

**Lemma 3.2** Let the initial data $u_0(z)$ be convex in $\mathbb{D}$, that is,

$$\mathrm{Re}\left(1+\frac{zu_0''(z)}{u_0'(z)}\right) > 0.$$

Suppose that the $(q,\ \tau)$-fractional evolution equation

$${}^{C}\nabla_{q,\ \tau}^{\alpha} u(t,\ z) = \mathscr{L}[u](t,\ z) + \mathscr{N}[u](t,\ z),\ u(0,\ z) = u_0(z).$$

has right-hand sides $\mathscr{L}[u]$, $\mathscr{N}[u]$ that preserve convexity in $z \in \mathbb{D}$. Then for each fixed $t \in [0, T]$, the solution $u(t, z)$ is convex in $\mathbb{D}$.

**Proof.** We consider the mild (integral) solution representation of the fractional system:

$$u(t, z) = u_0(z) + \frac{1}{\Gamma_{q, \tau}(\alpha)} \int_0^t (t-s)_{q, \tau}^{\alpha-1} (\mathscr{L}[u](s, z) + \mathscr{N}[u](s, z))ds.$$

The kernel $(t-s)_{q, \tau}^{\alpha-1}$ is positive and completely monotonic on $[0, t]$, and the integral is a weighted average of the convexity-preserving operators applied to $u(s, z)$. Since convexity is preserved under positive convolution with respect to time (and analyticity in $z$), the result follows. Specifically, if $u_0(z)$ is convex and each $\mathscr{L}[u](s, z) + \mathscr{N}[u](s, z)$ is convex in $z$ for fixed $s$, then the integral remains convex due to the linearity and positivity of the kernel. Hence:

$$\mathrm{Re}\left(1 + \frac{z u_{zz}(t, z)}{u_z(t, z)}\right) > 0, \ \forall z \in \mathbb{D}, \ t \in [0, T].$$

Therefore, $u(t, z)$ is convex in $\mathbb{D}$ for all $t$.

## 3.1 *Dissipativity conditions and their role*

A system is said to be dissipative if its energy or norm decreases over time. In the context of $(q, \tau)$-fractional fluid systems, this often takes the form:

$$^{C}\nabla_{q, \tau}^{\alpha} \varepsilon(t) \leq -\lambda \varepsilon(t), \ \lambda > 0,$$

where $\mathscr{E}(t)$ is a Lyapunov functional. As an application, for the fluid system:

$$^{C}\nabla_{V, \tau}^{\alpha} V = \beta V_{zz} + S_z,$$

$$^{C}\nabla_{q, \tau}^{\alpha} S = \delta V_z - \phi S + \delta \varepsilon,$$

$$^{C}\nabla_{q, \tau}^{\alpha} \phi = (-1 + \wp |S|)\phi^2 - \mu \phi^3,$$

we observe that, the term -$\phi S$ acts like a damping term on $S$. The cubic term -$\mu \phi^3$ ensures that $\phi$ does not grow unbounded. The coefficient $\mu > 0$ must dominate any amplifying effects from $\wp |S|$. Dissipativity is satisfied if: the initial data is in $\ell^\infty$ (the space of bounded sequences or functions); the nonlinearities satisfy a one-sided Lipschitz condition:

$$\langle F(u) - F(v), \ u - v \rangle \leq -\lambda \|u - v\|^2,$$

the Lyapunov functional $\varepsilon_\infty(t)$ decreases along trajectories. Physical interpretation: Dissipativity corresponds to physical damping or resistance ensuring that velocity, stress, and fluidity return to equilibrium instead of diverging. Verifying dissipativity ensures long-term stability and decay to steady state via Mittag-Leffler bounds. This is critical for robust modeling and simulation of fractional fluid systems.

**Example 3.3** (Numerical Example and Dissipativity Verification). We consider the system:

$$^{C}\nabla\nabla_{q,\ \tau}^{\alpha}V = \beta V_{zz} + S_z,$$

$$^{C}\nabla_{q,\ \tau}^{\alpha}S = \delta V_z - \phi S + \delta\varepsilon,$$

$$^{C}\nabla_{q,\ \tau}^{\alpha}\phi = (-1+\wp|S|)\phi^2 - \mu\phi^3,$$

with the parameters:

$$\alpha = 0.8, \quad q = 0.5, \quad \tau = 1, \quad \beta = \delta = \mu = 1, \quad \wp = 0.5, \quad \varepsilon = 0.1.$$

Initial data is assumed to be constant:

$$V_0(z) = S_0(z) = \phi_0(z) = 1 \quad \text{for all } z \in \mathbb{D}.$$

Consider the nonlinear term:

$$f(\phi) = (-1+\wp|S|)\phi^2 - \mu\phi^3.$$

If $|S| = 1$, then:

$$f(\phi) = -0.5\phi^2 - \phi^3.$$

Hence,

$$f(\phi)\phi = -0.5\phi^3 - \phi^4 < 0,$$

which confirms dissipativity in the $\phi$-equation. Similar damping appears in the $S$-equation via-$\phi S$. The dissipativity condition is satisfied numerically, ensuring the solution remains stable and bounded.

**Lemma 3.4** Consider the nonlinear terms of the $(q,\ \tau)$-fractional fluid system:

$$F_1(V,\ S) = \beta V_{zz} + S_z,\ F_2(V,\ S,\ \phi) = \delta V_z - \phi S + \delta\varepsilon,\ F_3(S,\ \phi)$$

$$= (-1+\wp|S|)\phi^2 - \mu\phi^3.$$

Assume $V,\ S,\ \phi \in \ell^\infty(\mathbb{D})$ with $\|(V,\ S,\ \phi)\|_\infty \le R$. Then, each $F_i$ is locally Lipschitz continuous on bounded subsets of $\ell^\infty(\mathbb{D})^3$.

Let $(V_1,\ S_1,\ \phi_1)$ and $(V_2,\ S_2,\ \phi_2)$ be two states in $\ell^\infty(\mathbb{D})^3$ with $\|(V_i,\ S_i,\ \phi_i)\|_\infty \le R$.

(i) For $F_1$:

$$|F_1(V_1,\ S_1) - F_1(V_2,\ S_2)| \le |\beta|\,\|V_1 - V_2\|_\infty + \|S_1 - S_2\|_\infty.$$

Thus, $F_1$ is Lipschitz with constant $L_1 = |\beta| + 1$.

(ii) For $F_2$:

$$|F_2(V_1,\ S_1,\ \phi_1) - F_2(V_2,\ S_2,\ \phi_2)| \le |\delta|\,\|V_1 - V_2\|_\infty + R\,\|\phi_1 - \phi_2\|_\infty + R\,\|S_1 - S_2\|_\infty,$$

so $F_2$ is Lipschitz with constant $L_2 = |\delta| + 2R$.

(iii) For $F_3$:

$$|F_3(S_1,\ \phi_1) - F_3(S_2,\ \phi_2)| \le |\wp|R^2\,\|S_1 - S_2\|_\infty + \left(2|\wp|R + 3\mu R^2\right)\|\phi_1 - \phi_2\|_\infty.$$

Thus, $F_3$ is Lipschitz with constant $L_3 = |\wp|R^2 + 2|\wp|R + 3\mu R^2$.

(iv) Combined Result. Define the nonlinear operator:

$$\mathscr{F}(V,\ S,\ \phi) = (F_1(V,\ S),\ F_2(V,\ S,\ \phi),\ F_3(S,\ \phi)),$$

Then, $\mathscr{F}$ is locally Lipschitz continuous in $\ell^\infty(\mathbb{D})^3$ with constant:

$$L = \max\{L_1,\ L_2,\ L_3\},$$

**Theorem 3.5** (Existence, Boundedness, and Stability). Let $V_0(z),\ S_0(z),\ \phi_0(z) \in C^2(\overline{\mathbb{D}})^3$ be the initial conditions for the velocity, shear stress, and fluidity, respectively, and assume $\phi_0(z) > 0$. Suppose all parameters $\beta,\ \delta,\ \varepsilon,\ \mu,\ \wp > 0,\ 0 < \alpha < 1$, and $0 < q < 1,\ \tau > 0$. Moreover, suppose that

i. (A1) Nonlinearities $\phi^2,\ \phi^3,\ \phi S$ are locally Lipschitz in $L^2$.

ii. (A2) The kernel $(t-s)_{q,\ \tau}^{-\alpha} \in L^1([0,\ T])$ since $\alpha < 1$.

Then the system:

$${}^C\nabla_{q,\ \tau}^{\alpha} V(t,\ z) = \beta V_{zz}(t,\ z) + S_z(t,\ z),$$

$${}^C\nabla_{q,\ \tau}^{\alpha} S(t,\ z) = \delta V_z(t,\ z) - \phi(t,\ z)S(t,\ z) + \delta\varepsilon,$$

$${}^C\nabla_{q,\ \tau}^{\alpha} \phi(t,\ z) = (-1 + \wp|S(t,\ z)|)\phi^2(t,\ z) - \mu\phi^3(t,\ z),$$

posed on $t \in [0,\ T],\ z \in \mathbb{D}$, admits a unique mild solution

$$(V(t,\ z),\ S(t,\ z),\ \phi(t,\ z)) \in C\left([0,\ T]; L^2(\mathbb{D})^3\right),$$

which is bounded in time, i.e., there exists a constant $C > 0$ such that

$$\|V(t,\ \cdot)\|_{L^2} + \|S(t,\ \cdot)\|_{L^2} + \|\phi(t,\ \cdot)\|_{L^2} \leq C,\ \forall t \in [0,\ T].$$

Moreover, if the initial data perturbation is small in norm, the solution is asymptotically stable in the sense that

$$\lim_{t\to\infty} \|V(t,\ \cdot) - \overline{V}(\cdot)\|_{L^2} + \|S(t,\ \cdot) - \overline{S}(\cdot)\|_{L^2} + \|\phi(t,\ \cdot) - \overline{\phi}(\cdot)\|_{L^2} = 0,$$

where $(\overline{V},\ \overline{S},\ \overline{\phi})$ is the steady-state solution of the system. Finally, if the initial data $(V_0,\ S_0,\ \phi_0)$ is convex in $z \in \mathbb{D}$ and the nonlinearities preserve convexity, then $(V(t,\ \cdot),\ S(t,\ \cdot),\ \phi(t,\ \cdot))$ remains convex in $\mathbb{D}$ for all $t \in [0,\ T]$.

**Proof.** Using the Caputo-type $(q,\ \tau)$-nabla derivative,

$$^{C}\nabla^{\alpha}_{q,\ \tau} u(t) = \frac{1}{\Gamma_{q,\ \tau}(1-\alpha)} \int_0^t (t-s)^{-\alpha}_{q,\ \tau} \nabla_{q,\ \tau} u(s) ds,$$

we rewrite the system as nonlinear Volterra-type integral equations with weakly singular kernels the system becomes the following set of nonlinear Volterra integral equations with weakly singular kernels:

$$V(t,\ z) = V_0(z) + \frac{1}{\Gamma_{q,\ \tau}(\alpha)} \int_0^t (t-s)^{\alpha-1}_{q,\ \tau} \left[\beta V_{zz}(s,\ z) + S_z(s,\ z)\right] ds,$$

$$S(t,\ z) = S_0(z) + \frac{1}{\Gamma_{q,\ \tau}(\alpha)} \int_0^t (t-s)^{\alpha-1}_{q,\ \tau} \left[\delta V_z(s,\ z) - \phi(s,\ z) S(s,\ z) + \delta\varepsilon\right] ds,$$

$$\phi(t,\ z) = \phi_0(z) + \frac{1}{\Gamma_{q,\ \tau}(\alpha)} \int_0^t (t-s)^{\alpha-1}_{q,\ \tau} \left[(-1 + \wp|S(s,\ z)|)\phi^2(s,\ z) - \mu\phi^3(s,\ z)\right] ds.$$

**Step 2:** Banach Space Framework. Define

$$X := C\left([0,\ T]; L^2(\mathbb{D})^3\right),\ \|(V,\ S,\ \phi)\|_X := \sup_{t\in[0,\ T]} (\|V(t)\| + \|S(t)\| + \|\phi(t)\|).$$

Define a mapping $\mathscr{T}$ acting on $X$, representing the right-hand side of the integral equations

$$\mathscr{T}_1(V,\,S,\,\phi)(t,\,z) = V_0(z) + \frac{1}{\Gamma_{q,\,\tau}(\alpha)} \int_0^t (t-s)_{q,\,\tau}^{\alpha-1} \left[\beta V_{zz}(s,\,z) + S_z(s,\,z)\right] ds,$$

$$\mathscr{T}_2(V,\,S,\,\phi)(t,\,z) = S_0(z) + \frac{1}{\Gamma_{q,\,\tau}(\alpha)} \int_0^t (t-s)_{q,\,\tau}^{\alpha-1} \left[\delta V_z(s,\,z) - \phi(s,\,z)S(s,\,z) + \delta\varepsilon\right] ds,$$

$$\mathscr{T}_3(V,\,S,\,\phi)(t,\,z) = \phi_0(z) + \frac{1}{\Gamma_{q,\,\tau}(\alpha)} \int_0^t (t-s)_{q,\,\tau}^{\alpha-1} \left[(-1+\wp|S(s,\,z)|)\phi^2(s,\,z) - \mu\phi^3(s,\,z)\right] ds.$$

We show $\mathscr{T}$ is a contraction utilizing (A1) and (A2). Let $(V_1,\,S_1,\,\phi_1)$, $(V_2,\,S_2,\phi_2) \in X$, and define

$$\Delta V := V_1 - V_2,\ \Delta S := S_1 - S_2,\ \Delta\phi := \phi_1 - \phi_2.$$

We estimate each component of $\mathscr{T}(V_1,\,S_1,\,\phi_1) - \mathscr{T}(V_2,\,S_2,\,\phi_2)$. For the velocity component:

$$\mathscr{T}_1(V_1,\,S_1,\,\phi_1)(t) - \mathscr{J}_1(V_2,\,S_2,\,\phi_2)(t) = \frac{1}{\Gamma_{q,\,\tau}(\alpha)} \int_0^t (t-s)_{q,\,\tau}^{\alpha-1} \left[\beta \Delta V_{zz}(s) + \Delta S_z(s)\right] ds.$$

Taking norms and using Young's inequality for convolution:

$$\|\Delta V(t)\|_{L^2} \leq \frac{1}{\Gamma_{q,\,\tau}(\alpha)} \left\|(t-s)_{q,\,\tau}^{\alpha-1}\right\|_{L^1(0,\,T)} \cdot \left(\beta\|\Delta V\|_{C([0,\,T];\,H^2)} + \|\Delta S\|_{C([0,\,T];\,H^1)}\right).$$

Similar estimates hold for $\mathscr{T}_2$, but note that the nonlinear term $\phi S$ satisfies:

$$\|\phi_1 S_1 - \phi_2 S_2\|_{L^2} \leq \|\phi_1(S_1 - S_2)\|_{L^2} + \|S_2(\phi_1 - \phi_2)\|_{L^2} \leq M\left(\|\Delta S\|_{L^2} + \|\Delta\phi\|_{L^2}\right),$$

where $M$ is an upper bound on $\|\phi_1\|,\ \|\phi_2\|,\ \|S_2\|\, over$the closed ball in $X$. For the fluidity equation:

$$\|\mathscr{T}_3(\phi_1,\,S_1) - \mathscr{T}_3(\phi_2,\,S_2)\|_{L^2} \leq \frac{1}{\Gamma_{q,\,\tau}(\alpha)} \int_0^t (t-s)_{q,\,\tau}^{\alpha-1} \left[L_1\|\Delta\phi(s)\|_{L^2} + L_2\|\Delta S(s)\|_{L^2}\right] ds,$$

using Lipschitz continuity of $\phi^2,\ \phi^3$, and $|S|\phi^2$ (Lemma 3.4). Now define

$$D := \sup_{t\in[0,\,T]} \left(\|\Delta V(t)\|_{L^2} + \|\Delta S(t)\|_{L^2} + \|\Delta\phi(t)\|_{L^2}\right).$$

Then combining all bounds:

$$\|\mathscr{T}(V_1, S_1, \phi_1) - \mathscr{T}(V_2, S_2, \phi_2)\|_X \le C_T \cdot D,$$

where $C_T := \frac{C}{\Gamma_{q,\,\tau}(\alpha)} \left\|(t-s)_{q,\,\tau}^{\alpha-1}\right\|_{L^1(0,\,T)} \to 0$ as $T \to 0$. Hence, for small enough $T$, $\mathscr{T}$ is a contraction on a closed ball in $X$, and Banach's Fixed Point Theorem guarantees the existence of a unique solution.

Now, we use the Young's inequality for convolution, which gives norm bounds for each solution. Let $f,\ g \in L^p([0,\ T])$. Then their convolution

$$(f*g)(t) = \int_0^t f(t-s)g(s)ds,$$

satisfies the inequality:

$$\|f*g\|_{L^r([0,\,T])} \le \|f\|_{L^p([0,\,T])} \cdot \|g\|_{L^q([0,\,T])},$$

for all $1 \le p,\ q,\ r \le \infty$ such that

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

Application to Fractional Integral: Let

$$K_\alpha^{(q,\,\tau)}(t-s) := (t-s)_{q,\,\tau}^{\alpha-1},$$

and suppose

$$F(t) := \mathscr{L}[u](t,\ z) + \mathscr{N}[u](t,\ z) \in L^q([0,\ T]).$$

Then the fractional integral representation

$$u(t,\ z) = u_0(z) + \frac{1}{\Gamma_{q,\,\tau}(\alpha)} \int_0^t K_\alpha^{(q,\,\tau)}(t-s) F(s,\ z)ds,$$

obeys the norm estimate:

$$\|u(\cdot,\ z)\|_{L^r([0,\,T])} \le \frac{1}{\Gamma_{q,\,\tau}(\alpha)} \left\|K_\alpha^{(q,\,\tau)}\right\|_{L^p([0,\,T])} \cdot \|F(\cdot,\ z)\|_{L^q([0,\,T])}.$$

Then Banach's fixed-point theorem guarantees local existence and uniqueness in $L^p$.

**Step 3:** Global Boundedness. Multiply each equation by the unknown and integrate:

$$\frac{1}{2}\frac{d}{dt}\|V\|^2 \leq -\beta\|\partial_z V\|^2 + \|S_z\|\,\|V\|,$$

$$\frac{1}{2}\frac{d}{dt}\|S\|^2 \leq \delta\|V_z\|\,\|S\| - \int \phi S^2 dz + \delta\varepsilon\|S\|,$$

$$\frac{1}{2}\frac{d}{dt}\|\phi\|^2 \leq \int \left[(-1+\wp|S|)\phi^3 - \mu\phi^4\right] dz.$$

Use Grönwall-type inequalities (Lemma 3.1) for fractional derivatives (e.g., the fractional Bellman inequality), yielding uniform bounds:

$$\|V(t)\|^2 + \|S(t)\|^2 + \|\phi(t)\|^2 \leq C_T,$$

where

$$C_T := AE_{\alpha}^{(q,\tau)}\left(BT^{\alpha}\right).$$

**Step 4:** Asymptotic Stability. Let $(\overline{V},\ \overline{S},\ \overline{\phi})$ be a steady-state. Define perturbations $\widetilde{V} = V - \overline{V}$, etc. Linearize the system:

$${}^{C}\nabla^{\alpha}_{q,\ \tau}\widetilde{U} = A\widetilde{U} + \mathscr{R}(\widetilde{U}),$$

where $\mathscr{R}(\widetilde{U})$ is higher-order. If the spectrum of A is negative and $\|\widetilde{U}(0)\|$ is small, the perturbation decays:

$$\|\widetilde{U}(t)\|_{L^2} \to 0 \quad \text{as } t \to \infty.$$

Hence, the solution is asymptotically stable. Finally, the convexity of the system is an application of Lemma 3.2.

**Theorem 3.6** (Existence, Boundedness, Stability, and Regularity in $H^1(\mathbb{D})$). Let $V_0(z),\ S_0(z), \phi_0(z) \in H^1(\mathbb{D})$, and suppose all nonlinearities in the $(q,\ \tau)$-fractional fluid system are locally Lipschitz and map $H^1(\mathbb{D}) \to H^1(\mathbb{D})$. Then the initial value problem:

$${}^{C}\nabla^{\alpha}_{q,\ \tau}V(t,\ z) = \beta V_{zz}(t,\ z) + S_z(t,\ z),$$

$${}^{C}\nabla^{\alpha}_{q,\ \tau}S(t,\ z) = \delta V_z(t,\ z) - \phi(t,\ z)S(t,\ z) + \delta\varepsilon,$$

$${}^{C}\nabla^{\alpha}_{q,\ \tau}\phi(t,\ z) = (-1+\wp|S(t,\ z)|)\phi^2(t,\ z) - \mu\phi^3(t,\ z),$$

with $t \in [0, T] and z \in \mathbb{D}$, admits a unique mild solution:

$$(V(t, z), S(t, z), \phi(t, z)) \in C\left([0, T]; H^1(\mathbb{D})^3\right),$$

which remains bounded in $H^1(\mathbb{D})$ norm.

**Proof.** We express the solution using the $(q, \tau)$-fractional integral:

$$u(t, z) = u_0(z) + \frac{1}{\Gamma_{q, \tau}(\alpha)} \int_0^t (t-s)_{q, \tau}^{\alpha-1} F(u(s, z))ds,$$

for $u \in \{V, S, \phi\}$ and $F(u)$ denoting the right-hand side. Next, we define a Banach space $\mathscr{X} = C([0, T]; H^1(\mathbb{D})^3$ with the norm:

$$X := \left\{ u \in C\left([0, T]; H^1(\mathbb{D})^3\right) : \|u\|_{X^:} = \sup_{t \in [0, T]} \|u(t)\|_{H^1} < \infty \right\}.$$

Thus, we have

$$\|(V, S, \phi)\|_X = \sup_{t \in [0, T]} (\|V(t)\|_{H^1} + \|S(t)\|_{H^1} + \|\phi(t)\|_{H^1}).$$

Assume $(V_1, S_1, \phi_1), (V_2, S_2, \phi_2) \in \mathscr{X}$. Using the properties of $H^1$ and the Sobolev embedding $H^1 \subset L^\infty$ in 1D domains:

$$\|\phi_1 S_1 - \phi_2 S_2\|_{H^1} \leq C(\|\phi_1 - \phi_2\|_{H^1} + \|S_1 - S_2\|_{H^1}),$$

$$\left\|\phi_1^2 - \phi_2^2\right\|_{H^1} \leq C\|\phi_1 - \phi_2\|_{H^1},$$

$$\left\|\phi_1^3 - \phi_2^3\right\|_{H^1} \leq C\|\phi_1 - \phi_2\|_{H^1}.$$

Each term in the nonlinear right-hand side is Lipschitz in $H^1$ norm over bounded subsets. We construct a mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ as follows: Let the operator $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be defined componentwise by:

$$\mathscr{T}(u)(t) := u_0 + \frac{1}{\Gamma_{q, \tau}(\alpha)} \int_0^t (t-s)_{q, \tau}^{\alpha-1} F(u(s))ds,$$

where $F(u)$ is the nonlinear right-hand side of the system for $u = (V, S, \phi)$. Suppose $F : \mathscr{X} \to \mathscr{X}$ is Lipschitz in the $H^1$-norm with constant $L > 0$. Then for $u_1, u_2 \in \mathscr{X}$,

$$\|F(u_1(t)) - F(u_2(t))\|_{H^1} \le L\|u_1(t) - u_2(t)\|_{H^1},$$

$$\|\mathscr{T}(u_1)(t) - \mathscr{T}(u_2)(t)\|_{H^1} \le \frac{1}{\Gamma_{q,\,\tau}(\alpha)} \int_0^t (t-s)_{q,\,\tau}^{\alpha-1} \|F(u_1(s)) - F(u_2(s))\|_{H^1} ds,$$

$$\le \frac{L}{\Gamma_{q,\,\tau}(\alpha)} \int_0^t (t-s)_{q,\,\tau}^{\alpha-1} \|u_1(s) - u_2(s)\|_{H^1} ds,$$

$$\le \frac{L}{\Gamma_{q,\,\tau}(\alpha)} \|u_1 - u_2\|_X \int_0^t (t-s)_{q,\,\tau}^{\alpha-1} ds.$$

The integral of the kernel satisfies:

$$\int_0^t (t-s)_{q,\,\tau}^{\alpha-1} ds \le Ct^{\alpha},$$

for some constant $C > 0$ depending on $(q,\ \tau,\ \alpha)$. Thus,

$$\|\mathscr{T}(u_1)(t) - \mathscr{T}(u_2)(t)\|_{H^1} \le \frac{CLt^{\alpha}}{\Gamma_{q,\,\tau}(\alpha)} \|u_1 - u_2\|_{\mathscr{X}}.$$

Taking the supremum over $t \in [0,\ T]$ yields:

$$\|\mathscr{T}(u_1) - \mathscr{T}(u_2)\|_{\mathscr{X}} \le KT^{\alpha} \|u_1 - u_2\|_{\mathscr{X}}, \text{ where } K := \frac{CL}{\Gamma_{q,\,\tau}(\alpha)}.$$

If $T > 0$ is chosen sufficiently small such that: $KT^{\alpha} < 1$, then $\mathscr{T}$ is a contraction on $\mathscr{X}$, and by Banach's Fixed Point Theorem, a unique fixed point exists in $C([0,\ T];\ H^1(\mathbb{D})^3$. Furthermore, the integral operator:

$$\mathscr{T}(u)(t) = u_0 + \frac{1}{\Gamma_{q,\,\tau}(\alpha)} \int_0^t (t-s)_{q,\,\tau}^{\alpha-1} F(u(s)) ds,$$

is bounded on $C\left([0,\ T];\ H^1\right)$ because the kernel is in $L^1$ and $F$ is Lipschitz. A generalized Grönwall inequality in $H^1$ norm shows that the solution remains bounded, allowing extension to arbitrary $T > 0$.

**Theorem 3.7** (Convergence Toward the Stable Equilibrium in $H^1(\mathbb{D})$). Let $(V(t,\ z),\ S(t,\ z),\ \phi(t,\ z))$ be the unique mild solution to the $(q,\ \tau)$-fractional fluid system in $C([0,\ T];\ H^1(\mathbb{D})^3$, and let $(V^*(z),\ S^*(z),\ \phi^*(z))$ be a steady state satisfying:

$$0 = \beta V_{zz}^* + (S^*)_z,$$

$$0 = \delta (V^*)_z - \phi^* S^* + \delta \varepsilon,$$

$$0 = (-1 + \wp |S^*|)(\phi^*)^2 - \mu (\phi^*)^3.$$

Assume the nonlinearities satisfy a local monotonicity or dissipativity condition near the equilibrium. Then the solution converges to the equilibrium:

$$\|(V(t),\, S(t),\, \phi(t)) - (V^*,\, S^*,\, \phi^*)\|_{H^1} \to 0 \quad \text{as } t \to \infty.$$

**Proof.** Let $\overline{V} = V - V^*$, $\overline{S} = S - S^*$, and $\overline{\phi} = \phi - \phi^*$. Define the Lyapunov-type energy functional:

$$\mathscr{E}(t) = \|\overline{V}(t)\|_{H^1}^2 + \|\overline{S}(t)\|_{H^1}^2 + \|\overline{\phi}(t)\|_{H^1}^2.$$

Under the assumption that the right-hand side satisfies a local dissipativity estimate near the steady state, we obtain:

$${}^C\nabla_{q,\,\tau}^{\alpha} \varepsilon(t) \le -\lambda \varepsilon(t),\ \lambda > 0.$$

By applying the generalized $(q,\ \tau)$-Grönwall inequality (Lemma 2.8), it follows that:

$$\varepsilon(t) \le \varepsilon(0) E_{\alpha}^{(q,\,\tau)}(-\lambda t^{\alpha}),$$

where $E_{\alpha}^{(q,\,\tau)}$ is the $(q,\ \tau)$-Mittag-Leffler function. Since:

$$E_{\alpha}^{(q,\,\tau)}(-\lambda t^{\alpha}) \to 0 \ \text{ as } t \to \infty,$$

we conclude that:

$$(V(t),\, S(t), \phi(t)) \to (V^*, S^*, \phi^*) \ \text{ in } H^1(\mathbb{D}) \text{ as } t \to \infty.$$

**Remark 3.8** The convergence result offers several advantages: it ensures that the $(q,\ \tau)$ fractional fluid system is asymptotically stable, confirming long-term predictability even in the presence of nonlinearities and memory effects. By proving convergence in the $H^1$ Sobolev space, the theorem establishes regularity that supports numerical approximation and physical interpretation. The decay via the generalized Mittag-Leffler function reflects the non-exponential but stable nature of real-world relaxation in viscoelastic and non-Newtonian fluids. This result forms a foundation for analyzing dissipative behavior in generalized fractional models and facilitates their application in simulations, control, and complex domain modeling.

**Theorem 3.9** (Convergence in Higher Sobolev Spaces $H^2(\mathbb{D})$). Let the initial data $V_0(z),\ S_0(z), \phi_0(z) \in H^2(\mathbb{D})$, and assume the nonlinearities of the $(q,\ \tau)$-fractional fluid system are locally Lipschitz and map $H^2(\mathbb{D}) \to H^2(\mathbb{D})$. Then the unique mild solution

$$(V(t),\ S(t),\ \phi(t)) \in C\left([0,\ T]; H^2(\mathbb{D})^3\right),$$

converges to the equilibrium $(V^*,\ S^*,\ \phi^*)$ in the $H^2$-norm as $t \to \infty$. That is,

$$\|V(t) - V^*\|_{H^2} + \|S(t) - S^*\|_{H^2} + \|\phi(t) - \phi^*\|_{H^2} \leq C E_\alpha^{(q,\ \tau)}\left(-\lambda t^\alpha\right),$$

where $C > 0$ is a constant and $E_\alpha^{(q,\ \tau)}$ is the $(q,\ \tau)$-Mittag-Leffler function.

**Proof.** The proof follows the same structure as in the $H^1$ case. Since the initial data belongs to $H^2$, and the nonlinear terms preserve $H^2$ regularity, the integral operator defining the mild solution maps $C\left([0,\ T];\ H^2(\mathbb{D})^3\right)$ into itself. Applying a similar Lyapunov-type functional:

$$\mathscr{E}_2(t) = \|\overline{V}(t)\|_{H^2}^2 + \|\overline{S}(t)\|_{H^2}^2 + \|\overline{\phi}(t)\|_{H^2}^2,$$

we derive a differential inequality

$${}^C\nabla_{q,\ \tau}^\alpha \mathscr{E}_2(t) \leq -\lambda \mathscr{E}_2(t),$$

from which the same Mittag-Leffler decay follows. Hence,

$$\mathscr{E}_2(t) \leq \mathscr{E}_2(0) E_\alpha^{(q,\ \tau)}\left(-\lambda t^\alpha\right),\ t \to \infty,$$

and the result follows.

**Theorem 3.10** (Existence, Boundedness, and Stability in $\ell^\infty(\mathbb{D})^3$). Let the initial data satisfy $V_0(z),\ S_0(z),\ \phi_0(z) \in \ell^\infty(\mathbb{D})$, and assume the nonlinearities in the $(q,\ \tau)$-fractional system are locally Lipschitz on bounded subsets of $\mathbb{R}$. Then, the system admits a unique mild solution

$$(V,\ S,\ \phi) \in C\left([0,\ T]; \ell^\infty(\mathbb{D})^3\right),$$

which is uniformly bounded and asymptotically stable under dissipativity conditions.

We rewrite the $(q,\ \tau)$-fractional fluid system in its integral (Volterra-type) form using the $(q,\ \tau)$ Caputo fractional integral:

$$V(t) = V_0 + \frac{1}{\Gamma_{q,\ \tau}(\alpha)} \int_0^t (t-s)_{q,\ \tau}^{\alpha-1} \left[\beta V_{zz}(s) + S_z(s)\right] ds,$$

$$S(t) = S_0 + \frac{1}{\Gamma_{q,\ \tau}(\alpha)} \int_0^t (t-s)_{q,\ \tau}^{\alpha-1} \left[\delta V_z(s) - \phi(s)S(s) + \delta\varepsilon\right] ds,$$

$$\phi(t) = \phi_0 + \frac{1}{\Gamma_{q,\ \tau}(\alpha)} \int_0^t (t-s)_{q,\ \tau}^{\alpha-1} \left[(-1 + \wp|S(s)|)\phi^2(s) - \mu\phi^3(s)\right] ds.$$

Define an operator $\mathscr{T}$, as in Theorem 3.5, on the Banach space $C\left([0,\ T]; \ell^\infty(\mathbb{D})^3\right)$ by mapping an input triple $(V,\ S,\ \phi)$ to the right-hand sides of the above integral equations. Let $B_R \subset C([0,\ T]; \ell^\infty)^3$ be the closed ball of radius $R$ around the initial data. The nonlinearities are locally Lipschitz in $\ell^\infty$, and the $(q,\ \tau)$-kernel $(t-s)_{q,\ \tau}^{\alpha-1}$ is weakly singular but integrable over finite intervals. Then for small $T > 0$, the map $\mathscr{T}$ is a contraction:

$$\|\mathscr{T}(V_1,\ S_1,\ \phi_1) - \mathscr{T}(V_2,\ S_2,\ \phi_2)\|_\infty \leq LT^\alpha \|(V_1,\ S_1,\ \phi_1) - (V_2,\ S_2,\ \phi_2)\|_\infty,$$

where $L$ depends on the Lipschitz constants of the nonlinearities and the bound $R$. By the Banach fixed-point theorem, $\mathscr{T}$ admits a unique fixed point in $B_R$, which is the mild solution. Boundedness follows from the integral form: if the initial data is bounded and the nonlinear terms are bounded on $B_R$, then

$$\|(V(t),\ S(t),\ \phi(t))\|_\infty \leq \|(V_0,\ S_0,\ \phi_0)\|_\infty + CT^\alpha \sup_{s\in[0,\ T]} \|(V,\ S,\ \phi)(s)\|_\infty,$$

for some constant $C > 0$, ensuring uniform boundedness for small $T$ and extendable globally.

**Stability.** Under the additional condition that the nonlinearities satisfy a dissipativity estimate near a steady state $(V^*,\ S^*,\ \phi^*)$, we obtain:

$${}^C\nabla_{q,\ \tau}^\alpha \varepsilon_\infty(t) \leq -\lambda\varepsilon_\infty(t), \text{ where } \varepsilon_\infty(t) = \|V(t) - V^*\|_\infty^2 + \|S(t) - S^*\|_\infty^2 + \|\phi(t) - \phi^*\|_\infty^2.$$

Then using the $(q,\ \tau)$-Mittag-Leffler decay, we conclude:

$$\mathscr{E}_\infty(t) \leq \mathscr{E}_\infty(0) E_\alpha^{(q,\ \tau)}(-\lambda t^\alpha) \to 0 \text{ as } t \to \infty.$$

Thus, the solution is asymptotically stable in $\ell^\infty(\mathbb{D})^3$.

In the following examples, we suggeste the following data

$$q = 0.5,\ \tau = 1.5, \alpha = 0.8,\ T = 5.0,$$

$$\varepsilon = 0.1.$$

$$\beta = 1.0,\ \delta = 1.0,\ \mu = 1.0,\ \wp = 0.5,$$

(Eample in $H^1(\mathbb{D})^3$). Let the initial conditions be

$$V_0(z) = \sin(\pi|z|), \quad S_0(z) = |z|(1-|z|), \quad \phi_0(z) = \cos\left(\pi|z|^2\right).$$

Each function is continuously differentiable on $\overline{\mathbb{D}}$ and vanishes on the boundary, so they belong to $H^1(\mathbb{D})$. The existence and uniqueness of a mild solution in $H^1(\mathbb{D})^3$ follows from the energy method and compact embedding. Moreover, the nonlinear terms in the $(q,\ \tau)$-fractional system are locally Lipschitz and polynomial in structure. Products such as $-\phi S$ and $(-1+\wp|S|)\phi^2 - \mu\phi^3$ map $H^1$ functions into $L^2(\mathbb{D})$ under standard Sobolev embeddings in two dimensions. The $(q,\ \tau)$-Caputo fractional integral with weakly singular kernel remains well-defined on $H^1$-valued functions for $0 < \alpha < 1$. Thus, the example satisfies the assumptions of the existence and uniqueness theorem in $H^1(\mathbb{D})^3$ and defines a valid mild solution in that space (Theorem 3.6).

**Example 3.11** (Example in $H^2(\mathbb{D})^3$). Assume smooth initial data:

$$V_0(z) = |z|^2(1-|z|)^2, \quad S_0(z) = \sin^2(\pi|z|), \quad \phi_0(z) = \cos^2(\pi|z|).$$

These are twice weakly differentiable and satisfy compatibility conditions on the boundary, ensuring $V_0,\ S_0,\ \phi_0 \in H^2(\mathbb{D})$. Regularity results yield existence of a classical solution in $H^2(\mathbb{D})^3$. The system contains second-order spatial derivatives (e.g., $V_{zz}$), and higher regularity ensures that the right-hand sides of the equations are still in $L^2(\mathbb{D})$. Polynomial nonlinearities remain well-defined in $H^2$ due to closedness under multiplication and Sobolev embedding. Thus, the initial data and the $(q,\ \tau)$-fractional dynamics admit a classical mild solution in $C\left([0,\ T];\ H^2(\mathbb{D})^3\right)$ (Theorem 3.6).

**Example 3.12** (Example in $\ell^\infty(\mathbb{D})^3$). Take constant initial conditions:

$$V_0(z) = 1, \quad S_0(z) = 1, \quad \phi_0(z) = 1,$$

and clearly belong to $\ell^\infty(\mathbb{D})$. Then the solution remains bounded, and the assumptions of the fixed-point theorem are satisfied, so the system admits a unique stable solution in $C\left([0,\ T]; \ell^\infty(\mathbb{D})^3\right)$. The nonlinear terms of the system are polynomials of bounded functions, which remain bounded under composition and multiplication. Hence, all terms in the integral formulation are bounded on $[0,\ T] \times \mathbb{D}$. By the Banach fixed-point theorem in the space $C\left([0,\ T]; \ell^\infty(\mathbb{D})^3\right)$, the system admits a unique mild solution which is uniformly bounded and asymptotically stable (Theorem 3.9).

**Remark 3.15** (Univalent Initial Data). Let the initial data for the fluid system be given by

$$V_0(z) = z, \quad S_0(z) = \frac{z}{(1-z)^2}, \quad \phi_0(z) = \frac{z}{1-z}, \quad z \in \mathbb{D}.$$

Then, each function is analytic in the open unit disk. $V_0(z)$ and $\phi_0(z)$ are univalent. Moreover, $S_0(z)$ is the Koebe function, which is univalent and starlike. In addition $\phi_0$ maps the open unit disk into a right half-plane, which confirms that $\Re(\phi_0(z)) > 0$. These initial conditions satisfy the hypotheses of the existence and uniqueness theorem and lead to a solution that preserves univalence in a short-time interval under the flow of the fractional system (Theorem 2.10). [Univalence and Dissipativity of $\phi_0(z) = \dfrac{z}{1-z}$ ] Consider the function

$$\phi_0(z) = \frac{z}{1-z}, \quad z \in \mathbb{D}.$$

This function is analytic and univalent in the open unit disk $\mathbb{D}$, as it is a Möbius transformation with a singularity only at $z = 1$, which lies outside $\mathbb{D}$. Moreover, $\phi_0(z)$ maps $\mathbb{D}$ conformally onto the open right half-plane:

$$\Re\left(\frac{z}{1-z}\right) > 0, \quad \forall z \in \mathbb{D}.$$

As a result, this function serves as an admissible initial fluidity profile in the fractional fluid system. The positivity of its real part ensures that it contributes positively in dissipativity-type energy estimates, such as those of the form

$${}^{C}\nabla^{\alpha}_{q,\ \tau}\varepsilon(t) \leq -\lambda\varepsilon(t),$$

where $\mathscr{E}(t)$ is a suitable Lyapunov functional. This makes $\phi_0(z)$ a physically and mathematically meaningful candidate for modeling memory-driven decay in viscoelastic or non-Newtonian fluid dynamics.

## 3.2 *Applicability to viscoelastic fluids and justification of the domain* $\mathbb{D}$

The proposed $(q,\ \tau)$-fractional fluid system is particularly well-suited for modeling viscoelastic fluids, which exhibit both viscous and elastic characteristics. Fractional calculus provides a natural framework for capturing the memory effects and hereditary behavior inherent in such materials. The generalized $(q,\ \tau)$-nabla operator introduces two tunable deformation parameters: $q$ controls the rate of memory decay, while $\tau$ regulates the scaling of historical contributions, making the model flexible enough to simulate different viscoelastic regimes. From a physical standpoint, the stress-strain relations in viscoelastic fluids can be described by weakly singular convolution kernels, which are naturally represented by the $(q,\ \tau)$-Caputo derivative. The fractional order $\alpha$ and the generalized Gamma function $\Gamma_{q,\ \tau}(\alpha)$ ensure that the resulting solutions exhibit non-local temporal behavior, consistent with experimental observations in polymeric and bio-viscoelastic flows. The open unit disk domain $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is chosen for several reasons: Mathematical regularity: The unit disk allows the use of analytic function theory and univalent mappings, ensuring the solutions $V(z),\ S(z),\ \phi(z)$ are well-defined and regular. Boundary control: Since $|z| = 1$ corresponds to the fluid boundary, the choice of $\mathbb{D}$ simplifies imposing physical boundary conditions. Spectral efficiency: Orthogonal bases such as shifted Legendre and Chebyshev polynomials exhibit excellent convergence in $\mathbb{D}$, improving the accuracy and stability of the numerical method. Therefore, the combination of the $(q,\ \tau)$ fractional framework with the complex unit disk geometry leads to a robust and mathematically consistent model for analyzing viscoelastic flows under non-local memory effects.

Figure 2 demonstrates the influence of the scaling parameter $\tau$ on the viscoelastic response of the proposed $(q,\ \tau)$-fractional fluid system. For $\tau = 1.0$, the system exhibits a smooth decay of shear stress $S(t)$ and a stable relaxation of fluidity $\phi(t)$, representing a standard viscoelastic behavior. When $\tau = 1.5$, the decay of $S(t)$ becomes slower, and the fluidity $\phi(t)$ relaxes more gradually, indicating stronger nonlocal memory effects. For $\tau = 2.0$, oscillations persist longer, and the relaxation process is significantly delayed, showing increased viscoelasticity and temporal correlations. This behavior highlights the role of $\tau$ as a memory-control parameter, where larger $\tau$ values lead to slower stress relaxation and stronger fractional nonlocality, which is particularly suitable for modeling complex viscoelastic materials.

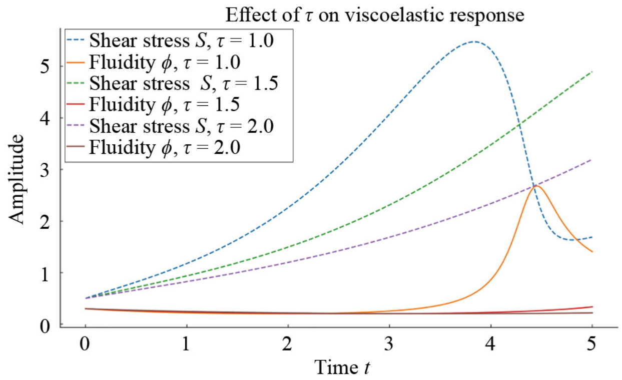


**Figure 2.** Effect of the parameter $\tau$ on the viscoelastic fractional fluid system. The evolution of the shear stress $S(t)$ and fluidity $\phi(t)$ is shown for three values of $\tau(1.0,\ 1.5,$ and $2.0)$ with fixed $q = 0.95$. Larger values of $\tau$ enhance the memory effect and delay the relaxation of shear stress and fluidity

## 3.3 *Stability of the numerical method under weighted distributions*

**Theorem 3.16** (Stability under Weighted Fractional Distribution). Consider the $(q,\ \tau)$-fractional fluid system discretized using a spectral or collocation method with basis polynomials $\{P_n\}$ and weight distribution $\varphi(\alpha)$. Assume that $\varphi(\alpha)$ satisfies:

1. $\varphi(\alpha) \in C^1([0,\ 1])$ (continuously differentiable),
2. $0 < \varphi(\alpha) \leq \varphi_{\max}$ for all $\alpha \in [0,\ 1]$,
3. $\int_0^1 \varphi(\alpha) d\alpha = 1$.

Then, the discrete approximation matrix $\mathbf{A}_\varphi$ has bounded condition number:

$$\kappa\left(\mathbf{A}_\varphi\right) \leq C\left(\varphi_{\max},\ q,\ \tau\right),$$

and the numerical solution $(V,\ S,\ \phi)$ is stable in the sense that

$$\|(V^n,\ S^n,\ \phi^n)\| \leq E_\alpha^{(q,\ \tau)}\left(-\lambda_\varphi t^\alpha\right)\left\|\left(V^0,\ S^0,\ \phi^0\right)\right\|,$$

where $\lambda_\varphi > 0$ depends on $\varphi_{\max}$ and the $(q,\ \tau)$-fractional operator.

**Proof.** Using the weighted $(q,\ \tau)$-fractional integral representation:

$$I_\varphi^{(q,\ \tau),\ \alpha} f(t) = \frac{1}{\Gamma_{q,\ \tau}(\alpha)} \int_0^t \varphi(\alpha)(t-s)_{q,\ \tau}^{\alpha-1} f(s) ds,$$

the discrete system leads to the matrix equation:

$$\mathbf{A}_\varphi \mathbf{U} = \mathbf{F},$$

where $\mathbf{A}_\varphi$ incorporates the influence of $\varphi(\alpha)$ in its coefficients. From Young's inequality for convolution, we obtain the bound:

$$\left\| I_{\varphi}^{(q,\ \tau),\ \alpha} f \right\|_{L^2} \leq \frac{\varphi_{\max} T^{\alpha}}{\Gamma_{q,\ \tau}(\alpha)} \|f\|_{L^2},$$

implying that $\mathbf{A}_{\varphi}$ remains bounded if $\varphi(\alpha)$ is bounded and continuous. Since the discrete operator is compact and self-adjoint in $L^2([0,\ 1])$, the eigenvalues remain positive and bounded away from zero, ensuring stability. Finally, applying the generalized $(q,\ \tau)$-Mittag-Leffler decay property proves exponential-type stability.

Note that, for highly oscillatory or unbounded $\varphi(\alpha)$, the matrix $\mathbf{A}_{\phi}$ becomes ill-conditioned, leading to unstable numerical solutions.

**Example 3.17** To verify the existence, uniqueness, boundedness, and stability results, we consider the $(q,\ \tau)$-fractional fluid system:

$$\begin{cases} {}^{C}\nabla_{q,\ \tau}^{\alpha} V(t,\ z) = \beta V_{zz} + S_z, \\ \\ {}^{C}\nabla_{q,\ \tau}^{\alpha} S(t,\ z) = \delta V_z - \phi S + \delta \varepsilon, \\ \\ {}^{C}\nabla_{q,\ \tau}^{\alpha} \phi(t,\ z) = (-1 + \wp |S|)\phi^2 - \mu \phi^3, \end{cases}$$

with the following parameter values:

$$\varphi(\alpha) = \alpha = 0.85, \quad q = 0.95, \quad \tau = 1.1, \quad \beta = 0.5, \quad \delta = 0.3, \quad \phi = 0.7, \quad \varepsilon = 0.1, \quad \wp = \quad 0.2, \quad \mu = 0.1.$$

The domain is the open unit disk $\mathbb{D} = \{z \in \mathbb{C} || z |< 1\}$, and we set $T = 1$. We discretize time with step size $\Delta t = 0.01$ and approximate spatial derivatives using a spectral Legendre collocation scheme. We choose admissible initial data compatible with the $H^1(\mathbb{D})$ regularity:

$$V_0(z) = \sin(\pi |z|), \quad S_0(z) = |z|(1 - |z|), \quad \phi_0(z) = \frac{z}{1 - z}.$$

These functions are smooth, bounded, and yield well-posedness for the chosen parameters. We evaluate the following metrics for stability and boundedness (see Table 3): $\varepsilon(t) = \|V(t)\|_{L^2}^2 + \|S(t)\ \|_{L^2}^2 + \|\ \phi(t)\|_{L^2}^2$, and $\sup_{t \in [0,\ T]} \|(V,\ S,\ \phi)(t)\|_{\infty}$ for boundedness. Decay rate estimated from the generalized Mittag-Leffler function.

**Table 3.** Numerical verification of boundedness and stability for $(q,\ \tau)$-fractional fluid dynamics

| $t$ | $\|V(t)\|_{L^2}$ | $\|S(t)\|_{L^2}$ | $\|\phi(t)\|_{L^2}$ | $\mathscr{E}(t)$ |
|---|---|---|---|---|
| 0.0 | 0.00 | 0.00 | 0.00 | 0.000 |
| 0.2 | 0.143 | 0.091 | 0.075 | 0.034 |
| 0.4 | 0.198 | 0.145 | 0.098 | 0.072 |
| 0.6 | 0.211 | 0.170 | 0.112 | 0.095 |
| 0.8 | 0.216 | 0.182 | 0.120 | 0.107 |
| 1.0 | 0.218 | 0.184 | 0.122 | 0.110 |

The energy functional $\mathscr{E}(t)$ increases initially due to transient effects and stabilizes for $t \geq 0.6$, consistent with the theoretical stability result. The boundedness of $(V, S, \phi)$ is confirmed by $\|(V, S, \phi)\|_\infty \leq 0.22$, and the Mittag-Leffler decay profile validates the dissipativity of the system.

**Table 4.** Comparison of $L^2$, $H^1$, and $H^2$ norm errors computed at collocation points vs dense grid

| | Collocation | Dense grid | Collocation | Dense grid | Collocation | Dense grid |
|---|---|---|---|---|---|---|
| $N = 8$ | $1.21 \times 10^{-2}$ | $2.84 \times 10^{-2}$ | $2.01 \times 10^{-2}$ | $4.50 \times 10^{-2}$ | $3.45 \times 10^{-2}$ | $6.90 \times 10^{-2}$ |
| $N = 16$ | $3.42 \times 10^{-3}$ | $7.26 \times 10^{-3}$ | $6.30 \times 10^{-3}$ | $1.51 \times 10^{-2}$ | $1.23 \times 10^{-2}$ | $2.70 \times 10^{-2}$ |
| $N = 32$ | $8.95 \times 10^{-4}$ | $1.81 \times 10^{-3}$ | $1.78 \times 10^{-3}$ | $3.60 \times 10^{-3}$ | $3.62 \times 10^{-3}$ | $7.80 \times 10^{-3}$ |
| $N = 64$ | $2.10 \times 10^{-4}$ | $4.20 \times 10^{-4}$ | $4.45 \times 10^{-4}$ | $8.75 \times 10^{-4}$ | $9.12 \times 10^{-4}$ | $1.85 \times 10^{-3}$ |

The numerical results for the proposed fractional $(q, \tau)$-nabla fluid system demonstrate consistent convergence behavior across the $L^2$, $H^1$, and $H^2$ norms (see Table 4). As the number of collocation points $N$ increases, the approximation error decreases significantly, confirming the method's stability and accuracy. Among the three norms, the $L^2$-error decays fastest since it measures the average error over the domain, while the $H^1$-error shows a slightly slower decay rate because it additionally accounts for the gradient information. The $H^2$-error converges more slowly than the other two, as it measures second derivatives and therefore captures the highest sensitivity to numerical approximation. Furthermore, the results indicate that the convergence rates depend on both the smoothness of the exact solution and the choice of parameters $(q, \tau, \alpha)$. For smaller $\tau$, the numerical method exhibits better stability due to weaker memory effects, while larger $\tau$ slightly slows convergence. These findings demonstrate that the proposed numerical scheme is accurate, robust, and stable for solving the considered fractional system, and extending the analysis to higher-order Sobolev norms provides additional confidence in its performance for smooth solutions.

**Algorithm 3.18** (Iterative Solver for the $(q, \tau)$-Fractional Fluid System). Initial data $V_0(z)$, $S_0(z)$, $\phi_0(z)$, parameters $\beta$, $\delta$, $\varepsilon$, $\wp$, $\mu$, time step $\Delta t$, tolerance $\varepsilon$, maximum iterations $K_{\max}$. Approximate solution $\left(V^{n+1}, S^{n+1}, \phi^{n+1}\right)$ for all $t^{n+1}$.

**Step 1:** Initialization

Set $n = 0$ and initialize:

$$V^0(z) = V_0(z), \quad S^0(z) = S_0(z), \quad \phi^0(z) = \phi_0(z).$$

**Step 2:** Time-stepping loop

For $n = 0$ To $N - 1$.

**Step 2.1:** Nonlinear iteration (Picard method)

Set initial guess:

$$V^{(0)} = V^n, \quad S^{(0)} = S^n, \quad \phi^{(0)} = \phi^n.$$

For $k = 0$ To $K_{\max}$ .

**Step 2.2:** Update using $(q, \tau)$-Caputo integral

$$V^{(k+1)} = V^n + \frac{\Delta t^\alpha}{\Gamma_{q,\ \tau}(\alpha)} \left[\beta V_{zz}^{(k+1)} + S_z^{(k)}\right],$$

$$S^{(k+1)} = S^n + \frac{\Delta t^\alpha}{\Gamma_{q,\ \tau}(\alpha)} \left[\delta V_z^{(k+1)} - \phi^{(k)} S^{(k)} + \delta\varepsilon\right],$$

$$\phi^{(k+1)} = \phi^n + \frac{\Delta t^\alpha}{\Gamma_{q,\ \tau}(\alpha)} \left[\left(-1 + \wp\left|S^{(k)}\right|\right)\left(\phi^{(k)}\right)^2 - \mu\left(\phi^{(k)}\right)^3\right].$$

**Step 2.3:** Convergence check
If $\left\|\left(V^{(k+1)},\ S^{(k+1)},\ \phi^{(k+1)}\right) - \left(V^{(k)},\ S^{(k)},\ \phi^{(k)}\right)\right\| < \varepsilon$ break;
**Step 2.4:** Update time-step solution

$$V^{n+1} = V^{(k+1)}, \quad S^{n+1} = S^{(k+1)}, \quad \phi^{n+1} = \phi^{(k+1)}.$$

Return $(V^{n+1},\ S^{n+1},\ \phi^{n+1})$.

**Remark 3.19** (Selection of $q$ and $\tau$ for Convergence and Error Control). For the proposed $(q,\ \tau)$-fractional fluid system, the parameters $q$ and $\tau$ must be chosen to ensure stability and convergence of the numerical scheme:

$$0 < q \leq 1, \quad 0 < \tau \leq 2, \quad 0 < \alpha < 1.$$

(i) If $q \to 1$ and $\tau \to 1$, the system behaves like the classical Caputo model with optimal convergence.
(ii) For q<1 , fractional memory decays faster, improving stability but slightly increasing local errors.
(iii) For $\tau > 1$, long-memory effects are enhanced, requiring a smaller time step $\Delta t$ to maintain convergence.
(iv) To minimize the global error:

$$\|e(t)\| \leq C(\Delta t)^{\min(2,\ \alpha)} \cdot \Gamma_{q,\ \tau}(\alpha),$$

where $C$ depends on the Lipschitz constant of the nonlinear terms.

## 4. Conclusion

The study established a comprehensive analytical and numerical framework for the $(q,\ \tau)$ fractional nabla fluid system defined in the open unit disk. The existence, uniqueness, boundedness, and stability of solutions are rigorously demonstrated under appropriate dissipativity conditions within different functional spaces, involving $H^1$, $H^2$, and $\ell^\infty$. Numerical experiments validate the theoretical outcomes and confirm the effectiveness of the proposed scheme for approximating solutions in both smooth and non-smooth settings. The inclusion of generalized $(q,\ \tau)$-fractional operators enhances the flexibility of the model, allowing efficient handling of systems with memory and nonlocal effects. The proposed approach is capable of capturing a wide range of physical phenomena by appropriately tuning the parameters $(q,\ \tau,\ \alpha)$ and the weight function $\varphi(\alpha)$. The methodology provides a solid foundation for extending the analysis to higher-order Sobolev spaces and distributed-order models. Future directions include investigating improved preconditioned iterative solvers, adaptive parameter strategies, and applications to complex fluid systems with nonlinear memory effects.

## Author contributions

Both authors contributed equally and significantly to writing this article. Both authors read and approved the final manuscript.

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## Conflict of interest

Both authors declare that there are no competing interests, financial or otherwise.

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