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Research Article

Fractional-Order Analysis of HTLV, HPV, and HIV Co-Infection Dynamics

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Abstract: Fractional calculus concept has proven to be a great, powerful, and effective tool in analyzing mathematical models across diverse various fields of scientific and engineering domains. A significant feature of this article is to investigate the novel Human T-cell Lymphotropic Virus (HTLV)/Human Papillomavirus (HPV)/Human Immunodeficiency Virus (HIV) multi-infection model, along with a computational numerical study and stability analysis to describe the modified Atangana-Baleanu-Caputo fractional order framework. The model performed stability analysis based on the Ulam-Hyers stability concept can be established by using the solution of existence and uniqueness conditions derived from the fixed-point techniques for the recommended problem. The multi-infection dynamical system behavior is expressed on the approximate solution of a two-step Lagrange interpolation polynomials numerical scheme utilizing a modified Atangana-Baleanu-Caputo fractional order framework, with all implementation and simulations conducted in Matrix Laboratory (MATLAB). Overall judgment shows that the numerical results of the recommended method significantly impact the multi-infection model behavior.

***Keywords*:** fixed-point approach, fractional dynamical system, Human T-cell Lymphotropic Virus (HTLV)/Human Papillomavirus (HPV)/Human Immunodeficiency Virus (HIV) multi-infection model, modified Atangana-Baleanu Caputo fractional order derivative, Ulam-Hyers stability

MSC: 26A33, 34A08, 92D25, 34K20, 65L07

1. Introduction

1.1 *Review of historical literature*

Sexually Transmitted Diseases (STDs) spread through genital, oral, or anal contact and remain a major global public health issue. Nearly one million new STD infections occur daily. These infections are caused by bacteria, viruses, protozoa, and fungi, with viral STDs posing the most significant long-term health risks. Among these, Human Immunodeficiency

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Virus (HIV), Human T-cell Lymphotropic Virus (HTLV), and Human Papillomavirus (HPV) are particularly concerning, as they are linked to Acquired Immune Deficiency Syndrome (AIDS), HTLV-associated diseases, and various cancers, respectively. Despite preventive measures such as vaccination and safe sex practices, these infections continue to spread rapidly [1].

HPV is one of the most widespread STDs, affecting both men and women globally. It is a small, double-stranded DNA virus, with most infections being asymptomatic and clearing naturally. However, persistent HPV infections can lead to genital warts or cancers, with high-risk strains such as HPV-16 and HPV-18 being linked to cervical and anogenital malignancies [2]. The introduction of HPV vaccines has significantly reduced infection rates, though disparities in vaccine access remain a challenge. As cancer cases continue to rise, mathematical models play a crucial role in optimizing vaccination strategies and predicting long-term HPV trends [3].

HTLV-1 is a retrovirus closely related to HIV, with significant health implications worldwide. It is associated with Adult T-cell Leukemia/Lymphoma (ATLL) and neurological disorders such as HTLV-associated myelopathy [4, 5]. The virus is transmitted through sexual contact, breastfeeding, and blood transfusions, with endemic regions in Japan, South America, and Africa. Unlike HIV, HTLV-1 spreads primarily through infected cell proliferation rather than rapid viral replication [6]. Since there is no vaccine or cure, prevention efforts focus on blood screening, safe sex practices, and public health awareness.

HIV is the causative agent of AIDS and affects over 38 million people globally. HIV targets $CD4^+$ T-cells, gradually weakening the immune system and increasing susceptibility to infections [7]. If left untreated, the progressive decline in $CD4^+$ T-cells leads to AIDS, severely compromising immune function. The introduction of Antiretroviral Therapy (ART) has transformed HIV into a manageable chronic condition, significantly reducing mortality rates. However, challenges such as stigma, healthcare access barriers, and drug resistance persist, making mathematical modeling essential for understanding transmission patterns and improving treatment strategies.

Fractional calculus, with a history as extensive as classical differential calculus, has become a pivotal area of research. Fractional Differential Equations (FDEs) extend ordinary differential equations by incorporating non-integer (fractional) orders of integration and differentiation. This advanced mathematical framework has proven highly effective in modeling complex phenomena across various scientific and engineering disciplines, including electromagnetics and control systems. Fundamental contributions by researchers like Riemann, Liouville, Atangana, Baleanu, and Caputo have provided a rigorous basis for fractional order derivatives and integrals, significantly advancing both theory and applications [8–11].

The stability analysis of the Ulam-Hyers concept originated in the mid-20^{th} century from the foundational contributions of Ulam [12] and Hyers [13]. Over time, Ulam-Hyers stability in fractional differential equations has emerged as a vital research area, bridging theoretical mathematics with real-world applications. Its growing significance underscores its role in ensuring the stability and reliability of mathematical models used in diverse fields.

1.2 *Review of related literature*

Mathematical modeling plays a vital role in evaluating the effectiveness of intervention strategies to control the spread of infectious diseases and deepen our understanding of their transmission dynamics. Over the years, both classical and contemporary studies have employed mathematical frameworks to analyze the spread of various infectious diseases. Several researchers have developed models to explore the dynamics of HPV, offering valuable insights into its transmission patterns [14, 15]. Similarly, extensive modeling efforts have focused on HIV, with studies such as [16, 17] contributing to the design of effective control strategies and enhancing the understanding of disease progression.

Beyond single-infection models, multiple studies have addressed the dynamics of co-infections involving combinations of these viruses. For example, the interaction between HPV and HIV has been explored in [18], showing that a reduced basic reproduction number for HPV may curb its spread and potentially delay the onset of AIDS. Additionally, the relationship between HPV and HSV-II has been examined in [19], highlighting that infection with one virus increases susceptibility to the other. Mhlanga [20] also proposed a deterministic co-infection model for HIV and HSV-II that incorporates both biological features and the impact of poor adherence to HSV-II treatment protocols. In [21], the authors presented the fractional order framework for the modified Atangana-Baleanu-Caputo type, which has attracted interest in the literature for its precise depiction of real-world systems and applicability across several fields.

In a recent study [22], the authors developed a model using Hermite polynomials for the HIV-1/HTLV-I co-infection. Similarly, the study in [23] proposed a fractional HIV/AIDS and pneumonia model that assesses the effect of treatment population, highlighting different approaches to modeling the spread of the disease. The authors [24] proposed a fractional order model of breast cancer with the fractional order framework for the modified Atangana-Baleanu-Caputo type. This modified fractional order derivative has demonstrated its use as an effective instrument for the investigation and modeling of intricate processes [25–27]. The findings in this research validate that the fractional order framework for the modified Atangana-Baleanu-Caputo type employed enhances comprehension of the disease's dynamics and features. Kalipeni et al. [28], exploitative illnesses are instances that lead to extremely significant illness and, if left untreated, mortality in HIV/AIDS survivors. In order to illustrate the significant influence of a probabilistic framework condition mentioned in [29], we performed this work to create this paper. Baleanu et al. [30] investigated the approximate solution of Goursat problem by using local fractional operators. In [31], the author explored the Burger's and coupled Burger's equations based on the fractional derivatives.

1.3 *Research gap of the study*

The VOSviewer analysis and the Scopus data search clearly indicate that Ulam-Hyers stability has been extensively researched in mathematical models; however, there has been comparatively little investigation into Ulam-Hyers stability in fractional order models regarding the dynamics of HTLV, HPV, and HIV multi-infection. Most existing studies either concentrate on fractional order models in general disease dynamics or focus on stability analyses within integer-order systems. However, the combined approach of fractional calculus, stability analysis, and multiple disease transmission modeling remains underexplored. This gap highlights the need for a rigorous stability analysis to ensure the reliability of fractional order models in describing real-world multi-infection scenarios.

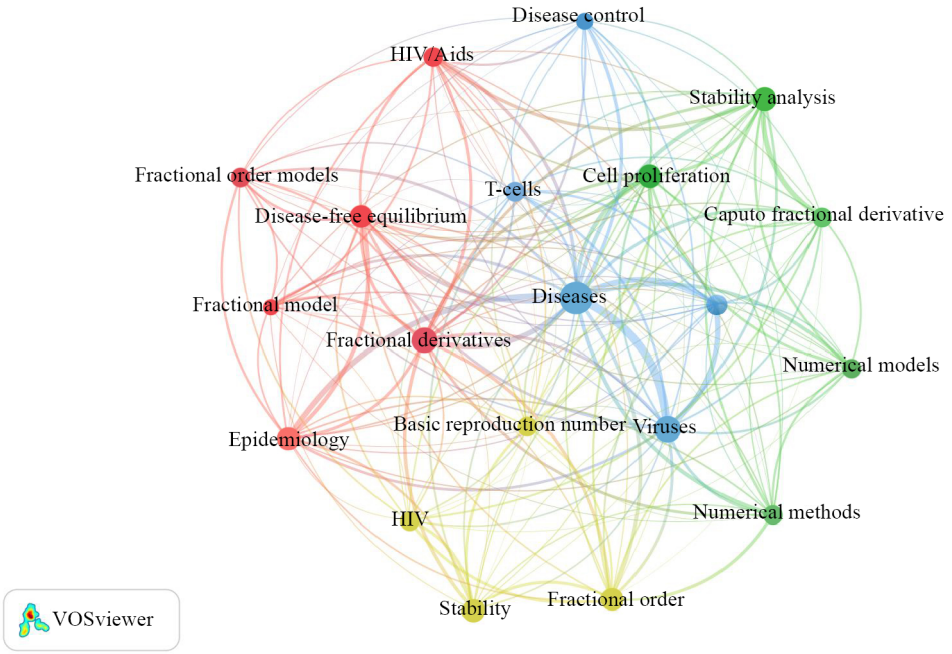

Figure 1. VOSviewer software visualization of research trends related to fractional derivatives, stability analysis, and epidemiological modeling of HIV, HPV, and HTLV

We conducted a Scopus data search using the following keywords:

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(TITLE-ABS-KEY("Hyers-Ulam stability" OR "Ulam-Hyers stability" OR "Stability")

AND TITLE-ABS-KEY("fractional" OR "co-infection")

AND TITLE-ABS-KEY("HTLV" OR "HPV" OR "HIV")).
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This search yielded 121 documents from various fields and methodologies. Using VOSviewer software, we analyzed and identified the research gaps. Figure 1 shows the variation in existing studies based on keyword comparisons.

1.4 *Identified research gaps*

We used VOSviewer software and Scopus data to identify research gaps as follows:

• The fractional calculus concept has been mostly used in previous literature on bio-epidemiology dynamic systems for single disease models. Other than multi-infection scenarios, few studies have investigated the Ulam-Hyers stability concept. Particularly, the research gap identified the Ulam-Hyers stability analysis of the HTLV-HPV-HIV multi-infection model.

• To explore the modified Atangana-Baleanu-Caputo fractional derivative framework for the TLV-HPV-HIV multi-infection model remains largely unexplored.

• Most of the studies discussed fractional calculus based numerical schemes without proof of the theoretical validation.

Our present work aims to address the research gap in stability analysis for the HTLV/HPV/HIV multi-infection model using fractional calculus.

1.5 *Novelty of the study*

While stability analysis of HTLV/HPV/HIV multi-infection models has been explored, the Ulam-Hyers stability of such models using fractional-order derivatives—specifically within the modified Atangana-Baleanu-Caputo framework—remains unaddressed in prior literature. This study aims to bridge this research gap by making the following key contributions:

• This study highlights and addresses the key research gaps in the present literature regarding the stability analysis of an HTLV/HPV/HIV multi-infection model. In particular, we introduce a novel concept that applies the modified Atangana-Baleanu-Caputo fractional order framework to study the Ulam-Hyers stability concept.

• We examine and present a fresh viewpoint on Ulam-Hyers stability that can be performed by using the solution of existence and uniqueness conditions to derive from the fixed-point theorem for the HTLV/HPV/HIV multi-infection model within the fractional calculus framework.

• A numerical scheme based on a two-step Lagrange interpolation polynomial is developed to simulate the multi-infection model under the modified Atangana-Baleanu-Caputo fractional order framework with different fractional orders and parameter values.

• Overall judgment shows that the proposed numerical method achieves significant accuracy and efficiency, profoundly impacting the analysis of system dynamics.

This work identified the previous literature to address the modified Atangana-Baleanu-Caputo fractional order framework of an HTLV/HPV/HIV multi-infection model in the concept of Ulam-Hyers stability.

1.6 *Fundamental concepts of the study*

Here, we provide the fundamental concepts for the recommended method techniques needed in depth in the present work, as described in the previous literature.

Definition 1 [21, 32] Let $0 < \nu_1 < 1$ and function $\ell \in H^1(0, T)$, the fractional order framework for modified Atangana-Baleanu-Caputo type is defined as:

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}(\ell(t)) = \frac{\mathscr{Z}(\nu_1)}{1-\nu_1}\left[\ell(t) - \mathbb{E}_{\nu_1}\Big(-\vartheta_{\nu_1}t^{\nu_1}\Big)\ell(0) + \vartheta_{\nu_1}\int_0^t \ell(x)(t-x)^{\nu_1-1}\mathbb{E}_{\nu_1,\nu_1}(-\vartheta_{\nu_1}(t-x)^{\nu_1})dx\right].$$

Here, it $\mathscr{Z}(\nu_1)$ denoted the normalization term satisfies the $\mathscr{Z}(0) = 1 = \mathscr{Z}(1)$, and the corresponding integral is expressed as follows: Let $0 < \nu_1 < 1$ and function $\ell \in H^1(0, T)$, the fractional order framework for modified Atangana-Baleanu integral is defined as:

$$^{mo-\mathscr{AB}}\mathscr{I}_0^{\nu_1}(\ell(t)) = \frac{\mathscr{Z}(1-\nu_1)}{\mathscr{Z}(\nu_1)}[\ell(t) - \ell(0)] + \vartheta_{\nu_1}\Big[^{\mathscr{RL}}\mathscr{I}_0^{\nu_1}(\ell(t) - \ell(0))\Big].$$

Lemma 1 [33] Let $0 < \nu_1 < 1$ and function be $\ell' \in H^1(0, \infty)$

$$^{mo-\mathscr{AB}}\mathscr{I}_0^{\nu_1}\ ^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}(\ell(t)) = \ell(t) - \ell(0),$$

given by

$$\ell(t) = \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\Big(\ell(t)\Big) + \frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1)}\int_0^t (t-x)^{\nu_1-1}\Big(\ell(x)\Big)dx - \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\Big(\ell(0)\Big)\left(1 + \frac{\vartheta_{\nu_1}t^{\nu_1}}{\Gamma(\nu_1+1)}\right).$$

1.7 *Structure of the study*

This research work is organized as follows:

- Section 2 presents the model formulation and describes the problem under investigation.
- Section 3 establishes the criteria for the existence and uniqueness of the recommended system using fixed-point approaches.
- Section 4 analyzes the Ulam-Hyers stability of the system.
- Section 5 develops a numerical method expressed based on the fractional calculus of a Lagrange polynomial interpolation.
- Section 6 provides the conclusions, summarizing key findings, highlighting contributions, and discussing the broader implications of this study.

This structured approach ensures a systematic investigation of the proposed model, its stability, and numerical validation.

2. Model formulation and description

To analyze the interactions among various compartments in the transmission dynamics of HILV/HPV/HIV multiple infections, the total human population is categorized into twenty-two compartments. These categories include susceptible

individuals (S), exposed individuals for HPV (E_p), HIV (E_h), and HILV (E_l), as well as those exposed to both infections: HPV and HIV (E_{ph}), HILV and HPV (E_{lp}), and HILV and HIV (E_{lh}). Also, the infected compartments consist of individuals with HPV (I_p), HIV (I_h), and HILV (I_l) along with those co-infected with infected HPV and HIV (I_{ph}), HILV and HPV (I_{lp}), and HILV and HIV (I_{lh}). Individuals with cervical cancer (C), AIDS (A), and HILV (L) fall into the disease category. Additionally, some individuals experience multiple diseases, suffering from cervical cancer and AIDS (CA), cervical cancer and HILV (CL), AIDS and HILV (AL), and all three diseases combined (CAL), recovered individuals from HPV (R_p), HILV (R_l). Figure 1 shows the schematic representation of the present study. We consider the HILV/HPV/HIV multiple infection model in ordinary differential equation form as follows:

$$\frac{dS}{dt} = \Pi + \chi_p R_p + \chi_l R_l - (\tau_p + \tau_{ph} + \tau_h + \tau_{lp} + \tau_{lh} + \tau_l + \mu)S,$$

$$\frac{dE_p}{dt} = \tau_p S - (\eta_p + \rho_{17} + \rho_1 + \mu)E_p,$$

$$\frac{dE_{ph}}{dt} = \tau_{ph} S + \rho_1 E_p + \rho_2 E_h - (\eta_{ph} + \mu)E_{ph},$$

$$\frac{dE_{lp}}{dt} = \tau_{lp} S + \rho_{17} E_p + \rho_{18} E_l - (\eta_{lp} + \mu)E_{lp},$$

$$\frac{dE_h}{dt} = \tau_h S - (\eta_h + \rho_2 + \rho_3 + \mu)E_h,$$

$$\frac{dE_{lh}}{dt} = \tau_{lh} S + \rho_3 E_h + \rho_4 E_l - (\eta_{lh} + \mu)E_{lh},$$

$$\frac{dE_l}{dt} = \tau_l S - (\eta_l + \rho_4 + \rho_{18} + \mu)E_l,$$

$$\frac{dI_p}{dt} = \eta_p E_p - (\alpha_p + \rho_{13} + \rho_5 + \omega_p + \mu)I_p,$$

$$\frac{dI_{ph}}{dt} = \eta_{ph} E_{ph} + \rho_5 I_p + \rho_6 I_h - (\alpha_{ph} + \mu)I_{ph},$$

$$\frac{dI_{lp}}{dt} = \eta_{lp} E_{lp} + \rho_{13} I_p + \rho_{14} I_l - (\alpha_{lp} + \mu)I_{lp},$$

$$\frac{dI_h}{dt} = \eta_h E_h - (\alpha_h + \rho_6 + \rho_7 + \mu)I_h,$$

$$\frac{dI_{lh}}{dt} = \eta_{lh} E_{lh} + \rho_7 I_h + \rho_8 I_l - (\alpha_{lh} + \mu)I_{lh},$$

$$\frac{dI_l}{dt} = \eta_l E_l - (\omega_l + \rho_8 + \rho_1 6 + \mu) I_l,$$

$$\frac{dC}{dt} = \alpha_p I_p - (\rho_9 + \rho_{15} + \varphi + \mu + \xi) C,$$

$$\frac{dCA}{dt} = \alpha_{ph} I_{ph} + \rho_9 C + \rho_{10} A - (\theta + \mu + \xi) CA,$$

$$\frac{dCL}{dt} = \alpha_{lp} I_{lp} + \rho_{15} C + \rho_{16} L - (\pi + \mu + \xi) CL,$$

$$\frac{dA}{dt} = \alpha_h I_h - (\rho_{10} + \rho_{11} + \psi + \mu + \xi) A,$$

$$\frac{dAL}{dt} = \alpha_{lh} I_{lh} + \rho_{11} A + \rho_{12} L - (\delta + \mu + \xi) AL,$$

$$\frac{dL}{dt} = \alpha_l I_l - (\rho_{12} + \rho_{16} + \gamma + \mu + \xi) L,$$

$$\frac{dCAL}{dt} = \varphi C + \theta CA + \pi CL + \psi A + \delta AL + \gamma L - (\mu + \xi) CAL,$$

$$\frac{dR_p}{dt} = \omega_p I_p - (\chi_p + \mu) R_p,$$

$$\frac{dR_l}{dt} = \omega_l I_l - (\chi_l + \mu) R_l. \tag{1}$$

The modified Atangana-Baleanu Caputo sense of fractional HILV/HPV/HIV multiple infection model, which is presented in the following form:

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(S(t)\Big) = \Pi + \chi_p R_p + \chi_l R_l - (\tau_p + \tau_{ph} + \tau_h + \tau_{lp} + \tau_{lh} + \tau_l + \mu) S,$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(E_p(t)\Big) = \tau_p S - (\eta_p + \rho_{17} + \rho_1 + \mu) E_p,$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(E_{ph}(t)\Big) = \tau_{ph} S + \rho_1 E_p + \rho_2 E_h - (\eta_{ph} + \mu) E_{ph},$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(E_{lp}(t)\Big) = \tau_{lp} S + \rho_{17} E_p + \rho_{18} E_l - (\eta_{lp} + \mu) E_{lp},$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(E_h(t)\Big) = \tau_h S - (\eta_h + \rho_2 + \rho_3 + \mu)E_h,$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(E_{lh}(t)\Big) = \tau_{lh} S + \rho_3 E_h + \rho_4 E_l - (\eta_{lh} + \mu)E_{lh},$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(E_l(t)\Big) = \tau_l S - (\eta_l + \rho_4 + \rho_{18} + \mu)E_l,$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(I_p(t)\Big) = \eta_p E_p - (\alpha_p + \rho_{13} + \rho_5 + \omega_p + \mu)I_p,$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(I_{ph}(t)\Big) = \eta_{ph} E_{ph} + \rho_5 I_p + \rho_6 I_h - (\alpha_{ph} + \mu)I_{ph},$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(I_{lp}(t)\Big) = \eta_{lp} E_{lp} + \rho_{13} I_p + \rho_{14} I_l - (\alpha_{lp} + \mu)I_{lp},$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(I_h(t)\Big) = \eta_h E_h - (\alpha_h + \rho_6 + \rho_7 + \mu)I_h,$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(I_{lh}(t)\Big) = \eta_{lh} E_{lh} + \rho_7 I_h + \rho_8 I_l - (\alpha_{lh} + \mu)I_{lh},$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(I_l(t)\Big) = \eta_l E_l - (\omega_l + \rho_8 + \rho_{16} + \mu)I_l,$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(C(t)\Big) = \alpha_p I_p - (\rho_9 + \rho_{15} + \varphi + \mu + \xi)C,$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(CA(t)\Big) = \alpha_{ph} I_{ph} + \rho_9 C + \rho_{10} A - (\theta + \mu + \xi)CA,$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(CL(t)\Big) = \alpha_{lp} I_{lp} + \rho_{15} C + \rho_{16} L - (\pi + \mu + \xi)CL,$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(A(t)\Big) = \alpha_h I_h - (\rho_{10} + \rho_{11} + \psi + \mu + \xi)A,$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(AL(t)\Big) = \alpha_{lh} I_{lh} + \rho_{11} A + \rho_{12} L - (\delta + \mu + \xi)AL,$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(L(t)\Big) = \alpha_l I_l - (\rho_{12} + \rho_{16} + \gamma + \mu + \xi)L,$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(CAL(t)\Big) = \varphi C + \theta CA + \pi CL + \psi A + \delta AL + \gamma L - (\mu + \xi)CAL,$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(R_p(t)\Big) = \omega_p I_p - (\chi_p + \mu)R_p,$$

$$^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}\Big(R_l(t)\Big) = \omega_l I_l - (\chi_l + \mu)R_l. \tag{2}$$

with the initial condition of the model (2) is given as: $S(0) = S_0$, $E_p(0) = E_{p_0}$, $E_{ph}(0) = E_{ph_0}$, $E_{lp}(0) = E_{lp_0}$, $E_h(0) = E_{h_0}$, $E_{lh}(0) = E_{lh_0}$, $E_l(0) = E_{l_0}$, $I_p(0) = I_{p_0}$, $I_{ph}(0) = I_{ph_0}$, $I_{lp}(0) = I_{lp_0}$, $I_h(0) = I_{h_0}$, $I_{lh}(0) = I_{lh_0}$, $I_l(0) = I_{l_0}$, $C(0) = C_0$, $CA(0) = CA_0$, $CL(0) = CL_0$, $A(0) = A_0$, $AL(0) = AL_0$, $L(0) = L_0$, $CAL(0) = CAL_0$, $R_p(0) = R_{p_0}$, $R_l(0) = R_{l_0}$, and $^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}$ denotes the fractional order framework for modified Atangana-Baleanu-Caputo type with fractional order (ν_1).

Based on these parameters, we establish the following assumptions:

(1) **Susceptible population**

The total human population is divided into multiple compartments based on infection status. The susceptible individuals (S) increase due to new births or recruitment at a rate Π and lost their temporary immunity for recovery from HILV and HPV infections at rates $\chi_p R_p$ and $\chi_l R_l$, respectively. However, this population decreases due to disease transmission at rates:

$$\tau_p = \frac{\beta_p I_p}{N_p},\ \tau_h = \frac{\beta_h I_h}{N_h},\ \tau_l = \frac{\beta_l I_l}{N_l},\ \tau_{ph} = \frac{\beta_{ph} I_{ph}}{N_{ph}},\ \tau_{lp} = \frac{\beta_{lp} I_{lp}}{N_{lp}},\ \tau_{lh} = \frac{\beta_{lh} I_{lh}}{N_{lh}}.$$

Where, β_p, β_h, β_l, β_{ph}, β_{lp} and β_{lh} denote the disease contact rates. The total populations of the given model: HPV total populations N_p, HPV and HIV co-infection total populations N_{ph}, HILV and HPV co-infection total populations N_{lp}, HIV total populations N_l, HILV total populations N_h, and HILV and HIV co-infection total populations N_{lh} are defined as:

$$N_p = S + E_p + I_p + C + R_p,$$

$$N_h = S + E_h + I_h + A,$$

$$N_l = S + E_l + I_l + L + R_l,$$

$$N_{ph} = S + E_{ph} + I_{ph} + CA,$$

$$N_{lp} = S + E_{lp} + I_{lp} + CL,$$

$$N_{lh} = S + E_{lh} + I_{lh} + AL.$$

The natural death rate is given by μ.

(2) **Exposed population**

The exposed compartments represent individuals who have been infected but are not yet symptomatic. These include:

• Exposed to HPV (E_p): Individuals transition from S to E_p at a rate $\tau_p S$. They either progress to the infected stage at a rate $\eta_p E_p$ or recover through contact with another disease at the rate (ρ_{17}, ρ_1), with a natural mortality rate μ.

• Exposed to HPV and HIV (E_{ph}): Individuals contract both HPV and HIV at a rate $\tau_{ph} S$ and transition to infection at $\eta_{ph} E_{ph}$.

• Exposed to HILV and HPV (E_{lp}): Individuals acquire both HILV and HPV at a rate $\tau_{lp} S$ and become infected at a rate $\eta_{lp} E_{lp}$.

• Exposed to HIV (E_h): Individuals contract HIV at a rate $\tau_h S$ and progress to infection at $\eta_h E_h$.

• Exposed to HILV and HIV (E_{lh}): Individuals contract both infections at a rate $\tau_{lh} S$ and become fully infected at a rate $\eta_{lh} E_{lh}$.

• Exposed to HILV (E_l): Individuals acquire HILV at a rate $\tau_l S$ and progress to active infection at a rate $\eta_l E_l$.

(3) **Infected population**

The infected compartments include symptomatic individuals actively transmitting the disease:

• Infected with HPV (I_p): Progress from E_p at a rate η_p, with possible transitions to cervical cancer $(\alpha_p I_p)$, co-infection (ρ_{13}, ρ_5), or recovery $(\omega_p I_p)$.

• HPV and HIV co-infected (I_{ph}): Progress from E_{ph}, potentially leading to cervical cancer $(\alpha_{ph} I_{ph})$ or further disease stages.

• HILV and HPV co-infected (I_{lp}): Transition from E_{lp} at a rate η_{lp}, possibly developing cancer $(\alpha_{lp} I_{lp})$.

• Infected with HIV (I_h): Progress from E_h, potentially advancing to AIDS $(\alpha_h I_h)$, co-infection (ρ_6, ρ_7), or natural mortality.

• HILV and HIV co-infected (I_{lh}): Progress from E_{lh}, with potential advancement to AIDS $(\alpha_{lh} I_{lh})$.

• Infected with HILV (I_l): Progress from E_l, potentially developing severe complications $(\omega_l I_l)$.

(4) **Advanced disease states**

The following conditions arise due to disease progression:

• Cervical cancer (C): Develops from HPV infections $(\alpha_p I_p)$, progressing to severe states or mortality.

• AIDS (A): Develops from HIV infections $(\alpha_h I_h)$, with progression to advanced AIDS-HILV (AH).

• HILV complications (L): Arise from HILV infections $(\alpha_l I_l)$, potentially coexisting with other diseases.

• Multiple disease states: (CAL): Includes combinations such as cervical cancer and AIDS (CA), cervical cancer and HILV (CL), AIDS and HILV (AL), and all three diseases (CAL).

(5) **Recovered population**

Recovered individuals may regain susceptibility over time:

• Recovered from HPV (R_p): Individuals recover from HPV infection but may become susceptible again.

• Recovered from HILV (R_l): Similarly, individuals recover from HILV infection but remain at risk of reinfection.

3. Main results of the study

To establish the criteria for existence and uniqueness in this context, we employ the fixed-point techniques. Applying the integral term for fractional order framework in the modified Atangana-Baleanu-Caputo type (as defined in Definition 1) to both sides of the fractional multi-infection model, we obtain that

$$\mathscr{J}_1\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \Pi + \chi_p R_p + \chi_l R_l - (\tau_p + \tau_{ph} + \tau_h + \tau_{lp} + \tau_{lh} + \tau_l + \mu) S,$$

$$\mathscr{J}_2\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \tau_p S - (\eta_p + \rho_{17} + \rho_1 + \mu)E_p,$$

$$\mathscr{J}_3\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \tau_{ph} S + \rho_1 E_p + \rho_2 E_h - (\eta_{ph} + \mu)E_{ph},$$

$$\mathscr{J}_4\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \tau_{lp} S + \rho_{17} E_p + \rho_{18} E_l - (\eta_{lp} + \mu)E_{lp},$$

$$\mathscr{J}_5\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \tau_h S - (\eta_h + \rho_2 + \rho_3 + \mu)E_h,$$

$$\mathscr{J}_6\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \tau_{lh} S + \rho_3 E_h + \rho_4 E_l - (\eta_{lh} + \mu)E_{lh},$$

$$\mathscr{J}_7\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \tau_l S - (\eta_l + \rho_4 + \rho_{18} + \mu)E_l,$$

$$\mathscr{J}_8\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \eta_p E_p - (\alpha_p + \rho_{13} + \rho_5 + \omega_p + \mu)I_p,$$

$$\mathscr{J}_9\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \eta_{ph} E_{ph} + \rho_5 I_p + \rho_6 I_h - (\alpha_{ph} + \mu)I_{ph},$$

$$\mathscr{J}_{10}\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \eta_{lp} E_{lp} + \rho_{13} I_p + \rho_{14} I_l - (\alpha_{lp} + \mu)I_{lp},$$

$$\mathscr{J}_{11}\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \eta_h E_h - (\alpha_h + \rho_6 + \rho_7 + \mu) I_h,$$

$$\mathscr{J}_{12}\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \eta_{lh} E_{lh} + \rho_7 I_h + \rho_8 I_l - (\alpha_{lh} + \mu) I_{lh},$$

$$\mathscr{J}_{13}\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \eta_l E_l - (\omega_l + \rho_8 + \rho_1 6 + \mu) I_l,$$

$$\mathscr{J}_{14}\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \alpha_p I_p - (\rho_9 + \rho_{15} + \varphi + \mu + \xi) C,$$

$$\mathscr{J}_{15}\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \alpha_{ph} I_{ph} + \rho_9 C + \rho_{10} A - (\theta + \mu + \xi) CA,$$

$$\mathscr{J}_{16}\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \alpha_{lp} I_{lp} + \rho_{15} C + \rho_{16} L - (\pi + \mu + \xi) CL,$$

$$\mathscr{J}_{17}\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \alpha_h I_h - (\rho_{10} + \rho_{11} + \psi + \mu + \xi) A,$$

$$\mathscr{J}_{18}\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \alpha_{lh} I_{lh} + \rho_{11} A + \rho_{12} L - (\delta + \mu + \xi) AL,$$

$$\mathscr{J}_{19}\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \alpha_l I_l - (\rho_{12} + \rho_{16} + \gamma + \mu + \xi) L,$$

$$\mathscr{J}_{20}\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \varphi C + \theta CA + \pi CL + \psi A + \delta AL + \gamma L - (\mu + \xi)CAL,$$

$$\mathscr{J}_{21}\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \omega_p I_p - (\chi_p + \mu)R_p,$$

$$\mathscr{J}_{22}\big(t, S, E_p, E_{ph}, E_{lp}, E_h, E_{lh}, E_l, I_p, I_{ph}, I_{lp}, I_h, I_{lh}, I_l, C, CA, CL, A, AL, L, CAL, R_p, R_l\big)$$

$$= \omega_l I_l - (\chi_l + \mu)R_l.$$

Based on the twenty-two equations mentioned above, our projected model (2) can be expressed as:

$$\begin{cases} {}^{mo-\mathscr{ABC}}\mathscr{D}_0^{\nu_1}(\mathscr{J}(t)) = \mathscr{H}(t), \quad t \in [0, T], \quad 0 < \nu_1 \leq 1, \\ \\ \mathscr{J}(0) = \mathscr{J}_0, \end{cases} \tag{3}$$

where,

$$\mathscr{J}(t) = \begin{cases} S(t),\ E_p(t),\ E_{ph}(t),\ E_{lp}(t), \\ E_h(t),\ E_{lh}(t),\ E_l(t),\ I_p(t), \\ I_{ph}(t),\ I_{lp}(t),\ I_h(t),\ I_{lh}(t), \\ I_l(t),\ C(t),\ CA(t),\ CL(t), \\ A(t),\ AL(t),\ L(t),\ CAL(t), \\ R_p(t),\ R_l(t), \end{cases} \qquad \mathscr{J}_0 = \begin{cases} S(0),\ E_p(0),\ E_{ph}(0),\ E_{lp}(0), \\ E_h(0),\ E_{lh}(0),\ E_l(0),\ I_p(0), \\ I_{ph}(0),\ I_{lp}(0),\ I_h(0),\ I_{lh}(0), \\ I_l(0),\ C(0),\ CA(0),\ CL(0), \\ A(0),\ AL(0),\ L(0),\ CAL(0), \\ R_p(0),\ R_l(0), \end{cases}$$

and

$$\mathscr{H}(t,\ \mathscr{J}(t)) = \begin{cases} \mathscr{H}_1(t, S(t)),\ \mathscr{H}_2(t, E_p(t)),\ \mathscr{H}_3(t, E_{ph}(t)),\ \mathscr{H}_4(t, E_{lp}(t)), \\ \mathscr{H}_5(t, E_h(t)),\ \mathscr{H}_6(t, E_{lh}(t)),\ \mathscr{H}_7(t, E_l(t)),\ \mathscr{H}_8(t, I_p(t)), \\ \mathscr{H}_9(t, I_{ph}(t)),\ \mathscr{H}_{10}(t, I_{lp}(t)),\ \mathscr{H}_{11}(t, I_h(t)),\ \mathscr{H}_{12}(t, I_{lh}(t)), \\ \mathscr{H}_{13}(t, I_l(t)),\ \mathscr{H}_{14}(t, C(t)),\ \mathscr{H}_{15}(t, CA(t)),\ \mathscr{H}_{16}(t, CL(t)), \\ \mathscr{H}_{17}(t, A(t)),\ \mathscr{H}_{18}(t, AL(t)),\ \mathscr{H}_{19}(t, L(t)),\ \mathscr{H}_{20}(t, CAL(t)), \\ \mathscr{H}_{21}(t, R_p(t)),\ \mathscr{H}_{22}(t, R_l(t)). \end{cases} \tag{4}$$

Applying the fractional integral term from Definition 1 and utilizing Lemma 1 to the equation (3) yields the following result:

$$\mathscr{J}(t) = \mathscr{J}(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}(t,\ \mathscr{J}(t)) + \frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1)}\int_0^t (t-x)^{\nu_1-1}\mathscr{H}(x,\ \mathscr{J}(x))dx$$

$$-\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_0(t,\ \mathscr{J}(0))\left(1+\frac{\vartheta_{\nu_1}t^{\nu_1}}{\Gamma(\nu_1+1)}\right).$$

3.1 *Solution existence*

In this subsection, it is necessary to examine the existence criteria for the fractional order derivative in the modified Atangana-Baleanu Caputo sense mentioned below:

($\mathscr{M}$): To verify the accuracy of our findings, we consider the following assumptions: Regarding the $\mathscr{J}(t)$, $\widetilde{\mathscr{J}}(t)$ belong to $\mathfrak{L}[0,\ 1)$, be a continuous function, with $\|\mathscr{J}(t)\| \leq \mathfrak{L}$ being non-negative constants denoted as $\mathfrak{L}$. The kernels $\mathscr{H}$ and then there exists $\mathscr{L} > 0$ such that if $\mathscr{J}$, $\widetilde{\mathscr{J}} \in \mathscr{B}$, then hold the Lipchitz condition and contraction if the following inequality holds:

$$\left|\mathscr{H}(t,\ \mathscr{J}(t)) - \mathscr{H}(t,\ \widetilde{\mathscr{J}}(t))\right|$$

$$\leq \mathscr{L}\left[\left|\mathscr{J} - \widetilde{\mathscr{J}}\right|\right].$$

Let's assume:

$$\mathscr{J}(t) = \mathscr{J}(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}(t,\ \mathscr{J}(t)) + \frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1)}\int_0^t (t-x)^{\nu_1-1}\mathscr{H}(x,\ \mathscr{J}(x))dx$$

$$-\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_0(t,\ \mathscr{J}(0))\left(1+\frac{\vartheta_{\nu_1}t^{\nu_1}}{\Gamma(\nu_1+1)}\right).$$

Now, we will proceed to the following stage, which involves establishing the recursive formula for the system. These formulas are as follows:

$$\mathscr{J}_n(t) - \mathscr{J}(0) = \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}(t,\ \mathscr{J}_{n-1}(t)) + \frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1)}\int_0^t (t-x)^{\nu_1-1}\mathscr{H}(x,\ \mathscr{J}_{n-1}(x))dx$$

$$-\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_0(t,\ \mathscr{J}_{n-1}(0))\left(1+\frac{\vartheta_{\nu_1}t^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

and

$$\mathscr{J}_{n+1}(t)-\mathscr{J}(0)=\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}(t,\,\mathscr{J}_n(t))+\frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1)}\int_0^t(t-x)^{\nu_1-1}\mathscr{H}(x,\,\mathscr{J}_n(x))dx$$

$$-\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_0(t,\,\mathscr{J}_n(0))\left(1+\frac{\vartheta_{\nu_1}t^{\nu_1}}{\Gamma(\nu_1+1)}\right).$$

Theorem 1 The considered model (2) has a solution provided that holds true:

$$\Lambda=\max\mathscr{L}<1.$$

Proof. We define the functions

$$\mathscr{Q}1_n(t)=\mathscr{J}_{n+1}(t)-\mathscr{J}(t).$$

Next, we can derive the following expressions for the difference between consecutive terms:

$$\begin{aligned}\mathscr{Q}1_n(t)&=\mathscr{J}_{n+1}(t)-\mathscr{J}(t)\\&=\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}(t,\,\mathscr{J}_n(t))+\frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1)}\int_0^t(t-x)^{\nu_1-1}\mathscr{H}(x,\,\mathscr{J}_n(x))dx\\&\quad-\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_0(t,\,\mathscr{J}_n(0))\left(1+\frac{\vartheta_{\nu_1}t^{\nu_1}}{\Gamma(\nu_1+1)}\right)\\&\quad-\left(\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}(t,\,\mathscr{J}(t))+\frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1)}\int_0^t(t-x)^{\nu_1-1}\mathscr{H}(x,\,\mathscr{J}(x))dx\right.\\&\quad\left.-\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_0(t,\,\mathscr{J}(0))\left(1+\frac{\vartheta_{\nu_1}t^{\nu_1}}{\Gamma(\nu_1+1)}\right)\right),\end{aligned}$$

then,

$$\mathscr{Q}1_n(t)=\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\Big(\mathscr{H}(t,\,\mathscr{J}_n(t))-\mathscr{H}(t,\,\mathscr{J}(t))\Big)+\frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1)}\int_0^t(t-x)^{\nu_1-1}\Big(\mathscr{H}(x,\,\mathscr{J}_n(x))-\mathscr{H}(x,\,\mathscr{J}(x))\Big)dx$$

$$-\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\Big(\mathscr{H}_0(t,\,\mathscr{J}_n(0))-\mathscr{H}(t,\,\mathscr{J}(0))\Big)\left(1+\frac{\vartheta_{\nu_1}t^{\nu_1}}{\Gamma(\nu_1+1)}\right).$$

Then, we find that

$$\begin{aligned}\left\|\mathscr{Q}1_n(t)\right\| &= \left\| \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\Big(\mathscr{H}(t,\ \mathscr{J}_n(t)) - \mathscr{H}(t,\ \mathscr{J}(t))\Big)\right.\\ &\quad + \frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1)}\int_0^t (t-x)^{\nu_1-1}\Big(\mathscr{H}(x,\ \mathscr{J}_n(x)) - \mathscr{H}(x,\ \mathscr{J}(x))\Big)dx\\ &\quad \left. - \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\Big(\mathscr{H}_0(t,\ \mathscr{J}_n(0)) - \mathscr{H}(t,\ \mathscr{J}(0))\Big)\left(1+\frac{\vartheta_{\nu_1}t^{\nu_1}}{\Gamma(\nu_1+1)}\right)\right\|,\\ &\leq \left[\frac{1-\nu_1}{\mathscr{Z}(\nu_1)} + \frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1+1)}\right]\mathscr{L}\|\mathscr{J}_n - \mathscr{J}\|,\\ &\leq \left[\frac{1-\nu_1}{\mathscr{Z}(\nu_1)} + \frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1+1)}\right]^n \mathscr{L}^n\|\mathscr{J}_n - \mathscr{J}\|.\end{aligned}$$

Thus, from the above functions, when $n \to \infty$, then $\mathscr{Q}1(t)_n \to 0$, and $\mathscr{L} < 1$, which completes the proof. □

3.2 *Solution unique*

In this subsection, for our modified Atangana-Baleanu in the fractional order derivative Caputo sense model (2), we establish the uniqueness criteria in Theorem 2, which reads as follows:

Theorem 2 For the model (2), there is a unique solution if

$$\left[\frac{1-\nu_1}{\mathscr{Z}(\nu_1)} + \frac{1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1+1)}\right]\mathscr{L} \leq 1,$$

holds true.

Proof. For the unique solution of the model (2), we assume that there existing anther solution say $\widetilde{\mathscr{J}}(t)$ the another solution with

$$\begin{aligned}\widetilde{\mathscr{J}}(t) - \widetilde{\mathscr{J}}(0) &= \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}(t,\ \widetilde{\mathscr{J}}(t)) + \frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1)}\int_0^t (t-x)^{\nu_1-1}\mathscr{H}(x,\ \widetilde{\mathscr{J}}(x))dx\\ &\quad - \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_0(t,\ \widetilde{\mathscr{J}}(0))\left(1+\frac{\vartheta_{\nu_1}t^{\nu_1}}{\Gamma(\nu_1+1)}\right).\end{aligned}$$

Now, we write

$$\| \mathscr{J}(t) - \widetilde{\mathscr{J}}(t) \| \leq \Bigg\| \frac{1-\nu_1}{\mathscr{Z}(\nu_1)} \Big(\mathscr{H}(t, \mathscr{J}(t)) - \mathscr{H}(t, \widetilde{\mathscr{J}}(t)) \Big)$$

$$+ \frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1)} \int_0^t (t-x)^{\nu_1 - 1} \Big(\mathscr{H}(x, \mathscr{J}(x)) - \mathscr{H}(x, \widetilde{\mathscr{J}}(x)) \Big) dx$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)} \Big(\mathscr{H}_0(t, \mathscr{J}(0)) - \mathscr{H}_0(t, \widetilde{\mathscr{J}}(0)) \Big) \left(1 + \frac{\vartheta_{\nu_1} t^{\nu_1}}{\Gamma(\nu_1 + 1)} \right) \Bigg\|,$$

$$\leq \frac{1-\nu_1}{\mathscr{Z}(\nu_1)} \mathscr{L} \| \mathscr{J} - \widetilde{\mathscr{J}} \| + \frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1 + 1)} \int_0^t (t-x)^{\nu_1 - 1} \mathscr{L} \| \mathscr{J} - \widetilde{\mathscr{J}} \| dx,$$

$$\leq \left[\frac{1-\nu_1}{\mathscr{Z}(\nu_1)} + \frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1 + 1)} \right] \mathscr{L} \| \mathscr{J} - \widetilde{\mathscr{J}} \|, \tag{5}$$

and so

$$\left[\left[\frac{1-\nu_1}{\mathscr{Z}(\nu_1)} \mathscr{L} + \frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1 + 1)} \mathscr{L} \right] - 1 \right] \| \mathscr{J} - \widetilde{\mathscr{J}} \| \geq 0. \tag{6}$$

The above inequality (6) is true if

$$\| \mathscr{J}(t) - \widetilde{\mathscr{J}}(t) \| = 0.$$

Hence,

$$\mathscr{J}(t) = \widetilde{\mathscr{J}}(t).$$

Hence, the uniqueness of the solution is proved. As a result, the solution for the model (2) is unique. □

4. Stability analysis of the study

This study explores the Ulam-Hyers stability of the fractional order framework for the modified Atangana-Baleanu-Caputo type fractional model. We introduce a modified definition that is related to the proposed methodology, providing a more refined approach to stability investigation.

Definition 2 The HTLV/HPV/HIV multi-infection model is considered to be Ulam-Hyers stable, if for a $\mathscr{E} > 0$ and $\mathscr{J}(t)$ satisfying

$$\left|{}^{mo-\mathscr{A}\mathscr{B}\mathscr{C}}_{0}\mathscr{D}_{t}^{\nu_1}\mathscr{J}(t)-\mathscr{H}(t,\mathscr{J}(t))\right|\leq\mathscr{E}, \tag{7}$$

there exist constants $\mathscr{K}>0$ and a solution $\widetilde{\mathscr{J}}(t)$ of the equation (2), such that

$$\left\|\mathscr{J}(t)-\widetilde{\mathscr{J}}(t)\right\|\leq\mathscr{K}\mathscr{E}.$$

Remark 1 Assume that, if a continuous function h_1 exists such that, the function $\mathscr{J}(t)$ be a solution of first inequality (7),

- $|h_1(t)|<\mathscr{E}_1$, and
- ${}^{mo-\mathscr{A}\mathscr{B}\mathscr{C}}_{0}\mathscr{D}_{t}^{\nu_1}\mathscr{J}(t)=\mathscr{H}(t,\mathscr{J}(t))+h_1(t)$.

Theorem 3 The HTLV/HPV/HIV multi-infection model is considered Ulam-Hyers stable if the hypotheses ($\mathscr{M}$) hold and a specific inequality is satisfied:

$$\left[\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}+\frac{1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1+1)}\right]\mathscr{L}\leq 1.$$

Proof. Let $\mathscr{E}>0$ and the function $\mathscr{J}(t)$ be arbitrary so that

$$\left|{}^{mo-\mathscr{A}\mathscr{B}\mathscr{C}}_{0}\mathscr{D}_{t}^{\nu_1}\mathscr{J}(t)-\mathscr{H}(t,\mathscr{J}(t))\right|\leq\mathscr{E}.$$

Based on the remark mentioned above, there exists a function h_1 such that $|h_1(t)|<\mathscr{E}$, we obtain

$${}^{mo-\mathscr{A}\mathscr{B}\mathscr{C}}_{0}\mathscr{D}_{t}^{\nu_1}\mathscr{J}(t)=\mathscr{H}(t,\mathscr{J}(t))+h_1(t).$$

Consequently,

$$\mathscr{J}(t)=\mathscr{J}(0)+\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}(t,\mathscr{J}(t))+\frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1)}\int_0^t(t-x)^{\nu_1-1}\mathscr{H}(x,\mathscr{J}(x))dx,$$

$$+\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}h_1(t)+\frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1)}\int_0^t(t-x)^{\nu_1-1}h_1(x)dx.$$

If we consider the unique solution of projected system as $\widetilde{\mathscr{J}}(t)$, next

$$\widetilde{\mathscr{J}}(t)=\widetilde{\mathscr{J}}(0)+\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}(t,\widetilde{\mathscr{J}}(t))+\frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1)}\int_0^t(t-x)^{\nu_1-1}\mathscr{H}(x,\widetilde{\mathscr{J}}(x))dx.$$

Hence,

$$\left|\mathscr{J}(t)-\widetilde{\mathscr{J}}(t)\right| \leq \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\left|\mathscr{H}(t,\mathscr{J}(t))-\mathscr{H}(t,\widetilde{\mathscr{J}}(t))\right|$$

$$+\frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1)}\int_0^t (t-x)^{\nu_1-1}\left|\mathscr{H}(x,\mathscr{J}(x))-\mathscr{H}(x,\widetilde{\mathscr{J}}(x))\right|dx,$$

$$+\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\left|h_1(t)\right|+\frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1)}\int_0^t (t-x)^{\nu_1-1}\left|h_1(x)\right|dx,$$

$$\leq \left[\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}+\frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1+1)}\right]\mathscr{L}|\mathscr{J}-\widetilde{\mathscr{J}}|+\left[\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}+\frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1+1)}\right]\mathscr{E}.$$

In consequence,

$$\left\|\mathscr{J}(t)-\widetilde{\mathscr{J}}(t)\right\| \leq \frac{\left[\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}+\frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1+1)}\right]\mathscr{E}}{1-\left[\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}+\frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1+1)}\right]\mathscr{L}}.$$

Then

$$\left\|\mathscr{J}(t)-\widetilde{\mathscr{J}}(t)\right\| \leq \mathscr{K}\mathscr{E}.$$

Here,

$$\mathscr{K}=\frac{\left[\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}+\frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1+1)}\right]\mathscr{E}}{1-\left[\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}+\frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1+1)}\right]\mathscr{L}}.$$

Therefore, the HTLV/HPV/HIV multi-infection model has Ulam-Hyers stability. □

5. Numerical scheme and simulations

To apply the Atangana-Baleanu-Caputo fractional order framework, given as follows,

$$^{mo-\mathscr{A}\mathscr{B}\mathscr{C}}\mathscr{D}_0^{\nu_1}\mathscr{J}(t)=\mathscr{H}(t,\mathscr{J}(t)). \tag{8}$$

By utilizing the basic definition of fractional calculus, we get

$$\mathscr{J}(t) = \mathscr{J}(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}(t,\ \mathscr{J}_{\chi-1}(t)) + \frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1)}\int_0^t (t-x)^{\nu_1-1}\mathscr{H}(x,\ \mathscr{J}(x))dx$$

$$-\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_0(0,\ \mathscr{J}(0))\left(1+\frac{\vartheta_{\nu_1}t^{\nu_1}}{\Gamma(\nu_1+1)}\right). \tag{9}$$

Using Lagrange interpolation polynomials, we develop a numerical scheme for this system.
Replacing t by $t_{\chi+1}$, we get

$$\mathscr{J}(t_{n+1}) = \mathscr{J}(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}(t_\chi,\ \mathscr{J}(t_\chi)) + \frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1)}\int_0^{t_{\chi+1}} (t_{\chi+1}-x)^{\nu_1-1}\mathscr{H}(x,\ \mathscr{J}_{\chi-1}(x))dx$$

$$-\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_0(0,\ \mathscr{J}_{\chi-1}(0))\left(1+\frac{\vartheta_{\nu_1}t_\chi^{\nu_1}}{\Gamma(\nu_1+1)}\right). \tag{10}$$

The function $\mathscr{H}(\chi,\ \mathscr{J}(\chi))$ is an approximate solution based on the two-step Lagrange polynomial interpolation scheme as follows:

$$\mathscr{H}(t,\ \mathscr{J}(t)) = \frac{\mathscr{H}(t_\chi,\ \mathscr{J}_\chi)}{h}(x-t_{\chi-1}) - \frac{\mathscr{H}(t_{\chi-1},\ \mathscr{J}_{\chi-1})}{h}(x-t_\chi). \tag{11}$$

Equation (11) is now substituted into (10) to derive

$$\mathscr{J}_{\chi+1}(t) = \mathscr{J}(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}(t_\chi,\ \mathscr{J}(t_\chi))$$

$$+\frac{\nu_1}{\mathscr{Z}(\nu_1)\Gamma(\nu_1)}\sum_{\chi=0}^{n}\left(\frac{\mathscr{H}(t_\chi,\ \mathscr{J}_\chi)}{h}\int_{t_\chi}^{t_{\chi+1}}(x-t_{\chi+1})(t_{n+1}-x)^{\nu_1-1}dx,\right.$$

$$\left.-\frac{\mathscr{H}(t_{\chi-1},\ \mathscr{J}_{\chi-1})}{h}\int_{t_\chi}^{t_{\chi+1}}(x-t_\chi)(t_{\chi+1}-x)^{\nu_1-1}dx\right)$$

$$-\frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}(0,\ \mathscr{J}_\varepsilon(0))\left(1+\frac{\vartheta_{\nu_{1\varepsilon}}t_\chi^{\nu_1}}{\Gamma(\nu_1+1)}\right). \tag{12}$$

Equation (12), we get the following approximation

$$\mathscr{J}(t_{\chi+1}) = \mathscr{J}(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}(t_\chi,\, \mathscr{J}(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}(t_\chi,\, \mathscr{J}(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}(t_{\chi-1},\, \mathscr{J}(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}(0,\, \mathscr{J}_\varepsilon(0))\left(1 + \frac{\vartheta_{\nu_{1_\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right).$$

Finally, the fractional HTLV/HPV/HIV multiple-infection model in the modified Atangana-Baleanu-Caputo type is ultimately formulated as a numerical representation as follows:

$$S(t_{\chi+1}) = S(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_1(t_\chi, S(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}\times\left(\mathscr{H}_1(t_\chi, S(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1}\right.$$

$$\left. - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big) - \mathscr{H}_1(t_{\chi-1}, S(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_1(0, S(0))\left(1 + \frac{\vartheta_{\nu_{1_\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$E_p(t_{\chi+1}) = E_p(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_2(t_\chi, E_p(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_2(t_\chi, E_p(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_2(t_{\chi-1}, E_p(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_2(0, E_p(0))\left(1 + \frac{\vartheta_{\nu_{1_\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$E_{ph}(t_{\chi+1}) = E_{ph}(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_3(t_\chi, E_{ph}(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_3(t_\chi, E_{ph}(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_3(t_{\chi-1}, E_{ph}(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_3(0, E_{ph}(0))\left(1 + \frac{\vartheta_{\nu_{1_\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$E_{lp}(t_{\chi+1}) = E_{lp}(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_4(t_\chi, E_{lp}(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_4(t_\chi, E_{lp}(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_4(t_{\chi-1}, E_{lp}(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_4(0, E_{lp}(0))\left(1 + \frac{\vartheta_{\nu_{1_\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$E_h(t_{\chi+1}) = E_h(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_5(t_\chi, E_h(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_5(t_\chi, E_h(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_5(t_{\chi-1}, E_h(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_5(0, E_h(0))\left(1 + \frac{\vartheta_{\nu_{1_\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$E_{lh}(t_{\chi+1}) = E_{lh}(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_6(t_\chi, E_{lh}(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_6(t_\chi, E_{lh}(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_6(t_{\chi-1}, E_{lh}(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_6(0, E_{lh}(0))\left(1 + \frac{\vartheta_{\nu_{1_\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$E_l(t_{\chi+1}) = E_l(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_7(t_\chi, E_l(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_7(t_\chi, E_l(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_7(t_{\chi-1}, E_l(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_7(0, E_l(0))\left(1 + \frac{\vartheta_{\nu_{1_\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$I_p(t_{\chi+1}) = I_p(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_8(t_\chi, I_p(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_8(t_\chi, I_p(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_8(t_{\chi-1}, I_p(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_8(0, I_p(0))\left(1 + \frac{\vartheta_{\nu_{1_\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$I_{ph}(t_{\chi+1}) = I_{ph}(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_9(t_\chi, I_{ph}(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_9(t_\chi, I_{ph}(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_9(t_{\chi-1}, I_{ph}(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_9(0, I_{ph}(0))\left(1 + \frac{\vartheta_{\nu_{1_\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$I_{lp}(t_{\chi+1}) = I_{lp}(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{10}(t_\chi, I_{lp}(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_{10}(t_\chi, I_{lp}(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_{10}(t_{\chi-1}, I_{lp}(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{10}(0, I_{lp}(0))\left(1 + \frac{\vartheta_{\nu_{1_\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$I_h(t_{\chi+1}) = I_h(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{11}(t_\chi, I_h(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_{11}(t_\chi, I_h(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_{11}(t_{\chi-1}, I_h(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{11}(0, I_h(0))\left(1 + \frac{\vartheta_{\nu_{1_\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$I_{lh}(t_{\chi+1}) = I_{lh}(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{12}(t_\chi, I_{lh}(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_{12}(t_\chi, I_{lh}(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_{12}(t_{\chi-1}, I_{lh}(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{12}(0, I_{lh}(0))\left(1 + \frac{\vartheta_{\nu_{1\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$I_l(t_{\chi+1}) = I_l(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{13}(t_\chi, I_l(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_{13}(t_\chi, I_l(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_{13}(t_{\chi-1}, I_l(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{13}(0, I_l(0))\left(1 + \frac{\vartheta_{\nu_{1\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$C(t_{\chi+1}) = C(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{14}(t_\chi, C(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_{14}(t_\chi, C(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_{14}(t_{\chi-1}, C(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{14}(0, C(0))\left(1 + \frac{\vartheta_{\nu_{1\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$CA(t_{\chi+1}) = CA(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{15}(t_\chi, CA(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_{15}(t_\chi, CA(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_{15}(t_{\chi-1}, CA(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{15}(0, CA(0))\left(1 + \frac{\vartheta_{\nu_{1_\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$CL(t_{\chi+1}) = CL(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{16}(t_\chi, CL(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_{16}(t_\chi, CL(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_{16}(t_{\chi-1}, CL(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{16}(0, CL(0))\left(1 + \frac{\vartheta_{\nu_{1_\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$A(t_{\chi+1}) = A(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{17}(t_\chi, A(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_{17}(t_\chi, A(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_{17}(t_{\chi-1}, A(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{17}(0, A(0))\left(1 + \frac{\vartheta_{\nu_{1_\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$AL(t_{\chi+1}) = AL(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{18}(t_\chi, AL(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_{18}(t_\chi, AL(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_{18}(t_{\chi-1}, AL(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{18}(0, AL(0))\left(1 + \frac{\vartheta_{\nu_{1\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$L(t_{\chi+1}) = L(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{19}(t_\chi, L(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_{19}(t_\chi, L(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_{19}(t_{\chi-1}, L(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{19}(0, L(0))\left(1 + \frac{\vartheta_{\nu_{1\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$CAL(t_{\chi+1}) = CAL(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{20}(t_\chi, CAL(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_{20}(t_\chi, CAL(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_{20}(t_{\chi-1}, CAL(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{20}(0, CAL(0))\left(1 + \frac{\vartheta_{\nu_{1\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$R_p(t_{\chi+1}) = R_p(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{21}(t_\chi, R_p(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_{21}(t_\chi, R_p(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_{21}(t_{\chi-1}, R_p(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{21}(0, R_p(0))\left(1 + \frac{\vartheta_{\nu_{1_\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right),$$

$$R_l(t_{\chi+1}) = R_l(0) + \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{22}(t_\chi, R_l(t_\chi)) + \frac{\vartheta h^{\nu_1}}{\vartheta(\nu_1+2)}\sum_{\chi=0}^{\kappa}$$

$$\times\left(\mathscr{H}_{22}(t_\chi, R_l(t_\chi))\Big((\kappa+2+\nu_1-\chi)(\kappa+1-\chi)^{\nu_1} - (\kappa+2+2\nu_1-\chi)(\kappa-\chi)^{\nu_1}\Big)\right.$$

$$\left. - \mathscr{H}_{22}(t_{\chi-1}, R_l(t_{\chi-1}))\Big((\kappa+1-\chi)^{\nu_1} - (\kappa-\chi+1+\nu_1)(\kappa-\chi)^{\nu_1}\Big)\right)$$

$$- \frac{1-\nu_1}{\mathscr{Z}(\nu_1)}\mathscr{H}_{22}(0, R_l(0))\left(1 + \frac{\vartheta_{\nu_{1_\varepsilon}}(\chi h)^{\nu_1}}{\Gamma(\nu_1+1)}\right).$$

The numerical simulations of the proposed model are shown in Figures 2-23. They use the Lagrange polynomial interpolation method combined with the Atangana-Baleanu-Caputo sense with fractional derivative. These simulations are conducted across various fractional orders to provide deeper insights into the system's dynamic behavior. The fractional orders $\nu_1 = 0.75, 0.80, 0.85, 0.90$, and 0.95 are analyzed over a time period of $t = 100$. The initial conditions for the model are assumed as follows: $S(0) = 600$; $E_p(0) = 170$; $E_{ph}(0) = 250$; $E_{lp}(0) = 200$; $E_h(0) = 200$; $E_{lh}(0) = 240$; $E_l(0) = 250$; $I_p(0) = 140$; $I_{ph}(0) = 140$; $I_{lp}(0) = 140$; $I_h(0) = 160$; $I_{lh}(0) = 180$; $I_l(0) = 160$; $C(0) = 60$; $CA(0) = 40$; $CL(0) = 50$; $A(0) = 40$; $AL(0) = 50$; $L(0) = 50$; $CAL(0) = 30$; $R_p(0) = 120$ and $R_l(0) = 130$. Some parameter values are adopted from the World Health Organization, while others are assumed. The numerical simulations consider the following parameter values: $\Pi = 0.004$; $\pi = 0.01$; $\beta_l = 0.0018$; $\psi = 0.3$; $\beta_h = 0.042$; $\beta_p = 0.042$; $\beta_{ph} = 0.019$; $\beta_{lp} = 0.03$; $\beta_{lh} = 0.02$; $\delta = 0.12$; $\gamma = 0.14$; $\chi_p = 0.045$; $\chi_l = 0.045$; $\rho_1 = 0.04$; $\rho_2 = 0.02$; $\rho_3 = 0.04$; $\rho_4 = 0.05$; $\rho_5 = 0.03$; $\rho_6 = 0.04$; $\rho_7 = 0.02$; $\rho_8 = 0.02$; $\rho_9 = 0.03$; $\rho_{10} = 0.05$; $\rho_{11} = 0.03$; $\rho_{12} = 0.04$; $\rho_{13} = 0.02$; $\rho_{14} = 0.03$; $\rho_{15} = 0.04$; $\rho_{16} = 0.02$; $\rho_{17} = 0.02$; $\rho_{18} = 0.03$; $\eta_s = 0.02$; $\eta_h = 0.02$; $\eta_p = 0.02$; $\eta_{ph} = 0.02$; $\eta_{lp} = 0.02$; $\eta_{lh} = 0.02$; $\alpha_l = 0.03$; $\alpha_h = 0.03$; $\alpha_p = 0.03$; $\alpha_{ph} = 0.03$; $\alpha_{lp} = 0.03$; $\alpha_{lh} = 0.03$; $\omega_p = 0.035$; $\omega_l = 0.045$; $\varphi = 0.1$; $\xi = 0.0001$; $\theta = 0.2$ and $\mu = 0.025$.

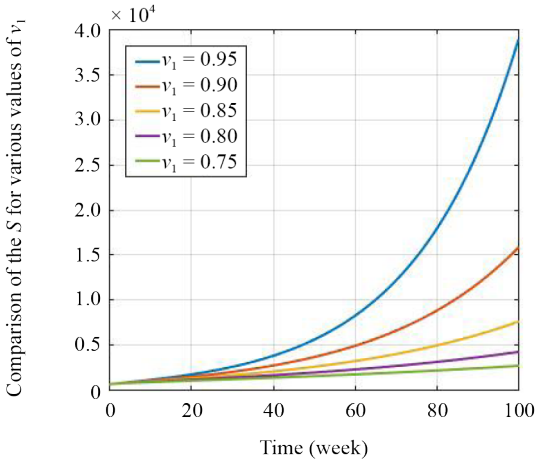

Figure 2. Comparison of S over time for varying values of v_1

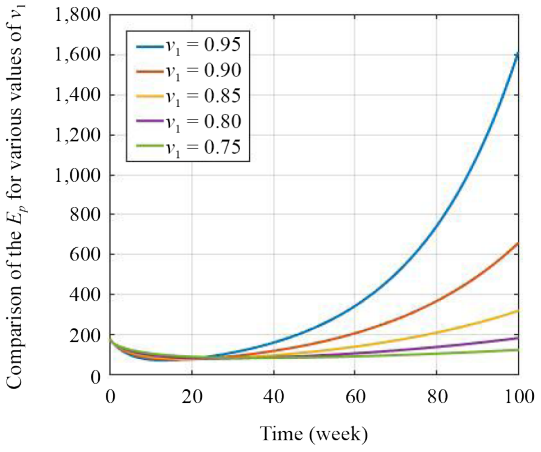

Figure 3. Comparison of E_p over time for varying values of v_1

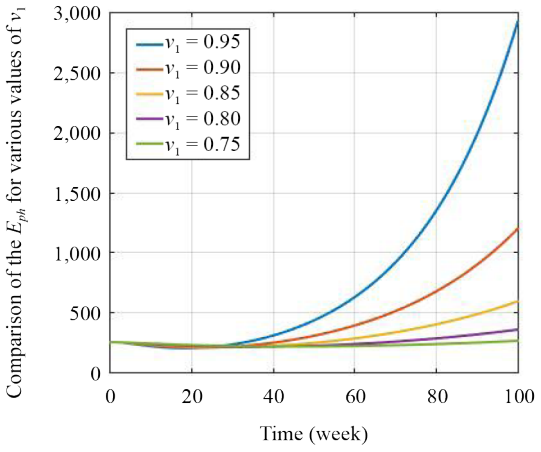

Figure 4. Comparison of E_{ph} over time for varying values of v_1

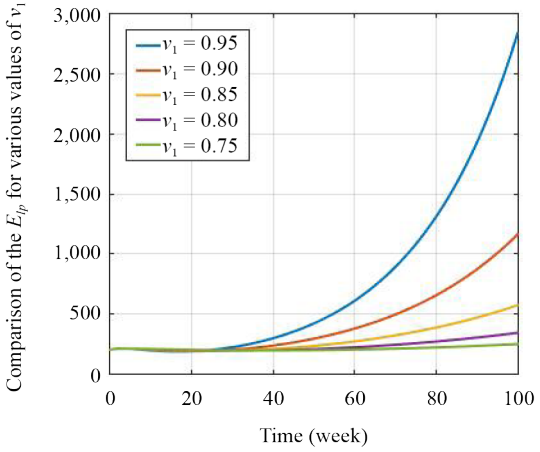

Figure 5. Comparison of E_{lp} over time for varying values of v_1

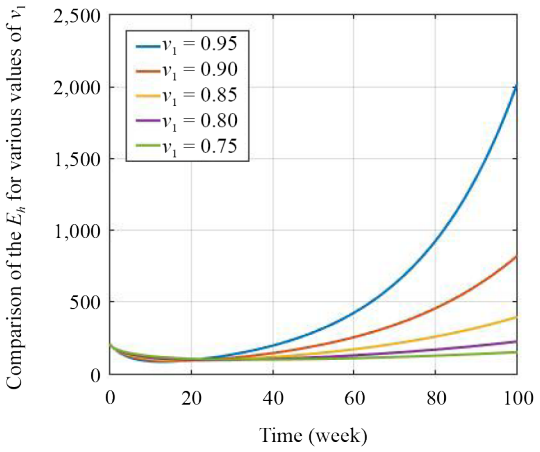

Figure 6. Comparison of E_h over time for varying values of v_1

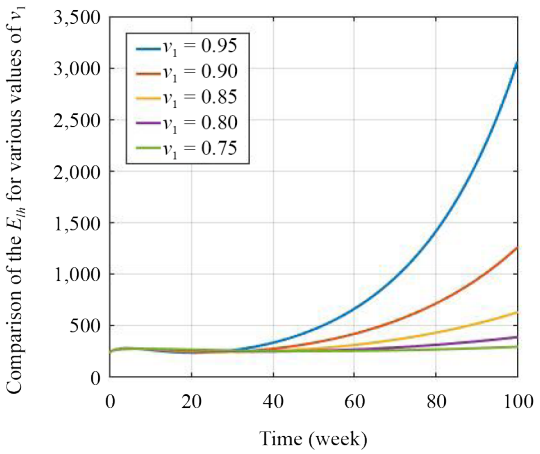

Figure 7. Comparison of E_{lh} over time for varying values of v_1

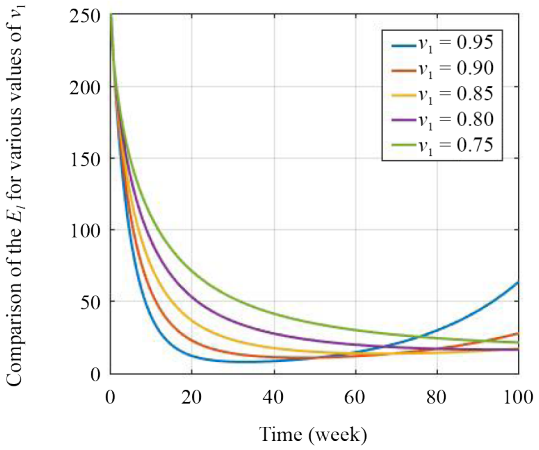

Figure 8. Comparison of E_l over time for varying values of v_1

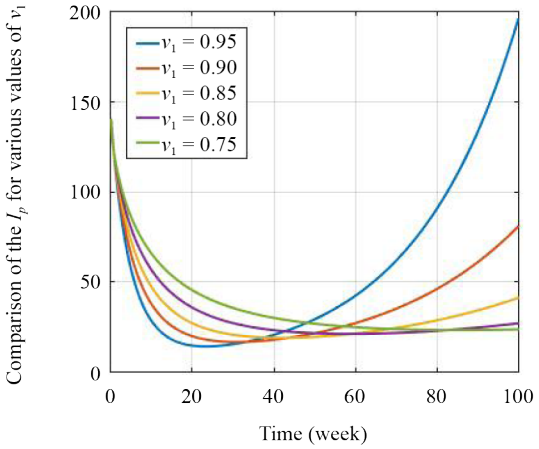

Figure 9. Comparison of I_p over time for varying values of v_1

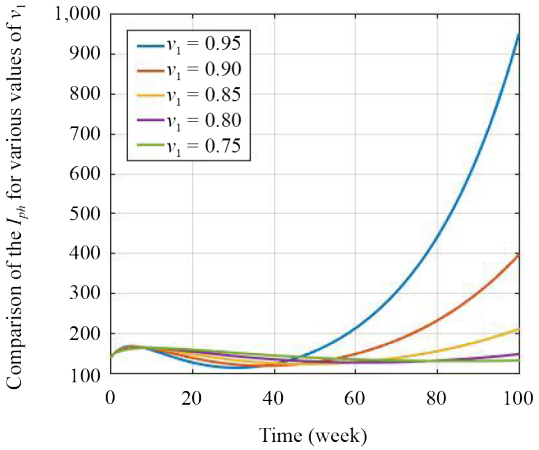

Figure 10. Comparison of I_{ph} over time for varying values of v_1

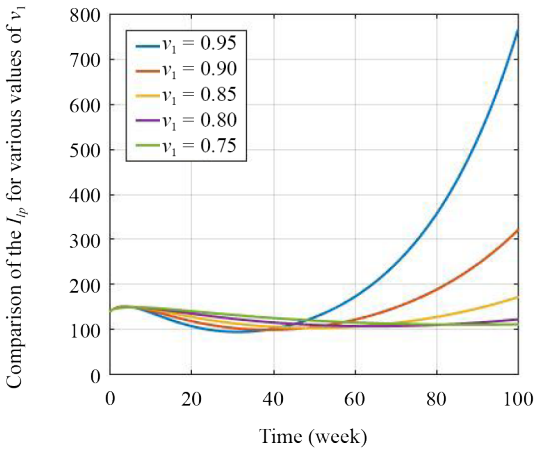

Figure 11. Comparison of I_{lp} over time for varying values of v_1

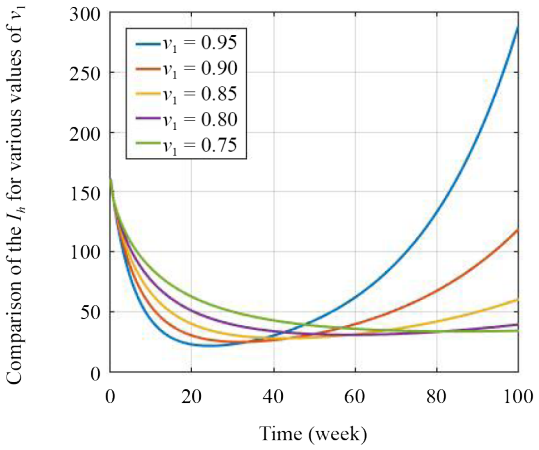

Figure 12. Comparison of I_h over time for varying values of v_1

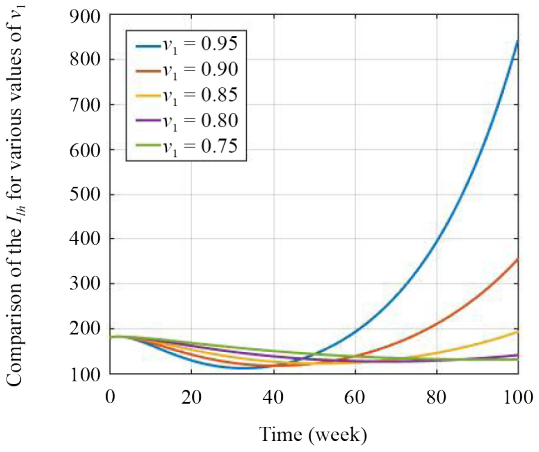

Figure 13. Comparison of I_{lh} over time for varying values of v_1

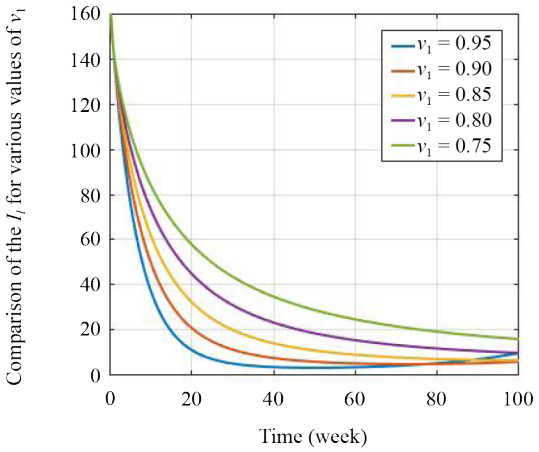

Figure 14. Comparison of I_l over time for varying values of v_1

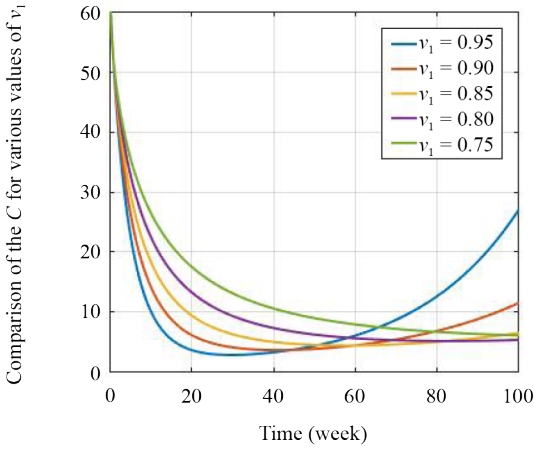

Figure 15. Comparison of C over time for varying values of v_1

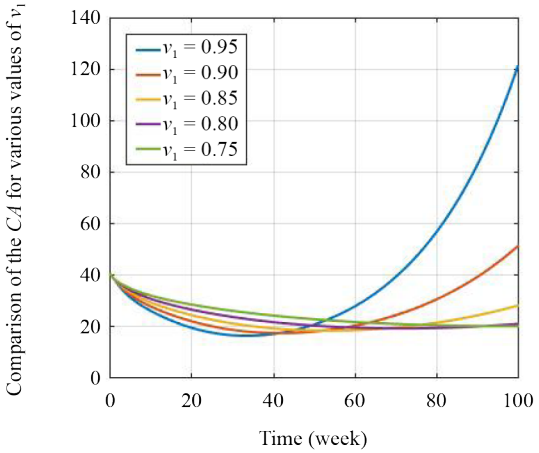

Figure 16. Comparison of CA over time for varying values of v_1

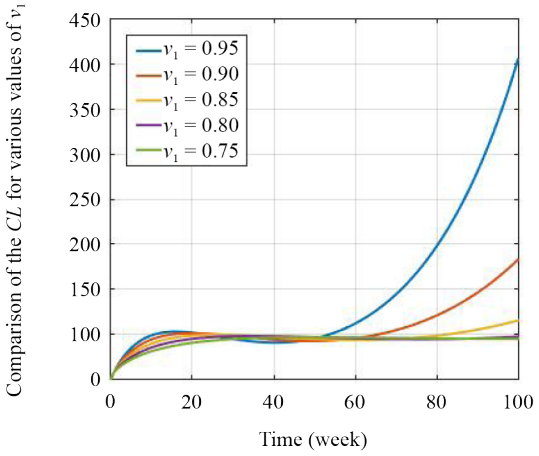

Figure 17. Comparison of CL over time for varying values of v_1

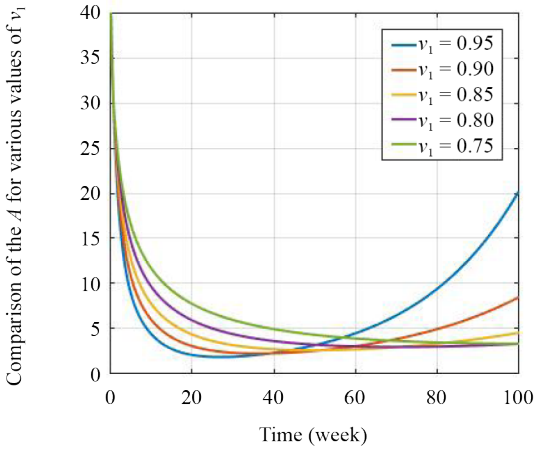

Figure 18. Comparison of A over time for varying values of v_1

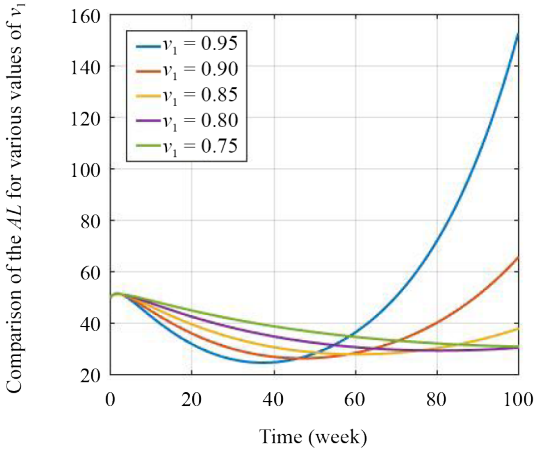

Figure 19. Comparison of AL over time for varying values of v_1

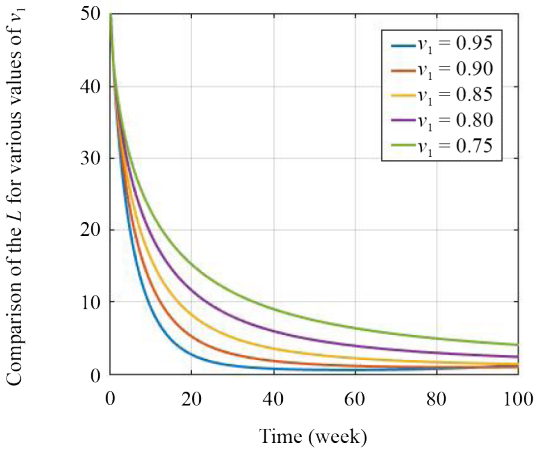

Figure 20. Comparison of L over time for varying values of v_1

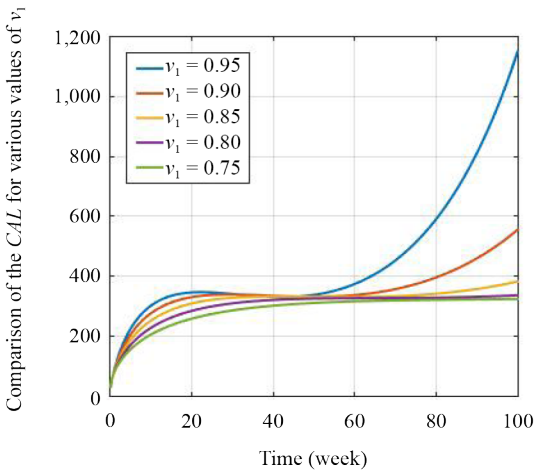

Figure 21. Comparison of CAL over time for varying values of v_1

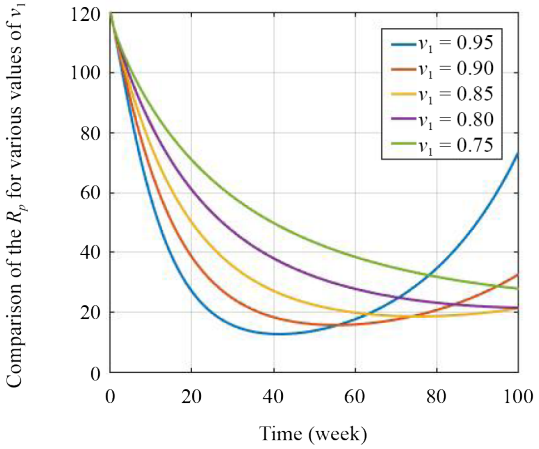

Figure 22. Comparison of R_p over time for varying values of v_1

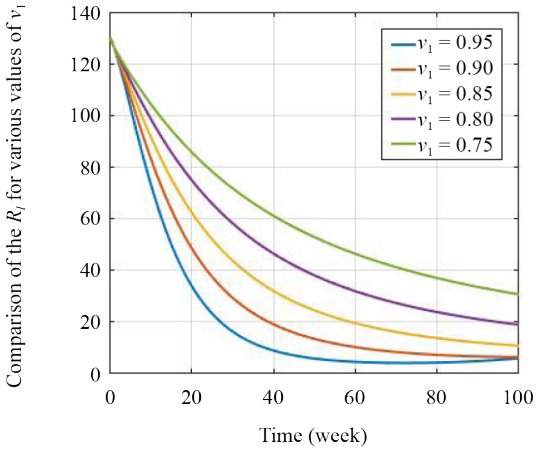

Figure 23. Comparison of R_l over time for varying values of ν_1

The influence of varying fractional orders on system dynamics is depicted in Figures 2-7, illustrating the temporal variations of S, E_p, E_{ph}, E_{lp}, E_h, and E_{lh} for different values of ν_1. It is observed that all these compartments increase as ν_1 decreases, indicating a direct relationship between the fractional order and the growth rate of susceptible and exposed individuals. Figures 8 and 9 demonstrate the effect of different ν_1 values on E_l and I_p. The results indicate that E_l and I_p initially decrease rapidly before stabilizing, with their values increasing as ν_1 decreases. This suggests that lower fractional orders contribute to an initial decline followed by a resurgence in exposed and infected populations. Figures 10 and 11 illustrate the behavior of I_{ph} and I_{lp} under varying fractional orders. Initially, these compartments exhibit growth, followed by a stabilization phase before increasing more rapidly as ν_1 decreases. This trend highlights the role of fractional order in regulating the dynamics of co-infected individuals.

Figure 12 examines the effect of ν_1 on I_h over time. The results reveal that I_h declines initially before increasing with decreasing ν_1, reaffirming the connection between fractional order and the initial suppression followed by an accelerated growth phase of the infected population. Figure 13 presents the behavior of I_{lh} under different fractional orders. Initially, I_{lh} increases, stabilizes, and grows more rapidly as ν_1 decreases. This demonstrates that lower fractional orders intensify infection dynamics after an initial plateau. Figure 14 illustrates the variation of I_l over time for different ν_1 values. It is observed that I_l increases as ν_1 decreases, eventually tending towards zero, indicating that higher fractional orders accelerate the reduction of infected individuals. Figures 15 and 16 analyze the effects of ν_1 on C and CA over time. The results suggest that C and CA initially decline but subsequently increase with decreasing ν_1, indicating that a lower fractional order prolongs the presence of these compartments. Figure 17 shows the evolution of CL under varying fractional orders. Initially, CL increases, stabilizes for some time, and then grows rapidly as ν_1 decreases, reinforcing the impact of fractional orders on infection persistence.

Figures 18 and 19 examine the effects of ν_1 on A and AL. Similar to previous results, both compartments exhibit an initial decline followed by an increase as ν_1 decreases, emphasizing the influence of fractional order on the dynamics of these populations. Figure 20 presents the variations of L over time for different ν_1 values. It is observed that L increases as ν_1 decreases, ultimately approaching zero, suggesting that higher fractional orders expedite the reduction of latent individuals. Figure 21 depicts the behavior of CAL under different fractional orders. Initially, CAL increases, stabilizes, and then rapidly rises as ν_1 decreases, reinforcing the role of fractional calculus in infection progression. Figure 22 investigates the effect of ν_1 on R_p over time. The results indicate that R_p initially decreases before increasing with decreasing ν_1, highlighting a complex interaction between fractional order and recovery dynamics. Figure 23 illustrates the variations of R_l over time for different ν_1 values. Similar to I_l and L, R_l increases as ν_1 decreases, ultimately tending towards zero, further demonstrating how fractional order derivatives influence disease progression and recovery rates. Overall, these findings underscore the significant impact of fractional orders on the system's dynamics, particularly in determining infection persistence, recovery trends, and population stability. We believe that fractional order modeling is

a useful way to understand the fractional HTLV/HPV/HIV multiple-infection model based on the numerical results. The results demonstrate how fractional calculus captures the disease dynamics more effectively than classical integer-order approaches.

6. Conclusion

In this study, we successfully investigated the stability analysis of a fractional HTLV/HPV/HIV multiple-infection model, modified using the Atangana-Baleanu-Caputo derivative. We derived the sufficient conditions for solution existence and uniqueness via fixed-point theory and prove Ulam-Hyers stability properties. To analyze numerical simulations, we developed computational schemes based on fractional calculus principles and employed Lagrange polynomial interpolation. The results of this study contribute to a deep understanding of the stability analysis of fractional order epidemiological models. Future research should extend this framework to similar infectious disease models, addressing additional complexities and real-world applications in a more comprehensive manner.

Author contributions

The authors equally conceived of the study participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Availablity of data and materials

Data sharing does not apply to this article as no datasets were generated or analysed during the current study.

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Conflict of interest

The authors declare no competing financial interest.

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