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Research Article

New Approach of Fractional Integral Inequalities Involving Interval-Valued Mappings with Applications to Matrix Analysis

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Abstract: Convexity via fractional calculus is a widely accepted concept that has attracted considerable interest in the field of applied mathematics. The aim of this paper is to develop new types of fractional Hermite-Hadamard-Mercer inequalities related to the generalized k-fractional conformable integral operator within the framework of interval analysis. To highlight the applicability of the concepts covered in this study, several illustrative examples are presented. The ideas and methods presented in this study could serve as a foundation for further research in this area.

Keywords: Hermite-Hadamard inequality, convexity, HH's Mercer inequality, generalized fractional $\mathtt{k}$-operator, interval-valued mapping

MSC: 26A33, 26A51, 26D07, 26D10, 26D15

1. Introduction

Convexity and optimization have a significant impact on various fields of applied sciences, including computer science and data analysis [1], control systems [2], finance [3], modeling optimization [4, 5], and signal processing and estimation [6]. The study of convexity in conjunction with integral problems is a fascinating and important area of research. Consequently, convex function applications have been used to represent multiple inequalities. Among these, the Hermite-Hadamard (H-H) inequality represents a fundamental result in convex analysis. The concept of convexity has showed to be a useful domain of inspiration in both the pure and applied sciences.

The modern theory of convex functions emerged in the early 20$^{\text{th}}$ century through the work of mathematicians like Jacques Hadamard and Hermann Minkowski. Hadamard provided rigorous definitions and explored the properties of convex functions in mathematical analysis, while Minkowski integrated convexity into geometry and functional analysis. Convex functions became central to optimization theory due to the pioneering contributions of Leonid Kantorovich and Harold Kuhn, who developed methods such as linear programming and the Karush-Kuhn-Tucker (KKT) conditions. Today, convex functions are a cornerstone of mathematical optimization, physics, information technology, probability

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theory, economics, and machine learning, reflecting their broad applicability and historical evolution. See [7–14] for the literature.

Over the past 20 years, fractional calculus has been essential to the fields of applied mathematics. Due to its wide range of applications, it continues to receive significant attention in ongoing research. The Hermite-Hadamard (H-H) inequality via fractional operator was first investigated by Sarikaya et al. [15]. Numerous fields, including transform theory, mathematical biology, modeling, engineering, fluid flow, finance, healthcare, and image processing, use fractional calculus. Additional information on this subject can be found in the references [16–19].

The novelty and objective of this paper lie in utilizing the Generalized k-Fractional Conformable Integral Operator (GkFCIO) in interval analysis to investigate a new type of Hermite-Hadamard-Mercer inequality. Additionally, we investigated a few matrix applications using GkFCIO.

The structure of this manuscript is constructed as follows: In Section 2, presents some related theorems, concepts, and remarks that are essential for the upcoming portions. In Section 3, we provide several interval analysis indications and basic data. In Section 4, we utilize the GkFCIO to establish several new types of Hermite-Hadamard-Mercer inequalities. In Section 5, we present additional applications. In Section 6, we offer a conclusion.

2. Preliminaries

It would be pertinent to highlight and analyze a few definitions, findings, and theorems in this section for the benefit of the reader's interest and the article's standard.

Definition 1 [20] The function $\Im\colon [\mathfrak{c}_1, \mathfrak{d}_1] \to \mathbb{R}$ is convex, if

$$\Im\left(\zeta\rho_1 + (1-\zeta)\rho_2\right) \leq \zeta\Im\left(\rho_1\right) + (1-\zeta)\Im\left(\rho_2\right) \tag{1}$$

where $\rho_1, \rho_2 \in [\mathfrak{c}_1, \mathfrak{d}_1]$ and $\zeta \in [0, 1]$.

If $\Im\colon \mathscr{I} \to \mathbb{R}$ is convex, then for $\rho_1, \rho_2 \in \mathscr{I}$ and $\rho_2 > \rho_1$, then H-H inequality,

$$\Im\left(\frac{\rho_1+\rho_2}{2}\right) \leq \frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\Im(\zeta)d\zeta \leq \frac{\Im(\rho_1)+\Im(\rho_2)}{2}, \tag{2}$$

holds.

Interested readers can refer to [21] and [22].

If $\Im$ is concave, both inequalities hold in the opposite direction.

Jensen's inequality holds a prominent place among the various inequalities related to convexity found in the literature. It is widely utilized by mathematicians in fields such as inequality and information theory, owing to its applicability under relatively simple conditions. Below, we present the formal statement of Jensen's inequality.

With considering $0 < \aleph_1 \leq \aleph_2 \leq \ldots \leq \aleph_n$ and suppose $\varsigma = \left(\varsigma_1, \varsigma_2, \ldots, \varsigma_n\right)$ is positive weights such that $\sum_{d=1}^{n} \varsigma_d = 1$, then the Jensen Inequality (JI) (see [23]) states that if $\Im$ is convex on the $\left[\rho_1, \rho_2\right]$, then

$$\Im\left(\sum_{d=1}^{n}\varsigma_d\ \aleph_d\right) \leq \left(\sum_{d=1}^{n}\varsigma_d\ \Im\left(\aleph_d\right)\right), \tag{3}$$

for all $\aleph_d \in [\rho_1, \rho_2]$, $\varsigma_d \in [0, 1]$ and $(d = 1, 2, \ldots, n)$.

The extraction of information-theoretic bounds for practical distances is facilitated by this fundamental inequality (see [24]).

Even though Jensen's inequality has been the focus of many research, Mercer's version is the most interesting and unique of the studies on the topic. Mercer presented in 2003 a novel-version of "Jensen's inequality" in [25], which is as follows:

If $\Im\colon [\rho_1, \rho_2] \to \mathbb{R}$ is a convex function, then

$$\Im\left(\rho_1+\rho_2-\sum_{d=1}^{n}\varsigma_d\ \aleph_d\right)\leq\Im\left(\rho_1\right)+\Im\left(\rho_2\right)-\sum_{d=1}^{n}\varsigma_d\ \Im\left(\aleph_d\right), \tag{4}$$

holds for all $\aleph_d \in [\rho_1, \rho_2]$, $\varsigma_d \in [0, 1]$ and $(d = 1, 2, \ldots, n)$.

In [26] author proposed many of Jensen-Mercer Inequalities (JMI). Mercer's type inequalities have been extended to higher dimensions in a number of ways by Niezgoda [27]. Jensen-Mercer's type inequality has recently significantly advanced inequality theory because of its well-known characterizations. Kian [28] explored superquadratic functions in relation to Jensen's inequality.

The following H-H-Mercer inequality was shown in [29]:

$$\Im\left(\rho_1+\rho_2-\frac{u_1+u_2}{2}\right)\leq\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\Im\left(\rho_1+\rho_2-\zeta\right)d\zeta \tag{5}$$

$$\leq\frac{\Im\left(\rho_1+\rho_2-u_1\right)+\Im\left(\rho_1+\rho_2-u_2\right)}{2}\leq\Im\left(\rho_1\right)+\Im\left(\rho_2\right)-\frac{\Im\left(u_1\right)+\Im\left(u_2\right)}{2}, \tag{6}$$

where $\Im$ is convex on $[\mathfrak{c}, \mathfrak{d}]$. For JMI, one can refer [27, 30, 31].

Despite being around for as long as classical analysis, fractional analysis has only lately started to draw more attention from academics. Its continual evolution is driven by its usefulness in natural phenomena, its applications across various fields such as engineering and the sciences, its adaptability, and the additional perspectives it offers to mathematical theories.

Definition 2 [32] Consider $\Im \in \mathscr{L}[\mathfrak{c}_1, \mathfrak{d}_1]$. The Riemann-Liouville (R-L) fractional-integrals with order $\Delta > 0$ is defined by

$${}_{\mathfrak{c}_1}\mathfrak{J}^{\Delta}\Im(\tau)=\frac{1}{\Gamma(\Delta)}\int_{\mathfrak{c}_1}^{\tau}(\tau-\mu)^{\Delta-1}\Im(\mu)d\mu, \quad \mathfrak{c}_1<\tau, \tag{7}$$

and

$$\mathfrak{J}^{\Delta}_{\mathfrak{d}_1}\Im(\tau)=\frac{1}{\Gamma(\Delta)}\int_{\tau}^{\mathfrak{d}_1}(\mu-\tau)^{\Delta-1}\Im(\mu)d\mu, \quad \tau<\mathfrak{d}_1. \tag{8}$$

Gamma function is defined as $\Gamma(\Delta)=\int_0^{\infty}e^{-u}u^{\Delta-1}du$.

An integral operator of fractional stated by Fahad et al. [33]. They also provided this operator's characteristics. Suppose that $\gamma\in\mathscr{C},\ Re(\gamma)>0,$ then the left and right sided generalized fractional conformable integral operators has defined respectively, as follows

$$
{}_{\mathfrak{c}_1}^{\gamma}\mathfrak{J}^{\Delta}\Im(x) = \frac{1}{\Gamma(\gamma)}\int_{\mathfrak{c}_1}^{x}\left(\frac{(x-\mathfrak{c}_1)^{\Delta}-(\zeta-\mathfrak{c}_1)^{\Delta}}{\Delta}\right)^{\gamma-1}\frac{\Im(\zeta)}{(\zeta-\mathfrak{c}_1)^{1-\Delta}}\,d\zeta \tag{9}
$$

and

$$
{}^{\gamma}\mathfrak{J}_{\mathfrak{d}_1}^{\Delta}\Im(x) = \frac{1}{\Gamma(\gamma)}\int_{x}^{\mathfrak{d}_1}\left(\frac{(\mathfrak{d}_1-x)^{\Delta}-(\mathfrak{d}_1-\zeta)^{\Delta}}{\Delta}\right)^{\gamma-1}\frac{\Im(\zeta)}{(\mathfrak{d}_1-\zeta)^{1-\Delta}}\,d\zeta, \tag{10}
$$

where $\mathfrak{d}_1 > \mathfrak{c}_1$, $\Delta \in [0, 1]$.

The fractional integral in (9) coincides with the Riemann–Liouville fractional integral (7), when $\mathfrak{c}_1 = 0$ and $\Delta = 1$. It also coincides with the Hadamard fractional (see [34]) once $\mathfrak{c}_1 = 0$ and $\Delta \to 0$. Moreover (10) coincides with the Riemann–Liouville fractional integral (8), when $\mathfrak{d}_1 = 0$ and $\Delta = 1$. It also coincides with the Hadamard fractional (see [34]) once $\mathfrak{d}_1 = 0$ and $\Delta \to 0$.

The generalized k-fractional conformable integrals are defined as

$$
{}_{\mathfrak{c}_1}^{(\gamma,\mathrm{k})}\mathfrak{J}^{\Delta}\Im(x) = \frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(\gamma)}\int_{\mathfrak{c}_1}^{x}\left(\frac{(x-\mathfrak{c}_1)^{\Delta}-(\zeta-\mathfrak{c}_1)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{\mathrm{k}}-1}\frac{\Im(\zeta)}{(\zeta-\mathfrak{c}_1)^{1-\Delta}}\,d\zeta, \tag{11}
$$

and

$$
{}^{(\gamma,\mathrm{k})}\mathfrak{J}_{\mathfrak{d}_1}^{\Delta}\Im(x) = \frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(\gamma)}\int_{x}^{\mathfrak{d}_1}\left(\frac{(\mathfrak{d}_1-x)^{\Delta}-(\mathfrak{d}_1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{\mathrm{k}}-1}\frac{\Im(\zeta)}{(\mathfrak{d}_1-\zeta)^{1-\Delta}}\,d\zeta, \tag{12}
$$

where $\mathfrak{d}_1 > \mathfrak{c}_1$, $\Delta > 0$, $\Re(\gamma) > 0$, and $\Gamma_{\mathrm{k}}(x)$ is defined in [35–37] as

$$
\Gamma_{\mathrm{k}}(x) = \lim_{\eta\longrightarrow\infty}\frac{\Theta!\mathrm{k}^{\Theta}(n\mathrm{k})^{\frac{x}{\mathrm{k}}-1}}{(x)_{\Theta,\mathrm{k}}},
$$

where

$$
(\pi)_{\Theta,\mathrm{k}} = \begin{cases} 1, & \Theta = 0; \\ \pi(\pi+\mathrm{k})\ldots(\pi+(\Theta-1)\mathrm{k}), & \Theta \in N. \end{cases}
$$

Additionally, several noteworthy inequalities for Interval-Valued Functions (IVFs) have been studied recently, including those of H-H, Ostrowski, Simpson, and others. For further results, see the literature in [38–41].

3. Interval calculus

Interval analysis is utilized for exploring interval-valued variational control challenges in merely some of the fields, including neural processing, artificial intelligence, information theory, fuzzy logic, and evolutionary algorithms. Moore [42] used the Riemann integral to study functions in the interval value frame. Bhurjee and Panda [43] developed a method for solving large-scale multi-objective fractional programming problems. Zhang et al. [44] adapted the concepts of interval-valued functions via preinvexity. Zhao et al. [45] introduced integral inequalities for interval-valued functions, specifically the Chebyshev-type, in their investigation of the interval double integral. Among the domains where interval analysis may be useful are signal processing, beam physics, economics, control circuit design, computer graphics, chemical engineering, global optimization, robotics, and error analysis. The right-sided R-L fractional integral with interval values was studied by Budak et al. [38], who also constructed new H-H inequalities for these integrals. Sharma et al. [46] defined preinvexity and gave improvements of fractional H-H-type inequalities for interval analysis.

3.1 *Definitions*

An interval $Z = [a_1, b_1]$ is defined as:

$$[a_1, b_1] = \{x \in \mathbb{R} \mid a_1 \leq x \leq b_1\}.$$

3.2 *Basic operations*

Let $\mathbf{X}_1 = [a_1, b_1]$ and $\mathbf{Y}_1 = [c_1, d_1]$. The following operations are defined:

- **By Addition:**

$$\mathbf{X}_1 + \mathbf{Y}_1 = [a_1 + c_1, b_1 + d_1]$$

- **By Subtraction:**

$$\mathbf{X}_1 - \mathbf{Y}_1 = [a_1 - d_1, b_1 - c_1]$$

- **By Multiplication:**

$$\mathbf{X}_1 \cdot \mathbf{Y}_1 = [\min(a_1c_1, a_1d_1, b_1c_1, b_1d_1), \max(a_1c_1, a_1d_1, b_1c_1, b_1d_1)]$$

- **By Division (if $\mathbf{0} \notin \mathbf{Y}_1$):**

$$\frac{\mathbf{X}_1}{\mathbf{Y}_1} = [\min(a_1/c_1, a_1/d_1, b_1/c_1, b_1/d_1), \max(a_1/c_1, a_1/d_1, b_1/c_1, b_1/d_1)]$$

3.3 *Other concepts*

Intersection of Intervals: The intersection of $\mathbf{X}_1 = [a_1, b_1]$ and $\mathbf{Y}_1 = [c_1, d_1]$ is:

$$\mathbf{X} \cap \mathbf{Y}_1 = \begin{cases} [\max(a_1, c_1), \min(b_1, d_1)], & \text{if } \max(a_1, c_1) \leq \min(b_1, d_1), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Length of an Interval: The length of $\mathbf{X}_1 = [a_1, b_1]$ is:

$$\text{Length}(\mathbf{X}_1) = b_1 - a_1.$$

Midpoint of an Interval: The midpoint of $\mathbf{X}_1 = [a_1, b_1]$ is:

$$\text{Midpoint}(\mathbf{X}_1) = \frac{a_1 + b_1}{2}.$$

Let $\mathbf{X}_1 = [a_1, b_1]$ and $\mathbf{Y}_1 = [c_1, d_1]$ be intervals. The following algebraic properties hold for interval arithmetic:

3.4 *Remarks*

1. The set of intervals is closed under addition, subtraction, multiplication, and division (if $0 \notin \mathbf{Y}_1$):

$$\mathbf{X}_1 + \mathbf{Y}_1, \quad \mathbf{X}_1 - \mathbf{Y}_1, \quad \mathbf{X}_1 \cdot \mathbf{Y}_1, \quad \frac{\mathbf{X}_1}{\mathbf{Y}_1} \quad \text{are all intervals.}$$

2. Addition and multiplication of intervals are commutative:

$$\mathbf{X}_1 + \mathbf{Y}_1 = \mathbf{Y}_1 + \mathbf{X}_1,$$

$$\mathbf{X}_1 \cdot \mathbf{Y}_1 = \mathbf{Y}_1 \cdot \mathbf{X}_1.$$

3. Addition and multiplication of intervals are associative:

$$(\mathbf{X}_1 + \mathbf{Y}_1) + \mathbf{Z}_1 = \mathbf{X}_1 + (\mathbf{Y}_1 + \mathbf{Z}_1),$$

$$(\mathbf{X}_1 \cdot \mathbf{Y}_1) \cdot \mathbf{Z}_1 = \mathbf{X}_1 \cdot (\mathbf{Y}_1 \cdot \mathbf{Z}_1).$$

4. Multiplication distributes over addition:

$$\mathbf{X}_1 \cdot (\mathbf{Y}_1 + \mathbf{Z}_1) = (\mathbf{X}_1 \cdot \mathbf{Y}_1) + (\mathbf{X}_1 \cdot \mathbf{Z}_1).$$

5. -The additive identity is $[0, 0]$:

$$\mathbf{X}_1 + [0, 0] = \mathbf{X}_1.$$

-The multiplicative identity is $[1, 1]$:

$$\mathbf{X}_1 \cdot [1, 1] = \mathbf{X}_1.$$

6. The additive inverse of $\mathbf{X}_1 = [a_1, b_1]$ is $-\mathbf{X}_1 = [-b_1, -a_1]$, such that:

$$\mathbf{X}_1 + (-\mathbf{X}_1) = [0, 0].$$

7. (If $0 \notin \mathbf{X}$), the multiplicative inverse of $\mathbf{X}_1 = [a_1, b_1]$ is:

$$\mathbf{X}_1^{-1} = \left[b_1^{-1}, a_1^{-1}\right], \quad \text{if } 0 \notin [a_1, b_1].$$

8. Interval multiplication is subdistributive over interval addition:

$$\mathbf{X}_1 \cdot (\mathbf{Y}_1 + \mathbf{Z}_1) \subseteq (\mathbf{X}_1 \cdot \mathbf{Y}_1) + (\mathbf{X}_1 \cdot \mathbf{Z}_1).$$

This arises because interval arithmetic accounts for all possible combinations of endpoints, leading to potential overestimation.

9. If $\mathbf{X}_1 = [a_1, b_1]$ and $\mathbf{Y}_1 = [c_1, d_1]$, then:

$$\mathbf{X}_1 \subseteq \mathbf{Y}_1 \implies \mathbf{X}_1 + \mathbf{Z}_1 \subseteq \mathbf{Y}_1 + \mathbf{Z}_1,$$

$$\mathbf{X}_1 \subseteq \mathbf{Y}_1 \implies \mathbf{X}_1 \cdot \mathbf{Z}_1 \subseteq \mathbf{Y}_1 \cdot \mathbf{Z}_1.$$

Ramon elaborate the Riemann–integral for mapping with interval values in [42]. $IR_{([\rho_1, \rho_2])}$ forms the sets of R-L integrable IV-mappings and $R_{([\rho_1, \rho_2])}$ signify the real-valued mappings on $[\rho_1, \rho_2]$, respectively (see [47], p. 131).

3.5 *Integral of interval-valued functions.*

The idea of an integral for interval-valued functions is discussed in this section. The following essential concepts will be presented prior to the definition of integral. A mapping $\Im$ is interval-valued function of ζ on $[\rho_1, \rho_2]$ if $\zeta \in [\rho_1, \rho_2]$

$$\Im = [\underline{\Im}(\zeta), \bar{\Im}(\zeta)],$$

where $\underline{\Im}$ and $\bar{\Im}$ are real-valued mappings. Let

$$\mathscr{P}\colon \rho_1 = \zeta_0 < \zeta_1 < \ldots < \zeta_d = \rho_2,$$

where $\mathscr{P}$ is defined by,

$$\text{mesh}(\mathscr{P}) = \max\{\zeta_i - \zeta_{i-1}\colon i = 1, 2, \ldots, d\}.$$

Here the notation $\mathscr{P}([\rho_1, \rho_2])$ the set of all partitions of $[\rho_1, \rho_2]$. Suppose that $\mathscr{P}(\delta, [\rho_1, \rho_2])$ is the set of all $P = \mathscr{P}([\rho_1, \rho_2])$ such as $\text{mesh}(P) < \delta$.

Choose $\xi_i \in [\zeta_{i-1} - \zeta_i]$, $i = 1, 2, \ldots, d$ and define sum

$$\mathscr{S}(\Im, P, \delta) = \sum_{i=1}^{d} \Im(\delta_i)[\zeta_{i-1} - \zeta_i],$$

where $\Im\colon [\rho_1, \rho_2] \to \mathbb{R}_I$ and $\mathscr{S}(\Im, P, \delta)$ is Riemann sum of $\Im$ corresponding to $P \in \mathscr{P}([\rho_1, \rho_2])$.

Definition 3 A mapping $\Im\colon [\rho_1, \rho_2] \to \mathbb{R}_I$ is interval Riemann integrable (IR-integrable) on $[\rho_1, \rho_2]$, if $\mathscr{A} \in \mathbb{R}_I$ such as for $\varepsilon > 0$, there exist $\delta > 0$ as

$$d(\mathscr{S}(\Im, P, \delta), \mathscr{A}) < \varepsilon$$

for every Riemann sum $\mathscr{S}$ of $\Im$ corresponding to each $P \in \mathscr{P}(\delta, [\rho_1, \rho_2])$ and independent of choice of $\xi_i \in [\zeta_{i-1}, \zeta_i]$ for $1 \le i \le d$. In this case, $\mathscr{A}$ is called the $IR-$integral of $\Im$ on $[\rho_1, \rho_2]$ and is denoted by

$$\mathscr{A} = (IR)\int_{\rho_1}^{\rho_2} \Im(\zeta)\, d\zeta.$$

The following theorem gives a relation between IR-integrable and Riemann integrable (R-integrable) ([47, p. 131]).

Theorem 1 An IV mapping $\Im\colon [\rho_1, \rho_2] \to \mathbb{R}_I$ with $\Im(\zeta) = [\underline{\Im}(\zeta), \bar{\Im}(\zeta)]$. The mapping $\Im \in IR_{([\rho_1, \rho_2]} \Leftrightarrow \underline{\Im}(\zeta), \bar{\Im}(\zeta) \in IR_{([\rho_1, \rho_2]}$ and

$$(IR)\int_{\rho_1}^{\rho_2} \Im(\zeta)\, d\zeta = \left[(R)\int_{\rho_1}^{\rho_2} \underline{\Im}(\zeta)\, d\zeta,\, (R)\int_{\rho_1}^{\rho_2} \bar{\Im}(\zeta)\, d\zeta\right],$$

where $R_{([\rho_1, \rho_2])}$ denotes the R-integrable function. The following IVC mapping was defined in [41, 48].

Definition 4 $\forall\, x_1, y_1 \in [\rho_1, \rho_2]$ and $\delta \in (0, 1)$, the mapping $\Im\colon [\rho_1, \rho_2] \to \mathbb{R}_I^+$ is $\hbar$-convex if

$$\hbar(\delta)\Im(x_1) + \hbar(1-\delta)\Im(y_1) \subseteq \Im(\delta x_1 + (1-\delta) y_1), \tag{13}$$

where $\hbar$: $[\mathfrak{c}_1, \mathfrak{d}_1] \to \mathbb{R}$ is a positive function with $\hbar \neq 0$ and $(0, 1) \subseteq [\mathfrak{c}_1, \mathfrak{d}_1]$. We indicate the set of all $\hbar$-IVC mapping with $SX(\hbar, [\rho_1, \rho_2], \mathbb{R}_I^+)$.

The standard definition of an IVC mapping is (13) with $\hbar(\delta) = \delta$ (see [49]). If $\hbar(\delta) = \delta^s$ in (13), then we have the definition of s-IVC mapping (see [50]).

Theorem 2 ([51]) If $SX(\hbar, [\rho_1, \rho_2], \mathbb{R}_I^+$ and $\hbar(\frac{1}{2}) \neq 0,$ then we get

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\Im\left(\frac{\rho_1+\rho_2}{2}\right) \supseteq \frac{1}{\rho_2-\rho_1}(IR)\int_{\rho_1}^{\rho_2}\Im(\zeta)\,d\zeta \supseteq [\Im(\rho_1)+\Im(\rho_2)]\int_0^1 \hbar(\zeta)\,d\zeta. \tag{14}$$

Remark 1 In (14), if $\hbar(\delta) = \delta$, the inclusion becomes the following

$$\Im\left(\frac{\rho_1+\rho_2}{2}\right) \supseteq \frac{1}{\rho_2-\rho_1}(IR)\int_{\rho_1}^{\rho_2}\Im(x)\,dx \supseteq \frac{\Im(\rho_1)+\Im(\rho_2)}{2}, \tag{15}$$

which was initiated by Elżbieta in [49].

In (14), if $\hbar(\delta) = \delta^s$, the inclusion becomes

$$2^{s-1}\Im\left(\frac{\rho_1+\rho_2}{2}\right) \supseteq \frac{1}{\rho_2-\rho_1}(IR)\int_{\rho_1}^{\rho_2}\Im(x)\,dx \supseteq \frac{\Im(\rho_1)+\Im(\rho_2)}{s+1}, \tag{16}$$

which initiated by Osuna et al. in [52].

Definition 5 [49] A mapping $\Im$: $[\rho_1, \rho_2] \to \mathbb{R}_I$ is said to be IVCF, if for all $x_1, y_1 \in [\rho_1, \rho_2]$, $\delta \in (0, 1),$ we have

$$\delta\Im(x_1) + (1-\delta)\Im(y_1) \subseteq \Im(\delta x_1 + (1-\delta)y_1). \tag{17}$$

Theorem 3 [49] $\Im$: $[\rho_1, \rho_2] \to \mathbb{R}_I$ is an IVCF, if and only if $\underline{\Im}$ is convex on $[\rho_1, \rho_2]$ and $\bar{\Im}$ is concave on $[\rho_1, \rho_2]$.

Theorem 4 Jensen Inclusion-Interval-Valued Function (JIIVF): [53] Let $0 < \aleph_1 \leq \aleph_2 \leq \ldots \leq \aleph_n$ and $\Im$ be an IVCF containing ς_k, then we have:

$$\Im\left(\sum_{j=1}^{n}\varsigma_d\,\aleph_d\right) \supseteq \left(\sum_{d=1}^{n}\varsigma_d\,\Im(\aleph_d)\right), \tag{18}$$

where $\sum_{d=1}^{n}\varsigma_d = 1,\ \varsigma_d \in [0, 1]$.

The inequality (18) was extended to IVCF in [53] as follows:

Theorem 5 (Inclusion with IV mapping in Mercer) [53] If $\Im$ is an IVCF on $[\rho_1, \rho_2]$ and $\mathscr{L}(\rho_2) \geq \mathscr{L}(\rho_0)\ \forall\ \rho_0 \in [\rho_1, \rho_2],$ then

$$\Im\left(\rho_1+\rho_2-\sum_{d=1}^{n}\varsigma_d\,\aleph_j\right) \supseteq \Im(\rho_1)+\Im(\rho_2)\ominus_g\sum_{j=1}^{n}\varsigma_d\,\aleph_j. \tag{19}$$

Euler Gamma function [54, p.53], is defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

This function is of great importance in mathematical statistics, physics and probability theory, just to mention a few areas only. Thoroughly, we take the $\Gamma(.)$ is Euler Gamma. Also that Λ, γ, $\Bbbk > 0$. Let $\Delta \in (0, 1)$ and $\Im$: $[\rho_1, \rho_2] \to \mathbb{R}_I^+$ is an IVCF such as $\Im(\zeta) = [\underline{\Im}(\zeta), \bar{\Im}(\zeta)]$ and $\mathscr{L}(\rho_2) \geq \mathscr{L}(\varepsilon_0)$, $\forall \varepsilon_0 \in [\rho_1, \rho_2]$.

4. HH-Mercer like inequality involving IVCF via GkCFIO

Numerous inequalities have been established in convex theory since the concept of convexity was first introduced more than a decade ago. With discipline of convexity, the "H-H inequality" is the most commonly used and acknowledged inequality. This inequality was initially proposed by Hermite and Hadamard. The concept of this inequality inspired numerous mathematicians to use the multiple convexity senses to study and evaluate classical inequalities.

Now, we will prove the Mercer's-Hadamard like inclusion for an IVCF via GkFIO.

In the following, we assume that:

(A1) Let $\Delta \in (0, 1)$. Suppose that $\Im$: $[\rho_1, \rho_2] \to \mathbb{I}_c^+$ is an CIVF such that $\Im(\Delta) = [\underline{\Im}(\Delta), \overline{\Im}(\Delta)]$ and $0 \leq \rho_1 < \rho_2$.

Theorem 6 With consideration in (A1), then for all $\flat_1, \flat_2 \in [\rho_1, \rho_2]$, the following inclusions for G$\Bbbk$FIO holds:

$$\begin{aligned}
&\Im\left(\rho_1 + \rho_2 - \frac{\flat_1 + \flat_2}{2}\right) \\
&\supseteq \frac{\Delta^{\frac{\gamma}{\Bbbk}} \Gamma_{\Bbbk}(\gamma + \Bbbk)}{2(\flat_2 - \flat_1)^{\frac{\Delta\gamma}{\Bbbk}}} \left\{ {}^{(\gamma, \Bbbk)}_{(\rho_1+\rho_2-\flat_2)}\mathfrak{J}^{\Delta} \Im(\rho_1 + \rho_2 - \flat_1) \ + \ {}^{(\gamma, \Bbbk)}\mathfrak{J}^{\Delta}_{(\rho_1+\rho_2-\flat_1)} \Im(\rho_1 + \rho_2 - \flat_2) \right\} \\
&\supseteq \frac{\Im(\rho_1 + \rho_2 - \flat_2) \ + \ \Im(\rho_1 + \rho_2 - \flat_1)}{2} \\
&\supseteq (\Im(\rho_1) + \Im(\rho_2)) \ominus_g \frac{\Im(\flat_1) + \Im(\flat_2)}{2}.
\end{aligned} \tag{20}$$

Proof. In the context of interval-valued (IV) functions, if $\Im$ is a convex mapping, then $\forall u, v \in [\rho_1, \rho_2]$, we attain:

$$\begin{aligned}
\Im\left(\rho_1 + \rho_2 - \frac{u + v}{2}\right) &= \Im\left(\frac{(\rho_1 + \rho_2 - u) + (\rho_1 + \rho_2 - v)}{2}\right) \\
&\supseteq \frac{1}{2}\left\{\Im(\rho_1 + \rho_2 - u) + \Im(\rho_1 + \rho_2 - v)\right\}.
\end{aligned}$$

Usually

$$\rho_1 + \rho_2 - u = \zeta(\rho_1 + \rho_2 - \flat_1) + (1 - \zeta)(\rho_1 + \rho_2 - \flat_2)$$

and

$$\rho_1+\rho_2-v=\zeta\left(\rho_1+\rho_2-\flat_2\right)+(1-\zeta)\left(\rho_1+\rho_2-\flat_1\right),$$

$\forall\, \flat_1, \flat_2 \in [\rho_1, \rho_2]$ and $\zeta \in [0, 1]$, we get

$$\Im\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right) \supseteq \frac{1}{2}\Big\{\Im\left(\zeta\left(\rho_1+\rho_2-\flat_1\right)+(1-\zeta)\left(\rho_1+\rho_2-\flat_2\right)\right)$$

$$+\Im\left(\zeta\left(\rho_1+\rho_2-\flat_2\right)+(1-\zeta)\left(\rho_1+\rho_2-\flat_1\right)\right)\Big\}.$$

Using (21), the product of the term $\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{k}-1}(1-\zeta)^{\Delta-1}$ and its anti-derivative by inclusion w.r.t. ζ over the interval [0, 1] yields:

$$\frac{k}{\gamma\Delta^{\frac{\gamma}{k}}}\Im\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)$$

$$\supseteq \frac{1}{2}\Big\{\int_0^1\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{k}-1}(1-\zeta)^{\Delta-1}\Im\left(\zeta\left(\rho_1+\rho_2-\flat_1\right)+(1-\zeta)\left(\rho_1+\rho_2-\flat_2\right)\right)d\zeta$$

$$+\int_0^1\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{k}-1}(1-\zeta)^{\Delta-1}\Im\left(\zeta\left(\rho_1+\rho_2-\flat_2\right)+(1-\zeta)\left(\rho_1+\rho_2-\flat_1\right)\right)d\zeta\Big\}$$

$$\supseteq\frac{1}{2}\Big\{\frac{1}{\left(\flat_2-\flat_1\right)^{\frac{\Delta\gamma}{k}}}$$

$$\times\int_{\rho_1+\rho_2-\flat_2}^{\rho_1+\rho_2-\flat_1}\left(\frac{\left(\flat_2-\flat_1\right)^{\Delta}-\left(\left(\rho_1+\rho_2-\flat_1\right)-z\right)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{k}-1}\frac{\Im(z)}{\left(\left(\rho_1+\rho_2-\flat_1\right)-z\right)^{1-\Delta}}dz$$

$$+\frac{1}{\left(\flat_2-\flat_1\right)^{\frac{\Delta\gamma}{k}}}$$

$$\times\int_{\rho_1+\rho_2-\flat_2}^{\rho_1+\rho_2-\flat_1}\left(\frac{\left(\flat_2-\flat_1\right)^{\Delta}-\left(z-\left(\rho_1+\rho_2-\flat_2\right)\right)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{k}-1}\frac{\Im(z)}{\left(z-\left(\rho_1+\rho_2-\flat_2\right)\right)^{1-\Delta}}dz\Big\}$$

$$\supseteq\frac{\Gamma_k(\gamma)}{2\left(\flat_2-\flat_1\right)^{\frac{\Delta\gamma}{k}}}\Big\{{}^{(\gamma,k)}_{(\rho_1+\rho_2-\flat_2)}\mathfrak{J}^{\Delta}\,\Im\left(\rho_1+\rho_2-\flat_1\right)+{}^{(\gamma,k)}\mathfrak{J}^{\Delta}_{(\rho_1+\rho_2-\flat_1)}\,\Im\left(\rho_1+\rho_2-\flat_2\right)\Big\}, \tag{21}$$

and the first part of the above inequality (20) is obtained.

Now for the second inclusion, we will focus that $\Im$ is an IVCF, so we have

$$\Im\left(\zeta\left(\rho_1+\rho_2-\flat_1\right)+(1-\zeta)\left(\rho_1+\rho_2-\flat_2\right)\right)$$

$$\supseteq \zeta\Im\left(\rho_1+\rho_2-\flat_1\right)+(1-\zeta)\Im\left(\rho_1+\rho_2-\flat_2\right) \tag{22}$$

and

$$\Im\left(\zeta\left(\rho_1+\rho_2-\flat_2\right)+(1-\zeta)\left(\rho_1+\rho_2-\flat_1\right)\right)$$

$$\supseteq (1-\zeta)\Im\left(\rho_1+\rho_2-\flat_1\right)+\zeta\Im\left(\rho_1+\rho_2-\flat_2\right). \tag{23}$$

By summing the inclusions of (22) and (23), we get

$$\Im\left(\zeta\left(\rho_1+\rho_2-\flat_1\right)+(1-\zeta)\left(\rho_1+\rho_2-\flat_2\right)\right)$$

$$+\Im\left(\zeta\left(\rho_1+\rho_2-\flat_2\right)+(1-\zeta)\left(\rho_1+\rho_2-\flat_1\right)\right)$$

$$\supseteq \zeta\Im\left(\rho_1+\rho_2-\flat_1\right)+(1-\zeta)\Im\left(\rho_1+\rho_2-\flat_2\right)$$

$$+ (1-\zeta)\Im\left(\rho_1+\rho_2-\flat_1\right)+\zeta\Im\left(\rho_1+\rho_2-\flat_2\right)$$

$$\supseteq\Im\left(\rho_1+\rho_2-\flat_1\right)+\Im\left(\rho_1+\rho_2-\flat_2\right)$$

$$\supseteq 2\{\Im\left(\rho_1\right)+\Im\left(\rho_2\right)\}\ominus_g\{\Im\left(\flat_1\right)+\Im\left(\flat_2\right)\}. \tag{24}$$

Using (24), the product of the term $\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{k}-1}(1-\zeta)^{\Delta-1}$ and its anti-derivative by inclusion with respect to ζ over the interval [0, 1] yields:

$$\int_0^1\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{k}-1}(1-\zeta)^{\Delta-1}\Im\left(\zeta\left(\rho_1+\rho_2-\flat_1\right)+(1-\zeta)\left(\rho_1+\rho_2-\flat_2\right)\right)d\zeta$$

$$+\int_0^1\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{k}-1}(1-\zeta)^{\Delta-1}\Im\left(\zeta\left(\rho_1+\rho_2-\flat_2\right)+(1-\zeta)\left(\rho_1+\rho_2-\flat_1\right)\right)d\zeta$$

$$\supseteq \left\{\Im\left(\rho_1+\rho_2-\flat_1\right)+\Im\left(\rho_1+\rho_2-\flat_2\right)\right\}\int_0^1\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{k}-1}(1-\zeta)^{\Delta-1}d\zeta$$

$$\supseteq \left\{2\{\Im\left(\rho_1\right)+\Im\left(\rho_2\right)\}\ominus_g\{\Im\left(\flat_1\right)+\Im\left(\flat_2\right)\}\right\}\int_0^1\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{k}-1}(1-\zeta)^{\Delta-1}d\zeta$$

Employing the GkCFIO, we get

$$\frac{\Gamma_k(\gamma)}{\left(\flat_2-\flat_1\right)^{\frac{\Delta\gamma}{k}}}\left\{{}^{(\gamma,k)}_{(\rho_1+\rho_2-\flat_2)}\mathfrak{J}^{\Delta}\,\Im\left(\rho_1+\rho_2-\flat_1\right)\ +\ {}^{(\gamma,k)}\mathfrak{J}^{\Delta}_{(\rho_1+\rho_2-\flat_1)}\,\Im\left(\rho_1+\rho_2-\flat_2\right)\right\}$$

$$\supseteq\frac{k}{\gamma\Delta^{\frac{\gamma}{k}}}\left\{\Im\left(\rho_1+\rho_2-\flat_1\right)+\Im\left(\rho_1+\rho_2-\flat_2\right)\right\}$$

$$\supseteq\frac{k}{\gamma\Delta^{\frac{\gamma}{k}}}\left\{2\{\Im\left(\rho_1\right)+\Im\left(\rho_2\right)\}\ominus_g\{\Im\left(\flat_1\right)+\Im\left(\flat_2\right)\}\right\}. \tag{25}$$

Dividing by 2 in above inclusion (25)

$$\frac{\Gamma_k(\gamma)}{2\left(\flat_2-\flat_1\right)^{\frac{\Delta\gamma}{k}}}\left\{{}^{(\gamma,k)}_{(\rho_1+\rho_2-\flat_2)}\mathfrak{J}^{\Delta}\,\Im\left(\rho_1+\rho_2-\flat_1\right)\ +\ {}^{\gamma}\mathfrak{J}^{\Delta}_{(\rho_1+\rho_2-\flat_1)}\,\Im\left(\rho_1+\rho_2-\flat_2\right)\right\}$$

$$\supseteq\frac{k}{2\gamma\Delta^{\frac{\gamma}{k}}}\left\{\Im\left(\rho_1+\rho_2-\flat_1\right)+\Im\left(\rho_1+\rho_2-\flat_2\right)\right\}$$

$$\supseteq\frac{k}{2\gamma\Delta^{\frac{\gamma}{k}}}\left\{2\{\Im\left(\rho_1\right)+\Im\left(\rho_2\right)\}\ominus_g\{\Im\left(\flat_1\right)+\Im\left(\flat_2\right)\}\right\}. \tag{26}$$

Combining (21) and (26), we can get (20). □

Corollary 1 In Theorem 6 with the assumptions $\underline{\Im}(\zeta)=\bar{\Im}(\zeta)$, we can get

$$\Im\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)\leq\frac{\Delta^{\frac{\gamma}{k}}\Gamma_k(\gamma+k)}{2\left(\flat_2-\flat_1\right)^{\frac{\Delta\gamma}{k}}}$$

$$\leq\frac{\Delta^{\frac{\gamma}{k}}\Gamma_k(\gamma+k)}{2\left(\flat_2-\flat_1\right)^{\frac{\Delta\gamma}{k}}}\left\{{}^{(\gamma,k)}_{(\rho_1+\rho_2-\flat_2)}\mathfrak{J}^{\Delta}\Im\left(\rho_1+\rho_2-\flat_1\right)\ +\ {}^{(\gamma,k)}\mathfrak{J}^{\Delta}_{(\rho_1+\rho_2-\flat_1)}\Im\left(\rho_1+\rho_2-\flat_2\right)\right\}$$

$$\leq \frac{\Im(\rho_1+\rho_2-\flat_1) + \Im(\rho_1+\rho_2-\flat_2)}{2}$$

$$\leq (\Im(\rho_1)+\Im(\rho_2)) - \frac{\Im(\flat_1)+\Im(\flat_2)}{2}.$$

Corollary 2 In Theorem 6 with the assumptions $\flat_1 = \rho_1$ and $\flat_2 = \rho_2$, we can get

$$\Im\left(\frac{\rho_1+\rho_2}{2}\right) \supseteq \frac{\Delta^{\frac{\gamma}{\mathtt{k}}}\Gamma_{\mathtt{k}}(\gamma+\mathtt{k})}{2(\rho_2-\rho_1)^{\frac{\Delta\gamma}{\mathtt{k}}}}\left\{ {}^{(\gamma,\,\mathtt{k})}_{\rho_1}\mathfrak{J}^{\Delta}\Im(\rho_2) + {}^{(\gamma,\,\mathtt{k})}\mathfrak{J}^{\Delta}_{\rho_2}\Im(\rho_1)\right\}$$

$$\supseteq \frac{\Im(\rho_1) + \Im(\rho_2)}{2}.$$

Theorem 7 With consideration in (A1), then for all $\flat_1, \flat_2 \in [\rho_1, \rho_2]$, the following inclusions for GkFIO holds:

$$\Im\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)$$

$$\supseteq \frac{1}{2}\left(\frac{2}{\flat_2-\flat_1}\right)^{\frac{\Delta\gamma}{\mathtt{k}}}\Delta^{\frac{\Delta\gamma}{\mathtt{k}}}\Gamma_{\mathtt{k}}(\gamma+\mathtt{k})\left\{ {}^{(\gamma,\,\mathtt{k})}_{\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)}\mathfrak{J}^{\Delta}\,\Im(\rho_1+\rho_2-\flat_1)\right.$$

$$\left. + {}^{(\gamma,\,\mathtt{k})}\mathfrak{J}^{\Delta}_{\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)}\,\Im(\rho_1+\rho_2-\flat_2)\right\}$$

$$\supseteq (\Im(\rho_1)+\Im(\rho_2)) \ominus_g \frac{\Im(\flat_1)+\Im(\flat_2)}{2}. \tag{27}$$

Proof. In the context of interval-valued (IV) functions, if $\Im$ is a convex mapping, then $\forall u, v \in [\rho_1, \rho_2]$, we attain:

$$\Im\left(\rho_1+\rho_2-\frac{u+v}{2}\right) = \Im\left(\frac{(\rho_1+\rho_2-u)+(\rho_1+\rho_2-v)}{2}\right)$$

$$\supseteq \frac{1}{2}\left\{\Im(\rho_1+\rho_2-u)+\Im(\rho_1+\rho_2-v)\right\}.$$

Usually,

$$u = \frac{\zeta}{2}\flat_1 + \frac{2-\zeta}{2}\flat_2$$

and

$$v = \frac{2-\zeta}{2}\flat_1 + \frac{\zeta}{2}\flat_2.$$

$\forall\, \flat_1, \flat_2 \in [\rho_1, \rho_2]$ and $\zeta \in [0, 1]$, we get

$$\Im\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)$$

$$\supseteq \frac{1}{2}\left\{ \Im\left(\rho_1+\rho_2-[\frac{\zeta}{2}\flat_1+\frac{2-\zeta}{2}\flat_2]\right)+\Im\left(\rho_1+\rho_2-[\frac{2-\zeta}{2}\flat_1+\frac{\zeta}{2}\flat_2]\right)\right\}. \tag{28}$$

Using (28), product of the term $\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{k}-1}(1-\zeta)^{\Delta-1}$ and anti-derivative by inclusion w.r.t. ζ over $[0, 1]$, we obtain

$$\frac{k}{\gamma\Delta^{\frac{\gamma}{k}}}\Im\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)$$

$$\supseteq \frac{1}{2}\left\{\int_0^1\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{k}-1}(1-\zeta)^{\Delta-1}\Im\left(\rho_1+\rho_2-[\frac{\zeta}{2}\flat_1+\frac{2-\zeta}{2}\flat_2]\right)d\zeta\right.$$

$$\left.+\int_0^1\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{k}-1}(1-\zeta)^{\Delta-1}\Im\left(\rho_1+\rho_2-[\frac{2-\zeta}{2}\flat_1+\frac{\zeta}{2}\flat_2]\right)d\zeta\right\}.$$

Employing the GkCFIO, we get

$$\frac{k}{\gamma\Delta^{\frac{\gamma}{k}}}\Im\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)$$

$$\supseteq\frac{1}{2}\left(\frac{2}{\flat_2-\flat_1}\right)^{\frac{\Delta\gamma}{k}}\Gamma_k(\gamma)\left\{{}^{(\gamma,k)}_{\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)}\mathfrak{J}^{\Delta}\,\Im\left(\rho_1+\rho_2-\flat_1\right)\right.$$

$$\left.+\ {}^{(\gamma,k)}\mathfrak{J}^{\Delta}_{\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)}\ \Im\left(\rho_1+\rho_2-\flat_2\right)\right\}, \tag{29}$$

and the first part of the above inequality (27) is obtained.

Now for the second inclusion, we focus that $\Im$ is a IVCF, we have

$$\Im\left(\rho_1+\rho_2-[\frac{\zeta}{2}\flat_1+\frac{2-\zeta}{2}\flat_2]\right)\supseteq\Im(\rho_1)+\Im(\rho_2)\ominus_g[\frac{\zeta}{2}\Im(\flat_1)+\frac{2-\zeta}{2}\Im(\flat_2)] \tag{30}$$

and

$$\Im\left(\rho_1+\rho_2-[\frac{2-\zeta}{2}\flat_1+\frac{\zeta}{2}\flat_2]\right)\supseteq\Im(\rho_1)+\Im(\rho_2)\ominus_g[\frac{2-\zeta}{2}\Im(\flat_1)+\frac{\zeta}{2}\Im(\flat_2)]. \tag{31}$$

By summing the above inclusions (30) and (31), we get

$$\Im\left(\rho_1+\rho_2-[\frac{\zeta}{2}\flat_1+\frac{2-\zeta}{2}\flat_2]\right)+\Im\left(\rho_1+\rho_2-[\frac{2-\zeta}{2}\flat_1+\frac{\zeta}{2}\flat_2]\right)$$

$$\supseteq 2\{\Im(\rho_1)+\Im(\rho_2)\}\ominus_g\{\Im(\flat_1)+\Im(\flat_2)\}. \tag{32}$$

Using (32), product of the term $\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{\mathrm{k}}-1}(1-\zeta)^{\Delta-1}$ and anti-derivative by inclusion w.r.t. ζ over $[0, 1]$, we have

$$\left\{\int_0^1\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{\mathrm{k}}-1}(1-\zeta)^{\Delta-1}\Im\left(\rho_1+\rho_2-[\frac{\zeta}{2}\flat_1+\frac{2-\zeta}{2}\flat_2]\right)d\zeta\right.$$

$$\left.+\int_0^1\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{\mathrm{k}}-1}(1-\zeta)^{\Delta-1}\Im\left(\rho_1+\rho_2-[\frac{2-\zeta}{2}\flat_1+\frac{\zeta}{2}\flat_2]\right)d\zeta\right\}$$

$$\supseteq\left(2\{\Im(\rho_1)+\Im(\rho_2)\}\ominus_g\{\Im(\flat_1)+\Im(\flat_2)\}\right)\int_0^1\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{\mathrm{k}}-1}(1-\zeta)^{\Delta-1}d\zeta. \tag{33}$$

Dividing by 2 in above inclusion (33), we get

$$\frac{1}{2}\left(\frac{2}{\flat_2-\flat_1}\right)^{\frac{\Delta\gamma}{\mathrm{k}}}\Gamma_{\mathrm{k}}(\gamma)\left\{{}^{(\gamma,\mathrm{k})}_{\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)}\mathfrak{J}^{\Delta}\,\Im((\rho_1+\rho_2-\flat_1))\right.$$

$$\left.+\ {}^{(\gamma,\mathrm{k})}\mathfrak{J}^{\Delta}_{\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)}\ \Im((\rho_1+\rho_2-\flat_2))\right\}$$

$$\supseteq\frac{\mathrm{k}}{2\gamma\Delta^{\frac{\gamma}{\mathrm{k}}}}\left(2\{\Im(\rho_1)+\Im(\rho_2)\}\ominus_g\{\Im(\flat_1)+\Im(\flat_2)\}\right). \tag{34}$$

Combining the (29) and (34), we can get (27). □

Corollary 3 In Theorem 7 with the assumptions $\underline{\mathfrak{I}}(\zeta) = \bar{\mathfrak{I}}(\zeta)$, we can get

$$\mathfrak{I}\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)$$

$$\leq \frac{1}{2}\left(\frac{2}{\flat_2-\flat_1}\right)^{\frac{\Delta\gamma}{\mathrm{k}}} \Delta^{\frac{\gamma}{\mathrm{k}}}\Gamma_{\mathrm{k}}(\gamma+\mathrm{k})\left\{ {}^{(\gamma,\mathrm{k})}_{\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)}\mathfrak{J}^{\Delta}\,\mathfrak{I}(\rho_1+\rho_2-\flat_1)\right.$$

$$\left. + {}^{(\gamma,\mathrm{k})}\mathfrak{J}^{\Delta}_{\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)}\ \mathfrak{I}(\rho_1+\rho_2-\flat_2)\right\}$$

$$\leq (\mathfrak{I}(\rho_1)+\mathfrak{I}(\rho_2)) - \frac{\mathfrak{I}(\flat_1)+\mathfrak{I}(\flat_2)}{2}.$$

Remark 2 If we letting $\flat_1 = \rho_1$ and $\flat_2 = \rho_2$ and $\mathrm{k} = 1$ in Corollary 3, it gives to [55, Theorem 2.1].

Remark 3 If we letting $\flat_1 = \rho_1$ and $\flat_2 = \rho_2$, $\Delta = 1$ and $\mathrm{k} = 1$ in Corollary 3, it gives to [56, Theorem 4].

Theorem 8 With consideration in (A1), then for all $\flat_1, \flat_2 \in [\rho_1, \rho_2]$, the following inclusions for GkFIO holds:

$$\mathfrak{I}\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)$$

$$\supseteq \frac{\Delta^{\frac{\gamma}{\mathrm{k}}}}{2}\left(\frac{2}{\flat_2-\flat_1}\right)^{\frac{\Delta\gamma}{\mathrm{k}}} \Gamma_{\mathrm{k}}(\gamma+\mathrm{k})\left\{ {}^{(\gamma,\mathrm{k})}_{(\rho_1+\rho_2-\flat_2)}\mathfrak{J}^{\Delta}\,\mathfrak{I}\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)\right.$$

$$\left. + {}^{(\gamma,\mathrm{k})}\mathfrak{J}^{\Delta}_{(\rho_1+\rho_2-\flat_1)}\ \mathfrak{I}\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)\right\}$$

$$\supseteq (\mathfrak{I}(\rho_1)+\mathfrak{I}(\rho_2)) \ominus_g \frac{\mathfrak{I}(\flat_1)+\mathfrak{I}(\flat_2)}{2}. \tag{35}$$

Proof. In the context of interval-valued (IV) functions, if $\mathfrak{I}$ is a convex mapping, then $\forall u, v \in [\rho_1, \rho_2]$, we get:

$$\mathfrak{I}\left(\rho_1+\rho_2-\frac{u+v}{2}\right) = \mathfrak{I}\left(\frac{(\rho_1+\rho_2-u)+(\rho_1+\rho_2-v)}{2}\right)$$

$$\supseteq \frac{1}{2}\left\{\mathfrak{I}(\rho_1+\rho_2-u)+\mathfrak{I}(\rho_1+\rho_2-v)\right\}.$$

$$u = \frac{1-\zeta}{2}\flat_1 + \frac{1+\zeta}{2}\flat_2$$

and

$$v=\frac{1+\zeta}{2}\flat_1+\frac{1-\zeta}{2}\flat_2.$$

$\forall\,\flat_1,\,\flat_2\in[\rho_1,\,\rho_2]$ and $\zeta\in[0,\,1]$, we get

$$\Im\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)\supseteq\frac{1}{2}\left\{\Im\left(\rho_1+\rho_2-[\frac{1-\zeta}{2}\flat_1+\frac{1+\zeta}{2}\flat_2]\right)\right.$$
$$\left.+\Im\left(\rho_1+\rho_2-[\frac{1+\zeta}{2}\flat_1+\frac{1-\zeta}{2}\flat_2]\right)\right\}. \tag{36}$$

Using (41), product of the term $\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{\mathrm{k}}-1}(1-\zeta)^{\Delta-1}$ and anti-derivative by inclusion w.r.t. ζ over $[0,\,1]$, we obtain

$$\frac{\mathrm{k}}{\gamma\Delta^{\frac{\gamma}{\mathrm{k}}}}\Im\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)$$

$$\supseteq\frac{1}{2}\left\{\int_0^1\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{\mathrm{k}}-1}(1-\zeta)^{\Delta-1}\Im\left(\rho_1+\rho_2-[\frac{1-\zeta}{2}\flat_1+\frac{1+\zeta}{2}\flat_2]\right)d\zeta\right.$$

$$\left.+\int_0^1\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{\mathrm{k}}-1}(1-\zeta)^{\Delta-1}\Im\left(\rho_1+\rho_2-[\frac{1+\zeta}{2}\flat_1+\frac{1-\zeta}{2}\flat_2]\right)d\zeta\right\}$$

$$\supseteq\frac{1}{2}\left(\frac{2}{\flat_2-\flat_1}\right)^{\frac{\Delta\gamma}{\mathrm{k}}}$$

$$\times\left\{\int_{\rho_1+\rho_2-\flat_2}^{\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}}\left(\frac{\left(\frac{2}{\flat_2-\flat_1}\right)^{\Delta}-(z-(\rho_1+\rho_2-\flat_2))^{\Delta}}{\Delta}\right)^{\frac{\gamma}{\mathrm{k}}-1}\frac{\Im(z)}{(z-(\rho_1+\rho_2-\flat_2))^{1-\Delta}}dz\right.$$

$$\left.+\int_{\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}}^{\rho_1+\rho_2-\flat_1}\left(\frac{\left(\frac{2}{\flat_2-\flat_1}\right)^{\Delta}-((\rho_1+\rho_2-\flat_1)-z)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{\mathrm{k}}-1}\frac{\Im(z)}{((\rho_1+\rho_2-\flat_1)-z)^{1-\Delta}}dz\right\}.$$

Employing the GkCFIO, we get

$$\frac{\mathtt{k}}{\gamma\Delta^{\frac{\gamma}{\mathtt{k}}}}\Im\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)$$

$$\supseteq \frac{\Gamma_{\mathtt{k}}(\gamma)}{2}\left(\frac{2}{\flat_2-\flat_1}\right)^{\frac{\Delta\gamma}{\mathtt{k}}}\left\{{}^{(\gamma,\,\mathtt{k})}_{(\rho_1+\rho_2-\flat_2)}\mathfrak{J}^{\Delta}\,\Im\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)\right.$$

$$\left. + \;{}^{(\gamma,\,\mathtt{k})}\mathfrak{J}^{\Delta}_{(\rho_1+\rho_2-\flat_1)}\;\Im\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)\right\}. \tag{37}$$

The first part of the above inequality (35) is obtained.

Now to prove the second inclusion from JMI, we have

$$\Im\left(\rho_1+\rho_2-[\frac{1-\zeta}{2}\flat_1+\frac{1+\zeta}{2}\flat_2]\right)\supseteq \Im(\rho_1)+\Im(\rho_2)\ominus_g[\frac{1-\zeta}{2}\Im(\flat_1)+\frac{1+\zeta}{2}\Im(\flat_2)] \tag{38}$$

and

$$\Im\left(\rho_1+\rho_2-[\frac{1+\zeta}{2}\flat_1+\frac{1-\zeta}{2}\flat_2]\right)\supseteq \Im(\rho_1)+\Im(\rho_2)\ominus_g[\frac{1+\zeta}{2}\Im(\flat_1)+\frac{1-\zeta}{2}\Im(\flat_2)]. \tag{39}$$

By summing the above inclusions (38) and (39), we get

$$\Im\left(\rho_1+\rho_2-[\frac{1-\zeta}{2}\flat_1+\frac{1+\zeta}{2}\flat_2]\right)+\Im\left(\rho_1+\rho_2-[\frac{1+\zeta}{2}\flat_1+\frac{1-\zeta}{2}\flat_2]\right)$$

$$\supseteq\Im(\rho_1)+\Im(\rho_2)\ominus_g(\Im(\flat_1)+\Im(\flat_2)). \tag{40}$$

With (40) product of the term $\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{\mathtt{k}}-1}(1-\zeta)^{\Delta-1}$ and anti-derivative by inclusion w.r.t. ζ over [0, 1], we obtain

$$\left\{\int_0^1\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{\mathtt{k}}-1}(1-\zeta)^{\Delta-1}\Im\left(\rho_1+\rho_2-[\frac{1-\zeta}{2}\flat_1+\frac{1+\zeta}{2}\flat_2]\right)d\zeta\right.$$

$$\left.+\int_0^1\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{\mathtt{k}}-1}(1-\zeta)^{\Delta-1}\Im\left(\rho_1+\rho_2-[\frac{1+\zeta}{2}\flat_1+\frac{1-\zeta}{2}\flat_2]\right)d\zeta\right\}$$

$$\supseteq\left(2\{\Im(\rho_1)+\Im(\rho_2)\}\ominus_g\{\Im(\flat_1)+\Im(\flat_2)\}\right)\int_0^1\left(\frac{1-(1-\zeta)^{\Delta}}{\Delta}\right)^{\frac{\gamma}{\mathtt{k}}-1}(1-\zeta)^{\Delta-1}d\zeta.$$

Employing the GkCFIO, we get

$$\Gamma_{\mathtt{k}}(\gamma)\left(\frac{2}{\flat_2-\flat_1}\right)^{\Delta\gamma}$$

$$\times\left\{{}^{(\gamma,\mathtt{k})}_{(\rho_1+\rho_2-\flat_2)}\mathfrak{J}^{\Delta}\,\mathfrak{S}\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)+{}^{(\gamma,\mathtt{k})}\mathfrak{J}^{\Delta}_{(\rho_1+\rho_2-\flat_1)}\;\mathfrak{S}\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)\right\}$$

$$\supseteq\frac{\mathtt{k}}{\gamma\Delta^{\frac{\gamma}{\mathtt{k}}}}\left(2\{\mathfrak{S}(\rho_1)+\mathfrak{S}(\rho_2)\}\ominus_g\{\mathfrak{S}(\flat_1)+\mathfrak{S}(\flat_2)\}\right). \tag{41}$$

Dividing by 2 the above inequality (41), we get

$$\frac{\Gamma_{\mathtt{k}}(\gamma)}{2}\left(\frac{2}{\flat_2-\flat_1}\right)^{\frac{\Delta\gamma}{\mathtt{k}}}$$

$$\times\left\{{}^{(\gamma,\mathtt{k})}_{(\rho_1+\rho_2-\flat_2)}\mathfrak{J}^{\Delta}\,\mathfrak{S}\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)+{}^{(\gamma,\mathtt{k})}\mathfrak{J}^{\Delta}_{(\rho_1+\rho_2-\flat_1)}\;\mathfrak{S}\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)\right\}$$

$$\supseteq\frac{\mathtt{k}}{2\gamma\Delta^{\frac{\gamma}{\mathtt{k}}}}\left(2\{\mathfrak{S}(\rho_1)+\mathfrak{S}(\rho_2)\}\ominus_g\{\mathfrak{S}(\flat_1)+\mathfrak{S}(\flat_2)\}\right). \tag{42}$$

Combining the (37) and (42), we can get (35). □

Corollary 4 In Theorem 8 with the assumptions $\underline{\mathfrak{S}}(\zeta)=\bar{\mathfrak{S}}(\zeta)$, we can get

$$\mathfrak{S}\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)$$

$$\leq\frac{\Delta^{\frac{\gamma}{\mathtt{k}}}}{2}\left(\frac{2}{\flat_2-\flat_1}\right)^{\frac{\Delta\gamma}{\mathtt{k}}}\Gamma_{\mathtt{k}}(\gamma+\mathtt{k})\left\{{}^{(\gamma,\mathtt{k})}_{(\rho_1+\rho_2-\flat_2)}\mathfrak{J}^{\Delta}\,\mathfrak{S}\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)\right.$$

$$\left.+\;{}^{(\gamma,\mathtt{k})}\mathfrak{J}^{\Delta}_{(\rho_1+\rho_2-\flat_1)}\;\mathfrak{S}\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)\right\}$$

$$\leq(\mathfrak{S}(\rho_1)+\mathfrak{S}(\rho_2))-\frac{\mathfrak{S}(\flat_1)+\mathfrak{S}(\flat_2)}{2}.$$

Corollary 5 In Theorem 8 with the assumptions $\flat_1=\rho_1$ and $\flat_2=\rho_2$, we can get

$$\Im\left(\frac{\rho_1+\rho_2}{2}\right)$$

$$\supseteq \frac{\Delta^{\frac{\gamma}{k}}}{2}\left(\frac{2}{\rho_2-\rho_1}\right)^{\frac{\Delta\gamma}{k}}\Gamma_k(\gamma+k)\left\{ {}^{(\gamma,\,k)}_{\rho_1}\mathfrak{J}^{\Delta}\,\Im\left(\frac{\rho_1+\rho_2}{2}\right)\right.$$

$$\left.+\,{}^{(\gamma,\,k)}\mathfrak{J}^{\Delta}_{\rho_2}\,\Im\left(\frac{\rho_1+\rho_2}{2}\right)\right\}\supseteq\frac{\Im(\rho_1)+\Im(\rho_2)}{2}.$$

5. Application to matrix

Analysis regarding convex theory and fractional operator are employing in the applied sciences. The study shows that these ideas have many possible uses in a variety of scientific fields, such as fluid dynamics and optimization. We offer a few applications pertaining to matrices that will improve accuracy. The classical Jensen's inequality is extended to matrix-valued functions by the Jensen-Mercer type inequality involving matrices. Researchers can examine and estimate matrix-valued expressions thanks to this inequality, which gives bounds for convex functions of matrices. It aids in the bounding of matrix norms and eigenvalues in matrix analysis. It makes filter design and matrix-valued signal analysis easier in signal processing. Jensen-Mercer type inequalities are also useful in control theory for complex system controller design and stability analysis. Researchers can improve performance and efficiency in a variety of applications by using these inequalities to obtain tighter bounds and more precise estimates.

The symbol $\mathscr{T}^{n_*}$ the set of $n_*\times n_*$ matrices in format of complex, B_{n_*} the algebraic notation of $n_*\times n_*$ matrices in complex and by $B^+_{n_*}$ the strictly positive matrices in B_{n_*}. That is, $\mathscr{G}\in B^+_{n_*}$ if $\langle\mathscr{D}x,\,x\rangle>0\ \forall$ nonzero $x\in\mathscr{T}^{n_*}$. In [57], the author created that the mapping $\Im(\Delta)=\left\|\mathscr{D}^{\Delta}\mathscr{X}\mathscr{K}^{1-\Delta}+\mathscr{D}^{1-\Delta}\mathscr{X}\mathscr{K}^{\Delta}\right\|$, $\mathscr{D},\ \mathscr{K}\ \in\mathscr{B}^+_{n_*},\ \ \mathscr{X}\in\mathscr{B}_{n_*},$ is convex for all $\Delta\in[0,\,1]$.

Example 1 Under the conditions of Theorem 6, we have

$$\left\|\mathscr{D}^{\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}}\mathscr{X}\mathscr{K}^{1-(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2})}+\mathscr{D}^{1-(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2})}\mathscr{X}\mathscr{K}^{\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}}\right\|$$

$$\supseteq\frac{\Delta^{\frac{\gamma}{k}}\Gamma_k(\gamma+k)}{2(\flat_2-\flat_1)^{\frac{\Delta\gamma}{k}}}\times\left[{}^{(\gamma,\,k)}_{\rho_1+\rho_2-\flat_2}\mathfrak{J}^{\Delta}\left\|\mathscr{D}^{\rho_1+\rho_2-\flat_1}\mathscr{X}\mathscr{K}^{1-(\rho_1+\rho_2-\flat_1)}+\mathscr{D}^{1-(\rho_1+\rho_2-\flat_1)}\mathscr{X}\mathscr{K}^{\rho_1+\rho_2-\flat_1}\right\|\right.$$

$$\left.+\,{}^{(\gamma,\,k)}\mathfrak{J}^{\Delta}_{\rho_1+\rho_2-\flat_1}\left\|\mathscr{D}^{\rho_1+\rho_2-\flat_2}\mathscr{X}\mathscr{K}^{1-(\rho_1+\rho_2-\flat_2)}+\mathscr{D}^{1-(\rho_1+\rho_2-\flat_2)}\mathscr{X}\mathscr{K}^{\rho_1+\rho_2-\flat_2}\right\|\right]$$

$$\supseteq\frac{1}{2}\left\{\left\|\mathscr{D}^{\rho_1+\rho_2-\flat_1}\mathscr{X}\mathscr{K}^{1-(\rho_1+\rho_2-\flat_1)}+\mathscr{D}^{1-(\rho_1+\rho_2-\flat_1)}\mathscr{X}\mathscr{K}^{\rho_1+\rho_2-\flat_1}\right\|\right.$$

$$\left.+\left\|\mathscr{D}^{\rho_1+\rho_2-\flat_2}\mathscr{X}\mathscr{K}^{1-(\rho_1+\rho_2-\flat_2)}+\mathscr{D}^{1-(\rho_1+\rho_2-\flat_2)}\mathscr{X}\mathscr{K}^{\rho_1+\rho_2-\flat_2}\right\|\right\}$$

$$\supseteq\left\|\mathscr{D}^{\rho_1}\mathscr{X}\mathscr{K}^{1-\rho_1}+\mathscr{D}^{1-\rho_1}\mathscr{X}\mathscr{K}^{\rho_1}\right\|+\left\|\mathscr{D}^{\rho_2}\mathscr{X}\mathscr{K}^{1-\rho_2}+\mathscr{D}^{1-\rho_2}\mathscr{X}\mathscr{K}^{\rho_2}\right\|$$

$$\ominus_g \frac{1}{2}\left\{ \left\| \mathscr{D}^{\flat_1}\mathscr{X}\mathscr{K}^{1-\flat_1} + \mathscr{D}^{1-\flat_1}\mathscr{X}\mathscr{K}^{\flat_1} \right\| + \left\| \mathscr{D}^{\flat_2}\mathscr{X}\mathscr{K}^{1-\flat_2} + \mathscr{D}^{1-\flat_2}\mathscr{X}\mathscr{K}^{\flat_2} \right\| \right\}.$$

Example 2 Under the conditions of Example 1, Theorem 7, we have

$$\left\| \mathscr{D}^{\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}}\mathscr{X}\mathscr{K}^{1-\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)} + \mathscr{D}^{1-\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)}\mathscr{X}\mathscr{K}^{\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}} \right\|$$

$$\supseteq \frac{1}{2}\left(\frac{2}{\flat_2-\flat_1}\right)^{\frac{\Delta\gamma}{k}} \Delta^{\frac{\gamma}{k}} \Gamma_k(\gamma+k)$$

$$\times \left[{}^{(\gamma,k)}\mathfrak{I}^{\Delta}_{\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}} \left\| \mathscr{D}^{\rho_1+\rho_2-\flat_2}\mathscr{X}\mathscr{K}^{1-(\rho_1+\rho_2-\flat_2)} + \mathscr{D}^{1-(\rho_1+\rho_2-\flat_2)}\mathscr{X}\mathscr{K}^{\rho_1+\varpi_2-\flat_2} \right\| \right.$$

$$\left. + {}^{(\gamma,k)}_{\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}}\mathfrak{I}^{\Delta} \left\| \mathscr{D}^{\rho_1+\rho_2-\flat_1}\mathscr{X}\mathscr{K}^{1-(\rho_1+\rho_2-\flat_1)} + \mathscr{D}^{1-(\rho_1+\rho_2-\flat_1)}\mathscr{X}\mathscr{K}^{\rho_1+\rho_2-\flat_1} \right\| \right]$$

$$\supseteq \left\{ \left\| \mathscr{D}^{\rho_1}\mathscr{X}\mathscr{K}^{1-\rho_1} + \mathscr{D}^{1-\rho_1}\mathscr{X}\mathscr{K}^{\rho_1} \right\| + \left\| \mathscr{D}^{\rho_2}\mathscr{X}\mathscr{K}^{1-\rho_2} + \mathscr{D}^{1-\rho_2}\mathscr{X}\mathscr{K}^{\rho_2} \right\| \right\}$$

$$\ominus_g \frac{1}{2}\left\{ \left\| \mathscr{D}^{\flat_1}\mathscr{X}\mathscr{K}^{1-\flat_1} + \mathscr{D}^{1-\flat_1}\mathscr{X}\mathscr{K}^{\flat_1} \right\| + \left\| \mathscr{D}^{\flat_2}\mathscr{X}\mathscr{K}^{1-\flat_2} + \mathscr{D}^{1-\flat_2}\mathscr{X}\mathscr{K}^{\flat_2} \right\| \right\}.$$

Example 3 Under the conditions of Example 1, Theorem 8, we have

$$\left\| \mathscr{D}^{\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}}\mathscr{X}\mathscr{K}^{1-\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)} + \mathscr{D}^{1-\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)}\mathscr{X}\mathscr{K}^{\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}} \right\|$$

$$\supseteq \frac{\Delta^{\frac{\gamma}{k}}}{2}\left(\frac{2}{\flat_2-\flat_1}\right)^{\frac{\Delta\gamma}{k}} \Gamma_k(\gamma+k)$$

$$\times \left[{}^{(\gamma,k)}_{\rho_1+\rho_2-\flat_2}\mathfrak{I}^{\Delta} \right.$$

$$\times \left\| \mathscr{D}^{\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}}\mathscr{X}\mathscr{K}^{1-\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)} + \mathscr{D}^{1-\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)}\mathscr{X}\mathscr{K}^{\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}} \right\|$$

$$+ {}^{(\gamma,k)}\mathfrak{I}^{\Delta}_{\rho_1+\rho_2-\flat_1}$$

$$\times \left\| \mathscr{D}^{\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}} \mathscr{X} \mathscr{K}^{1-\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)} + \mathscr{D}^{1-\left(\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}\right)} \mathscr{X} \mathscr{K}^{\rho_1+\rho_2-\frac{\flat_1+\flat_2}{2}} \right\|$$

$$\supseteq \left\{ \left\| \mathscr{D}^{\rho_1} \mathscr{X} \mathscr{K}^{1-\rho_1} + \mathscr{D}^{1-\rho_1} \mathscr{X} \mathscr{K}^{\rho_1} \right\| + \left\| \mathscr{D}^{\rho_2} \mathscr{X} \mathscr{K}^{1-\rho_2} + \mathscr{D}^{1-\rho_2} \mathscr{X} \mathscr{K}^{\rho_2} \right\| \right\}$$

$$\ominus_g \frac{1}{2} \left\{ \left\| \mathscr{D}^{\flat_1} \mathscr{X} \mathscr{K}^{1-\flat_1} + \mathscr{D}^{1-\flat_1} \mathscr{X} \mathscr{K}^{\flat_1} \right\| + \left\| \mathscr{D}^{\flat_2} \mathscr{X} \mathscr{K}^{1-\flat_2} + \mathscr{D}^{1-\flat_2} \mathscr{X} \mathscr{K}^{\flat_2} \right\| \right\}.$$

6. Conclusions

Fractional calculus have numerous applications in modeling, inequality theory, mathematical biology, and engineering in applied mathematics. A number of researchers from a variety of scientific disciplines have shown a great deal of interest in fractional calculus. In this paper, we employed GkCFIO to obtain some novel sorts of H-H-Mercer inequalities for CIVF. We have included the validations to raise the overall quality. Finally, some well-known matrix applications are discussed. In the context of inspiring, further research in a wide range of applied and pure disciplines, the concept of convexity via interval analysis may be adopted to achieve a range of outcomes in special functions and quantum mechanics, associated mathematical inequalities, and optimization theory.

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Conflict of interest

The authors declare no competing financial interest.

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