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Research Article

Application of a Generalized Balloon-Shaped Domain on Bi-Bounded Turning Functions

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Abstract: This study addresses the contemporary scholarly interest in analytic functions tailored for specific domains and functions within the unit disk. These functions hold significant relevance across diverse technological and scientific disciplines, including but not limited to electromagnetic theory, plasma physics, mathematical biology, and optics. We introduce a novel subclass of bi-univalent functions formed by the convolution of bounded turning functions and generalized balloon-shaped domains. Our primary focus is on the investigation of coefficient-related problems, encompassing the second Hankel determinant, the Fekete-Szegö inequality, and initial coefficient bounds. This research provides precise bounds for each of these analytical challenges. To validate our findings, rather than a comparative analysis with existing results, a method that may be subject to inaccuracies stemming from previous methodologies, we leverage the extremal function for this newly defined class to confirm the precise limits for the characteristics examined herein.

Keywords: univalent functions, generalized balloon-shaped, bi-bounded turning functions, Hankel determinant

MSC: 30C50, 30C80, 30C45

1. Introduction and definitions

In the present day, complex analysis has significantly contributed to the field of mathematics, becoming a crucial element today. The study of complex analysis, specifically focusing on complex variables, has made significant contributions to both practical and theoretical applications and has applications in physics and engineering, offering solutions to real-world problems. Geometric Function Theory ($\mathscr{GFT}$) has seen a notable increase in complex analysis because of its many uses in different scientific areas. The latest developments in the geometric characteristics of Analytic Functions ($\mathfrak{AF}s$) have led to a rise in research contributions [1–5].

Certain domains of intricate analysis can also serve as essential instruments in advanced mathematical fields; for further information, refer to [6]. Research on univalent functions has been a major focus in complex analysis since the early 1900s, attracting numerous researchers. In the domain of univalent functions ($\mathfrak{UF}s$), being the most basic, we have seen the growth of many key properties in the last hundred years, leading to a wealth of elegant theorems and highlighting

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several unresolved issues. Using derivatives and integrals of different orders is becoming more popular as a creative method to address real-world problems in various fields like geometry and physics [7]. This method is shown through its application in simulating the human liver and investigating the dissemination of the dengue virus [8]. Moreover, research investigated the capability of a novel fractional derivative operator in heat transfer simulations. Additionally, fractional operators are anticipated to be helpful in solving different real-world challenges, as indicated in [9].

The set:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

consists of complex-valued functions denoted by $\mathscr{A}$ that are analytic and bounded in the Unit Disk ($\mathfrak{U}_D$) and normalized with the conditions $f(0) = f'(0) - 1 = 0$ with the expression given below

$$\mathscr{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

The function $u(z)$, defined by the equation

$$u(z) = 1 + \sum_{n=1}^{\infty} u_n z^n.$$

When z belongs to the set $\mathfrak{U}_D$ and has $\Re(u(z)) > 0$, it is designated as $\mathscr{P}$ in $\mathfrak{U}_D$, given that $|u_n| < 2$ as stated in reference [10].

It is well know that $\mathfrak{U}\mathfrak{F}s$ are functions that does not repeat any values in $\mathfrak{U}_D$. The set $\mathscr{S}$ represents the subset of functions in class $\mathscr{A}$ that are univalent.

The well-known types of $\mathfrak{U}\mathfrak{F}s$, such as convex functions ($\mathscr{C}\mathscr{F}s$) ($f \in \mathscr{C}$), starlike functions ($\mathscr{S}\mathscr{F}s$) ($f \in \mathscr{S}^\star$) and bounded turning functions ($\mathfrak{B}\mathfrak{F}s$) ($f \in \mathscr{B}\mathscr{T}$), can be described in the following manner:

$$\Re(\mathscr{S}\mathscr{T}) > 0, \;\; \Re(1 + \mathscr{C}\mathscr{V}) > 0 \text{ and } \Re(f'(z)) > 0, \tag{2}$$

where

$$\mathscr{S}\mathscr{T} = \frac{zf'(z)}{f(z)}, \;\; \mathscr{C}\mathscr{V} = \frac{zf''(z)}{f'(z)}.$$

Using the function $f(z)$ from equation (1), we can define the k^{cth} Hankel determinant introduced in 1973 by Noonan and Thomas [11] when $k \geq 1$, $\wp \geq 1$, and $a_1 = 1$.

$$\mathscr{H}_k(\wp) = \begin{vmatrix} a_\wp & a_{\wp+1} & a_{\wp+2} & \cdots & a_{\wp+k-1} \\ a_{\wp+1} & a_{\wp+2} & a_{\wp+3} & \cdots & a_{\wp+k} \\ a_{\wp+2} & a_{\wp+3} & a_{\wp+4} & \cdots & a_{\wp+k+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{\wp+k-1} & a_{\wp+k} & a_{\wp+k+1} & \cdots & a_{\wp+2(k-1)} \end{vmatrix}. \tag{3}$$

Using the formula from equation (3) with $k=2$ and $\wp=1$, we obtain the well-known Fekete-Szego functional ($\mathscr{F}\mathscr{S}f$).

$$\mathscr{H}_2(1) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = |a_3 - a_2^2|. \tag{4}$$

This feature is further expanded in a broader manner as:

$$|a_3 - \rho a_2^2| \tag{5}$$

where ρ represents a number that can be either real or complex.

The Hankel determinant is crucial in singularity theory [12] and is useful for examining power series with integer coefficients (see [13]). Many researchers have determined maximum limits for $\mathscr{H}_k(\wp)$ with varying values of k and c in different categories of $\mathfrak{AF}s$ (refer to [14–19], for more information).

Every function f belonging to set $\mathscr{S}$ has a corresponding inverse defined as follows:

$$f^{-1}(f(z)) = z, \quad z \in \mathscr{U}$$

and

$$f(f^{-1}(w)) = w, \ |w| < x_0(f), \ x_0(f) \geq \frac{1}{4}.$$

The following series illustrates the inverse of $f(z)$, denoted as $f^{-1}(w)$.

$$f^{-1}(w) = h(w) = w + W_2 w^2 + w_3 W^3 + w_4 W^4 + \cdots, \tag{6}$$

where

$$W_2 = -a_2, \ \ W_3 = 2a_2^2 - a_3 \ \text{ and } \ W_4 = 5a_2 a_3 - 5a_2^3 - a_4. \tag{7}$$

The defined class below:

$$\mathscr{S}^\star(\Delta) = \{f \in \mathscr{A} : \ \mathscr{S}\mathscr{T} \prec \Delta(z), \ \Re(\Delta(z)) > 0 \ (z \in \mathscr{U})\} \tag{8}$$

was introduced by Ma and Minda [20] in 1992 using the concept of subordination. By the help of this definition, numerous researchers were able to define some subclasses of $\mathscr{S}\mathscr{F}s$ by making use of different expressions of $\Delta(z)$, which will be discussed in the next paragraph.

Firstly, in 1996, Sokol and Stankiewicz [21] extended the class defined in (8) by making use of $\Delta(z) = \sqrt{1+z}$, which is define as follows:

$$\mathcal{S}_B^{\star} = \left\{ \mathcal{S}\mathcal{T} \prec \sqrt{1+z} \right\}$$

and obtained some interesting results. Afterwards, Raina and Sokol [22] in 2015 made use of another interesting domain that extended the work of Ma and Minda [20], which is defined as follows:

$$\mathcal{S}\mathcal{T} \prec z + \sqrt{1+z^2}$$

and obtained some nice results. In 2018, Priya and Sharma [23] extended the work of Raina and Sokol [22] by defining the following class:

$$\mathcal{S}\mathcal{T} \prec z + \sqrt[3]{1+z^2}.$$

This is also an extension of the work of Ma and Minda [20] by considering $\Delta(z) = z + \sqrt[3]{1+z^2}$. Also, in 2019, Mahmoud et al. [24] chose $\Delta(z) = e^z$ in (8) and obtained some interesting results in their research paper [24]. In 2020, Geol et al. [25] extended the work of Ma and Minda [20] by defining the following class:

$$\mathcal{S}_{Sig}^{\star} = \left\{ \mathcal{S}\mathcal{T} \prec \frac{2}{1+e^{-z}} \right\}.$$

For more details on sigmoid functions see [26, 27].

Then in 2021, Ullah et al. [28] extended the work of Ma and Minda [20] by taking $\Delta(z) = 1 + \tanh z$, and after that Shi et al. [29] further extended it by taking $\Delta(z) = 1 + \sin z$ and obtained some interesting results.

Recently, Masih et al. [30] in 2022 extended the work of Sokol and Stankiewicz [21] by generalizing the Bernoulli Lemniscate and defined the following class:

$$\mathcal{S}_{\zeta}^{\star} = \left\{ \mathcal{S}\mathcal{T} \prec (1+z)^{\zeta}, \quad 0 < \zeta < 1 \right\}$$

and obtained some coefficients bounds and analyzed the class.

Ahmad et al. [31] in 2023 extended the works of Sokol and Stankiewicz [21], and Geol et al. by making use of the quotient of Bernoulli Lemniscate and modified sigmoid function and defined the class of $\mathfrak{B}\mathfrak{F}s$ as follows:

$$\mathcal{R}_{sl} = \left\{ f'(z) \prec \frac{2\sqrt{1+z}}{1+e^{-z}} \right\}$$

and obtained some coefficients bounds which are all sharp.

The bi-univalent functions ($\mathcal{B}\mathcal{U}s$) in the field of $\mathcal{G}\mathcal{F}\mathcal{T}$, denoted by Ξ, was first researched by Lewin [32], who proved that $|a_2| \leq 4 \times 3^{-1}$. Later, following Lewin's research, Brannan and Clunie [33] expanded on the concept to establish $|a_2| \leq (2)^{1/2}$ and Netanyahu [34] demonstrated that $|a_2| < 4/3$. The Bi-$\mathcal{C}\mathcal{F}s$ and Bi-$\mathcal{S}\mathcal{F}s$ were first presented by Brannan and Clunie [33] in 1985. Study on subclasses Ξ has been a major scholarly interest for the past decade. Finding the initial boundaries of coefficients for particular subclasses piqued the curiosity of Ξ. The importance of coefficient problems in the context of $\mathscr{BU}s$ has been emphasized by Srivastava et al. [35]. In 2010, Srivastava et al. [35] discovered two remarkable subclasses within the family of Ξ functions. He indicated that for functions in these subclasses, $|a_2|$ and $|a_3|$ act as boundaries. Frasin and Aouf [36] began calculating the precise values of $|a_2|$ and $|a_3|$ for functions within two new subclasses of the function category.

In recent times, researchers in the field of $\mathscr{GFT}$ have started working on a class of $\mathscr{BU}s$ linked with domains found in the positive right half-plane. Mustafa and Murugusundaramoorthy [37] in 2021 defined a class of $\mathscr{BU}s$ which can be represented as:

$$(1-\iota)\mathscr{ST}+\iota(1+\mathscr{CV})\prec z+\sqrt{1+z^2}\ \ (\iota\geq 0)$$

and

$$(1-\iota)\frac{wh'(w)}{h(w)}+\iota\left(1+\frac{wh''(w)}{h'(w)}\right)\prec w+\sqrt{1+w^2}\ \ (\iota\geq 0)$$

where $h(w)$ can be found in (6) and sharp coefficient bounds are obtained.

In 2023, Shaba et al. [38] extended the work of Mustafa and Murugusundaramoorthy [37] by defining a class of $\mathscr{BU}s$ associated with a four-leaf domain with the means of $\mathfrak{BF}s$, which is defined as follows:

$$f'(z)\prec 1+\frac{5}{6}z+\frac{1}{6}z^4$$

and

$$h'(z)\prec 1+\frac{5}{6}w+\frac{1}{6}w^4$$

where $h(w)$ can be found in (6) and sharp coefficient bounds are obtained.

1.1 *Motivation*

Shaba et al. [38] successfully established a novel subclass of $\mathscr{BU}s$ within $\mathscr{U}$ and identified the extremal function for each new subclass. Their approach to classifying coefficient problems concerning $\mathscr{BU}s$ diverged from traditional methods, enabling the precise determination of sharp coefficients. The consistent use of the extremal function for each subclass in their research substantiated the accuracy of their findings, providing significant motivation for continued research in this area. This innovative approach, which involves addressing coefficient issues and verifying them with the extremal function rather than relying on potentially unreliable comparisons with other subclasses, is particularly noteworthy. Inspired by the works of [31, 38], our research endeavors to develop a new subclass of. This will be achieved through the application of a new differential operator associated with the Gregory Coefficients. Our study will specifically focus on analyzing Hankel determinants of the second order, $\mathscr{FS}f$, by examining the coefficients $(|a_n|)$ $(n=2,\ 3,\ 4)$ and thoroughly exploring these mathematical objects.

2. The recently specified class and lemmas

A novel subclass of bi-$\mathfrak{B}\mathfrak{F}s$ linked to a generalized balloon-shaped domain is introduced.

Definition 1 If the restrictions $z \in \mathscr{U}$, $0 \leq \zeta < 1$ and $h(w)$ are satisfied according to (6), then $f \in \mathscr{G}\mathscr{B}_{\Xi}^{Sig}(\zeta)$ given the condition:

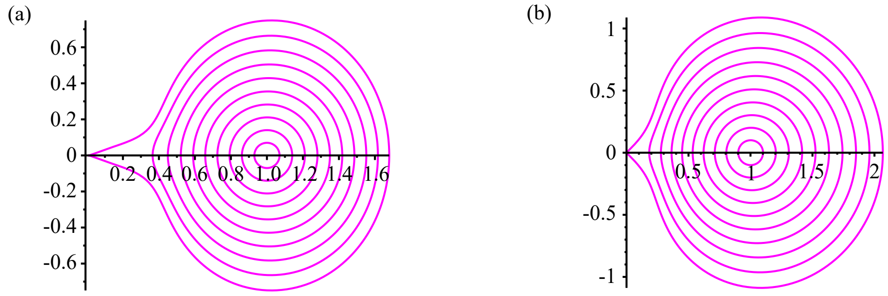
$$f'(z) \prec \frac{(z+1)^{\zeta} \times 2}{1+e^{-z}} \tag{9}$$

and

$$h'(w) \prec \frac{(w+1)^{\zeta} \times 2}{1+e^{-w}} \tag{10}$$

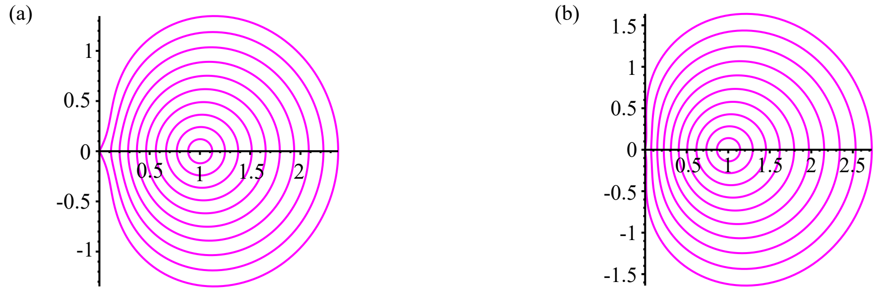
are satisfied.

Figures 1 and 2 below show the description of the generalized balloon-shaped domain as it appears on the complex plane, which lies in the positive right half-plane.



When $\zeta = 0.2$ When $\zeta = 0.5$

Figure 1. The diagrams for different point in the domain $\frac{2(z+1)^{\zeta}}{1+e^{-z}}$



When $\zeta = 0.7$ When $\zeta = 0.9$

Figure 2. The diagrams for different point in the domain $\frac{2(z+1)^{\zeta}}{1+e^{-z}}$

Remark 1 This expression presents the bi-$\mathfrak{BF}$ class with subordination at $\zeta = \frac{1}{2}$, which is symbolized by $\mathscr{G}\mathscr{B}_{\Xi}$ and meets the specified criteria.

$$f'(z) \prec \frac{2\sqrt{1+z}}{1+e^{-z}} \tag{11}$$

and

$$h'(w) \prec \frac{2\sqrt{1+w}}{1+e^{-w}}. \tag{12}$$

Remark 2 This expression introduces a new class of bi-$\mathfrak{BF}$ which consists of a $\mathfrak{BF}s$ linked to a modified sigmoid function, which satisfies the conditions for subordination at $\zeta = 0$ and is denoted as $\mathscr{G}\mathscr{B}_{\Xi}^{Sig}$.

$$f'(z) \prec \frac{2}{1+e^{-z}} \tag{13}$$

and

$$h'(w) \prec \frac{2}{1+e^{-w}}. \tag{14}$$

The incorporation of the following lemmas will enhance the development of our main findings.

Lemma 1 [38] Assume that $u \in \mathscr{P}$ has the form of a series

$$u(z) = 1 + \sum_{n=1}^{+\infty} u_n z^n \tag{15}$$

where $\Re(u(z)) > 0$ and z in $\mathfrak{UD}$, then

$$|u_n| \leq 2, \quad n \in \mathbb{N}.$$

Lemma 2 [38] Assume that $u \in \mathscr{P}$ has the form of a series given in (15), and $\Re(u(z)) > 0$ where z in $\mathfrak{U}_D$, then

$$u_2 = \frac{u_1^2 + (4-u_1^2)e_1}{2},$$

$$u_3 = \frac{u_1^3 + 2(4-u_1^2)u_1e_1 - (4-u_1^2)u_1e_1^2 + 2(4-u_1^2)(1-|e_1|^2)z}{4}.$$

3. Coefficients bounds findings

Theorem 1 Suppose $f \in \mathcal{GB}_{\Xi}^{Sig}(\zeta)$. Then,

$$|a_2| \leq \frac{2\zeta+1}{4},$$

$$|a_3| \leq \frac{2\zeta+1}{6},$$

$$|a_4| \leq \frac{2\zeta+1}{8}.$$

The distinctiveness of the mathematical equation was substantiated by the precisely specified upper bounds above the given function.

$$f_1(z) = z + \frac{2\zeta+1}{4} z^2 + \cdots,$$

$$f_2(z) = z + \frac{2\zeta+1}{6} z^3 + \cdots,$$

$$f_3(z) = z + \frac{2\zeta+1}{8} z^4 + \cdots,$$

$$f_4(z) = z + \frac{2\zeta+1}{10} z^5 + \cdots.$$

Proof. If $f \in \mathcal{GB}_{\Xi}^{Sig}(\zeta)$, $0 \leq \zeta < 1$. Following that, the designated criteria

$$f'(z) = \frac{2\left(1+\upsilon_1(z)\right)^{\zeta}}{1+e^{-\upsilon_1(z)}} \tag{16}$$

and

$$h'(w) = \frac{2\left(1+\upsilon_2(w)\right)^{\zeta}}{1+e^{-\upsilon_2(w)}} \tag{17}$$

are met by the $\mathfrak{AF}s$ $\upsilon_1 : \mathcal{U} \longrightarrow \mathcal{U}$ and $\upsilon_2 : \mathcal{U}_{x_0} \longrightarrow \mathcal{U}_{x_0}$ together with the initial condition $\upsilon_1(0) = 0 = \upsilon_2(0)$, $|\upsilon_1(z)| < 1$, and $|\upsilon_2(w)| \leq 1$.

By employing (16) and (17) and carrying out simple computations yields

$$f'(z) = 1 + \frac{2\zeta+1}{4} u_1 z + \left(\frac{2\zeta+1}{4} u_2 + \frac{\zeta^2 - 2\zeta - 1}{8} u_1^2\right) z^2$$

$$+\left(\frac{4\zeta^3-30\zeta^2+26\zeta+11}{192}u_1^3+\frac{\zeta^2-2\zeta-1}{4}u_2u_1+\frac{2\zeta+1}{4}u_3\right)z^3+\cdots \tag{18}$$

and

$$h'(w)=1+\frac{2\zeta+1}{4}b_1w+\left(\frac{2\zeta+1}{4}b_2+\frac{\zeta^2-2\zeta-1}{8}b_1^2\right)w^2$$

$$+\left(\frac{4\zeta^3-30\zeta^2+26\zeta+11}{192}b_1^3+\frac{\zeta^2-2\zeta-1}{4}b_2b_1+\frac{2\zeta+1}{4}b_3\right)w^3+\cdots \tag{19}$$

where $u,\ b\in\mathscr{P}$.

To find the values of a_2, a_3, and a_4, it is necessary to compare and equate equations (18) and (19), which gives

$$2a_2=\frac{2\zeta+1}{4}u_1, \tag{20}$$

$$3a_3=\frac{2\zeta+1}{4}u_2+\frac{\zeta^2-2\zeta-1}{8}u_1^2, \tag{21}$$

$$4a_4=\frac{4\zeta^3-30\zeta^2+26\zeta+11}{192}u_1^3+\frac{\zeta^2-2\zeta+1}{4}u_2u_1+\frac{2\zeta+1}{4}u_3 \tag{22}$$

also,

$$-2a_2=\frac{2\zeta+1}{4}b_1, \tag{23}$$

$$6a_2^2-3a_3=\frac{2\zeta+1}{4}b_2+\frac{\zeta^2-2\zeta-1}{8}b_1^2, \tag{24}$$

$$-20a_2^3+20a_2a_3-4a_4=\frac{4\zeta^3-30\zeta^2+26\zeta+11}{192}b_1^3+\frac{\zeta^2-2\zeta+1}{4}b_2b_1+\frac{2\zeta+1}{4}b_3. \tag{25}$$

The following equation is obtained by using equations (20) and (23) in the manner described below

$$\frac{2\zeta+1}{8}u_1=-\frac{2\zeta+1}{8}b_1\ \Rightarrow\ u_1=-b_1\Rightarrow u_1^2=b_1^2\Rightarrow u_1^3=-b_1^3. \tag{26}$$

By applying Lemma 1 and performing elementary calculations on the concluding equation, the theorem's conclusion becomes clear:

$$|a_2|\leq\frac{2\zeta+1}{4}. \tag{27}$$

By utilizing equations (21) and (24), we can calculate the boundary for a_3, with $u_1 = -b_1$ being established, which gives

$$a_3 = a_2^2 + \frac{(2\zeta+1)(u_2-b_2)}{24},$$

so that

$$a_3 = \frac{(2\zeta+1)^2}{64}u_1^2 + \frac{(2\zeta+1)(u_2-b_2)}{24}. \tag{28}$$

In the same way, utilizing equations (22) and (25) with regard to equations (26) and (28) to determine a_4 leads to the subsequent equation:

$$a_4 = \frac{4\zeta^3-30\zeta^2+26\zeta+1}{768}u_1^3 + \frac{5(2\zeta+1)^2(u_2-b_2)u_1}{384} + \frac{(2\zeta+1)(u_3-b_3)}{32}$$

$$+\frac{(\zeta^2-2\zeta+1)u_1(u_2+b_2)}{32}. \tag{29}$$

It can be seen from (26) and applying Lemma 2 with $|x| \le 1$, $|y| \le 1$, $|z| \le 1$, and $|w| \le 1$ results in:

$$u_2 - b_2 = \frac{4-u_1^2}{2}(x-y), \tag{30}$$

$$u_2 + b_2 = u_1^2 + \frac{4-u_1^2}{2}(x+y), \tag{31}$$

$$u_3 - b_3 = \frac{u_1^3}{2} + \frac{(4-u_1^2)u_1}{2}(x+y) - \frac{(4-u_1^2)u_1}{4}(x^2+y^2)$$

$$+\frac{4-u_1^2}{2}\left([1-|x|^2]z - [1-|y|^2]w\right). \tag{32}$$

Utilizing equations (28) and (30) leads to

$$a_3 = \frac{(2\zeta+1)^2}{64}u_1^2 + \frac{(2\zeta+1)(4-u_1^2)(x-y)}{48}. \tag{33}$$

Since $u \in \mathscr{P}$, we have $|u_1| \le 2$. Letting $u_1 = \gamma$, we may assume without loss of generality that $\gamma \in [0,\ 2]$. Thus, substituting expressions (30) in (33), and letting $g = |x|$ and $h = |y|$, we have

$$|a_3| \le \frac{(2\zeta+1)^2}{64}\gamma^2 + \frac{(2\zeta+1)(4-\gamma^2)(g+h)}{48},\ \ (g,\ h) \in [0,\ 1]^2.$$

Additionally, there is a requirement to define a function $Q : \mathbb{R} \longrightarrow \mathbb{R}$ that must be evaluated for its maximum value within the closed square $E = \{(g,\ h) : (g,\ h) \in [0,\ 1]^2\}$ as depicted:

$$Q(g,\ h) = \frac{(2\zeta+1)^2}{64}\gamma^2 + \frac{(2\zeta+1)(4-\gamma^2)(g+h)}{48},\ \ (g,\ h) \in [0,\ 1]^2.$$

The highest value of $Q(g,\ h)$ is found on the boundary of E, and by taking the derivative of $Q(g,\ h)$ with respect to g, it can be concluded:

$$Q_g(g,\ h) = \frac{(2\zeta+1)(4-\gamma^2)}{48}.$$

Assuming $Q_g(g,\ h) \geq 0$, given that b_5 is within the range of $[0,\ 1]$ and γ falls within the range of $[0,\ 2]$. It can be inferred that as u_5 increases, the function $Q(g,\ h)$ also increases and peaks at $g = 1$, suggesting that:

$$\max[Q(g,\ h) : g \in [0,\ 1]] = Q(1,\ h) = \frac{(2\zeta+1)^2}{64}\gamma^2 + \frac{(2\zeta+1)(4-\gamma^2)(1+h)}{48}.$$

Taking further derivative of $Q(1,\ h)$ leads to

$$Q'(1,\ h) = \frac{(2\zeta+1)(4-\gamma^2)}{48}.$$

Assuming $Q'(1,\ h) \geq 0$, given that $\gamma \in [0,\ 2]$. It is evident that the function $Q(1,\ h)$ rises and peaks at $h = 1$, indicating that:

$$\max[Q(g,\ h) : u_5 \in [0,\ 1]] = Q(1,\ 1) = \frac{(2\zeta+1)^2}{64}\gamma^2 + \frac{(2\zeta+1)(4-\gamma^2)}{24}.$$

Furthermore,

$$Q(g,\ h) \leq \max[Q(g,\ h) : g \in [0,\ 1]] = Q(1,\ 1) = \frac{(2\zeta+1)^2}{64}\gamma^2 + \frac{(2\zeta+1)(4-\gamma^2)}{24}.$$

If a_3 has an absolute value that is smaller or equal to $Q(g,\ h)$, it is easy to see that.

$$|a_3| \leq S(\zeta)\gamma^2 + \frac{2\zeta+1}{6},\ \ \gamma \in [0,\ 2],$$

taking

$$S(\zeta) = \frac{1}{4}\left[\frac{(2\zeta+1)^2}{16} - \frac{2\zeta+1}{6}\right].$$

We present a function $Q_1 : \mathbb{R} \longrightarrow \mathbb{R}$ that is intended to find its highest possible value, given by:

$$Q_1(\gamma) = S(\zeta)\gamma^2 + \frac{2\zeta+1}{6}, \ \ \gamma \in [0,\ 2].$$

The equation $Q_1'(\gamma) = 2S(\zeta)\gamma$ is true when evaluating the derivative of $Q_1(\gamma)$ with γ values in the range of $[0,\ 2]$. When $S(\zeta) \leq 0$, $Q_1'(\gamma) \leq 0$, causing $Q_1(\gamma)$ to decrease and reach a peak at $\gamma = 0$. Thus,

$$\max[Q_1(\gamma) : \gamma \in [0,\ 2]] = Q_1(0) = \frac{2\zeta+1}{6}.$$

If $S(\zeta) \geq 0$ holds, then $Q_1'(\gamma) \geq 0$, leading to $Q_1(\gamma)$ becoming a increasing function with its peak at $\gamma = 2$. Hence

$$\max[Q_1(\gamma) : \gamma \in [0,\ 2]] = Q_1(2) = \frac{(2\zeta+1)^2}{16}.$$

Therefore, a successful calculation of the precise maximum value for $|a_3|$ is presented in the following manner:

$$|a_3| \leq \frac{2\zeta+1}{6}. \tag{34}$$

By using equations (29), (30), (31), and (32) in conjunction with the familiar triangular inequality, we can express the inequality for the magnitude of a_4 as follows:

$$|a_4| \leq s_1(\gamma) + s_2(\gamma)(g+h) + s_3(\gamma)(g^2+h^2) = Q_2(g,\ h),$$

so that

$$s_1(\gamma) = \frac{4\zeta^2 - 6\zeta^2 + 2\zeta + 37}{768}\gamma^3 + \frac{(2\gamma+1)(4-\gamma^2)}{32},$$

$$s_2(\gamma) = \frac{5(2\zeta+1)^2(4-\gamma^2)\gamma}{64} + \frac{\zeta^2(4-\gamma^2)\gamma}{64},$$

$$s_3(\gamma) = \frac{(2\zeta+1)(4-\gamma^2)(\gamma-2)}{128}.$$

The parameters $s_1(\gamma)$, $s_2(\gamma)$, and $s_3(\gamma)$ of $Q_2(g,\ h)$ change depending on the parameter γ, necessitating the maximization of $Q_2(g,\ h)$ on E for every $\gamma \in [0,\ 2]$. Subsequently, we have to identify identify the maximum value of $Q_2(g,\ h)$ for distinct γ values. Due to the fact that gamma is equal to zero, as s_2 at 0 equals 0,

$$s_1(0) = \frac{2\zeta+1}{8}, \ \ s_3(0) = -\frac{2\zeta+1}{16},$$

and gives the following:

$$Q_2(g,\ h) = \frac{2\zeta+1}{8} - \frac{2\zeta+1}{16}(g^2+h^2), \quad (g,\ h) \in [0,\ 1]^2.$$

Consequently, we possess

$$Q_2(g,\ h) \leq \max[Q(g,\ h) : (g,\ h) \in E] = Q(0,\ 0) = \frac{2\zeta+1}{8}.$$

Let's consider γ to be equal to 2. If $s_2(2) = s_3(2) = 0$, it follows that

$$s_1(2) = \frac{4\gamma^2 - 6\gamma^2 + 2\gamma + 37}{96}.$$

As a result, we have a set value for the function $Q_2(g,\ h)$ as stated:

$$Q_2(g,\ h) = s_1(2) = \frac{4\gamma^2 - 6\gamma^2 + 2\gamma + 37}{96}.$$

Noticing that $Q_2(g,\ h)$ does not achieve a highest value on E when γ is in the range of $[0,\ 2]$, we can infer that

$$|a_4| \leq \frac{2\zeta+1}{8}. \tag{35}$$

Basically, we can verify that the outcomes achieved in equations (27), (34) and (35) are applicable to the functions mentioned here:

$$f_1(z) = z + \frac{2\zeta+1}{4}z^2 + \frac{\zeta^2}{6}z^3 + \cdots,$$

$$f_2(z) = z + \frac{2\zeta+1}{6}z^3 + \cdots,$$

$$f_3(z) = z + \frac{2\zeta+1}{8}z^4 + \cdots. \qquad \square$$

The specified Corollary is valid if the condition $\zeta = 1$ is met as stated in Theorem 1.

Corollary 1 If $f \in \mathscr{G}\mathscr{B}_{\Xi}$. Then,

$$|a_2| \leq \frac{1}{2}, \quad |a_3| \leq \frac{1}{3} \quad |a_4| \leq \frac{1}{4}.$$

The functions listed below are used to verify the accuracy of the results.

$$f_1 = z + \frac{1}{2}z^2 + \frac{1}{24}z^3 + \cdots,$$

$$f_2 = z + \frac{1}{3}z^3 + \cdots,$$

$$f_3 = z + \frac{1}{4}z^4 + \cdots.$$

For Theorem 1, when $\zeta = 0$, we can derive another novel finding, which will yield a new subclass of $\mathscr{B}\mathscr{U}s$ consisting of $\mathfrak{B}\mathfrak{F}s$ that are subordinate to the modified sigmoid function.

Corollary 2 If $f \in \mathscr{G}\mathscr{B}_{\Xi}^{Sig}$. Then,

$$|a_2| \leq \frac{1}{4}, \quad |a_3| \leq \frac{1}{6} \quad |a_4| \leq \frac{1}{8}.$$

The functions listed below are used to verify the accuracy of the results.

$$f_1 = z + \frac{1}{4}z^2 + \cdots,$$

$$f_2 = z + \frac{1}{6}z^3 + \cdots,$$

$$f_3 = z + \frac{1}{8}z^4 + \cdots.$$

4. Estimations for the second Hankel determinant

Theorem 2 Suppose $f \in \mathscr{G}\mathscr{B}_{\Xi}^{Sig}(\zeta)$. Then,

$$|a_3^2 - a_2a_4| \leq \left(\frac{2\zeta+1}{6}\right)^2$$

where

$$f_2(z) = z + \frac{2\zeta+1}{6}z^3 + \cdots.$$

Proof. If $f \in \mathscr{G}\mathscr{B}_{\Xi}^{Sig}(\zeta)$ and $0 \leq \zeta < 1$, then let's consider the situation. By using equations (26), (28), and (29), we can show that $a_2a_4 - a_3^2$ is equivalent, that is

$$a_2a_4 - a_3^2 = -\frac{(2\zeta+1)(16\zeta^3+96\zeta^2+58\zeta+1)}{12,288}u_1^4 + \frac{(2\zeta+1)^3}{3,072}u_1^2(u_2-b_2)$$

$$+\frac{(2\zeta+1)^2}{256}u_1(u_3-b_3) + \frac{(2\zeta+1)(\zeta^2-2\zeta+1)}{256}u_1^2(u_2+b_2)$$

$$-\frac{(2\zeta+1)^2}{576}(u_2-b_2)^2.$$

By utilizing equations (30), (31) and (32), and choosing γ values between 0 and 2 while employing the triangle inequality leads to $|x| = g$ and $|y| = h$, the outcome is:

$$|a_2a_4 - a_4^2| \leq S_1(\gamma) + S_2(\gamma)(g+h) + S_3(\gamma)(g^2+h^2) + S_4(\gamma)(g+h)^2, \tag{36}$$

so that

$$S_1(\gamma) = \frac{(2\zeta+1)(16\zeta^3+48\zeta^2+106\zeta-71)}{12,288}\gamma^4 + \frac{(2\zeta+1)^2(4-\gamma^2)\gamma}{256} \geq 0,$$

$$S_2(\gamma) = \frac{(2\zeta+1)(16\zeta^2+4\zeta+25)(4-\gamma^2)}{6,144}\gamma^2 \geq 0,$$

$$S_3(\gamma) = \frac{(2\zeta+1)^2(4-\gamma^2)(\gamma-2)\gamma}{1,024} \leq 0,$$

$$S_4(\gamma) = \frac{(2\zeta+1)^2(4-\gamma^2)^2}{2,304} \geq 0.$$

We define the function $Q_3 : \mathbb{R} \longrightarrow \mathbb{R}$ with the purpose of establishing its maximum value with respect to γ within the interval from 0 to 2, as specified by the expression:

$$Q_3(g,\ h) = S_1(\gamma) + S_2(\gamma)(g+h) + S_3(\gamma)(g^2+h^2) + S_4(\gamma)(g+h)^2,\ \ (g,\ h) \in [0,\ 1]^2.$$

The values of $Q_3(g,\ h)$ coefficients $S_1(\gamma)$, $S_2(\gamma)$, $S_3(\gamma)$, and $S_4(\gamma)$ change with the parameter γ, calling for optimization of $Q_3(g,\ h)$ over E for every γ in the range $[0,\ 2]$. Afterwards, we must find the maximum parameter of $Q_3(g,\ h)$ for different γ values.

(a) With γ equal to zero, since $S_1(0) = S_2(0) = S_3(0) = 0$,

$$S_4(0) = \frac{(2\zeta+1)^2}{144}$$

which gives

$$Q_3(g,\ h) = \frac{(2\zeta+1)^2}{144}(g+h)^2, \quad (g,\ h) \in E.$$

One can conclude that the highest value of $Q_3(g,\ h)$ is situated at the boundary of E, and by taking the derivative of $Q_3(g,\ h)$ with respect to u_5, it can be deduced that:

$$(Q_3)_g(g,\ h) = \frac{(2\zeta+1)^2}{72}(g+h), \quad h \in [0,\ 1].$$

Assuming that $(Q_3)_g(g,\ h) \geq 0$, with h ranging from 0 to 1 and γ ranging from 0 to 2. The function $Q_3(g,\ h)$ is found to rise as u_5 increases and reaches its maximum at $h = 1$, suggesting that:

$$\max[Q_3(g,\ h) : g \in [0,\ 1]] = Q_3(1,\ h) = \frac{(2\zeta+1)^2}{144}(1+h)^2,\ h \in [0,\ 1].$$

Further derivation of $Q_3(1,\ h)$ leads to

$$Q_3'(1,\ h) = \frac{(2\zeta+1)^2}{72}(1+h).$$

If $(Q_3)_g(1,\ h)$ is greater than or equal to 0, for γ in the range of $[0,\ 2]$. The function $Q_3(1,\ h)$ is observed to rise and reach its maximum at $h = 1$, suggesting that:

$$\max[Q_3(1,\ h) : h \in [0,\ 1]] = Q_3(1,\ 1) = \left(\frac{2\zeta+1}{6}\right)^2.$$

Therefore, when $\gamma = 0$, we get:

$$Q_3(g,\ h) \leq \max[Q_3(g,\ h); (g,\ h) \in [0,\ 1]^2] = Q_3(1,\ 1) = \left(\frac{2\zeta+1}{6}\right)^2.$$

Since $|a_3^2 - a_2a_4| \leq Q_3(g,\ h)$, yields

$$|a_3^2 - a_2a_4| \leq \left(\frac{2\zeta+1}{6}\right)^2.$$

(b) Let γ be equal to 2. If $S_2(2) = S_3(2) = S_4(2) = 0$, then

$$S_1(2)=\frac{(2\zeta+1)(16\zeta^3+48\zeta^2+106\zeta-71)}{768}$$

which offers the consistent function displayed underneath:

$$Q_3(g,\ h)=S_1(2)=\frac{(2\zeta+1)(16\zeta^3+48\zeta^2+106\zeta-71)}{768}.$$

Hence, yields

$$|a_3^2-a_2a_4|\leq\frac{(2\zeta+1)(16\zeta^3+48\zeta^2+106\zeta-71)}{768},$$

for $\gamma=2$.

(c) To examine the maximum point of $Q_3(g,\ h)$ in the interval $\gamma\in(0,\ 2)$, we will employ the function $\Xi(Q_3)=(Q_3)_{gg}(g,\ h)(Q_3)_{hh}(g,\ h)-((Q_3)_{gh}(g,\ h))^2$.

Furthermore, we will analyze two situations to establish the preferred result for the expression $\Xi(Q_3)=4S_3(\gamma)\{S_3(\gamma)+2S_4(\gamma)\}$ in this case.

(i) If $s_3(\gamma)+2S_4(\gamma)\leq 0$ for $\gamma\in(0,\ 2)$, then the function Q_3 will not possess a maximum on E since $(Q_3)_{g,\ h}(g,\ h)=(Q_3)_{g,\ h}(g,\ h)=2S_4(\gamma)\geq 0$, and $\Xi(Q_3)\geq 0$.

(ii) In order for the maximum value of function Q_3 on E to occur, the condition $E(Q_3)\leq 0$ must hold when $S_3(\gamma)+2S_4(\gamma)\geq 0$ for $\gamma\in(0,\ 2)$.

As a result of the results from the three cases, we develop

$$|a_3^2-a_2a_4|\leq\left(\frac{2\zeta+1}{6}\right)^2. \tag{37}$$

In essence, we can verify that the outcome derived in equation (37) is applicable to the function mentioned:

$$f_2(z)=z+\frac{2\zeta+1}{6}z^3+\cdots.$$ □

The Corollary discussed in Theorem 2 is valid when $\gamma=1/2$ is met.

Corollary 3 If $f\in\mathcal{G}\mathcal{B}_{\Xi}$. Then,

$$|a_3^2-a_2a_4|\leq\left(\frac{1}{3}\right)^2.$$

The function listed below is used to verify the accuracy of the result.

$$f_2=z+\frac{1}{3}z^3+\cdots.$$

For Theorem 2, when $\zeta = 0$, we can derive another novel finding, which will yield a new subclass of $\mathscr{BU}\,s$ consisting of $\mathfrak{BF}s$ that are subordinate to the modified sigmoid function.

Corollary 4 If $f \in \mathscr{GB}_{\Xi}^{Sig}$. Then,

$$|a_3^2 - a_2a_4| \leq \left(\frac{1}{6}\right)^2.$$

The function listed below is used to verify the accuracy of the result.

$$f_2 = z + \frac{1}{6}z^3 + \cdots.$$

5. Estimations for the Fekete-Szego inequality

Theorem 3 Suppose $f \in \mathscr{GB}_{\Xi}^{Sig}(\zeta)$. Then,

$$\left|a_3 - \rho a_2^2\right| \leq \begin{cases} \dfrac{(2\zeta+1)^2\rho(\zeta)}{16}, & |1-\rho| \leq \rho(\zeta) \\[2ex] \dfrac{(2\zeta+1)^2|1-\rho|}{16}, & |1-\rho| \geq \rho(\zeta), \end{cases}$$

where

$$\rho(\zeta) = \frac{16}{6(2\zeta+1)}. \tag{38}$$

The precise maximum bounds on the specified function validated the accuracy of the equation:

$$f_2(z) = z + \frac{2\zeta+1}{6}z^3 + \cdots.$$

Proof. If $f \in \mathscr{GB}_{\Xi}^{Sig}(\zeta)$ and $0 \leq \zeta < 1$, then let's consider the situation. By using equations (26), (28), (30) and (31), we can show that $a_3 - \rho a_2^2$ is equivalent, that is

$$a_3 - \rho a_2^2 = \frac{(2\zeta+1)^2}{64}\left[1-\rho\right]u_1^2 + \frac{(2\zeta+1)(4-u_1^2)(x+y)}{48}. \tag{39}$$

Using equation (39), we can choose γ between 0 and 2 and apply the triangle inequality to obtain $|x| = g$, and $|y| = h$, gives

$$\left|a_3 - \rho a_2^2\right| \leq \frac{|1-\rho|(2\zeta+1)^2}{64}\gamma^2 + \frac{(2\zeta+1)(4-\gamma^2)(g+h)}{48}.$$

We aim to identify the peak value of the function $Q_4 : \mathbb{R} \to \mathbb{R}$ within the specified range of γ values, namely $[0,\ 2]$. The function in question is described by the given expression:

$$Q_4(g,\ h) = \frac{|1-\rho|(2\zeta+1)^2}{64}\gamma^2 + \frac{(2\zeta+1)(4-\gamma^2)(g+h)}{48},\ (g,\ h) \in E,\ \gamma \in [0,\ 2].$$

The highest value of $Q_4(g,\ h)$ is found at the boundary of E, and by taking the derivative of $Q_4(g,\ h)$ with respect to g, we can ascertain that:

$$(Q_4)_g(g,\ h) = \frac{(2\zeta+1)(4-\gamma^2)}{48},\ \ \gamma \in [0,\ 2].$$

Assuming $(Q_4)_g(g,\ h) \geq 0$, with b_5 ranging from 0 to 1 and γ ranging from 0 to 2. One can ascertain that the function $Q_4(g,\ h)$ grows as g increases and peaks at $g = 1$, which suggests that:

$$\max[Q_4(g,\ h) : g \in [0,\ 1]] = Q_4(1,\ h) = \frac{|1-\rho|(2\zeta+1)^2}{64}\gamma^2$$
$$+ \frac{(2\zeta+1)(4-\gamma^2)(1+h)}{48},\ \ h \in [0,\ 1],\ \ \gamma \in [0,\ 2].$$

Further elaboration of $Q_4(1,\ h)$ leads to

$$Q_4'(1,\ h) = \frac{(2\zeta+1)(4-\gamma^2)}{48},\ \ \gamma \in [0,\ 2].$$

Given that $(Q_4)_g(1,\ h) \geq 0$ for $\gamma \in [0,\ 2]$. It is evident that the function $Q_4(1,\ h)$ rises steadily and peaks at $h = 1$, suggesting that:

$$\max[Q_4(1,\ h) : g \in [0,\ 1]] = Q(1,\ 1) = \frac{|1-\rho|(2\zeta+1)^2}{64}\gamma^2$$
$$+ \frac{(2\zeta+1)(4-\gamma^2)}{24},\ \ \gamma \in [0,\ 2].$$

Futhermore, we have

$$Q_4(g,\ h) \leq \max[(g,\ h) : (g,\ h) \in E] = Q_4(1,\ 1)$$
$$= \frac{|1-\rho|(2\zeta+1)^2}{64}\gamma^2 + \frac{(2\zeta+1)(4-\gamma^2)}{24}.$$

Since $\left|a_3 - \rho a_2^2\right| \leq Q_3(g,\ h)$, gives

$$\left|a_3-\rho a_2^2\right| \leq \frac{(2\zeta+1)^2}{4}\left[\frac{|1-\rho|-\rho(\zeta)}{16}\right]\gamma^2+\frac{(2\zeta+1)^2\rho(\zeta)}{16},$$

where

$$\rho(\zeta)=\frac{16}{6(2\zeta+1)}.$$

Function $Q_5:[0,\ 2]\longrightarrow\mathbb{R}$ is introduced to find the highest value it can take for γ in the interval $[0,\ 2]$, defined as:

$$Q_5(\gamma)=\frac{(2\zeta+1)^2}{4}\left[\frac{|1-\rho|-\rho(\zeta)}{16}\right]\gamma^2+\frac{(2\zeta+1)^2\rho(\zeta)}{16}.$$

Further differentiation of $Q_5(\gamma)$ leads to

$$Q_5'(\gamma)=\frac{(2\zeta+1)^2}{2}\left[\frac{|1-\rho|-\rho(\zeta)}{16}\right]\gamma.$$

If $Q_5'(\gamma)\leq 0$, $Q_5(\gamma)$ will decrease. The peak of the function occurs at $\gamma=0$ if $|1-\rho|\leq\rho(\zeta)$. So

$$\max[Q_5(\gamma);\ \gamma\in[0,\ 2]]=Q_5(0)=\frac{(2\zeta+1)^2\rho(\zeta)}{16}.$$

If $Q_5'(\gamma)\geq 0$, $Q_5(\gamma)$ will experience an increase. The function reaches its maximum at $\gamma=2$ if $|1-\rho|\geq\rho(\zeta)$. Thus

$$\max[Q_5(\gamma);\ \gamma\in[0,\ 2]]=Q_5(2)=\frac{(2\zeta+1)^2|1-\rho|}{16}.$$

Hence, we get

$$\left|a_3-\rho a_2^2\right|\leq\begin{cases}\dfrac{(2\zeta+1)^2\rho(\zeta)}{16}, & |1-\rho|\leq\rho(\zeta)\\[2ex] \dfrac{(2\zeta+1)^2|1-\rho|}{16}, & |1-\rho|\geq\rho(\zeta).\end{cases} \tag{40}$$

We can confirm that the result obtained in equation (40) applies effectively to the function provided.

$$f_2(z)=z+\frac{2\zeta+1}{6}z^3+\cdots.$$

$\square$

According to Theorem 3, if the condition $\zeta = 1/2$ holds, then the corresponding Corollary is true.

Corollary 5 If $f \in \mathscr{G}\mathscr{B}_{\Xi}$. Then,

$$\left|a_3 - \rho a_2^2\right| \leq \begin{cases} \dfrac{\rho}{4}, & |1-\rho| \leq \rho \\ \dfrac{|1-\rho|}{4}, & |1-\rho| \geq \rho, \end{cases}$$

where

$$\rho = \frac{4}{3}.$$

The function listed below is used to verify the accuracy of the result.

$$f_2 = z + \frac{1}{3}z^3 + \cdots.$$

For Theorem 3, when $\zeta = 0$, we can derive another novel finding, which will yield a new subclass of $\mathscr{B}\mathscr{U}\,s$ consisting of $\mathfrak{B}\mathfrak{F}s$ that are subordinate to the modified sigmoid function.

Corollary 6 If $f \in \mathscr{G}\mathscr{B}_{\Xi}^{Sig}$. Then,

$$\left|a_3 - \rho a_2^2\right| \leq \begin{cases} \dfrac{\rho}{16}, & |1-\rho| \leq \rho \\ \dfrac{|1-\rho|}{16}, & |1-\rho| \geq \rho, \end{cases}$$

where

$$\rho = \frac{16}{6}.$$

The function listed below is used to verify the accuracy of the result.

$$f_2 = z + \frac{1}{6}z^3 + \cdots.$$

Theorem 4 Suppose $f \in \mathscr{G}\mathscr{B}_{\Xi}^{Sig}(\zeta)$. Then,

$$\left|a_3-\rho a_2^2\right| \leq \begin{cases} \dfrac{(2\zeta+1)^2(1-\rho)}{16} & \text{if } \rho \leq 1-\rho(\zeta) \\ \dfrac{(2\zeta+1)^2\rho(\zeta)}{16} & \text{if } 1-\rho(\zeta) \leq \rho \leq 1+\rho(\zeta) \\ \dfrac{(2\zeta+1)^2(\rho-1)}{16} & \text{if } 1+\rho(\zeta) \leq \rho, \end{cases} \tag{41}$$

where

$$\rho(\zeta)=\frac{16}{6(2\zeta+1)}.$$

Proof. Assuming $f \in \mathscr{G}\mathscr{B}_{\Xi}^{Sig}(\zeta)$, $0 \leq \zeta < 1$. We have $|1-\rho| \geq \rho(\zeta)$ and $|1-\rho| \leq \rho(\zeta)$ when $\rho \in \mathbb{R}$. Yields:

$$\rho \leq 1-\rho(\zeta) \text{ either } \rho \geq 1+\rho(\zeta)$$

and

$$1-\rho(\zeta) \leq \rho \leq 1+\rho(\zeta). \qquad \square$$

Furthermore, the outcomes for $\rho = 1$ can be obtained from Theorem 4.

Corollary 7 Suppose $f \in \mathscr{G}\mathscr{B}_{\Xi}^{Sig}(\zeta)$. Then,

$$\left|a_3-a_2^2\right| \leq \frac{2\zeta+1}{6}.$$

Corollary 8 If $f \in \mathscr{G}\mathscr{B}_{\Xi}$. Then,

$$|a_3-a_2^2| \leq \frac{1}{3}.$$

Corollary 9 If $f \in \mathscr{G}\mathscr{B}_{\Xi}^{Sig}$. Then,

$$|a_3-a_2^2| \leq \frac{1}{6}.$$

6. Conclusion

We examined a new subclass of $\mathscr{BU}s$ in the unit disk by utilizing subordination and the generalized balloon-shaped domain concept. The article is split into five parts. The opening section of this research establishes the fundamental background and fundamental terms, followed by the second section, which outlines previously proven lemmas and introduces a new class of $\mathscr{BU}$s characterized by a generalized domain with a balloon-like shape. In section 3, we investigated the fresh challenges for the recent subclass, particularly focusing on the precise coefficient bounds, all of which are precise. This was verified by applying the extremal function to the novel category of $\mathscr{BU}s$. In section 4, we achieved the precise limit for the Hankel determinant. In section 5, we have obtained the exact limit for the Fekete-Szego inequality for real and complex parameters. Because the new subclass of $\mathscr{BU}s$ is exceptional, it is not possible to corroborate our findings by comparing them to existing ones, given that the methods utilized in prior research may lack accuracy. Hence, we have accurately identified the extremal function for the new category to verify the exact limits for the characteristics studied in this study.

Open Problems and Future work: The study highlights certain issues that have not yet been resolved.

$$\frac{zf'(z)}{f(z)} \prec \frac{2(1+z)^{\zeta}}{1+e^{-z}} \tag{42}$$

and

$$\frac{wh'(w)}{h(w)} \prec \frac{2(1+w)^{\zeta}}{1+e^{-w}} \tag{43}$$

where $0 < \zeta < 1$. In this field of research, scholars can delve into this issue and take advantage of the latest developed operators, such as those presented in [39, 40] and also fractional calculus in Geometric Function Theory (GFT) particularly in the context of bounded turning functions and other subclasses, including starlike functions, convex functions, etc., to uncover additional properties, including the third Hankel determinant [41].

Conflict of interest

The authors declare no competing financial interest.

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