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Research Article

# An Enhance Vehicle Selection Problem for an Effective Transportation System: A Logarithmic-Based Distance Measure Approaches via q-Rung Orthopair Fuzzy Information

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**Abstract:** An effective transportation system is fundamental for socioeconomic development, public safety, and environmental sustainability. A critical component of such a system is the Vehicle Selection Problem (VSP), which is inherently a Complex Decision-Making (CDM) problem due to conflicting and uncertain criteria such as fuel efficiency, purchase cost, maintenance cost, and warranty. To address this complexity, this study develops two novel logarithmic-based distance measures within the q-Rung Orthopair Fuzzy (q-ROF) framework. The proposed distance metrics incorporate membership, non-membership, and hesitation degrees, along with the cardinality of the universe of discuss, ensuring a more comprehensive representation of uncertainty. Their metric properties are rigorously proven, and they are integrated with the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) to solve a VSP. The case study involving seven vehicle brands and seven evaluation criteria, assessed by domain experts, demonstrates the effectiveness of the proposed approach. Comparative analysis with existing logarithmic-based distance measures shows that the new methods provide superior accuracy, stability, and discrimination ability in ranking alternatives. The findings highlight the practical significance of the proposed q-ROF distance measures, offering a robust decision-support tool for vehicle selection and other CDM scenarios under uncertainty.

***Keywords***: q-rung orthopair fuzzy set, vehicle selection process, multi-criteria decision-making, q-rung orthopair fuzzy distance metric

**MSC:** 90B50, 90C70

## 1. Introduction

An Effective Transportation System (ETS) plays a critical role in the social-economic development, public safety and environmental sustainability of a country. The need for effective transportation has become increasingly pressing due to rapid urbanization, population growth and rising demand for mobility [1, 2]. One of the vital components of


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effective transportation planning is vehicle selection, which involves identifying the most suitable vehicle among various alternatives based on multiple conflicting and fuzzy criteria, such as fuel efficiency, safety, low price, suitable for everyday use, high quality, driving comfort, good warranty/costumer service, and design. All of these conflicting criteria make the Vehicle Selection Problem (VSP) a complex Multi-Criteria Decision-Making (MCDM) process, thereby rendering traditional crisp decision-making models inadequate in handling it [3, 4].

Owing to the vague nature of real-world decision-making problems, varying mathematical frameworks have been introduced to help in handling these uncertainties. Zadeh [5] introduced the Fuzzy Sets (FSs) theory which was later enhanced to Intuitionistic Fuzzy Sets (IFSs) [6], Pythagorean Fuzzy Sets (PFSs) [7] and Fermatean Fuzzy Sets (FFSs) [8]. While FSs, IFSs, PFSs, and FFSs provide varying degrees of flexibility, these models impose restrictive constraints on the sum of membership and non-membership degrees which limit the ability of the models to represent higher levels of uncertainty in decision-making contexts where experts' opinions are highly conflicting or imprecise. To tackle all the setbacks observed in IFSs, PFSs and FFSs, q-Rung Othorpair Fuzzy Sets (q-ROFSs) were introduced by Yager [9]. This model generalizes the existing models and permits the sum of the qth power of the Membership Degree (MD) and the Non-Membership Degree (NMD) to be at most one, thus enabling more refined and reliable decision-making in intricate and uncertain environments. In this study, we propose two logarithmic-based q-ROF Distance Metrics (q-ROFDMs) and apply the methods in a VSP, so as to enhance efficient transportation system based on the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) approach. The integration of q-ROFDM and TOPSIS is motivated by the need to handle conflicting, imprecise and uncertain criteria in real world Complex Decision-Making (CDM) scenarios, since the classical Multi-Criteria Decision-Making (MCDM) often falls short in ranking alternatives which are very close. TOPSIS approach determines the best alternative by measuring its distance from the Positive Ideal Solution (PIS) and Negative Ideal Solution (NIS), which when combined with q-ROFSs allows the distance method to incorporate MD, NMD and Hesitation Margin (HM) simultaneously, producing more reliable nearness coefficients, such that, the rankings reflect not only how good or bad an alternative is, but also the uncertainty in the evaluations. This ensures that alternatives with close valuations can still be effectively separated, making the decision-making process more precise, robust, and interpretable in complex scenarios such as vehicle selection.

### 1.1 *Literature review*

Decision making in complex environments such as vehicle selection, medical diagnosis, and renewable energy source selection, have greatly been enhanced by the FSs theory [5]. This theory enhanced the mathematical framework of handling uncertainties with a MD defined within the interval of 0 and 1. Despite the enhanced framework of FSs and its ability to handle varying uncertainties, it did not include the HM, which is inherent in real-world decision-making problems. To address this stumbling block, Atanassov [6] improved on the FSs theory by introducing IFSs, which incorporate the NMD and the HM together with MD, thereby providing an enhanced mathematical framework for handling uncertainties. Due to IFSs' efficiency and flexibility, they have been efficient in handling varying decision-making problems. In [10–13], some enhanced intuitionistic distance metrics were developed and applied in predicting maternity outcomes and other varying medical analysis. Some enhanced intuitionistic fuzzy similarity measures were developed in [14–16] and applied in face recognition patterns, emergency management, software quality evaluation, and pattern recognition problems. In [17], a correlation coefficient was proposed for IFSs and applied in medical emergency. IFSs were able to handle multifaceted decision-making problems but could not resolve cases in which the aggregate of the MD and NMD exceeds one.

To address the setback in IFSs, Yager [7] introduced PFSs, where the addition of the squares of MD and NMD ranges between 0 and 1. PFSs offer a more robust framework in handling real-world issues. Anum et al. [18] proposed an enhanced Pythagorean fuzzy distance metric and applied it in the selection of a renewable energy source based on MCDA. Ejegwa et al. [19] developed a novel Pythagorean fuzzy distance metric and applied it in pattern classification and diagnostic processes. Others can be seen in [20–22] where Pythagorean fuzzy distance metrics have been applied sundry CDM environments. Saikia et al. [23] proposed a novel similarity measure of PFSs based on the distance of the degree of membership, non-membership and hesitancy of PFSs. Prabha et al. [24] developed a geometric mean technique to resolve PFSs transportation system based on a Pythagorean fuzzy algorithm. In [25–27], Pythagorean fuzzy sets were applied in questionnaire analysis and decision-making. Despite the significance of PFSs, it lacks the ability to handle real-world decision-making problems, where the addition of the square of MD and NMD exceeds one, which is a setback to some practical CDM problems. This limitation inspired the introduction of FFSs [8]. FFSs improved the mathematical framework of PFSs such that, the sum of the third power of MD and NMD is within the interval of 0 and 1. This property makes FFSs a great tool for cracking real-world CDM problems CDM. Hence, Sahoo [28] proposed a novel FFSs method for solving transportation problems where supply, demand and transportation costs were captured as Fermatean Fuzzy (FF) numbers. Ejegwa et al. [29] proposed an enhanced FF composition relation based on a maximum-average approach with an application in diagnostic analysis. Kirisci [30] developed a correlation coefficient of FFSs and applied it to medical diagnosis based on MCDM. Ejegwa and Sarkar [31] proposed a FF approach to disease diagnosis based on correlation coefficients. In [32–37], sundry FF approaches ranging from aggregate operators, similarity and distance metrics were developed and applied in admission process, insecurity evaluation and clustering analysis. Despite the advantages of FFS over other existing frameworks, the constraint that the cube sum of the MD and NMD not to exceed one limited its applications.

To tackle all the setbacks observed in IFSs, PFSs and FFSs, q-ROFSs were introduced by Yager [9], defined as an enhanced framework which generalizes all the existing models and allows decision-makers to characterise and handle vague information with greater flexibility. Unlike FSs, IFSs, PFSs and FFSs, q-ROFSs allow the summation of the $q^{th}$ powers of MD and NMD to be bounded by one, thereby expanding the space of inclusion. This generalization not only improves q-ROFSs ability to model imprecision but also enables enhanced discrimination among alternatives in CDM environments. With the improved mathematical framework of q-ROFS, it has become an essential tool for researchers to make more informed and effective decisions under imprecision. In the area of diagnostic medicine, Ejegwa and Davvaz [38] proposed a mean of diagnosis using an improved composite relation to decide a patient's medical state where disease and patients were captured as q-ROF values based on some clinical manifestations. In [39], Dounis et al. proposed some mathematical methodologies for handling vagueness in medical diagnosis using q-ROFSs, where some aggregates were proposed to improve CDM in disease identification. Dounis and Stefopoulos [40] developed an intelligent diagnosis reasoning using aggregate operators, composite fuzzy relation, and similarity metrics of q-ROFSs.

Hussain et al. [41] proposed a novel method of combining the Muirhead mean with q-ROFSs, the integration which was valuable in medical diagnosis where the interplay and extreme importance of symptoms and diagnostic criteria are complex and critical. This approach showed not only improvement in patients' outcomes but also enhances the reliability of diagnostic procedures. Petchimuthu et al. [42] explored a novel approach of integrating artificial intelligence, MCDM and q-ROF fuzzy expologarithmic aggregation operators to address the complex dynamics of sustainable urban development. The study integrated a practical demonstration in real-time, with specific emphasis on enacting appropriate policies for sustainable urban innovation and resilience. This operator design improves aggregation semantics but their exposition is focused on aggregation and AI integration rather than on designing distance metrics that explicitly include hesitation and guarantee metric properties for TOPSIS. Zhang and Gao [43] proposed an interpretable robust TODIM (an acronym in Portuguese for interactive and multicriterial decision making) approach tailored for generalized orthopair fuzzy settings, by first extending these TODIM methods to accommodate generalized orthopair fuzzy sets, integrating it into a unified framework with an illustrative example to demonstrate the application of the TODIM approach. Though, TODIM is generally different from TOPSIS, it emphasizes decision maker risk attitude and prospect theory behaviour rather than geometric closeness to an ideal. Petchimuthu et al. [44] introduced an innovative methodology that combines MCDM with q-ROF Yager aggregation operator to address the multifaceted complexities of sustainable urban development, with a critical evaluation of the advantages and limitations of the proposed operators, underscoring their effectiveness in promoting urban resilience and minimizing environmental impact within CDM environment. However, the study used aggregation mechanics and prioritization for TOPSIS ranking not distance metrics. Akram et al. [45] proposed an integrated framework combining step-wise weight assessment ratio analysis and ELECTRE under interval rough environments for evaluating industrial machine tool. Akram et al. [46] employed an ELECTRE based outranking method with fuzzy information to optimize water supply management, and ELECTRE-based decision-making using spherical fuzzy information was presented for project evaluation [47]. However, these works [45–47] could not present the alternatives in a ranking order like the TOPSIS method.

In [48], some correlation coefficients for q-ROFSs were presented to discuss pattern recognition and medical diagnosis. Turkarslan et al. [49] presented pattern recognition-based q-ROF information. Ejegwa et al. [50] discussed the use of q-ROFSs in employment procedure and diagnosis using correlation coefficient-based MCDM under q-ROFSs. Hussain et al. [51] developed a technique of distance measure for q-ROFSs using Hausdorff metric, which was used to construct similarity metric for q-ROFSs. Du [52] presented the Minkowski-type distance method for q-ROFSs and utilized it for MCDM problem. Peng and Liu [53] developed some distance, similarity and inclusion metrics for q-ROFSs and applied the methods in cases of clustering, pattern recognition, and medical diagnosis. In [54], a distance method for q-ROFSs was constructed and used to discuss MCDM for supplier selection and some applications of q-ROF distance method were explored in clustering analysis, pattern recognition, smart manufacturing, and investment analysis [55]. In the area of transportation systems, Dutta et al. [56] by combining the concept of distance metrics and q-ROFSs, a novel distance metric under q-ROFSs was developed with an application to solving transportation problems. The idea of q-ROF distance methods have been explored based on logarithmic functions. In [57], Wang et al. presented two logarithmic-based q-ROF distance metrics inspired by the Jensen-Shannon divergence and applied the methods to practical scenarios of pattern recognition and MCDM. Suri et al. [58] proposed a distinct and robust logarithmic-based q-ROF distance metric to discuss selection of financial investment funds.

While numerous studies have been explored under IFSs based on TOPSIS [11–16], PFSs based on TOPSIS [22, 23] and FFSs based on TOPSIS [30–35], the power constraint on the sum of the MD and NMD of these sets limits the expressive supremacy of the techniques when judgments are conflicting. Existing q-ROFSs-TOPSIS variants show improved discrimination, yet some logarithmic-based distance metrics omit the indeterminacy component like in Suri et al. [58], and other ignores the effect of the universe's cardinality like in [57], both of which can yield bias rankings when choice alternatives are close. Motivated by these limitation, this study aims to propose two logarithmic-based q-ROF distance metrics by modifying the techniques in [57] to improve reliability and precision of distance outputs. Owing to the impact of transportation system to economic growth and the role of VSP in an efficient transport system, the new logarithmic-based q-ROF distance metrics are applied to VSP to enhance efficient transportation system based on TOPSIS approach. Singh and Yadav [59] presented a transportation problem under IFSs and a discussion on transportation was carried out under interval-valued IFSs [60]. The problems of transportation were discussed in [28, 61] under FFSs based on score functions. In addition, another studies on transportation were carried out, where the uncertainties of the transportation problems were addressed [62, 63]. In all these studies, none of them studied VSP in q-ROF framework based on distance metric. Because of the reliability of q-ROFSs in resolving imprecision, it is advantageous to discuss VSP using logarithmic-based q-ROFDM for effective results.

## 1.2 *Motivation and contributions*

Owing to the vital role q-ROFDMs play in solving real-world decision-making problems in complex fields like VSP, this work is motivated by the need to overcome key gaps in the existing distance models and application potentials: The research gaps include:

i. Traditional FSs, IFSs, PFSs and FFSs struggle to handle CDM scenarios where the sum of the DM and NDM exceeds 1.

ii. Existing logarithmic-based q-ROFDMs ignore the indeterminacy component of q-ROFSs, which is critical for handling uncertainty.

iii. Existing logarithmic-based q-ROFDMs lack cardinality integration, reducing accuracy in distance calculations.

iv. Classical MCDM techniques fail to rank alternatives with close evaluations.

v. Vehicle selection involves conflicting criteria (fuel efficiency, cost and reliability among others) with inherent vagueness.

To address these gaps, this study:

i. Modify existing q-ROFDMs to improve precision and reliability by including the indeterminacy components of q-ROFSs.

ii. Presents logarithmic-based q-ROFDMs that improve precision and reliability by normalizing the cardinality of the universe of discourse.

iii. Rigorously proves the new metrics to satisfy non-degeneracy, symmetry, boundedness and triangular inequality.

iv. Integrates the new distance metrics with the TOPSIS technique to handle real-world uncertainties in VSP using expert derived linguistic variables to rank vehicle alternatives, showcasing practical utility.

The proposed metrics bridge the gap between theoretical robustness and real-world applicability in a vehicle selection case study, offering a more flexible and precise tool for complex MCDM problems. The innovation in this study lies in the novel construction of two logarithmic-based q-ROFDMs that incorporate all the three defining parameters of q-ROFSs, which existing logarithmic-based metrics have failed to incorporate resulting in reduced accuracy and weaker interpretability in MCDM problems. By explicitly embedding all the three components, the proposed metrics minimizes information loss, finer discrimination among alternatives in CDM scenarios and yield a more complete and robust quantification of uncertainty. The TOPSIS approach is adopted in this study for the VSP because among the several techniques in the MCDM domain, it is the highly prominent and adopted MCDM method owing to its edge of selecting the best alternative. In fact, the TOPSIS scheme helps in completing decision-making realistically because of its inherent ability of ranking alternatives without ambiguity.

The rest of this work is structured as follows; Section 2 reviews some basics concepts relating to q-ROFSs and existing logarithmic-based q-ROF distance metrics. Section 3 discusses the two new logarithmic-based q-ROF distance metrics and analyses their metric properties. Section 4 gives the application of the logarithmic-based q-ROF distance metrics in VSP based on TOPSIS technique. In addition, both sensitivity and comparative analyses are presented in Section 4 for the purpose of validation. Finally, Section 5 concludes the article with suggestions for further investigations.

## 2. Preliminaries

In this section, we review some of the fundamental concepts relating to q-ROFSs, distance metrics and some existing logarithmic-based q-ROF distance metrics. All of which is aimed at enhancing our discussion in the upcoming sections. Throughout this study, let $\mathscr{R} = \{r_1, r_2, \ldots, r_x\}$ be a nonempty set with $x$ as its cardinality, and let q-ROFS($\mathscr{R}$) be the collection of all the possible q-ROFSs in $\mathscr{R}$.

**Definition 1** ([5]) A fuzzy set $\mathfrak{D}$ in $\mathscr{R}$ is described as follows: $\mathfrak{D} = \{r_i, \mathfrak{D}_{\mathfrak{m}}(r_i) \mid r_i \in \mathscr{R}\}$, where $\mathfrak{D}_{\mathfrak{m}}\colon \mathscr{R} \longrightarrow [0, 1]$ is the membership function representing the MD, for $i = 1, 2, \ldots, x$.

**Definition 2** ([6]) An IFS $\mathfrak{E}$ in $\mathscr{R}$ is described as follows: $\mathfrak{E} = \{(r_i, \mathfrak{E}_{\mathfrak{m}}(r_i), \mathfrak{E}_{\mathfrak{n}}(r_i)) \mid r_i \in \mathscr{R}\}$, where $\mathfrak{E}_{\mathfrak{m}}\colon \mathscr{R} \longrightarrow [0, 1]$ and $\mathfrak{E}_{\mathfrak{n}}\colon \mathscr{R} \longrightarrow [0, 1]$, such that $0 \leq \mathfrak{E}_{\mathfrak{m}}(r_i) + \mathfrak{E}_{\mathfrak{n}}(r_i) \leq 1$, for each $r_i \in \mathscr{R}$, where $\mathfrak{E}_{\mathfrak{m}}(r_i)$ is the MD and $\mathfrak{E}_{\mathfrak{n}}(r_i)$ is the NMD. The HM, denoted as $\mathfrak{E}_{\mathfrak{h}}(r_i)$ and described by $\mathfrak{E}_{\mathfrak{h}}(r_i) \in [0, 1] = 1 - (\mathfrak{E}_{\mathfrak{m}}(r_i) + \mathfrak{E}_{\mathfrak{n}}(r_i))$ is the grade of indeterminacy of $r_i \in \mathscr{R}$ to $\mathfrak{E}$.

**Definition 3** ([7]) A PFS $\mathfrak{F}$ in $\mathscr{R}$ is described as follows: $\mathfrak{F} = \{(r_i, \mathfrak{F}_{\mathfrak{m}}(r_i), \mathfrak{F}_{\mathfrak{n}}(r_i)) \mid r_i \in \mathscr{R}\}$, where $\mathfrak{F}_{\mathfrak{m}}\colon \mathscr{R} \longrightarrow [0, 1]$ and $\mathfrak{F}_{\mathfrak{n}}\colon \mathscr{R} \longrightarrow [0, 1]$, such that $0 \leq \mathfrak{F}^2_{\mathfrak{m}}(r_i) + \mathfrak{F}^2_{\mathfrak{n}}(r_i) \leq 1$, for all $r_i \in \mathscr{R}$, where $\mathfrak{F}_{\mathfrak{m}}(r_i)$ is the MD and $\mathfrak{F}_{\mathfrak{n}}(r_i)$ is the NMD. The HM denoted as $\mathfrak{F}_{\mathfrak{h}}(r_i)$ and described by $\mathfrak{F}_{\mathfrak{h}}(r_i) \in [0, 1] = \left(1 - \left(\mathfrak{F}^2_{\mathfrak{m}}(r_i) + \mathfrak{F}^2_{\mathfrak{n}}(r_i)\right)\right)^{\frac{1}{2}}$ is the grade of indeterminacy of $r_i \in \mathscr{R}$ to $\mathfrak{F}$.

**Definition 4** ([8]) A FFS $\mathfrak{H}$ in $\mathscr{R}$ is described as follows: $\mathfrak{H} = \{(r_i, \mathfrak{H}_{\mathfrak{m}}(r_i), \mathfrak{H}_{\mathfrak{n}}(r_i)) \mid r_i \in \mathscr{R}\}$, where $\mathfrak{H}_{\mathfrak{m}}\colon \mathscr{R} \longrightarrow [0, 1]$ and $\mathfrak{H}_{\mathfrak{n}}\colon \mathscr{R} \longrightarrow [0, 1]$, such that $0 \leq \mathfrak{H}^3_{\mathfrak{m}}(r_i) + \mathfrak{H}^3_{\mathfrak{n}}(r_i) \leq 1$, for all $r_i \in \mathscr{R}$, where $\mathfrak{H}_{\mathfrak{m}}(r_i)$ is the MD and $\mathfrak{H}_{\mathfrak{n}}(r_i)$ is the NMD. The HM denoted as $\mathfrak{H}_{\mathfrak{h}}(r_i)$ and defined as $\mathfrak{H}_{\mathfrak{h}}(r_i) \in [0, 1] = \left(1 - \left(\mathfrak{H}^3_{\mathfrak{m}}(r_i) + \mathfrak{H}^3_{\mathfrak{n}}(r_i)\right)\right)^{\frac{1}{3}}$ is the grade of indeterminacy of $r_i \in \mathscr{R}$ to $\mathfrak{H}$.

**Definition 5** ([9]) A q-ROFS $\mathfrak{J}$ in $\mathscr{R}$ is described as follows: $\mathfrak{J} = \{(r_i, \mathfrak{J}_{\mathfrak{m}}(r_i), \mathfrak{J}_{\mathfrak{n}}(r_i)) \mid r_i \in \mathscr{R}\}$, where $\mathfrak{J}_{\mathfrak{m}}\colon \mathscr{R} \to [0, 1]$ is the MD and $\mathfrak{J}_{\mathfrak{n}}\colon \mathscr{R} \to [0, 1]$ is the NMD, such that $0 \leq \mathfrak{J}^q_{\mathfrak{m}}(r_i) + \mathfrak{J}^q_{\mathfrak{n}}(r_i) \leq 1$ for all $r_i \in \mathscr{R}$ and $q \geq 1$. The HM denoted as $\mathfrak{J}_{\mathfrak{h}}(r_i)$ defined by $\mathfrak{J}_{\mathfrak{h}}(r_i) \in [0, 1] = \left(1 - \mathfrak{J}^q_{\mathfrak{m}}(r_i) - \mathfrak{J}^q_{\mathfrak{n}}(r_i)\right)^{\frac{1}{q}}$ is the grade of indeterminacy of $r_i \in \mathscr{R}$ to $\mathfrak{J}$. For simplicity, $\mathfrak{J} = (\mathfrak{J}_{\mathfrak{m}}(r_i), \mathfrak{J}_{\mathfrak{n}}(r_i))$ represent the q-Rung Orthopair Fuzzy Number (q-ROFN).

**Definition 6** ([56]) Let $\mathfrak{J}, \mathfrak{L}, \mathfrak{M} \in$ q-ROFS($\mathscr{R}$) and $\mathbb{D}$: q-ROFSs($\mathscr{R}$)$\times$ q-ROFSs($\mathscr{R}$) $\to [0, 1]$, then $\mathbb{D}(\mathfrak{J}, \mathfrak{L})$ is called a q-ROF distance metric between $\mathfrak{J}$ and $\mathfrak{L}$ if it satisfies the following:

i. Boundedness; $0 \leq \mathbb{D}(\mathfrak{J}, \mathfrak{L}) \leq 1$,

ii. Non-degeneracy; $\mathbb{D}(\mathfrak{J}, \mathfrak{L}) = 0$ if and only if $\mathfrak{J} = \mathfrak{L}$,
iii. Symmetry; $\mathbb{D}(\mathfrak{J}, \mathfrak{L}) = \mathbb{D}(\mathfrak{L}, \mathfrak{J})$,
iv. Triangular inequality; $\mathbb{D}(\mathfrak{M}, \mathfrak{J}) + \mathbb{D}(\mathfrak{J}, \mathfrak{L}) \geq \mathbb{D}(\mathfrak{M}, \mathfrak{L})$.

### 2.1 *Existing logarithmic-based q-ROF distance methods*

Here, we outline some existing logarithmic-based q-ROF Distance Methods (q-ROFDMs).

Given two q-ROFSs $\mathfrak{J}, \mathfrak{L} \in$ q-ROFS$(\mathscr{R})$, the existing q-ROFDMs between $\mathfrak{J}$ and $\mathfrak{L}$ in [57] are presented in (1) and (2).

$$\mathbb{D}_{We1}(\mathfrak{J}, \mathfrak{L}) = \left[ \frac{1}{2} \sum_{i=1}^{x} \begin{pmatrix} \mathfrak{J}_{\mathfrak{m}}^{q}(r_i) \log\left(\frac{2\mathfrak{J}_{\mathfrak{m}}^{q}(r_i)}{\mathfrak{J}_{\mathfrak{m}}^{q}(r_i)+\mathfrak{L}_{\mathfrak{m}}^{q}(r_i)}\right) + \mathfrak{L}_{\mathfrak{m}}^{q}(r_i) \log\left(\frac{2\mathfrak{L}_{\mathfrak{m}}^{q}(r_i)}{\mathfrak{J}_{\mathfrak{m}}^{q}(r_i)+\mathfrak{L}_{\mathfrak{m}}^{q}(r_i)}\right) + \\ \mathfrak{J}_{\mathfrak{n}}^{q}(r_i) \log\left(\frac{2\mathfrak{J}_{\mathfrak{n}}^{q}(r_i)}{\mathfrak{J}_{\mathfrak{n}}^{q}(r_i)+\mathfrak{L}_{\mathfrak{n}}^{q}(r_i)}\right) + \mathfrak{L}_{\mathfrak{n}}^{q}(r_i) \log\left(\frac{2\mathfrak{L}_{\mathfrak{n}}^{q}(r_i)}{\mathfrak{J}_{\mathfrak{n}}^{q}(r_i)+\mathfrak{L}_{\mathfrak{n}}^{q}(r_i)}\right) \end{pmatrix} \right]^{\frac{1}{2}}, \tag{1}$$

$$\mathbb{D}_{We2}(\mathfrak{J}, \mathfrak{L}) = \left[ \frac{1}{2} \sum_{i=1}^{x} \begin{pmatrix} \mathfrak{J}_{\mathfrak{m}}^{q}(r_i) \log\left(\frac{2\mathfrak{J}_{\mathfrak{m}}^{q}(r_i)}{\mathfrak{J}_{\mathfrak{m}}^{q}(r_i)+\mathfrak{L}_{\mathfrak{m}}^{q}(r_i)}\right) + \mathfrak{L}_{\mathfrak{m}}^{q}(r_i) \log\left(\frac{2\mathfrak{L}_{\mathfrak{m}}^{q}(r_i)}{\mathfrak{J}_{\mathfrak{m}}^{q}(r_i)+\mathfrak{L}_{\mathfrak{m}}^{q}(r_i)}\right) + \\ \mathfrak{J}_{\mathfrak{n}}^{q}(r_i) \log\left(\frac{2\mathfrak{J}_{\mathfrak{n}}^{q}(r_i)}{\mathfrak{J}_{\mathfrak{n}}^{q}(r_i)+\mathfrak{L}_{\mathfrak{n}}^{q}(r_i)}\right) + \mathfrak{L}_{\mathfrak{n}}^{q}(r_i) \log\left(\frac{2\mathfrak{L}_{\mathfrak{n}}^{q}(r_i)}{\mathfrak{J}_{\mathfrak{n}}^{q}(r_i)+\mathfrak{L}_{\mathfrak{n}}^{q}(r_i)}\right) + \\ \mathfrak{J}_{\mathfrak{h}}^{q}(r_i) \log\left(\frac{2\mathfrak{J}_{\mathfrak{h}}^{q}(r_i)}{\mathfrak{J}_{\mathfrak{h}}^{q}(r_i)+\mathfrak{L}_{\mathfrak{h}}^{q}(r_i)}\right) + \mathfrak{L}_{\mathfrak{h}}^{q}(r_i) \log\left(\frac{2\mathfrak{L}_{\mathfrak{h}}^{q}(r_i)}{\mathfrak{J}_{\mathfrak{h}}^{q}(r_i)+\mathfrak{L}_{\mathfrak{h}}^{q}(r_i)}\right) \end{pmatrix} \right]^{\frac{1}{2}}. \tag{2}$$

In (1), the grade of indeterminacy is not considered, while in both (1) and (2), the cardinality of $\mathscr{R}$ is not incorporated in the mathematical framework. All which can alter the accuracy and efficiency of their distance outputs.

In [58], another logarithmic-based q-RODM was presented, which is (3).

$$\mathbb{D}_{Se}(\mathfrak{J}, \mathfrak{L}) = \frac{1}{2x\log 2} \sum_{i=1}^{x} \begin{bmatrix} \left(\mathfrak{J}_{\mathfrak{m}}^{q}(r_i) - \mathfrak{L}_{\mathfrak{m}}^{q}(r_i)\right) \log \frac{1+\mathfrak{J}_{\mathfrak{m}}^{q}(r_i)}{1+\mathfrak{L}_{\mathfrak{m}}^{q}(r_i)} + \\ \left(\mathfrak{J}_{\mathfrak{n}}^{q}(r_i) - \mathfrak{L}_{\mathfrak{n}}^{q}(r_i)\right) \log \frac{1+\mathfrak{J}_{\mathfrak{n}}^{q}(r_i)}{1+\mathfrak{L}_{\mathfrak{n}}^{q}(r_i)} \end{bmatrix}. \tag{3}$$

In (3), the grade of indeterminacy is not considered, which can lead to error of omission.

## 3. The new logarithmic-based q-ROFDMs

In this section, we present the new logarithmic-based q-ROFDMs along with their metric properties. The new logarithmic-based q-ROFDMs modify the techniques in [57] to enhance precision and increase interpretation reliability.

**Definition 7** Let $\mathfrak{J} = \{(r_i, \mathfrak{J}_{\mathfrak{m}}(r_i), \mathfrak{J}_{\mathfrak{n}}(r_i)) \mid r_i \in \mathscr{R}\}$ and $\mathfrak{L} = \{(r_i, \mathfrak{L}_{\mathfrak{m}}(r_i), \mathfrak{L}_{\mathfrak{n}}(r_i)) \mid r_i \in \mathscr{R}\}$ be two q-ROFSs in $\mathscr{R} = \{r_1, r_2, \ldots, r_x\}$, then the new logarithmic-based q-ROFDMs are defined as follows:

$$\mathbb{D}_1(\mathfrak{J}, \mathfrak{L}) = \sqrt{\frac{1}{3x}\left(\mathbb{d}_{\mathfrak{m}}^{1}(\mathfrak{J}, \mathfrak{L}) + \mathbb{d}_{\mathfrak{n}}^{1}(\mathfrak{J}, \mathfrak{L}) + \mathbb{d}_{\mathfrak{h}}^{1}(\mathfrak{J}, \mathfrak{L})\right)}, \tag{4}$$

$$\mathbb{D}_2(\mathfrak{J},\, \mathfrak{L}) = \frac{1}{3x}\left(\mathbb{d}^1_{\mathfrak{m}}(\mathfrak{J},\, \mathfrak{L}) + \mathbb{d}^1_{\mathfrak{n}}(\mathfrak{J},\, \mathfrak{L}) + \mathbb{d}^1_{\mathfrak{h}}(\mathfrak{J},\, \mathfrak{L})\right). \tag{5}$$

where

$$\left.\begin{aligned}
\mathbb{d}^1_{\mathfrak{m}}(\mathfrak{J},\, \mathfrak{L}) &= \sum_{i=1}^{x}\left[\mathfrak{J}^q_{\mathfrak{m}}(r_i)\log\left(\frac{2\mathfrak{J}^q_{\mathfrak{m}}(r_i)}{\mathfrak{J}^q_{\mathfrak{m}}(r_i)+\mathfrak{L}^q_{\mathfrak{m}}(r_i)}\right) + \mathfrak{L}^q_{\mathfrak{m}}(r_i)\log\left(\frac{2\mathfrak{L}^q_{\mathfrak{m}}(r_i)}{\mathfrak{J}^q_{\mathfrak{m}}(r_i)+\mathfrak{L}^q_{\mathfrak{m}}(r_i)}\right)\right] \\
\mathbb{d}^1_{\mathfrak{n}}(\mathfrak{J},\, \mathfrak{L}) &= \sum_{i=1}^{x}\left[\mathfrak{J}^q_{\mathfrak{n}}(r_i)\log\left(\frac{2\mathfrak{J}^q_{\mathfrak{n}}(r_i)}{\mathfrak{J}^q_{\mathfrak{n}}(r_i)+\mathfrak{L}^q_{\mathfrak{n}}(r_i)}\right) + \mathfrak{L}^q_{\mathfrak{n}}(r_i)\log\left(\frac{2\mathfrak{L}^q_{\mathfrak{n}}(r_i)}{\mathfrak{J}^q_{\mathfrak{n}}(r_i)+\mathfrak{L}^q_{\mathfrak{n}}(r_i)}\right)\right] \\
\mathbb{d}^1_{\mathfrak{h}}(\mathfrak{J},\, \mathfrak{L}) &= \sum_{i=1}^{x}\left[\mathfrak{J}^q_{\mathfrak{h}}(r_i)\log\left(\frac{2\mathfrak{J}^q_{\mathfrak{h}}(r_i)}{\mathfrak{J}^q_{\mathfrak{h}}(r_i)+\mathfrak{L}^q_{\mathfrak{h}}(r_i)}\right) + \mathfrak{L}^q_{\mathfrak{h}}(r_i)\log\left(\frac{2\mathfrak{L}^q_{\mathfrak{h}}(r_i)}{\mathfrak{J}^q_{\mathfrak{h}}(r_i)+\mathfrak{L}^q_{\mathfrak{h}}(r_i)}\right)\right]
\end{aligned}\right\} \tag{6}$$

Next, we showing that the new q-ROFDMs satisfy the properties of a distance metric as given in Definition 6.

**Theorem 1** Given three q-ROFSs, $\mathfrak{J}$, $\mathfrak{L}$, and $\mathfrak{M}$ in $\mathscr{R} = \{r_1,\, r_2,\, \ldots,\, r_x\}$, then, $\mathbb{D}_1(\mathfrak{J},\, \mathfrak{L})$ and $\mathbb{D}_2(\mathfrak{J},\, \mathfrak{L})$ satisfy the distance metric properties.

**Proof.** Substituting (6) into (4) and (5), we obtain:

$$\mathbb{D}_1(\mathfrak{J},\, \mathfrak{L}) = \sqrt{\frac{1}{3x}\sum_{i=1}^{x}\left(\begin{array}{c}
\mathfrak{J}^q_{\mathfrak{m}}(r_i)\log\left(\frac{2\mathfrak{J}^q_{\mathfrak{m}}(r_i)}{\mathfrak{J}^q_{\mathfrak{m}}(r_i)+\mathfrak{L}^q_{\mathfrak{m}}(r_i)}\right) + \mathfrak{L}^q_{\mathfrak{m}}(r_i)\log\left(\frac{2\,\mathfrak{L}^q_{\mathfrak{m}}(r_i)}{\mathfrak{J}^q_{\mathfrak{m}}(r_i)+\mathfrak{L}^q_{\mathfrak{m}}(r_i)}\right) + \\
\mathfrak{J}^q_{\mathfrak{n}}(r_i)\log\left(\frac{2\mathfrak{J}^q_{\mathfrak{n}}(r_i)}{\mathfrak{J}^q_{\mathfrak{n}}(r_i)+\mathfrak{L}^q_{\mathfrak{n}}(r_i)}\right) + \mathfrak{L}^q_{\mathfrak{n}}(r_i)\log\left(\frac{2\,\mathfrak{L}^q_{\mathfrak{n}}(r_i)}{\mathfrak{J}^q_{\mathfrak{n}}(r_i)+\mathfrak{L}^q_{\mathfrak{n}}(r_i)}\right) + \\
\mathfrak{J}^q_{\mathfrak{h}}(r_i)\log\left(\frac{2\mathfrak{J}^q_{\mathfrak{h}}(r_i)}{\mathfrak{J}^q_{\mathfrak{h}}(r_i)+\mathfrak{L}^q_{\mathfrak{h}}(r_i)}\right) + \mathfrak{L}^q_{\mathfrak{h}}(r_i)\log\left(\frac{2\,\mathfrak{L}^q_{\mathfrak{h}}(r_i)}{\mathfrak{J}^q_{\mathfrak{h}}(r_i)+\mathfrak{L}^q_{\mathfrak{h}}(r_i)}\right)
\end{array}\right)},$$

$$\mathbb{D}_2(\mathfrak{J},\, \mathfrak{L}) = \frac{1}{3x}\sum_{i=1}^{x}\left(\begin{array}{c}
\mathfrak{J}^q_{\mathfrak{m}}(r_i)\log\left(\frac{2\mathfrak{J}^q_{\mathfrak{m}}(r_i)}{\mathfrak{J}^q_{\mathfrak{m}}(r_i)+\mathfrak{L}^q_{\mathfrak{m}}(r_i)}\right) + \mathfrak{L}^q_{\mathfrak{m}}(r_i)\log\left(\frac{2\,\mathfrak{L}^q_{\mathfrak{m}}(r_i)}{\mathfrak{J}^q_{\mathfrak{m}}(r_i)+\mathfrak{L}^q_{\mathfrak{m}}(r_i)}\right) + \\
\mathfrak{J}^q_{\mathfrak{n}}(r_i)\log\left(\frac{2\mathfrak{J}^q_{\mathfrak{n}}(r_i)}{\mathfrak{J}^q_{\mathfrak{n}}(r_i)+\mathfrak{L}^q_{\mathfrak{n}}(r_i)}\right) + \mathfrak{L}^q_{\mathfrak{n}}(r_i)\log\left(\frac{2\,\mathfrak{L}^q_{\mathfrak{n}}(r_i)}{\mathfrak{J}^q_{\mathfrak{n}}(r_i)+\mathfrak{L}^q_{\mathfrak{n}}(r_i)}\right) + \\
\mathfrak{J}^q_{\mathfrak{h}}(r_i)\log\left(\frac{2\mathfrak{J}^q_{\mathfrak{h}}(r_i)}{\mathfrak{J}^q_{\mathfrak{h}}(r_i)+\mathfrak{L}^q_{\mathfrak{h}}(r_i)}\right) + \mathfrak{L}^q_{\mathfrak{h}}(r_i)\log\left(\frac{2\,\mathfrak{L}^q_{\mathfrak{h}}(r_i)}{\mathfrak{J}^q_{\mathfrak{h}}(r_i)+\mathfrak{L}^q_{\mathfrak{h}}(r_i)}\right)
\end{array}\right).$$

If $x = 1$, then $\mathscr{R} = \{r\}$ and thus:

$$\mathbb{D}_1(\mathfrak{I},\,\mathfrak{L}) = \left[\frac{1}{3}\begin{pmatrix} \mathfrak{I}^q_{\mathfrak{m}}(r)\log\left(\frac{2\mathfrak{I}^q_{\mathfrak{m}}(r)}{\mathfrak{I}^q_{\mathfrak{m}}(r)+\mathfrak{L}^q_{\mathfrak{m}}(r)}\right)+\mathfrak{L}^q_{\mathfrak{m}}(r)\log\left(\frac{2\mathfrak{L}^q_{\mathfrak{m}}(r)}{\mathfrak{I}^q_{\mathfrak{m}}(r)+\mathfrak{L}^q_{\mathfrak{m}}(r)}\right)+ \\ \mathfrak{I}^q_{\mathfrak{n}}(r)\log\left(\frac{2\mathfrak{I}^q_{\mathfrak{n}}(r)}{\mathfrak{I}^q_{\mathfrak{n}}(r)+\mathfrak{L}^q_{\mathfrak{n}}(r)}\right)+\mathfrak{L}^q_{\mathfrak{n}}(r)\log\left(\frac{2\mathfrak{L}^q_{\mathfrak{n}}(r)}{\mathfrak{I}^q_{\mathfrak{n}}(r)+\mathfrak{L}^q_{\mathfrak{n}}(r)}\right)+ \\ \mathfrak{I}^q_{\mathfrak{h}}(r)\log\left(\frac{2\mathfrak{I}^q_{\mathfrak{h}}(r)}{\mathfrak{I}^q_{\mathfrak{h}}(r)+\mathfrak{L}^q_{\mathfrak{h}}(r)}\right)+\mathfrak{L}^q_{\mathfrak{h}}(r)\log\left(\frac{2\mathfrak{L}^q_{\mathfrak{h}}(r)}{\mathfrak{I}^q_{\mathfrak{h}}(r)+\mathfrak{L}^q_{\mathfrak{h}}(r)}\right) \end{pmatrix}\right]^{\frac{1}{2}}.$$

i. For the proof of boundedness: $0 \le \mathbb{D}_1(\mathfrak{I},\,\mathfrak{L}) \le 1$, we get

$$\mathbb{D}_1(\mathfrak{I},\;\mathfrak{L}) = \left[\frac{1}{3}\begin{pmatrix} \mathfrak{I}^q_{\mathfrak{m}}(r)\log\left(\frac{2\mathfrak{I}^q_{\mathfrak{m}}(r)}{\mathfrak{I}^q_{\mathfrak{m}}(r_i)+\mathfrak{L}^q_{\mathfrak{m}}(r)}\right)+\mathfrak{L}^q_{\mathfrak{m}}(r)\log\left(\frac{2\,\mathfrak{L}^q_{\mathfrak{m}}(r)}{\mathfrak{I}^q_{\mathfrak{m}}(r)+\mathfrak{L}^q_{\mathfrak{m}}(r)}\right)+ \\ \mathfrak{I}^q_{\mathfrak{n}}(r)\log\left(\frac{2\mathfrak{I}^q_{\mathfrak{n}}(r)}{\mathfrak{I}^q_{\mathfrak{n}}(r)+\mathfrak{L}^q_{\mathfrak{n}}(r)}\right)+\mathfrak{L}^q_{\mathfrak{n}}(r)\log\left(\frac{2\,\mathfrak{L}^q_{\mathfrak{n}}(r)}{\mathfrak{I}^q_{\mathfrak{n}}(r)+\mathfrak{L}^q_{\mathfrak{n}}(r)}\right)+ \\ \mathfrak{I}^q_{\mathfrak{h}}(r)\log\left(\frac{2\mathfrak{I}^q_{\mathfrak{h}}(r)}{\mathfrak{I}^q_{\mathfrak{h}}(r)+\mathfrak{L}^q_{\mathfrak{h}}(r)}\right)+\mathfrak{L}^q_{\mathfrak{h}}(r)\log\left(\frac{2\,\mathfrak{L}^q_{\mathfrak{h}}(r)}{\mathfrak{I}^q_{\mathfrak{h}}(r)+\mathfrak{L}^q_{\mathfrak{h}}(r)}\right) \end{pmatrix}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{3}\begin{pmatrix} \left(\mathfrak{I}^q_{\mathfrak{m}}(r)+\mathfrak{L}^q_{\mathfrak{m}}(r)\right)\begin{pmatrix} \frac{\mathfrak{I}^q_{\mathfrak{m}}(r)}{\mathfrak{I}^q_{\mathfrak{m}}(r)+\mathfrak{L}^q_{\mathfrak{m}}(r)}\log\frac{2\mathfrak{I}^q_{\mathfrak{m}}(r)}{\mathfrak{I}^q_{\mathfrak{m}}(r_i)+\mathfrak{L}^q_{\mathfrak{m}}(r)}+ \\ \frac{\mathfrak{L}^q_{\mathfrak{m}}(r)}{\mathfrak{I}^q_{\mathfrak{m}}(r)+\mathfrak{L}^q_{\mathfrak{m}}(r)}\log\frac{2\,\mathfrak{L}^q_{\mathfrak{m}}(r)}{\mathfrak{I}^q_{\mathfrak{m}}(r)+\mathfrak{L}^q_{\mathfrak{m}}(r)} \end{pmatrix}+ \\ \left(\mathfrak{I}^q_{\mathfrak{n}}(r)+\mathfrak{L}^q_{\mathfrak{n}}(r)\right)\begin{pmatrix} \frac{\mathfrak{I}^q_{\mathfrak{n}}(r)}{\mathfrak{I}^q_{\mathfrak{n}}(r)+\mathfrak{L}^q_{n}(r)}\log\frac{2\mathfrak{I}^q_{\mathfrak{n}}(r)}{\mathfrak{I}^q_{\mathfrak{n}}(r_i)+\mathfrak{L}^q_{\mathfrak{n}}(r)}+ \\ \frac{\mathfrak{L}^q_{\mathfrak{n}}(r)}{\mathfrak{I}^q_{\mathfrak{n}}(r)+\mathfrak{L}^q_{\mathfrak{n}}(r)}\log\frac{2\,\mathfrak{L}^q_{\mathfrak{n}}(r)}{\mathfrak{I}^q_{\mathfrak{n}}(r)+\mathfrak{L}^q_{\mathfrak{n}}(r)} \end{pmatrix}+ \\ \left(\mathfrak{I}^q_{\mathfrak{h}}(r)+\mathfrak{L}^q_{\mathfrak{h}}(r)\right)\begin{pmatrix} \frac{\mathfrak{I}^q_{\mathfrak{h}}(r)}{\mathfrak{I}^q_{\mathfrak{h}}(r)+\mathfrak{L}^q_{\mathfrak{h}}(r)}\log\frac{2\mathfrak{I}^q_{\mathfrak{h}}(r)}{\mathfrak{I}^q_{\mathfrak{h}}(r_i)+\mathfrak{L}^q_{\mathfrak{h}}(r)}+ \\ \frac{\mathfrak{L}^q_{\mathfrak{h}}(r)}{\mathfrak{I}^q_{\mathfrak{h}}(r)+\mathfrak{L}^q_{\mathfrak{h}}(r)}\log\frac{2\,\mathfrak{L}^q_{\mathfrak{h}}(r)}{\mathfrak{I}^q_{\mathfrak{h}}(r)+\mathfrak{L}^q_{\mathfrak{h}}(r)} \end{pmatrix} \end{pmatrix}\right]^{\frac{1}{2}}.$$

Each pair of term is of the form:

$$a\log\frac{2a}{a+b}+b\log\frac{2b}{a+b},$$

which is the Jensen-Shannon divergence component between two values $a$ and $b$, which is symmetric, bounded and always non-negative. Hence, $0 \le \mathbb{D}_1(\mathfrak{I},\,\mathfrak{L})$.

We denote, $a_1 = \mathfrak{I}^q_{\mathfrak{m}}(r)$, $a_2 = \mathfrak{I}^q_{\mathfrak{n}}(r)$, $a_3 = \mathfrak{I}^q_{\mathfrak{h}}(r)$, $b_1 = \mathfrak{L}^q_{\mathfrak{m}}(r)$, $b_2 = \mathfrak{L}^q_{\mathfrak{n}}(r)$ and $b_3 = \mathfrak{L}^q_{\mathfrak{h}}(r)$. So that the total divergence becomes:

$$\mathbb{D}_1(\mathfrak{J},\, \mathfrak{L}) = \left[\frac{1}{3}\sum_{i=1}^{3}\left(a_i \log\left(\frac{2a_i}{a_i+b_i}\right) + b_i \log\left(\frac{2b_i}{a_i+b_i}\right)\right)\right]^{\frac{1}{2}}.$$

To show the upper bound is 1, we introduce the midpoint $m_i$ of the points $a_i$, $b_i$ as;

$$m_i = \frac{a_i+b_i}{2}.$$

Then,

$$\frac{2a_i}{a_i+b_i} = \frac{a_i}{m_i}, \quad \frac{2b_i}{a_i+b_i} = \frac{b_i}{m_i}.$$

Then each component becomes:

$$a_i \log\left(\frac{a_i}{m_i}\right) + b_i \log\left(\frac{b_i}{m_i}\right) = 2m_i\left[\frac{a_i}{a_i+b_i}\log\left(\frac{a_i}{m_i}\right) + \frac{b_i}{a_i+b_i}\log\left(\frac{b_i}{m_i}\right)\right],$$

This is a weighted average of Kullback-Leibler divergence and simplifies to:

$$2m_i \cdot H\left(\frac{a_i}{a_i+b_i}, \frac{b_i}{a_i+b_i}\right),$$

where $H(p,\, q)$ is the Shannon entropy function. So, the total expression becomes;

$$\mathbb{D}_1(\mathfrak{J},\, \mathfrak{L}) = \left[\frac{1}{3}\sum_{i=1}^{3} 2m_i \cdot H\left(\frac{a_i}{a_i+b_i}, \frac{b_i}{a_i+b_i}\right)\right]^{\frac{1}{2}}.$$

Since entropy $H(p,\, 1-p) \le \log 2$, and $m_i \le 1$, then;

$$2m_i \cdot H\left(\frac{a_i}{a_i+b_i}, \frac{b_i}{a_i+b_i}\right) \le 2\log 2.$$

Thus, the sum inside the square root is bounded above and hence;

$$\mathbb{D}_1(\mathfrak{J},\, \mathfrak{L}) \le \left[\frac{1}{3}\cdot 3 \cdot \log 4\right]^{\frac{1}{2}} = [\log 4]^{\frac{1}{2}} = \sqrt{2}.$$

However, because q-ROFSs parameters are constrained i.e $\mathfrak{J}^q_{\mathfrak{m}}(r)+\mathfrak{L}^q_{\mathfrak{n}}(r)\leq 1$, then, the effective upper bound is normalized. Hence, $0\leq \mathbb{D}_1(\mathfrak{J},\,\mathfrak{L})\leq 1$ as required.

ii. For the proof of non-degeneracy: $\mathbb{D}_1(\mathfrak{J},\,\mathfrak{L})=0$ if and only if $\mathfrak{J}=\mathfrak{L}$. Suppose $\mathfrak{J}=\mathfrak{L}$, then

$$\mathfrak{J}^q_{\mathfrak{m}}(r)=\mathfrak{L}^q_{\mathfrak{m}}(r),\quad \mathfrak{J}^q_{\mathfrak{n}}(r)=\mathfrak{L}^q_{\mathfrak{n}}(r),\quad \text{and}\quad \mathfrak{J}^q_{\mathfrak{h}}(r)+\mathfrak{L}^q_{\mathfrak{h}}(r).$$

Thus

$$\mathbb{D}_1(\mathfrak{J},\,\mathfrak{L})=\left[\frac{1}{3}\left(\begin{array}{c}\mathfrak{J}^q_{\mathfrak{m}}(r)\log\left(\frac{2\mathfrak{J}^q_{\mathfrak{m}}(r)}{\mathfrak{J}^q_{\mathfrak{m}}(r)+\mathfrak{L}^q_{\mathfrak{m}}(r)}\right)+\mathfrak{L}^q_{\mathfrak{m}}(r)\log\left(\frac{2\mathfrak{L}^q_{\mathfrak{m}}(r)}{\mathfrak{J}^q_{\mathfrak{m}}(r)+\mathfrak{L}^q_{\mathfrak{m}}(r)}\right)+\\ \mathfrak{J}^q_{\mathfrak{n}}(r)\log\left(\frac{2\mathfrak{J}^q_{\mathfrak{n}}(r)}{\mathfrak{J}^q_{\mathfrak{n}}(r)+\mathfrak{L}^q_{\mathfrak{n}}(r)}\right)+\mathfrak{L}^q_{\mathfrak{n}}(r)\log\left(\frac{2\mathfrak{L}^q_{\mathfrak{n}}(r)}{\mathfrak{J}^q_{\mathfrak{n}}(r)+\mathfrak{L}^q_{\mathfrak{n}}(r)}\right)+\\ \mathfrak{J}^q_{\mathfrak{h}}(r)\log\left(\frac{2\mathfrak{J}^q_{\mathfrak{h}}(r)}{\mathfrak{J}^q_{\mathfrak{h}}(r)+\mathfrak{L}^q_{\mathfrak{h}}(r)}\right)+\mathfrak{L}^q_{\mathfrak{h}}(r)\log\left(\frac{2\mathfrak{L}^q_{\mathfrak{h}}(r)}{\mathfrak{J}^q_{\mathfrak{h}}(r)+\mathfrak{L}^q_{\mathfrak{h}}(r)}\right)\end{array}\right)\right]^{\frac{1}{2}}$$

$$=\left[\frac{1}{3}\left(\begin{array}{c}\mathfrak{J}^q_{\mathfrak{m}}(r)\log\left(\frac{2\mathfrak{J}^q_{\mathfrak{m}}(r)}{\mathfrak{J}^q_{\mathfrak{m}}(r)+\mathfrak{J}^q_{\mathfrak{m}}(r)}\right)+\mathfrak{J}^q_{\mathfrak{m}}(r)\log\left(\frac{2\mathfrak{J}^q_{\mathfrak{m}}(r)}{\mathfrak{J}^q_{\mathfrak{m}}(r)+\mathfrak{J}^q_{\mathfrak{m}}(r)}\right)+\\ \mathfrak{J}^q_{\mathfrak{n}}(r)\log\left(\frac{2\mathfrak{J}^q_{\mathfrak{n}}(r)}{\mathfrak{J}^q_{\mathfrak{n}}(r)+\mathfrak{J}^q_{\mathfrak{n}}(r)}\right)+\mathfrak{J}^q_{\mathfrak{n}}(r)\log\left(\frac{2\mathfrak{J}^q_{\mathfrak{n}}(r)}{\mathfrak{J}^q_{\mathfrak{n}}(r)+\mathfrak{J}^q_{\mathfrak{n}}(r)}\right)+\\ \mathfrak{J}^q_{\mathfrak{h}}(r)\log\left(\frac{2\mathfrak{J}^q_{\mathfrak{h}}(r)}{\mathfrak{J}^q_{\mathfrak{h}}(r)+\mathfrak{J}^q_{\mathfrak{h}}(r)}\right)+\mathfrak{J}^q_{\mathfrak{h}}(r)\log\left(\frac{2\mathfrak{J}^q_{\mathfrak{h}}(r)}{\mathfrak{J}^q_{\mathfrak{h}}(r)+\mathfrak{J}^q_{\mathfrak{h}}(r)}\right)\end{array}\right)\right]^{\frac{1}{2}}$$

$$=\left[\frac{1}{3}\left(\begin{array}{c}\mathfrak{J}^q_{\mathfrak{m}}(r)\log 1+\mathfrak{J}^q_{\mathfrak{m}}(r)\log 1+\\ \mathfrak{J}^q_{\mathfrak{n}}(r)\log 1+\mathfrak{J}^q_{\mathfrak{n}}(r)\log 1+\\ \mathfrak{J}^q_{\mathfrak{h}}(r)\log 1+\mathfrak{J}^q_{\mathfrak{h}}(r)\log 1\end{array}\right)\right]^{\frac{1}{2}}=0,$$

hence,

$$\mathbb{D}_1(\mathfrak{J},\,\mathfrak{L})=0.$$

Conversely, if $\mathbb{D}_1(\mathfrak{J},\,\mathfrak{L})=0$, we have:

$$\mathfrak{J}^q_{\mathfrak{m}}(r)=\mathfrak{L}^q_{\mathfrak{m}}(r),\quad \mathfrak{J}^q_{\mathfrak{n}}(r)=\mathfrak{L}^q_{\mathfrak{n}}(r),\quad \text{and}\quad \mathfrak{J}^q_{\mathfrak{h}}(r)=\mathfrak{L}^q_{\mathfrak{h}}(r)\Longrightarrow \mathfrak{J}_{\mathfrak{m}}(r)=\mathfrak{L}_{\mathfrak{m}}(r),\,\mathfrak{J}_{\mathfrak{n}}(r)=\mathfrak{L}_{\mathfrak{n}}(r),\ \text{and}\ \mathfrak{J}_{\mathfrak{h}}(r)=\mathfrak{L}_{\mathfrak{h}}(r).$$

Hence,

$$\mathfrak{J}=\mathfrak{L}.$$

Therefore, (ii) is proved.

iii. Next, we proof symmetry: $\mathbb{D}_1(\mathfrak{J}, \mathfrak{L}) = \mathbb{D}_1(\mathfrak{L}, \mathfrak{J})$. This is easy because

$$\mathbb{D}_1(\mathfrak{J}, \mathfrak{L}) = \left[\frac{1}{3}\left(\begin{array}{c} \mathfrak{J}_{\mathfrak{m}}^q(r)\log\left(\frac{2\mathfrak{J}_{\mathfrak{m}}^q(r)}{\mathfrak{J}_{\mathfrak{m}}^q(r)+\mathfrak{L}_{\mathfrak{m}}^q(r)}\right)+\mathfrak{L}_{\mathfrak{m}}^q(r)\log\left(\frac{2\mathfrak{L}_{\mathfrak{m}}^q(r)}{\mathfrak{J}_{\mathfrak{m}}^q(r)+\mathfrak{L}_{\mathfrak{m}}^q(r)}\right)+ \\ \mathfrak{J}_{\mathfrak{n}}^q(r)\log\left(\frac{2\mathfrak{J}_{\mathfrak{n}}^q(r)}{\mathfrak{J}_{\mathfrak{n}}^q(r)+\mathfrak{L}_{\mathfrak{n}}^q(r)}\right)+\mathfrak{L}_{\mathfrak{n}}^q(r)\log\left(\frac{2\mathfrak{L}_{\mathfrak{n}}^q(r)}{\mathfrak{J}_{\mathfrak{n}}^q(r)+\mathfrak{L}_{\mathfrak{n}}^q(r)}\right)+ \\ \mathfrak{J}_{\mathfrak{h}}^q(r)\log\left(\frac{2\mathfrak{J}_{\mathfrak{h}}^q(r)}{\mathfrak{J}_{\mathfrak{h}}^q(r)+\mathfrak{L}_{\mathfrak{h}}^q(r)}\right)+\mathfrak{L}_{\mathfrak{h}}^q(r)\log\left(\frac{2\mathfrak{L}_{\mathfrak{h}}^q(r)}{\mathfrak{J}_{\mathfrak{h}}^q(r)+\mathfrak{L}_{\mathfrak{h}}^q(r)}\right) \end{array}\right)\right]^{\frac{1}{2}},$$

$$= \left[\frac{1}{3}\left(\begin{array}{c} \mathfrak{L}_{\mathfrak{m}}^q(r)\log\left(\frac{2\mathfrak{L}_{\mathfrak{m}}^q(r)}{\mathfrak{L}_{\mathfrak{m}}^q(r)+\mathfrak{J}_{\mathfrak{m}}^q(r)}\right)+\mathfrak{J}_{\mathfrak{m}}^q(r)\log\left(\frac{2\mathfrak{J}_{\mathfrak{m}}^q(r)}{\mathfrak{L}_{\mathfrak{m}}^q(r)+\mathfrak{J}_{\mathfrak{m}}^q(r)}\right)+ \\ \mathfrak{L}_{\mathfrak{n}}^q(r)\log\left(\frac{2\mathfrak{L}_{\mathfrak{n}}^q(r)}{\mathfrak{L}_{\mathfrak{n}}^q(r)+\mathfrak{J}_{\mathfrak{n}}^q(r)}\right)+\mathfrak{J}_{\mathfrak{n}}^q(r)\log\left(\frac{2\mathfrak{J}_{\mathfrak{n}}^q(r)}{\mathfrak{L}_{\mathfrak{n}}^q(r)+\mathfrak{J}_{\mathfrak{n}}^q(r)}\right)+ \\ \mathfrak{L}_{\mathfrak{h}}^q(r)\log\left(\frac{2\mathfrak{L}_{\mathfrak{h}}^q(r)}{\mathfrak{L}_{\mathfrak{h}}^q(r)+\mathfrak{J}_{\mathfrak{h}}^q(r)}\right)+\mathfrak{J}_{\mathfrak{h}}^q(r)\log\left(\frac{2\mathfrak{J}_{\mathfrak{h}}^q(r)}{\mathfrak{L}_{\mathfrak{h}}^q(r)+\mathfrak{J}_{\mathfrak{h}}^q(r)}\right) \end{array}\right)\right]^{\frac{1}{2}}$$

$$= \mathbb{D}_1(\mathfrak{L}, \mathfrak{J}),$$

which proves (iii).

iv. Finally, we prove triangular inequality: $\mathbb{D}_1(\mathfrak{M}, \mathfrak{J}) + \mathbb{D}_1(\mathfrak{J}, \mathfrak{L}) \geq \mathbb{D}_1(\mathfrak{M}, \mathfrak{L})$.

Given three q-ROFSs, $\mathfrak{J}$, $\mathfrak{L}$ and $\mathfrak{M}$ in $\mathscr{R}$, considering the assumptions $\mathscr{B}_i$, $(i = 1, 2, 3, 4)$ for evaluation:

$$\mathscr{B}_1\colon \mathfrak{M}_{\mathfrak{m}}^q(r) \leq \mathfrak{J}_{\mathfrak{m}}^q(r) \leq \mathfrak{L}_{\mathfrak{m}}^q(r),$$

$$\mathscr{B}_2\colon \mathfrak{L}_{\mathfrak{m}}^q(r) \leq \mathfrak{J}_{\mathfrak{m}}^q(r) \leq \mathfrak{M}_{\mathfrak{m}}^q(r),$$

$$\mathscr{B}_3\colon \mathfrak{J}_{\mathfrak{m}}^q(r) \leq \min\left\{\mathfrak{M}_{\mathfrak{m}}^q(r), \mathfrak{L}_{\mathfrak{m}}^q(r)\right\},$$

$$\mathscr{B}_4\colon \mathfrak{J}_{\mathfrak{m}}^q(r) \geq \max\left\{\mathfrak{M}_{\mathfrak{m}}^q(r), \mathfrak{L}_{\mathfrak{m}}^q(r)\right\}.$$

Given the assumptions $\mathscr{B}_1$ and $\mathscr{B}_2$, it follows that the triangle inequality holds:

$$|\mathfrak{M}_{\mathfrak{m}}^q(r) - \mathfrak{L}_{\mathfrak{m}}^q(r)| \leq |\mathfrak{M}_{\mathfrak{m}}^q(r) - \mathfrak{J}_{\mathfrak{m}}^q(r)| + |\mathfrak{J}_{\mathfrak{m}}^q(r) - \mathfrak{L}_{\mathfrak{m}}^q(r)|.$$

With assumption $\mathscr{B}_3$, we get:

$$\mathfrak{M}_{\mathfrak{m}}^q(r) - \mathfrak{J}_{\mathfrak{m}}^q(r) \geq 0 \quad \text{and} \quad \mathfrak{L}_{\mathfrak{m}}^q(r) - \mathfrak{J}_{\mathfrak{m}}^q(r) \geq 0.$$

Then, the calculation can be determined:

$$|\mathfrak{M}^q_{\mathfrak{m}}(r)-\mathfrak{J}^q_{\mathfrak{m}}(r)|+|\mathfrak{J}^q_{\mathfrak{m}}(r)-\mathfrak{L}^q_{\mathfrak{m}}(r)|-|\mathfrak{M}^q_{\mathfrak{m}}(r)-\mathfrak{L}^q_{\mathfrak{m}}(r)|$$

$$=\begin{cases}\mathfrak{M}^q_{\mathfrak{m}}(r)-\mathfrak{J}^q_{\mathfrak{m}}(r)+\mathfrak{L}^q_{\mathfrak{m}}(r)-\mathfrak{J}^q_{\mathfrak{m}}(r)-\mathfrak{M}^q_{\mathfrak{m}}(r)+\mathfrak{L}^q_{\mathfrak{m}}(r), & \text{if } \mathfrak{M}^q_{\mathfrak{m}}(r)\geq\mathfrak{L}^q_{\mathfrak{m}}(r)\\ \mathfrak{J}^q_{\mathfrak{m}}(r)-\mathfrak{M}^q_{\mathfrak{m}}(r)+\mathfrak{L}^q_{\mathfrak{m}}(r)-\mathfrak{J}^q_{\mathfrak{m}}(r)+\mathfrak{M}^q_{\mathfrak{m}}(r)-\mathfrak{L}^q_{\mathfrak{m}}(r), & \text{if } \mathfrak{M}^q_{\mathfrak{m}}(r)\leq\mathfrak{L}^q_{\mathfrak{m}}(r)\end{cases}$$

$$=2\left(\min\{\mathfrak{M}^q_{\mathfrak{m}}(r),\,\mathfrak{L}^q_{\mathfrak{m}}(r)\}-\mathfrak{J}^q_{\mathfrak{m}}(r)\right)\geq 0.$$

Using $\mathscr{B}_4$, we infer that:

$$|\mathfrak{M}^q_{\mathfrak{m}}(r)-\mathfrak{J}^q_{\mathfrak{m}}(r)|+|\mathfrak{J}^q_{\mathfrak{m}}(r)-\mathfrak{L}^q_{\mathfrak{m}}(r)|-|\mathfrak{M}^q_{\mathfrak{m}}(r)-\mathfrak{L}^q_{\mathfrak{m}}(r)|$$

$$=\begin{cases}\mathfrak{J}^q_{\mathfrak{m}}(r)-\mathfrak{M}^q_{\mathfrak{m}}(r)+\mathfrak{J}^q_{\mathfrak{m}}(r)-\mathfrak{L}^q_{\mathfrak{m}}(r)-\mathfrak{M}^q_{\mathfrak{m}}(r)+\mathfrak{L}^q_{\mathfrak{m}}(r), & \text{if } \mathfrak{M}^q_{\mathfrak{m}}(r)\geq\mathfrak{L}^q_{\mathfrak{m}}(r)\\ \mathfrak{M}^q_{\mathfrak{m}}(r)-\mathfrak{J}^q_{\mathfrak{m}}(r)+\mathfrak{J}^q_{\mathfrak{m}}(r)-\mathfrak{L}^q_{\mathfrak{m}}(r)+\mathfrak{M}^q_{\mathfrak{m}}(r)-\mathfrak{L}^q_{\mathfrak{m}}(r), & \text{if } \mathfrak{M}^q_{\mathfrak{m}}(r)\leq\mathfrak{L}^q_{\mathfrak{m}}(r)\end{cases}$$

$$=2\left(\mathfrak{J}^q_{\mathfrak{m}}(r)-\max\{\mathfrak{M}^q_{\mathfrak{m}}(r),\,\mathfrak{L}^q_{\mathfrak{m}}(r)\}\right)\geq 0.$$

Thus, the triangle inequality is also true for $\mathscr{B}_3$ and $\mathscr{B}_4$, in which case:

$$|\mathfrak{M}^q_{\mathfrak{m}}(r)-\mathfrak{L}^q_{\mathfrak{m}}(r)|\leq|\mathfrak{M}^q_{\mathfrak{m}}(r)-\mathfrak{J}^q_{\mathfrak{m}}(r)|+|\mathfrak{J}^q_{\mathfrak{m}}(r)-\mathfrak{L}^q_{\mathfrak{m}}(r)|.$$

In the same vein, we have:

$$|\mathfrak{M}^q_{\mathfrak{n}}(r)-\mathfrak{L}^q_{\mathfrak{n}}(r)|\leq|\mathfrak{M}^q_{\mathfrak{n}}(r)-\mathfrak{J}^q_{\mathfrak{n}}(r)|+|\mathfrak{J}^q_{\mathfrak{n}}(r)-\mathfrak{L}^q_{\mathfrak{n}}(r)|,$$

$$|\mathfrak{M}^q_{\mathfrak{h}}(r)-\mathfrak{L}^q_{\mathfrak{h}}(r)|\leq|\mathfrak{M}^q_{\mathfrak{h}}(r)-\mathfrak{J}^q_{\mathfrak{h}}(r)|+|\mathfrak{J}^q_{\mathfrak{h}}(r)-\mathfrak{L}^q_{\mathfrak{h}}(r)|.$$

Hence, (iv) holds.
Similarly, $\mathbb{D}_2(\mathfrak{J},\,\mathfrak{L})$ also satisfies the conditions in Definition 6. □

## 3.1 *Numerical verification of the new logarithmic-based q-ROFDMs*

We present some numerical examples to illustrate the properties and efficiency of the newly logarithmic-based q-ROFDMs.

**Example 1** Let $\mathfrak{J}$, $\mathfrak{L}$ and $\mathfrak{M}$ be three q-ROFSs in $\mathscr{R}=\{r_1,\,r_2\}$, as follows:

$$\mathfrak{J} = \{\langle r_1,\ 0.85,\ 0.54\rangle, \langle r_2,\ 0.50,\ 0.62\rangle\},$$

$$\mathfrak{L} = \{\langle r_1,\ 0.85,\ 0.54\rangle, \langle r_2,\ 0.50,\ 0.62\rangle\},$$

$$\mathfrak{M} = \{\langle r_1,\ 0.48,\ 0.74\rangle, \langle r_2,\ 0.82,\ 0.50\rangle\}.$$

If $q = 4$, using (4) and (5), the distance between the three q-ROFSs $\mathfrak{J}$, $\mathfrak{L}$ and $\mathfrak{M}$ are computed as follows:

$$\mathbb{D}_1(\mathfrak{J}, \mathfrak{L}) = 0.0000, \quad \mathbb{D}_1(\mathfrak{L}, \mathfrak{J}) = 0.0000,$$

$$\mathbb{D}_1(\mathfrak{J}, \mathfrak{M}) = 0.1634, \quad \mathbb{D}_1(\mathfrak{M}, \mathfrak{J}) = 0.1634,$$

$$\mathbb{D}_1(\mathfrak{L}, \mathfrak{M}) = 0.1634, \quad \mathbb{D}_1(\mathfrak{M}, \mathfrak{L}) = 0.1634,$$

$$\mathbb{D}_2(\mathfrak{J}, \mathfrak{L}) = 0.0000, \quad \mathbb{D}_2(\mathfrak{L}, \mathfrak{J}) = 0.0000,$$

$$\mathbb{D}_2(\mathfrak{J}, \mathfrak{M}) = 0.0267, \quad \mathbb{D}_2(\mathfrak{M}, \mathfrak{J}) = 0.0267,$$

$$\mathbb{D}_2(\mathfrak{L}, \mathfrak{M}) = 0.0267, \quad \mathbb{D}_2(\mathfrak{M}, \mathfrak{L}) = 0.0267.$$

Hence, it can be verified that the proposed distance metrics satisfy the first three distance properties.

**Example 2** Given three q-ROFSs $\mathfrak{J}$, $\mathfrak{L}$ and $\mathfrak{M}$ in $\mathscr{R} = \{r_1, r_2\}$, expressed as follows:

$$\mathfrak{J} = \{\langle r_1, 0.6, 0.4\rangle, \langle r_2, 0.8, 0.5\rangle\},$$

$$\mathfrak{L} = \{\langle r_1, 0.9, 0.3\rangle, \langle r_2, 0.8, 0.2\rangle\},$$

$$\mathfrak{M} = \{\langle r_1, 0.7, 0.5\rangle, \langle r_2, 0.7, 0.4\rangle\}.$$

For $q = 4$, the computed values using $\mathbb{D}_1(\mathfrak{J}, \mathfrak{L})$ and $\mathbb{D}_2(\mathfrak{J}, \mathfrak{L})$ are presented as follows:

$$\mathbb{D}_1(\mathfrak{J}, \mathfrak{L}) = 0.1466, \quad \mathbb{D}_1(\mathfrak{M}, \mathfrak{J}) = 0.0581, \quad \mathbb{D}_1(\mathfrak{M}, \mathfrak{L}) = 0.1321,$$

$$\mathbb{D}_2(\mathfrak{J}, \mathfrak{L}) = 0.0215, \quad \mathbb{D}_2(\mathfrak{M}, \mathfrak{J}) = 0.0034, \quad \mathbb{D}_2(\mathfrak{M}, \mathfrak{L}) = 0.0174.$$

Hence, we see that:

$$\mathbb{D}_1(\mathfrak{M}, \mathfrak{J}) + \mathbb{D}_1(\mathfrak{J}, \mathfrak{L}) = 0.2047 \geq \mathbb{D}_1(\mathfrak{M}, \mathfrak{L}) = 0.1321,$$

$$\mathbb{D}_2(\mathfrak{M}, \mathfrak{J}) + \mathbb{D}_2(\mathfrak{J}, \mathfrak{L}) = 0.0249 \geq \mathbb{D}_2(\mathfrak{M}, \mathfrak{L}) = 0.0174.$$

Therefore, this result verifies the fourth property.

**Example 3** Let $\mathfrak{J}$ and $\mathfrak{L}$ be q-ROFSs in $\mathscr{R} = \{r_1, r_2\}$. To compare the efficiency of the newly logarithmic-based q-ROFDMs with the existing logarithmic-based q-ROFDMs, two cases of q-ROFSs are employed for $q = 8$, and the comparison results are presented in Table 1 and Figure 1.

**Table 1.** Comparison results based on Example 3

| q-ROFDMs | Case I<br>$\mathfrak{J} = \langle r_1, 0.2, 0.4\rangle, \langle r_2, 0.7, 0.4\rangle$<br>$\mathfrak{L} = \langle r_1, 0.5, 0.2\rangle, \langle r_2, 0.6, 0.3\rangle$ | Case II<br>$\mathfrak{J} = \langle r_1, 0.2, 0.4\rangle, \langle r_2, 0.7, 0.4\rangle$<br>$\mathfrak{L} = \langle r_1, 0.5, 0.4\rangle, \langle r_2, 0.6, 0.5\rangle$ |
|---|---|---|
| $\mathbb{D}_{We1}$ [57] | 0.0576 | 0.0586 |
| $\mathbb{D}_{We2}$ [57] | 0.0584 | 0.0593 |
| $\mathbb{D}_{Se}$ [58] | 0.0006 | 0.0006 |
| $\mathbb{D}_1$ | 0.0337 | 0.0342 |
| $\mathbb{D}_2$ | 0.0011 | 0.0012 |

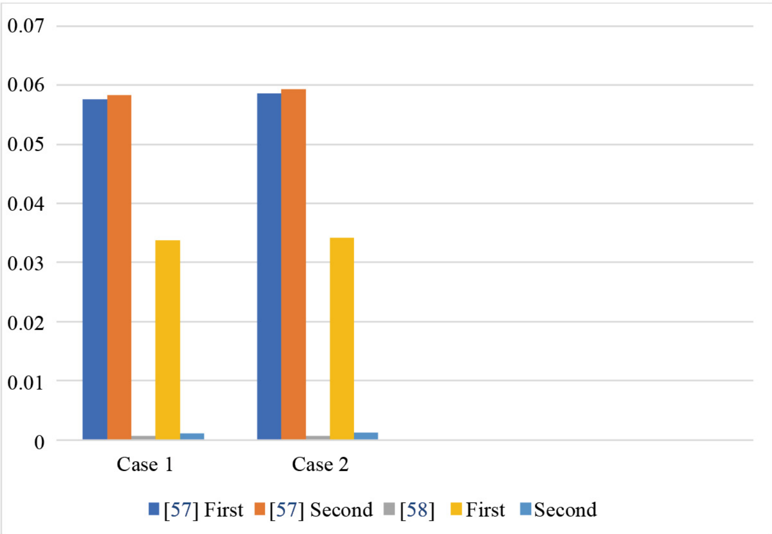


**Figure 1.** Pictorial view of the distance values

Table 1 and Figure 1 show that, the new logarithmic-based q-ROFDMs is efficient in distinguishing between closely related q-ROFSs unlike the method by Suri et al. [58]. It is also worthy to note that the new logarithmic-based methods produce more accurate results more than the methods in [57], which they modified.

## 4. Application in vehicle selection problem

VSP for an effective transportation system is inherently a CDM problem characterized by multiple conflicting criteria, uncertain data, and imprecise human judgments. In today's transportation landscape, selecting the most suitable vehicle for specific purposes, whether for public transit, logistics, emergency response, or private use requires careful consideration of numerous factors, which include: maintenance cost, fuel efficiency, terrain adaptability, resale value, purchase cost, reliability and passenger capacity. These factors are often difficult to quantify precisely and are subject to variability depending on technological advancements, user preferences, and regional infrastructure constraints. Conventional decision-making approaches may fall short when dealing with such ambiguity and vagueness. Hence, in this experiment, we adopt the new logarithmic-based q-ROFDMs based on TOPSIS approach to enhance the VSP for an efficient transportation system for $q = 4, 5, \ldots, 8$.

### 4.1 *TOPSIS algorithm*

**Step I**: Obtain the views of the vehicle maintenance professionals based on the LVs in Table 2.

**Step II**: Express the LVs as q-ROFNs to represent the q-ROF decision matrices represented by $\mathbb{M}_p$, which is defined as (7).

$$\mathbb{M}_p = (\omega_j(\mu_i))_{7\times 7} = \begin{array}{c} \\ \omega_1 \\ \vdots \\ \omega_7 \end{array} \begin{array}{c} \begin{array}{ccc} \mu_1 & \cdots & \mu_7 \end{array} \\ \begin{pmatrix} (\mathfrak{m}, \mathfrak{n})_{1,1} & \cdots & (\mathfrak{m}, \mathfrak{n})_{1,7} \\ \vdots & \ddots & \vdots \\ (\mathfrak{m}, \mathfrak{n})_{7,1} & \cdots & (\mathfrak{m}, \mathfrak{n})_{7,7} \end{pmatrix} \end{array}, \tag{7}$$

where $\omega_j$ represents the available vehicle brands $(j = 1, 2, \ldots, 7)$ and $\mu_i$ denotes the criteria $(i = 1, 2, \ldots, 7)$.

**Step III**: Obtain the average q-ROF decision matrix, $\widetilde{\mathbb{M}}$ using (8).

$$\widetilde{\mathbb{M}} = \frac{(\omega_j(\mu_i))_{7\times 7}}{p}, \tag{8}$$

where $p$ denotes the number of experts $(p = 1, 2, 3)$.

**Step IV**: Identify the least $\mu_i$ called the Cost Criterion (CC) and the rest of $\mu_i$ as the Benefit Criteria (BC). The CC is obtained using (9).

$$\min_{1\leq j\leq 7} \left\{ \sum_{i=1}^{7} \omega_j(\mu_i) \right\}. \tag{9}$$

**Step V**: Use $\widetilde{\mathbb{M}}$ to obtain the normalized q-ROF decision matrix $\widehat{\mathbb{M}}$, defined by (10).

$$\widehat{\mathbb{M}} = \begin{cases} \left(\omega_{j_{\mathrm{m}}}(\mu_i),\, \omega_{j_{\mathrm{n}}}(\mu_i)\right), & \text{for BC of } \omega_j, \\ \left(\omega_{j_{\mathrm{n}}}(\mu_i),\, \omega_{j_{\mathrm{m}}}(\mu_i)\right), & \text{for CC of } \omega_j. \end{cases} \tag{10}$$

**Step VI**: Find the Positive Ideal Solution (PIS), $\widehat{\mathbb{M}}^+ = \left\{\widehat{\mathbb{M}}_1^+, \ldots, \widehat{\mathbb{M}}_x^+\right\}$ and the Negative Ideal Solution (NIS), $\widehat{\mathbb{M}}^- = \left\{\widehat{\mathbb{M}}_1^-, \ldots, \widehat{\mathbb{M}}_x^-\right\}$ defined by (11).

$$\left.\begin{array}{l} \widehat{\mathbb{M}}^+ = \begin{cases} \left(\max_{1\le j\le 7}\left\{\omega_{j_{\mathrm{m}}}(\mu_i)\right\}, \min_{1\le j\le 7}\left\{\omega_{j_{\mathrm{n}}}(\mu_i)\right\}\right), & \text{if } \mu_i \text{ is the BC}, \\ \left(\min_{1\le j\le 7}\left\{\omega_{j_{\mathrm{m}}}(\mu_i)\right\}, \max_{1\le j\le 7}\left\{\omega_{j_{\mathrm{n}}}(\mu_i)\right\}\right), & \text{if } \mu_i \text{ is the CC}, \end{cases} \\ \widehat{\mathbb{M}}^- = \begin{cases} \left(\min_{1\le j\le 7}\left\{\omega_{j_{\mathrm{m}}}(\mu_i)\right\}, \max_{1\le j\le 7}\left\{\omega_{j_{\mathrm{n}}}(\mu_i)\right\}\right), & \text{if } \mu_i \text{ is the BC}, \\ \left(\max_{1\le j\le 7}\left\{\omega_{j_{\mathrm{m}}}(\mu_i)\right\}, \min_{1\le j\le 7}\left\{\omega_{j_{\mathrm{n}}}(\mu_i)\right\}\right), & \text{if } \mu_i \text{ is the CC}. \end{cases} \end{array}\right\} \tag{11}$$

**Step VII**: Compute $\mathbb{D}(\widehat{\mathbb{M}}^+, \omega_j)$ and $\mathbb{D}(\widehat{\mathbb{M}}^-, \omega_j)$.

**Step VIII**: Calculate the nearness coefficient, $\mathscr{C}(\omega_j)$ for each alternative $\omega_j$ using (12).

$$\mathscr{C}(\omega_j) = \frac{\mathbb{D}(\widehat{\mathbb{M}}^+, \omega_j)}{\mathbb{D}(\widehat{\mathbb{M}}^+, \omega_j) + \mathbb{D}(\widehat{\mathbb{M}}^-, \omega_j)}. \tag{12}$$

**Step IX**: Rank the vehicle brands in an ascending order of $\mathscr{C}(\omega_j)$ to determine the most appropriate vehicle to pick for an effective transportation system.

**Step X**: Pick the vehicle brand with the least $\mathscr{C}(\omega_j)$.

## 4.2 *Case study*

The data for this study is drawn from three randomly selected vehicle maintenance experts in Benue State, Nigeria. Seven brands of car vehicle considered, which are represented by the set of vehicles:

$$\omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7\}.$$

The criteria considered for the VSP are represented by the set:

$$\mu = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7\},$$

where, $\mu_1$ represents maintenance cost, $\mu_2$ represents fuel efficiency, $\mu_3$ represents terrain adaptability, $\mu_4$ represents resale value, $\mu_5$ represents purchase cost, $\mu_6$ represents reliability, and $\mu_7$ represents passenger capacity.

The mode of data collection adopted in the work is knowledge-based system via Linguistic Variables (LVs), which are linguistic terms use to represent numerical values. The LVs used are presented along with the q-ROFNs in Table 2.

**Table 2.** Linguistic variables for vehicle selection

| LVs | q-ROFNs |
|---|---|
| Extremely Good (EG) | (1.0, 0.0) |
| Very Good (VG) | (0.9, 0.1) |
| Good (G) | (0.7, 0.3) |
| Fair (F) | (0.5, 0.5) |
| Poor (P) | (0.3, 0.7) |
| Very Poor (VP) | (0.1, 0.9) |
| Extremely Poor (EP) | (0.0, 1.0) |

**Step I**: To determine the suitable vehicle to select based on expert knowledge, three vehicle maintenance Experts I, Expert II and Expert III were approached to give their professional views on the seven vehicle alternatives, and their views in terms of LVs are presented in Table 3.

**Table 3.** Linguistic variable from vehicle maintenance experts

| Vehicle brands | Main cost | Fuel efficiency | Terrain adapt. | Resale value | Purchase cost | Reliability | Passenger capacity |
|---|---|---|---|---|---|---|---|
| | | | Expert I | | | | |
| $\omega_1$ | G | G | P | EG | VG | F | G |
| $\omega_2$ | F | G | G | EG | P | G | EG |
| $\omega_3$ | EP | P | EG | EG | EG | EG | EP |
| $\omega_4$ | F | F | G | G | G | G | EG |
| $\omega_5$ | VG | EG | VP | VP | EP | P | VG |
| $\omega_6$ | G | EP | F | VP | VG | G | G |
| $\omega_7$ | P | P | VG | EP | EG | G | EG |
| | | | Expert II | | | | |
| $\omega_1$ | VG | G | G | G | G | VG | G |
| $\omega_2$ | EG | G | VG | G | VG | VG | G |
| $\omega_3$ | VP | F | VG | EP | VG | F | VP |
| $\omega_4$ | F | F | G | G | F | EG | F |
| $\omega_5$ | F | VG | G | G | G | VG | G |
| $\omega_6$ | G | EP | F | F | F | EG | EP |
| $\omega_7$ | EG | G | F | EG | F | P | EP |
| | | | Expert III | | | | |
| $\omega_1$ | G | G | P | VG | F | VG | G |
| $\omega_2$ | G | VG | G | G | VG | G | VG |
| $\omega_3$ | VP | F | VG | VG | G | EG | G |
| $\omega_4$ | EG | EG | G | G | F | G | F |
| $\omega_5$ | G | EG | VP | F | EG | P | F |
| $\omega_6$ | G | VG | G | G | P | VG | EP |
| $\omega_7$ | EG | G | F | F | EG | F | P |

**Step II:** Converted the LVs from the three experts in Table 3 to q-ROFNs (q-ROF decision matrices $\mathbb{M}_p$) using Table 2, as presented in Table 4.

**Table 4.** q-ROFNs from vehicle maintenance experts

| Vehicle brands | Main cost | Fuel efficiency | Terrain adapt. | Resale value | Purchase cost | Reliability | Passenger capacity |
|---|---|---|---|---|---|---|---|
| | | | Expert I | | | | |
| $\omega_1$ | (0.7,0.3) | (0.7,0.3) | (0.3,0.7) | (1.0,0.0) | (0.9,0.1) | (0.5,0.5) | (0.7,0.3) |
| $\omega_2$ | (0.5,0.5) | (0.7,0.3) | (0.7,0.3) | (1.0,0.0) | (0.3,0.7) | (0.7,0.3) | (1.0,0.0) |
| $\omega_3$ | (0.0,1.0) | (0.3,0.7) | (1.0,0.0) | (1.0,0.0) | (1.0,0.0) | (1.0,0.0) | (0.0,1.0) |
| $\omega_4$ | (0.5,0.5) | (0.5,0.5) | (0.7,0.3) | (0.7,0.3) | (0.7,0.3) | (0.7,0.3) | (1.0,0.0) |
| $\omega_5$ | (0.9,0.1) | (1.0,0.0) | (0.1,0.9) | (0.1,0.9) | (0.0,1.0) | (0.3,0.7) | (0.9,0.1) |
| $\omega_6$ | (0.7,0.3) | (0.0,1.0) | (0.5,0.5) | (0.1,0.9) | (0.9,0.1) | (0.7,0.3) | (0.7,0.3) |
| $\omega_7$ | (0.3,0.7) | (0.3,0.7) | (0.9,0.1) | (0.0,1.0) | (1.0,0.0) | (0.7,0.3) | (1.0,0.0) |
| | | | Expert II | | | | |
| $\omega_1$ | (0.9,0.1) | (0.7,0.3) | (0.7,0.3) | (0.7,0.3) | (0.7,0.3) | (0.9,0.1) | (0.7,0.3) |
| $\omega_2$ | (1.0,0.0) | (0.7,0.3) | (0.9,0.1) | (0.7,0.3) | (0.9,0.1) | (0.9,0.1) | (0.7,0.3) |
| $\omega_3$ | (0.1,0.9) | (0.5,0.5) | (0.9,0.1) | (0.0,1.0) | (0.9,0.1) | (0.5,0.5) | (0.1,0.9) |
| $\omega_4$ | (0.5,0.5) | (0.5,0.5) | (0.7,0.3) | (0.7,0.3) | (0.5,0.5) | (1.0,0.0) | (0.5,0.5) |
| $\omega_5$ | (0.5,0.5) | (0.9,0.1) | (0.7,0.3) | (0.7,0.3) | (0.7,0.3) | (0.9,0.1) | (0.7,0.3) |
| $\omega_6$ | (0.7,0.3) | (0.0,1.0) | (0.5,0.5) | (0.5,0.5) | (0.5,0.5) | (1.0,0.0) | (0.0,1.0) |
| $\omega_7$ | (1.0,0.0) | (0.7,0.3) | (0.5,0.5) | (1.0,0.0) | (0.5,0.5) | (0.3,0.7) | (0.0,1.0) |
| | | | Expert III | | | | |
| $\omega_1$ | (0.7,0.3) | (0.7,0.3) | (0.3,0.7) | (0.9,0.1) | (0.5,0.5) | (0.9,0.1) | (0.7,0.3) |
| $\omega_2$ | (0.7,0.3) | (0.9,0.1) | (0.7,0.3) | (0.7,0.3) | (0.9,0.1) | (0.7,0.3) | (0.9,0.1) |
| $\omega_3$ | (0.1,0.9) | (0.5,0.5) | (0.9,0.1) | (0.9,0.1) | (0.7,0.3) | (1.0,0.0) | (0.7,0.3) |
| $\omega_4$ | (1.0,0.0) | (1.0,0.0) | (0.7,0.3) | (0.7,0.3) | (0.5,0.5) | (0.7,0.3) | (0.5,0.5) |
| $\omega_5$ | (0.7,0.3) | (1.0,0.0) | (0.3,0.7) | (0.5,0.5) | (1.0,0.0) | (0.3,0.7) | (0.5,0.5) |
| $\omega_6$ | (0.7,0.3) | (0.9,0.1) | (0.7,0.3) | (0.7,0.3) | (0.3,0.7) | (0.9,0.1) | (0.0,1.0) |
| $\omega_7$ | (1.0,0.0) | (0.7,0.3) | (0.5,0.5) | (0.5,0.5) | (1.0,0.0) | (0.5,0.5) | (0.3,0.7) |

**Table 5.** Average q-ROFNs

| Vehicle brands | Main cost | Fuel efficiency | Terrain adapt | Resale value | Purchase cost | Reliability | Passenger capacity |
|---|---|---|---|---|---|---|---|
| $\omega_1$ | $\begin{pmatrix} 0.7667 \\ 0.2333 \end{pmatrix}$ | $\begin{pmatrix} 0.7000 \\ 0.3000 \end{pmatrix}$ | $\begin{pmatrix} 0.4333 \\ 0.5667 \end{pmatrix}$ | $\begin{pmatrix} 0.8667 \\ 0.1333 \end{pmatrix}$ | $\begin{pmatrix} 0.7000 \\ 0.3000 \end{pmatrix}$ | $\begin{pmatrix} 0.7667 \\ 0.2333 \end{pmatrix}$ | $\begin{pmatrix} 0.7000 \\ 0.3000 \end{pmatrix}$ |
| $\omega_2$ | $\begin{pmatrix} 0.7333 \\ 0.2667 \end{pmatrix}$ | $\begin{pmatrix} 0.7667 \\ 0.2333 \end{pmatrix}$ | $\begin{pmatrix} 0.7667 \\ 0.2333 \end{pmatrix}$ | $\begin{pmatrix} 0.8000 \\ 0.2000 \end{pmatrix}$ | $\begin{pmatrix} 0.7000 \\ 0.3000 \end{pmatrix}$ | $\begin{pmatrix} 0.7667 \\ 0.2333 \end{pmatrix}$ | $\begin{pmatrix} 0.8667 \\ 0.1333 \end{pmatrix}$ |
| $\omega_3$ | $\begin{pmatrix} 0.0667 \\ 0.9333 \end{pmatrix}$ | $\begin{pmatrix} 0.4333 \\ 0.5667 \end{pmatrix}$ | $\begin{pmatrix} 0.9333 \\ 0.0667 \end{pmatrix}$ | $\begin{pmatrix} 0.6333 \\ 0.3667 \end{pmatrix}$ | $\begin{pmatrix} 0.8667 \\ 0.1333 \end{pmatrix}$ | $\begin{pmatrix} 0.8333 \\ 0.1667 \end{pmatrix}$ | $\begin{pmatrix} 0.2667 \\ 0.7333 \end{pmatrix}$ |
| $\omega_4$ | $\begin{pmatrix} 0.6667 \\ 0.3333 \end{pmatrix}$ | $\begin{pmatrix} 0.6667 \\ 0.3333 \end{pmatrix}$ | $\begin{pmatrix} 0.7000 \\ 0.3000 \end{pmatrix}$ | $\begin{pmatrix} 0.6333 \\ 0.3667 \end{pmatrix}$ | $\begin{pmatrix} 0.5667 \\ 0.4333 \end{pmatrix}$ | $\begin{pmatrix} 0.8000 \\ 0.2000 \end{pmatrix}$ | $\begin{pmatrix} 0.6667 \\ 0.3333 \end{pmatrix}$ |
| $\omega_5$ | $\begin{pmatrix} 0.7000 \\ 0.3000 \end{pmatrix}$ | $\begin{pmatrix} 0.9667 \\ 0.0333 \end{pmatrix}$ | $\begin{pmatrix} 0.5667 \\ 0.4333 \end{pmatrix}$ | $\begin{pmatrix} 0.4333 \\ 0.5667 \end{pmatrix}$ | $\begin{pmatrix} 0.5667 \\ 0.4333 \end{pmatrix}$ | $\begin{pmatrix} 0.5000 \\ 0.5000 \end{pmatrix}$ | $\begin{pmatrix} 0.7000 \\ 0.3000 \end{pmatrix}$ |
| $\omega_6$ | $\begin{pmatrix} 0.7000 \\ 0.3000 \end{pmatrix}$ | $\begin{pmatrix} 0.3333 \\ 0.6667 \end{pmatrix}$ | $\begin{pmatrix} 0.5667 \\ 0.4333 \end{pmatrix}$ | $\begin{pmatrix} 0.4333 \\ 0.5667 \end{pmatrix}$ | $\begin{pmatrix} 0.5667 \\ 0.4333 \end{pmatrix}$ | $\begin{pmatrix} 0.8667 \\ 0.1333 \end{pmatrix}$ | $\begin{pmatrix} 0.2333 \\ 0.7667 \end{pmatrix}$ |
| $\omega_7$ | $\begin{pmatrix} 0.7667 \\ 0.2333 \end{pmatrix}$ | $\begin{pmatrix} 0.5667 \\ 0.4333 \end{pmatrix}$ | $\begin{pmatrix} 0.6333 \\ 0.3667 \end{pmatrix}$ | $\begin{pmatrix} 0.5000 \\ 0.5000 \end{pmatrix}$ | $\begin{pmatrix} 0.8333 \\ 0.1667 \end{pmatrix}$ | $\begin{pmatrix} 0.5000 \\ 0.5000 \end{pmatrix}$ | $\begin{pmatrix} 0.4333 \\ 0.5667 \end{pmatrix}$ |

**Step III:** Using (10) the q-ROF decision matrices $(\mathbb{M}_p)$ from the three professionals are expressed as an average q-ROF decision matrices $\widetilde{\mathbb{M}}$ as presented in Table 5.

**Step IV**: Using (9), the least criteria is $\mu_4$ (resale value) which is the CC, and the other criteria are the BC.

**Step V:** The normalized q-ROF decision matrix $(\widehat{\mathbb{M}})$ is obtained using (10) and presented in Table 6.

**Step VI**: the PIS $(\widehat{\mathbb{M}}^+)$ and NIS $(\widehat{\mathbb{M}}^-)$ are obtained using (11) and presented in Table 6.

**Table 6.** Normalized q-ROFDM and PIS/NIS

| Vehicle brands | Main cost | Fuel efficiency | Terrain adapt | Resale value | Purchase cost | Reliability | Passenger capacity |
|---|---|---|---|---|---|---|---|
| $\omega_1$ | $\begin{pmatrix} 0.7667 \\ 0.2333 \end{pmatrix}$ | $\begin{pmatrix} 0.7000 \\ 0.3000 \end{pmatrix}$ | $\begin{pmatrix} 0.4333 \\ 0.5667 \end{pmatrix}$ | $\begin{pmatrix} 0.1333 \\ 0.8667 \end{pmatrix}$ | $\begin{pmatrix} 0.7000 \\ 0.3000 \end{pmatrix}$ | $\begin{pmatrix} 0.7667 \\ 0.2333 \end{pmatrix}$ | $\begin{pmatrix} 0.7000 \\ 0.3000 \end{pmatrix}$ |
| $\omega_2$ | $\begin{pmatrix} 0.7333 \\ 0.2667 \end{pmatrix}$ | $\begin{pmatrix} 0.7667 \\ 0.2333 \end{pmatrix}$ | $\begin{pmatrix} 0.7667 \\ 0.2333 \end{pmatrix}$ | $\begin{pmatrix} 0.2000 \\ 0.8000 \end{pmatrix}$ | $\begin{pmatrix} 0.7000 \\ 0.3000 \end{pmatrix}$ | $\begin{pmatrix} 0.7667 \\ 0.2333 \end{pmatrix}$ | $\begin{pmatrix} 0.8667 \\ 0.1333 \end{pmatrix}$ |
| $\omega_3$ | $\begin{pmatrix} 0.0667 \\ 0.9333 \end{pmatrix}$ | $\begin{pmatrix} 0.4333 \\ 0.5667 \end{pmatrix}$ | $\begin{pmatrix} 0.9333 \\ 0.0667 \end{pmatrix}$ | $\begin{pmatrix} 0.3667 \\ 0.6333 \end{pmatrix}$ | $\begin{pmatrix} 0.8667 \\ 0.1333 \end{pmatrix}$ | $\begin{pmatrix} 0.8333 \\ 0.1667 \end{pmatrix}$ | $\begin{pmatrix} 0.2667 \\ 0.7333 \end{pmatrix}$ |
| $\omega_4$ | $\begin{pmatrix} 0.6667 \\ 0.3333 \end{pmatrix}$ | $\begin{pmatrix} 0.6667 \\ 0.3333 \end{pmatrix}$ | $\begin{pmatrix} 0.7000 \\ 0.3000 \end{pmatrix}$ | $\begin{pmatrix} 0.3667 \\ 0.6333 \end{pmatrix}$ | $\begin{pmatrix} 0.5667 \\ 0.4333 \end{pmatrix}$ | $\begin{pmatrix} 0.8000 \\ 0.2000 \end{pmatrix}$ | $\begin{pmatrix} 0.6667 \\ 0.3333 \end{pmatrix}$ |
| $\omega_5$ | $\begin{pmatrix} 0.7000 \\ 0.3000 \end{pmatrix}$ | $\begin{pmatrix} 0.9667 \\ 0.0333 \end{pmatrix}$ | $\begin{pmatrix} 0.5667 \\ 0.4333 \end{pmatrix}$ | $\begin{pmatrix} 0.5667 \\ 0.4333 \end{pmatrix}$ | $\begin{pmatrix} 0.5667 \\ 0.4333 \end{pmatrix}$ | $\begin{pmatrix} 0.5000 \\ 0.5000 \end{pmatrix}$ | $\begin{pmatrix} 0.7000 \\ 0.3000 \end{pmatrix}$ |
| $\omega_6$ | $\begin{pmatrix} 0.7000 \\ 0.3000 \end{pmatrix}$ | $\begin{pmatrix} 0.3333 \\ 0.6667 \end{pmatrix}$ | $\begin{pmatrix} 0.5667 \\ 0.4333 \end{pmatrix}$ | $\begin{pmatrix} 0.5667 \\ 0.4333 \end{pmatrix}$ | $\begin{pmatrix} 0.5667 \\ 0.4333 \end{pmatrix}$ | $\begin{pmatrix} 0.8667 \\ 0.1333 \end{pmatrix}$ | $\begin{pmatrix} 0.2333 \\ 0.7667 \end{pmatrix}$ |
| $\omega_7$ | $\begin{pmatrix} 0.7667 \\ 0.2333 \end{pmatrix}$ | $\begin{pmatrix} 0.5667 \\ 0.4333 \end{pmatrix}$ | $\begin{pmatrix} 0.6333 \\ 0.3667 \end{pmatrix}$ | $\begin{pmatrix} 0.5000 \\ 0.5000 \end{pmatrix}$ | $\begin{pmatrix} 0.8333 \\ 0.1667 \end{pmatrix}$ | $\begin{pmatrix} 0.5000 \\ 0.5000 \end{pmatrix}$ | $\begin{pmatrix} 0.4333 \\ 0.5667 \end{pmatrix}$ |
| | | | | PIS/NIS | | | |
| $\widehat{\mathbb{M}}^+$ | $\begin{pmatrix} 0.7667 \\ 0.2333 \end{pmatrix}$ | $\begin{pmatrix} 0.9667 \\ 0.0333 \end{pmatrix}$ | $\begin{pmatrix} 0.9333 \\ 0.0667 \end{pmatrix}$ | $\begin{pmatrix} 0.1333 \\ 0.8667 \end{pmatrix}$ | $\begin{pmatrix} 0.8667 \\ 0.1333 \end{pmatrix}$ | $\begin{pmatrix} 0.8667 \\ 0.1333 \end{pmatrix}$ | $\begin{pmatrix} 0.8667 \\ 0.1333 \end{pmatrix}$ |
| $\widehat{\mathbb{M}}^-$ | $\begin{pmatrix} 0.0667 \\ 0.9333 \end{pmatrix}$ | $\begin{pmatrix} 0.3333 \\ 0.6667 \end{pmatrix}$ | $\begin{pmatrix} 0.4333 \\ 0.5667 \end{pmatrix}$ | $\begin{pmatrix} 0.5667 \\ 0.4333 \end{pmatrix}$ | $\begin{pmatrix} 0.5667 \\ 0.4333 \end{pmatrix}$ | $\begin{pmatrix} 0.5000 \\ 0.5000 \end{pmatrix}$ | $\begin{pmatrix} 0.2333 \\ 0.7667 \end{pmatrix}$ |

**Step VII**: Using the new logarithmic-based distance schemes ($\mathbb{D}_1$ and $\mathbb{D}_2$) for $q = 7$, we obtain the following distance outcomes in Table 7.

**Table 7.** Distance indexes for the vehicle selection analysis

| q-ROFDMs | $(\widehat{\mathbb{M}}^+, \omega_1)$ | $(\widehat{\mathbb{M}}^+, \omega_2)$ | $(\widehat{\mathbb{M}}^+, \omega_3)$ | $(\widehat{\mathbb{M}}^+, \omega_4)$ | $(\widehat{\mathbb{M}}^+, \omega_5)$ | $(\widehat{\mathbb{M}}^+, \omega_6)$ | $(\widehat{\mathbb{M}}^+, \omega_7)$ |
|---|---|---|---|---|---|---|---|
| $\mathbb{D}_1$ | 0.2006 | 0.1619 | 0.2088 | 0.1888 | 0.1677 | 0.2124 | 0.2035 |
| $\mathbb{D}_2$ | 0.0402 | 0.0262 | 0.0436 | 0.0356 | 0.0281 | 0.0451 | 0.0414 |
| | $(\widehat{\mathbb{M}}^-, \omega_1)$ | $(\widehat{\mathbb{M}}^-, \omega_2)$ | $(\widehat{\mathbb{M}}^-, \omega_3)$ | $(\widehat{\mathbb{M}}^-, \omega_4)$ | $(\widehat{\mathbb{M}}^-, \omega_5)$ | $(\widehat{\mathbb{M}}^-, \omega_6)$ | $(\widehat{\mathbb{M}}^-, \omega_7)$ |
| $\mathbb{D}_1$ | 0.1596 | 0.1743 | 0.1438 | 0.1444 | 0.1831 | 0.1355 | 0.1384 |
| $\mathbb{D}_2$ | 0.0255 | 0.0304 | 0.0207 | 0.0208 | 0.0335 | 0.0184 | 0.0191 |
| | $\mathscr{C}(\omega_1)$ | $\mathscr{C}(\omega_2)$ | $\mathscr{C}(\omega_3)$ | $\mathscr{C}(\omega_4)$ | $\mathscr{C}(\omega_5)$ | $\mathscr{C}(\omega_6)$ | $\mathscr{C}(\omega_7)$ |
| $\mathbb{D}_1$ | 0.5569 | 0.4816 | 0.5921 | 0.5667 | 0.4780 | 0.6105 | 0.5952 |
| $\mathbb{D}_2$ | 0.6119 | 0.4630 | 0.6781 | 0.6312 | 0.4562 | 0.7102 | 0.6843 |

**Step VIII**: The nearness coefficients are computed using (12) and presented also in Table 7.

**Step IX**: Based on the computed nearness coefficients, the vehicle rankings are presented as follows:

$$\omega_5 \prec \omega_2 \prec \omega_1 \prec \omega_4 \prec \omega_3 \prec \omega_7 \prec \omega_6,$$

for both new q-ROFDMs. These nearness coefficients show that the new q-ROFDMs distinguished the closely related vehicle alternatives. This suggests that the inclusion of hesitation and cardinality components enhances the discriminatory power of the proposed metrics. From the vehicles' ordering, we see that the suitable vehicle to select for an effective transportation is the vehicle denoted by $\omega_5$, because it is the vehicle closest to the ideal solutions.

## 4.3 *Sensitivity analysis of the distance models*

Now, we evaluate the influence of the q-ROFSs' scales to decide the appropriate scale to apply for a reliable distance outcome using the new logarithmic-based q-ROFDMs. This analysis was conducted using $q = 4, 5, \ldots, 8$. The computed distance indexes between the vehicle alternatives and the PIS/NIS (using Java) are presented in Table 8. From the results, we notice that the distance accuracy increases as $q$ increases. This shows that, it is more reliable to use greater $q$ to enhance precision.

**Table 8.** Distance indexes for sensitivity analysis

| q-ROFDMs | q | $(\widehat{\mathbb{M}}^+, \omega_1)$ | $(\widehat{\mathbb{M}}^+, \omega_2)$ | $(\widehat{\mathbb{M}}^+, \omega_3)$ | $(\widehat{\mathbb{M}}^+, \omega_4)$ | $(\widehat{\mathbb{M}}^+, \omega_5)$ | $(\widehat{\mathbb{M}}^+, \omega_6)$ | $(\widehat{\mathbb{M}}^+, \omega_7)$ |
|---|---|---|---|---|---|---|---|---|
| $\mathbb{D}_1$ | 4 | 0.2114 | 0.1661 | 0.2406 | 0.1907 | 0.1765 | 0.2333 | 0.2130 |
| $\mathbb{D}_2$ | | 0.0447 | 0.0276 | 0.0579 | 0.0364 | 0.0312 | 0.0544 | 0.0454 |
| $\mathbb{D}_1$ | 5 | 0.2093 | 0.1653 | 0.2309 | 0.1924 | 0.1762 | 0.2280 | 0.2127 |
| $\mathbb{D}_2$ | | 0.0438 | 0.0273 | 0.0533 | 0.0370 | 0.0311 | 0.0520 | 0.0452 |
| $\mathbb{D}_1$ | 6 | 0.2054 | 0.1638 | 0.2198 | 0.1914 | 0.1729 | 0.2207 | 0.2091 |
| $\mathbb{D}_2$ | | 0.0422 | 0.0268 | 0.0483 | 0.0366 | 0.0299 | 0.0487 | 0.0437 |
| $\mathbb{D}_1$ | 7 | 0.2006 | 0.1619 | 0.2088 | 0.1888 | 0.1677 | 0.2124 | 0.2035 |
| $\mathbb{D}_2$ | | 0.0402 | 0.0262 | 0.0436 | 0.0356 | 0.0281 | 0.0451 | 0.0414 |
| $\mathbb{D}_1$ | 8 | 0.1952 | 0.1596 | 0.1985 | 0.1850 | 0.1612 | 0.2037 | 0.1969 |
| $\mathbb{D}_2$ | | 0.0381 | 0.0255 | 0.0394 | 0.0342 | 0.0260 | 0.0415 | 0.0388 |
| | | $(\widehat{\mathbb{M}}^-, \omega_1)$ | $(\widehat{\mathbb{M}}^-, \omega_2)$ | $(\widehat{\mathbb{M}}^-, \omega_3)$ | $(\widehat{\mathbb{M}}^-, \omega_4)$ | $(\widehat{\mathbb{M}}^-, \omega_5)$ | $(\widehat{\mathbb{M}}^-, \omega_6)$ | $(\widehat{\mathbb{M}}^-, \omega_7)$ |
| $\mathbb{D}_1$ | 4 | 0.1988 | 0.2199 | 0.1600 | 0.1841 | 0.2121 | 0.1581 | 0.1661 |
| $\mathbb{D}_2$ | | 0.0395 | 0.0484 | 0.0256 | 0.0339 | 0.0450 | 0.0250 | 0.0276 |
| $\mathbb{D}_1$ | 5 | 0.1860 | 0.2049 | 0.1565 | 0.1713 | 0.2030 | 0.1514 | 0.1573 |
| $\mathbb{D}_2$ | | 0.0346 | 0.0420 | 0.0245 | 0.0293 | 0.0412 | 0.0229 | 0.0248 |
| $\mathbb{D}_1$ | 6 | 0.1725 | 0.1892 | 0.1507 | 0.1575 | 0.1928 | 0.1435 | 0.1478 |
| $\mathbb{D}_2$ | | 0.0297 | 0.0358 | 0.0227 | 0.0248 | 0.0372 | 0.0206 | 0.0218 |
| $\mathbb{D}_1$ | 7 | 0.1596 | 0.1743 | 0.1438 | 0.1444 | 0.1831 | 0.1355 | 0.1384 |
| $\mathbb{D}_2$ | | 0.0255 | 0.0304 | 0.0207 | 0.0208 | 0.0335 | 0.0184 | 0.0191 |
| $\mathbb{D}_1$ | 8 | 0.1477 | 0.1606 | 0.1366 | 0.1326 | 0.1742 | 0.1278 | 0.1295 |
| $\mathbb{D}_2$ | | 0.0218 | 0.0258 | 0.0187 | 0.0176 | 0.0304 | 0.0163 | 0.0168 |

The nearness coefficients based on the distance outcomes in Table 8 are computed using (12) and presented in Table 9.

**Table 9.** Nearness coefficients for the vehicle selection analysis

| q-ROFDMs | q | $\mathscr{C}(\omega_1)$ | $\mathscr{C}(\omega_2)$ | $\mathscr{C}(\omega_3)$ | $\mathscr{C}(\omega_4)$ | $\mathscr{C}(\omega_5)$ | $\mathscr{C}(\omega_6)$ | $\mathscr{C}(\omega_7)$ |
|---|---|---|---|---|---|---|---|---|
| $\mathbb{D}_1$ | 4 | 0.5154 | 0.4303 | 0.6006 | 0.5088 | 0.4542 | 0.5961 | 0.5618 |
| $\mathbb{D}_2$ | | 0.5309 | 0.3632 | 0.6934 | 0.5177 | 0.4094 | 0.6851 | 0.6219 |
| $\mathbb{D}_1$ | 5 | 0.5295 | 0.4465 | 0.5960 | 0.5290 | 0.4647 | 0.6009 | 0.5749 |
| $\mathbb{D}_2$ | | 0.5587 | 0.3939 | 0.6851 | 0.5581 | 0.4302 | 0.6943 | 0.6457 |
| $\mathbb{D}_1$ | 6 | 0.5435 | 0.4640 | 0.5933 | 0.5486 | 0.4728 | 0.6060 | 0.5859 |
| $\mathbb{D}_2$ | | 0.5869 | 0.4281 | 0.6803 | 0.5631 | 0.4456 | 0.7027 | 0.6672 |
| $\mathbb{D}_1$ | 7 | 0.5569 | 0.4816 | 0.5921 | 0.5667 | 0.4780 | 0.6105 | 0.5952 |
| $\mathbb{D}_2$ | | 0.6119 | 0.4630 | 0.6781 | 0.6312 | 0.4562 | 0.7102 | 0.6843 |
| $\mathbb{D}_1$ | 8 | 0.5693 | 0.4984 | 0.5924 | 0.5825 | 0.4806 | 0.6145 | 0.6032 |
| $\mathbb{D}_2$ | | 0.6361 | 0.4971 | 0.6781 | 0.6602 | 0.4610 | 0.7180 | 0.6978 |

Using the computed nearness coefficients, the vehicle rankings are presented in Table 10. It is fascinating to see that both the new logarithmic-based distance schemes have the same order, which picks $\omega_2$ as the suitable vehicle brand for $q = 4,\ 5,\ 6$, and select $\omega_5$ as the suitable vehicle brand for $q = 7$ and 8. Since the accuracy of q-ROFS increases as q increases, it is reliable to consider the interpretation for $q = 7$ and 8 ahead of the cases for $q = 4,\ 5,\ 6$.

**Table 10.** Vehicle ordering for the vehicle selection analysis

| q-ROFDMs | q | Ranking | Selected Vehicle |
|---|---|---|---|
| $\mathbb{D}_1$ | 4 | $\omega_2 \prec \omega_5 \prec \omega_4 \prec \omega_1 \prec \omega_7 \prec \omega_6 \prec \omega_3$ | $\omega_2$ |
| $\mathbb{D}_2$ | | $\omega_2 \prec \omega_5 \prec \omega_4 \prec \omega_1 \prec \omega_7 \prec \omega_6 \prec \omega_3$ | $\omega_2$ |
| $\mathbb{D}_1$ | 5 | $\omega_2 \prec \omega_5 \prec \omega_4 \prec \omega_1 \prec \omega_7 \prec \omega_6 \prec \omega_3$ | $\omega_2$ |
| $\mathbb{D}_2$ | | $\omega_2 \prec \omega_5 \prec \omega_4 \prec \omega_1 \prec \omega_7 \prec \omega_6 \prec \omega_3$ | $\omega_2$ |
| $\mathbb{D}_1$ | 6 | $\omega_2 \prec \omega_5 \prec \omega_4 \prec \omega_1 \prec \omega_7 \prec \omega_3 \prec \omega_6$ | $\omega_2$ |
| $\mathbb{D}_2$ | | $\omega_2 \prec \omega_5 \prec \omega_4 \prec \omega_1 \prec \omega_7 \prec \omega_3 \prec \omega_6$ | $\omega_2$ |
| $\mathbb{D}_1$ | 7 | $\omega_5 \prec \omega_2 \prec \omega_1 \prec \omega_4 \prec \omega_3 \prec \omega_7 \prec \omega_6$ | $\omega_5$ |
| $\mathbb{D}_2$ | | $\omega_5 \prec \omega_2 \prec \omega_1 \prec \omega_4 \prec \omega_3 \prec \omega_7 \prec \omega_6$ | $\omega_5$ |
| $\mathbb{D}_1$ | 8 | $\omega_5 \prec \omega_2 \prec \omega_1 \prec \omega_4 \prec \omega_7 \prec \omega_3 \prec \omega_6$ | $\omega_5$ |
| $\mathbb{D}_2$ | | $\omega_5 \prec \omega_2 \prec \omega_1 \prec \omega_4 \prec \omega_7 \prec \omega_3 \prec \omega_6$ | $\omega_5$ |

The application of the proposed logarithmic-based q-ROF distance metrics within the TOPSIS framework yielded stable and discriminative results in the VSP. The sensitivity check across varying values of the parameter $q$ further confirmed the robustness of the proposed approach. As $q$ increased, the distance indexes in Table 8 became more precise, and the resulting rankings in Table 10 remained stable. This robustness is crucial in real-world decision-making, where experts' evaluations may fluctuate slightly due to subjective judgments.

## 4.4 *Comparative analysis based on TOPSIS approach*

We compare the new logarithmic-based q-ROFDMs with the existing logarithmic-based q-ROFDMs for $q = 7$ using the VSP data based on the TOPSIS approach, in order to show their advantage in terms of decision interpretation. Now, we compute the distance values between the vehicle brand alternatives and the ideal alternatives (NIS/PIS), to obtain the nearness coefficients and their orderings. The comparison analysis results are presented in Tables 11-13, and the comparative distance indexes are pictorially displayed in Figures 2 and 3.

**Table 11.** Comparative distance indexes

| q-ROFDMs | $(\widehat{\mathbb{M}}^{+},\ \omega_1)$ | $(\widehat{\mathbb{M}}^{+},\ \omega_2)$ | $(\widehat{\mathbb{M}}^{+},\ \omega_3)$ | $(\widehat{\mathbb{M}}^{+},\ \omega_4)$ | $(\widehat{\mathbb{M}}^{+},\ \omega_5)$ | $(\widehat{\mathbb{M}}^{+},\ \omega_6)$ | $(\widehat{\mathbb{M}}^{+},\ \omega_7)$ |
|---|---|---|---|---|---|---|---|
| $\mathbb{D}_{We1}$[57] | 0.5716 | 0.4480 | 0.6065 | 0.5206 | 0.4851 | 0.6084 | 0.5688 |
| $\mathbb{D}_{We2}$ [57] | 0.6499 | 0.5245 | 0.6767 | 0.6118 | 0.5433 | 0.6881 | 0.6595 |
| $\mathbb{D}_{Se}$ [58] | 0.1079 | 0.0711 | 0.1054 | 0.0951 | 0.0683 | 0.1118 | 0.1057 |
| $\mathbb{D}_1$ | 0.2006 | 0.1619 | 0.2088 | 0.1888 | 0.1677 | 0.2124 | 0.2035 |
| $\mathbb{D}_2$ | 0.0402 | 0.0262 | 0.0436 | 0.0356 | 0.0281 | 0.0451 | 0.0414 |
| q-ROFDMs | $(\widehat{\mathbb{M}}^{-},\ \omega_1)$ | $(\widehat{\mathbb{M}}^{-},\ \omega_2)$ | $(\widehat{\mathbb{M}}^{-},\ \omega_3)$ | $(\widehat{\mathbb{M}}^{-},\ \omega_4)$ | $(\widehat{\mathbb{M}}^{-},\ \omega_5)$ | $(\widehat{\mathbb{M}}^{-},\ \omega_6)$ | $(\widehat{\mathbb{M}}^{-},\ \omega_7)$ |
| $\mathbb{D}_{We1}$[57] | 0.4880 | 0.5368 | 0.4186 | 0.4352 | 0.5223 | 0.3999 | 0.4199 |
| $\mathbb{D}_{We2}$ [57] | 0.5170 | 0.5647 | 0.4660 | 0.4678 | 0.5932 | 0.4391 | 0.4484 |
| $\mathbb{D}_{Se}$ [58] | 0.0499 | 0.0568 | 0.0479 | 0.0387 | 0.0818 | 0.0426 | 0.0413 |
| $\mathbb{D}_1$ | 0.1596 | 0.1743 | 0.1438 | 0.1444 | 0.1831 | 0.1355 | 0.1384 |
| $\mathbb{D}_2$ | 0.0255 | 0.0304 | 0.0207 | 0.0208 | 0.0335 | 0.0184 | 0.0191 |

**Table 12.** Comparative nearness coefficients

| q-ROFDMs | $\mathscr{C}(\omega_1)$ | $\mathscr{C}(\omega_2)$ | $\mathscr{C}(\omega_3)$ | $\mathscr{C}(\omega_4)$ | $\mathscr{C}(\omega_5)$ | $\mathscr{C}(\omega_6)$ | $\mathscr{C}(\omega_7)$ |
|---|---|---|---|---|---|---|---|
| $\mathbb{D}_{We1}$[57] | 0.5394 | 0.4549 | 0.5916 | 0.5447 | 0.4815 | 0.6034 | 0.5753 |
| $\mathbb{D}_{We2}$ [57] | 0.5569 | 0.4815 | 0.5921 | 0.5667 | 0.4780 | 0.6105 | 0.5953 |
| $\mathbb{D}_{Se}$ [58] | 0.6838 | 0.5559 | 0.6875 | 0.7108 | 0.4550 | 0.7241 | 0.7190 |
| $\mathbb{D}_1$ | 0.5569 | 0.4816 | 0.5921 | 0.5667 | 0.4780 | 0.6105 | 0.5952 |
| $\mathbb{D}_2$ | 0.6119 | 0.4630 | 0.6781 | 0.6312 | 0.4562 | 0.7102 | 0.6843 |

**Table 13.** Comparative vehicle ordering

| q-ROFDMs | Ordering | Selected vehicle |
|---|---|---|
| $\mathbb{D}_{We1}$[57] | $\omega_2 \prec \omega_5 \prec \omega_1 \prec \omega_4 \prec \omega_7 \prec \omega_3 \prec \omega_6$ | $\omega_2$ |
| $\mathbb{D}_{We2}$ [57] | $\omega_5 \prec \omega_2 \prec \omega_4 \prec \omega_1 \prec \omega_3 \prec \omega_7 \prec \omega_6$ | $\omega_5$ |
| $\mathbb{D}_{Se}$ [58] | $\omega_5 \prec \omega_2 \prec \omega_1 \prec \omega_3 \prec \omega_4 \prec \omega_7 \prec \omega_6$ | $\omega_5$ |
| $\mathbb{D}_1$ | $\omega_5 \prec \omega_2 \prec \omega_1 \prec \omega_4 \prec \omega_3 \prec \omega_7 \prec \omega_6$ | $\omega_5$ |
| $\mathbb{D}_2$ | $\omega_5 \prec \omega_2 \prec \omega_1 \prec \omega_4 \prec \omega_3 \prec \omega_7 \prec \omega_6$ | $\omega_5$ |

To reinforce the innovativeness, the comparison results between the newly proposed and existing logarithmic q-ROFDMs in Tables 11-13 and Figures 2 and 3 show that, the new logarithmic-based q-ROFDMs produces more precise and efficient closeness coefficients and more stable rankings. In contrast, the existing methods in $\mathbb{D}_{We1}$ [57] and $\mathbb{D}_{Se}$ [58], which did not consider the HMs of the considered sets exhibit weaker separability of alternatives. It is also worthy to note that, the new logarithmic-based q-ROFDMs produce the same ordering like the methods $\mathbb{D}_{We2}$ [57] and $\mathbb{D}_{Se}$ [58], selecting $\omega_5$ as the best vehicle brand for an effective transportation system. However, the method in $\mathbb{D}_{we1}$ [57] has different ordering and selects $\omega_2$ as the suitable vehicle. This discrepancy is due to the omission of HMs in the scheme of $\mathbb{D}_{We1}$ [57]. The omission of hesitation margins in [57] and [58] resulted in weaker separability, whereas the proposed methods provided rankings that are both consistent and interpretable. This demonstrates that the integration of all three q-ROF's parameters minimizes information loss and provides a fuller representation of expert uncertainty. Thus, the proposed methods are not merely a theoretical extension, but a significant innovation that strengthens both the mathematical reliability and the decision-making effectiveness of the fuzzy MCDM.

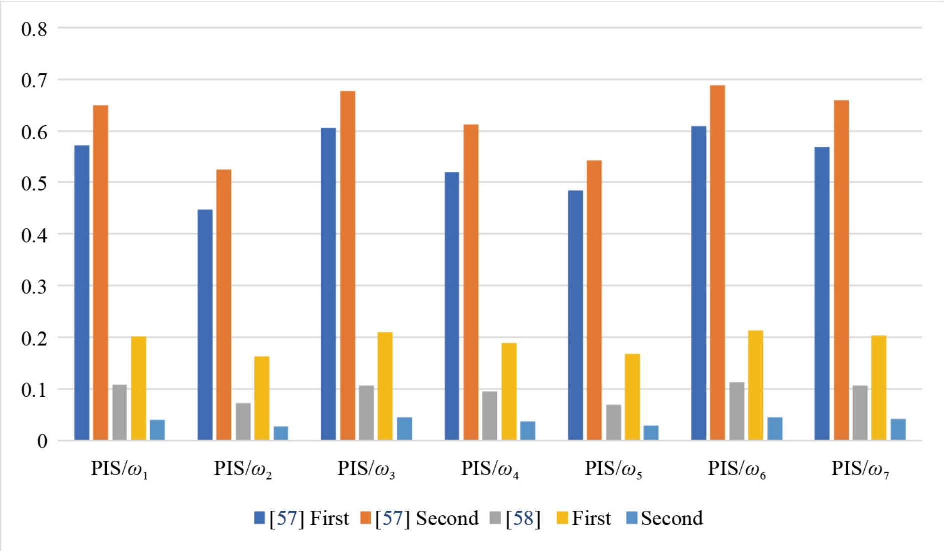


**Figure 2.** Distances between $\widehat{\mathbb{M}}^{+}$ and vehicle brands

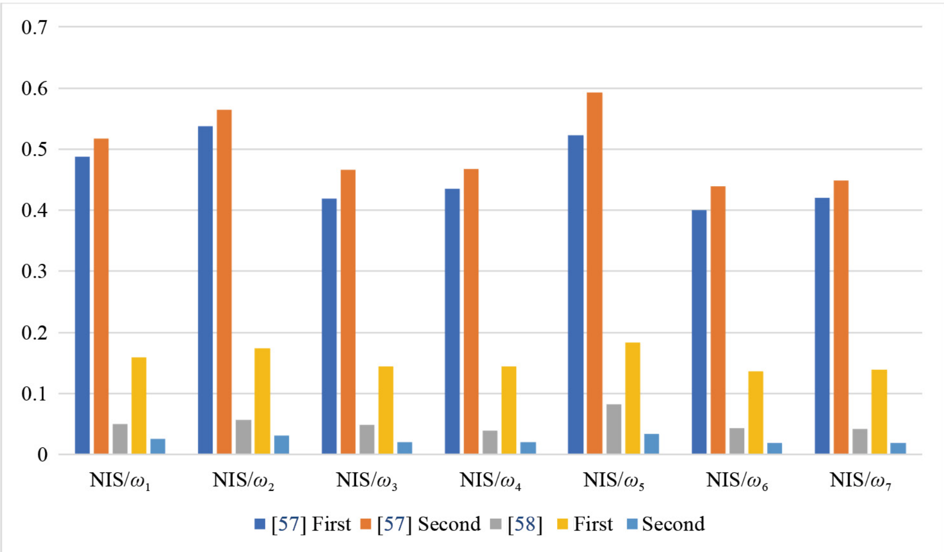


**Figure 3.** Distances between $\widehat{\mathbb{M}}^{-}$ and vehicle brands

## 5. Conclusion

In this study, we proposed two novel logarithmic-based distance measures within the q-ROF framework and demonstrated their theoretical soundness by proving that they satisfy all the axioms of the distance metric. Unlike existing logarithmic-based q-ROF distance measures, the proposed methods explicitly integrate MD, NMD, and HM, thereby minimizing information loss and improving interpretability in the MCDM context. To validate their applicability, the new distance matrices were embedded into the TOPSIS technique and applied to a real-world VSP involving seven vehicle brands and seven evaluation criteria derived from expert opinions. The results revealed that the proposed metrics outperform existing logarithmic-based q-ROF distance measures in terms of accuracy, stability, and their ability to discriminate between closely ranked alternatives. Specifically, the comparative analysis showed that the new metrics provide more consistent closeness coefficients and more reliable vehicle rankings, which underscores their robustness in decision-making under uncertainty. The implications of these findings extend beyond the transportation systems to practical decision-making problems. By offering a more flexible and comprehensive way to handle vagueness and conflicting criteria, the proposed q-ROF distance measures can be applied to a wide range of CDM problems, including renewable energy planning, medical diagnosis, financial investment, and smart manufacturing. This versatility highlights the broader relevance of the approach in both theoretical research and practical applications. Future research can build on this work in several directions. First, the proposed logarithmic-based distance metrics can be extended to other fuzzy environments, such as spherical fuzzy sets, picture fuzzy sets, and neutrosophic sets, to further enhance their utility. Second, integrating these distance measures with alternative MCDM methods (e.g., VIKOR, MABAC, MAIRCA, or RAFSI) could provide deeper insights into their comparative performance across different decision-making paradigms. Although the TOPSIS method is generally accepted due to its effectiveness in decision-making, its limitations have to do with the difficulty in selecting the appropriate one from multiple variants of TOPSIS and the issue of reliability [64, 65]. In subsequent studies, it is expedient to use the hybrid TOPSIS for better effectiveness. Finally, real-time decision-support systems could be developed by embedding the proposed methods into intelligent software platforms, thereby enabling practitioners to make more reliable and timely decisions in dynamic and uncertain environments.

## Confilict of interest

The authors declare no competing financial interest.

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## Appendix A. Java code for the two new logarithmic-based q-ROFDMs

```
public class LogarithmicMethod1and2 {
public static void main(String[] args) {
double n = 7;
double p=0.0;
double aA1P = 0.0;
double bA1P = 0.0;
double aA2P = 0.0;
double bA2P = 0.0;
double aA3P = 0.0;
double bA3P = 0.0;
double aA4P = 0.0;
double bA4P = 0.0;
double aA5P = 0.0;
double bA5P = 0.0;
double aA6P = 0.0;
double bA6P = 0.0;
double aA7P = 0.0;
double bA7P = 0.0;
double msumA1P =0.0;
double nsumA1P =0.0;
double hsumA1P =0.0;
double msumA2P =0.0;
double nsumA2P =0.0;
double hsumA2P =0.0;
double msumA3P =0.0;
double nsumA3P =0.0;
double hsumA3P =0.0;
double msumA4P =0.0;
double nsumA4P =0.0;
double hsumA4P =0.0;
double msumA5P =0.0;
double nsumA5P =0.0;
double hsumA5P =0.0;
double msumA6P =0.0;
double nsumA6P =0.0;
double hsumA6P =0.0;
double msumA7P =0.0;
double nsumA7P =0.0;
double hsumA7P =0.0;
double scale = Math.pow(10, 4);
double muA1[] = {0.7667, 0.7000, 0.4333, 0.8667, 0.7000, 0.7667, 0.7000};
double nuA1[] = {0.2333, 0.3000, 0.5667, 0.1333, 0.3000, 0.2333, 0.3000};
double piA1[] = new double[7];
double muA2[] = {0.7333, 0.7667, 0.7667, 0.8000, 0.7000, 0.7667, 0.8667};
double nuA2[] = {0.2667, 0.2333, 0.2333, 0.2000, 0.3000, 0.2333, 0.1333};
double piA2[] = new double[7];
```

```
double muA3[] = {0.0667, 0.4333, 0.9333, 0.6333, 0.8667, 0.8333, 0.2667};
double nuA3[] = {0.9333, 0.5667, 0.0667, 0.3667, 0.1333, 0.1667, 0.7333};
double piA3[] = new double[7];
double muA4[] = {0.6667, 0.6667, 0.7000, 0.6333, 0.5667, 0.8000, 0.6667};
double nuA4[] = {0.3333, 0.3333, 0.3000, 0.3667, 0.4333, 0.2000, 0.3333};
double piA4[] = new double[7];
double muA5[] = {0.7000, 0.9667, 0.5667, 0.4333, 0.5667, 0.5000, 0.7000};
double nuA5[] = {0.3000, 0.0333, 0.4333, 0.5667, 0.4333, 0.5000, 0.3000};
double piA5[] = new double[7];
double muA6[] = {0.7000, 0.3333, 0.5667, 0.4333, 0.5667, 0.8667, 0.2333};
double nuA6[] = {0.3000, 0.6667, 0.4333, 0.5667, 0.4333, 0.1333, 0.7667};
double piA6[] = new double[7];
double muA7[] = {0.7667, 0.5667, 0.6333, 0.5000, 0.8333, 0.5000, 0.4333};
double nuA7[] = {0.2333, 0.4333, 0.3667, 0.5000, 0.1667, 0.5000, 0.5667};
double piA7[] = new double[7];
double muP[] = {0.7667, 0.9667, 0.9333, 0.1333, 0.8667, 0.8667, 0.8667};
double nuP[] = {0.2333, 0.0333, 0.0667, 0.8667, 0.1333, 0.1333, 0.1333};
double piP[] = new double[7];
for(int i = 0; i<=n-1; i++)
{
piP[i] = Math.pow(1 - Math.pow(muP[i], 6.0) - Math.pow(nuP[i], 6.0), 0.16666666667);
piA1[i] = Math.pow(1 - Math.pow(muA1[i], 6.0) - Math.pow(nuA1[i], 6.0),0.16666666667);
piA2[i] = Math.pow(1 - Math.pow(muA2[i], 6.0) - Math.pow(nuA2[i], 6.0),0.16666666667);
piA3[i] = Math.pow(1 - Math.pow(muA3[i], 6.0) - Math.pow(nuA3[i], 6.0),0.16666666667);
piA4[i] = Math.pow(1 - Math.pow(muA4[i], 6.0) - Math.pow(nuA4[i], 6.0),0.16666666667);
piA5[i] = Math.pow(1 - Math.pow(muA5[i], 6.0) - Math.pow(nuA5[i], 6.0),0.16666666667);
piA6[i] = Math.pow(1 - Math.pow(muA6[i], 6.0) - Math.pow(nuA6[i], 6.0),0.16666666667);
piA7[i] = Math.pow(1 - Math.pow(muA7[i], 6.0) - Math.pow(nuA7[i], 6.0),0.16666666667);
}
for (int i = 0; i<=n-1; i++) {
msumA1P += Math.pow(muA1[i], 6.0) * Math.log10(2 * Math.pow(muA1[i], 6.0) / (Math.pow(muA1[i], 6.0) + Math.pow(muP[i], 6.0)))
+ Math.pow(muP[i], 6.0) * Math.log10(2 * Math.pow(muP[i], 6.0) / (Math.pow(muA1[i], 6.0) + Math.pow(muP[i], 6.0)));
nsumA1P += Math.pow(nuA1[i], 6.0) * Math.log10(2 * Math.pow(nuA1[i], 6.0) / (Math.pow(nuA1[i], 6.0) + Math.pow(nuP[i], 6.0)))
+ Math.pow(nuP[i], 6.0) * Math.log10(2 * Math.pow(nuP[i], 6.0) / (Math.pow(nuA1[i], 6.0) + Math.pow(nuP[i], 6.0)));
hsumA1P += Math.pow(piA1[i], 6.0) * Math.log10(2 * Math.pow(piA1[i], 6.0) / (Math.pow(piA1[i], 6.0) + Math.pow(piP[i], 6.0)))
+ Math.pow(piP[i], 6.0) * Math.log10(2 * Math.pow(piP[i], 6.0) / (Math.pow(piA1[i], 6.0) + Math.pow(piP[i], 6.0)));
msumA2P += Math.pow(muA2[i], 6.0) * Math.log10(2 * Math.pow(muA2[i], 6.0) / (Math.pow(muA2[i], 6.0) + Math.pow(muP[i], 6.0)))
+ Math.pow(muP[i], 6.0) * Math.log10(2 * Math.pow(muP[i], 6.0) / (Math.pow(muA2[i], 6.0) + Math.pow(muP[i], 6.0)));
nsumA2P += Math.pow(nuA2[i], 6.0) * Math.log10(2 * Math.pow(nuA2[i], 6.0) / (Math.pow(nuA2[i], 6.0) + Math.pow(nuP[i], 6.0)))
```

```
+ Math.pow(nuP[i], 6.0) * Math.log10(2 * Math.pow(nuP[i], 6.0) / (Math.pow(nuA2[i], 6.0) + Math.pow(nuP[i], 6.0)));
hsumA2P += Math.pow(piA2[i], 6.0) * Math.log10(2 * Math.pow(piA2[i], 6.0) / (Math.pow(piA2[i], 6.0) + Math.pow(piP[i], 6.0)))
+ Math.pow(piP[i], 6.0) * Math.log10(2 * Math.pow(piP[i], 6.0) / (Math.pow(piA2[i], 6.0) + Math.pow(piP[i], 6.0)));
msumA3P += Math.pow(muA3[i], 6.0) * Math.log10(2 * Math.pow(muA3[i], 6.0) / (Math.pow(muA3[i], 6.0) + Math.pow(muP[i], 6.0)))
+ Math.pow(muP[i], 6.0) * Math.log10(2 * Math.pow(muP[i], 6.0) / (Math.pow(muA3[i], 6.0) + Math.pow(muP[i], 6.0)));
nsumA3P += Math.pow(nuA3[i], 6.0) * Math.log10(2 * Math.pow(nuA3[i], 6.0) / (Math.pow(nuA3[i], 6.0) + Math.pow(nuP[i], 6.0)))
+ Math.pow(nuP[i], 6.0) * Math.log10(2 * Math.pow(nuP[i], 6.0) / (Math.pow(nuA3[i], 6.0) + Math.pow(nuP[i], 6.0)));
hsumA3P += Math.pow(piA3[i], 6.0) * Math.log10(2 * Math.pow(piA3[i], 6.0) / (Math.pow(piA3[i], 6.0) + Math.pow(piP[i], 6.0)))
+ Math.pow(piP[i], 6.0) * Math.log10(2 * Math.pow(piP[i], 6.0) / (Math.pow(piA3[i], 6.0) + Math.pow(piP[i], 6.0)));
msumA4P += Math.pow(muA4[i], 6.0) * Math.log10(2 * Math.pow(muA4[i], 6.0) / (Math.pow(muA4[i], 6.0) + Math.pow(muP[i], 6.0)))
+ Math.pow(muP[i], 6.0) * Math.log10(2 * Math.pow(muP[i], 6.0) / (Math.pow(muA4[i], 6.0) + Math.pow(muP[i], 6.0)));
nsumA4P += Math.pow(nuA4[i], 6.0) * Math.log10(2 * Math.pow(nuA4[i], 6.0) / (Math.pow(nuA4[i], 6.0) + Math.pow(nuP[i], 6.0)))
+ Math.pow(nuP[i], 6.0) * Math.log10(2 * Math.pow(nuP[i], 6.0) / (Math.pow(nuA4[i], 6.0) + Math.pow(nuP[i], 6.0)));
hsumA4P += Math.pow(piA4[i], 6.0) * Math.log10(2 * Math.pow(piA4[i], 6.0) / (Math.pow(piA4[i], 6.0) + Math.pow(piP[i], 6.0)))
+ Math.pow(piP[i], 6.0) * Math.log10(2 * Math.pow(piP[i], 6.0) / (Math.pow(piA4[i], 6.0) + Math.pow(piP[i], 6.0)));
msumA5P += Math.pow(muA5[i], 6.0) * Math.log10(2 * Math.pow(muA5[i], 6.0) / (Math.pow(muA5[i], 6.0) + Math.pow(muP[i], 6.0)))
+ Math.pow(muP[i], 6.0) * Math.log10(2 * Math.pow(muP[i], 6.0) / (Math.pow(muA5[i], 6.0) + Math.pow(muP[i], 6.0)));
nsumA5P += Math.pow(nuA5[i], 6.0) * Math.log10(2 * Math.pow(nuA5[i], 6.0) / (Math.pow(nuA5[i], 6.0) + Math.pow(nuP[i], 6.0)))
+ Math.pow(nuP[i], 6.0) * Math.log10(2 * Math.pow(nuP[i], 6.0) / (Math.pow(nuA5[i], 6.0) + Math.pow(nuP[i], 6.0)));
hsumA5P += Math.pow(piA5[i], 6.0) * Math.log10(2 * Math.pow(piA5[i], 6.0) / (Math.pow(piA5[i], 6.0) + Math.pow(piP[i], 6.0)))
+ Math.pow(piP[i], 6.0) * Math.log10(2 * Math.pow(piP[i], 6.0) / (Math.pow(piA5[i], 6.0) + Math.pow(piP[i], 6.0)));
msumA6P += Math.pow(muA6[i], 6.0) * Math.log10(2 * Math.pow(muA6[i], 6.0) / (Math.pow(muA6[i], 6.0) + Math.pow(muP[i], 6.0)))
+ Math.pow(muP[i], 6.0) * Math.log10(2 * Math.pow(muP[i], 6.0) / (Math.pow(muA6[i], 6.0) + Math.pow(muP[i], 6.0)));
nsumA6P += Math.pow(nuA6[i], 6.0) * Math.log10(2 * Math.pow(nuA6[i], 6.0) / (Math.pow(nuA6[i], 6.0) + Math.pow(nuP[i], 6.0)))
+ Math.pow(nuP[i], 6.0) * Math.log10(2 * Math.pow(nuP[i], 6.0) / (Math.pow(nuA6[i], 6.0) + Math.pow(nuP[i], 6.0)));
hsumA6P += Math.pow(piA6[i], 6.0) * Math.log10(2 * Math.pow(piA6[i], 6.0) / (Math.pow(piA6[i], 6.0) + Math.pow(piP[i], 6.0)))
```

```
+ Math.pow(piP[i], 6.0) * Math.log10(2 * Math.pow(piP[i], 6.0) / (Math.pow(piA6[i], 6.0) + Math.pow(piP[i], 6.0)));
msumA7P += Math.pow(muA7[i], 6.0) * Math.log10(2 * Math.pow(muA7[i], 6.0) / (Math.pow(muA7[i], 6.0) + Math.pow(muP[i], 6.0)))
+ Math.pow(muP[i], 6.0) * Math.log10(2 * Math.pow(muP[i], 6.0) / (Math.pow(muA7[i], 6.0) + Math.pow(muP[i], 6.0)));
nsumA7P += Math.pow(nuA7[i], 6.0) * Math.log10(2 * Math.pow(nuA7[i], 6.0) / (Math.pow(nuA7[i], 6.0) + Math.pow(nuP[i], 6.0)))
+ Math.pow(nuP[i], 6.0) * Math.log10(2 * Math.pow(nuP[i], 6.0) / (Math.pow(nuA7[i], 6.0) + Math.pow(nuP[i], 6.0)));
hsumA7P += Math.pow(piA7[i], 6.0) * Math.log10(2 * Math.pow(piA7[i], 6.0) / (Math.pow(piA7[i], 6.0) + Math.pow(piP[i], 6.0)))
+ Math.pow(piP[i], 6.0) * Math.log10(2 * Math.pow(piP[i], 6.0) / (Math.pow(piA7[i], 6.0) + Math.pow(piP[i], 6.0)));
}
aA1P = (double) Math.round(Math.sqrt((msumA1P+nsumA1P+hsumA1P)/(3*n)) * scale) / scale;
bA1P = (double) Math.round((msumA1P+nsumA1P+hsumA1P)/(3*n) * scale) / scale;
aA2P = (double) Math.round(Math.sqrt((msumA2P+nsumA2P+hsumA2P)/(3*n)) * scale) / scale;
bA2P = (double) Math.round((msumA2P+nsumA2P+hsumA2P)/(3*n) * scale) / scale;
aA3P = (double) Math.round(Math.sqrt((msumA3P+nsumA3P+hsumA3P)/(3*n)) * scale) / scale;
bA3P = (double) Math.round((msumA3P+nsumA3P+hsumA3P)/(3*n) * scale) / scale;
aA4P = (double) Math.round(Math.sqrt((msumA4P+nsumA4P+hsumA4P)/(3*n)) * scale) / scale;
bA4P = (double) Math.round((msumA4P+nsumA4P+hsumA4P)/(3*n) * scale) / scale;
aA5P = (double) Math.round(Math.sqrt((msumA5P+nsumA5P+hsumA5P)/(3*n)) * scale) / scale;
bA5P = (double) Math.round((msumA5P+nsumA5P+hsumA5P)/(3*n) * scale) / scale;
aA6P = (double) Math.round(Math.sqrt((msumA6P+nsumA6P+hsumA6P)/(3*n)) * scale) / scale;
bA6P = (double) Math.round((msumA6P+nsumA6P+hsumA6P)/(3*n) * scale) / scale;
aA7P = (double) Math.round(Math.sqrt((msumA7P+nsumA7P+hsumA7P)/(3*n)) * scale) / scale;
bA7P = (double) Math.round((msumA7P+nsumA7P+hsumA7P)/(3*n) * scale) / scale;
System.out.println("aA1P: " + aA1P);
System.out.println("aA2P: " + aA2P);
System.out.println("aA3P: " + aA3P);
System.out.println("aA4P: " + aA4P);
System.out.println("aA5P: " + aA5P);
System.out.println("aA6P: " + aA6P);
System.out.println("aA7P: " + aA7P);
System.out.println("");
System.out.println("");
System.out.println("bA1P: " + bA1P);
System.out.println("bA2P: " + bA2P);
System.out.println("bA3P: " + bA3P);
System.out.println("bA4P: " + bA4P);
System.out.println("bA5P: " + bA5P);
System.out.println("bA6P: " + bA6P);
System.out.println("bA7P: " + bA7P); }
}
```