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Research Article

# Dynamics of Prey Predator Model with Crowley-Martin Functional Response, Prey Refuge and Immigration of Both Species

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**Abstract:** In this paper, we introduce a novel prey-predator model with Crowley-Martin functional response, prey refuge, and immigration of both species. The existence, positivity, and boundedness of solutions are proved. The equilibrium points and stability are discussed. Theoretical analysis shows that if the model includes prey refuges but does not involve immigration, one or both species will go extinct. However, if the model includes both prey refuges and immigration, then no species will go extinct. In addition, the conditions for the stability of the system locally and globally are obtained, which show that the refuge of prey and the immigration of both species play important roles in the uniqueness of the coexistence equilibrium point, the stability of the system, and the existence of a limit cycle. Numerical simulations are performed to verify and explain the analytical results. It has been found that increasing the rate of refuges to a sufficient extent leads to the extinction of predators, but with the presence of constant immigration it leads to the survival of the system. The numerical results also show that prey refuges and constant immigration lead to system stability and increased species density. The results of this study are considered as conservation strategies for species survival and biodiversity continuity. Some ecological explanations are offered in this study.

***Keywords*:** prey-predator model, Crowley-Martin functional response, prey refuge, immigration, stability

**MSC:** 92B05, 34D05, 93C10

## 1. Introduction

Mathematical modeling is an important tool for describing many ecological phenomena as it gives a systematic form to simplify and analyze complex interactions, predict system behavior, and deduce key dynamics. The dynamics of many ecosystems can be represented through mathematical models, such as predator-prey interactions [1], competing species [2], disease transmission [3], etc. One of the most important topics in mathematical ecology is the predator-prey system, with its original model back to the basic contributions of Lotka [4] and Volterra [5]. Since then, numerous extensions of the Lotka-Volterra model have been developed to incorporate ecological and environmental factors that influence species interactions. Despite this progress, the study of predator-prey dynamics continues to represent a central and challenging problem in applied mathematics due to the intrinsic complexity and ecological significance of these systems.

The functional and numerical responses form the cornerstone of predator-prey systems, serving as key links within predator-prey equations. Functional responses describe how predators consume prey, while numerical responses reflect

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changes in predator populations resulting from this consumption. Various developments and forms of functional responses have been proposed. Holling introduced a prominent family of functional responses, namely types I, II, and III [6–8]. While Holling's responses are based exclusively on prey density, the Beddington-DeAngelis [9, 10] and Crowley-Martin [11] models incorporate both prey and predator densities. The Beddington-DeAngelis and Crowley-Martin types, which are extensions of Holling's type II, are widely applied in ecological studies. The Beddington-DeAngelis functional response accounts for prey and predator density; however, predator interference is not significant at high prey densities. In contrast, the Crowley-Martin functional response incorporates predator interference even at high prey densities.

The Crowley-Martin model has been little used in the literature due to its complicated mathematical analysis. However, some recent research has considered predator-prey models with the Crowley-Martin functional response in combination with various additional factors. Kong and Shao [12] suggested a predator prey model that involves fear, stage structure, time delays, and the Crowley-Martin functional response, and they mainly discussed the effects of these factors on the stability of the system. Mondal et al. [13] proposed the interaction between a generalist predator and prey with the Crowley-Martin response function, taking into account the effect of hunting cooperation, and they studied the System's dynamics and improved their model by introducing seasonal perturbations, which showed various patterns in response to seasonality. Anshu Sourav and Dubey [14] introduced a predator-prey model with a Crowley-Martin functional response involving the fear effect and hunting cooperation to study the effects of these factors on the dynamics of this system. Recently, Alebraheem [15] proposed seasonal carrying capacity predator-prey models with the Crowley-Martin functional response to investigate the impacts of periodic and sudden perturbations on the dynamics.

In the predator-prey interaction, habitat structure and selection play a crucial role. Refuges provide a state of safety for prey from predation. By reducing predation risk, the presence of refuge habitats can significantly alter predator-prey dynamics. In 2005, Kar [16] introduced a fixed proportion of prey refuges into a type II functional response function in a predator-prey system. Since then, subsequent research has examined the role of refuges in predator-prey systems, incorporating additional factors such as the fear effect [17–20], stage structure and harvesting [21], intraspecific competition and harvesting [22], and predator infection and harvesting [23]. Some theoretical and experimental studies based on prey refuges have explained that prey refuges can stabilize predator-prey interactions and prevent prey extinction [24–28]. While protecting prey may benefit these species and allow them to reach the carrying capacity of their environment, it can simultaneously harm predator populations, potentially leading to their extinction [29]. Consequently, this can harm the ecosystem and reduce biodiversity within these environments. Immigration is considered an important factor that can help achieve environmental stability and the survival of endangered species [30–32]. It may serve as a solution to this problem by protecting biodiversity through the so-called rescue phenomenon [33]. In the literature, few research papers have examined the immigration and refuge of prey within the same model. Mukherjee [34] incorporated constant prey refuges and the immigration of the competitor for the prey in two-prey, one-predator model, which use Holling type I functional response. They concluded that the refuge of the prey contributes to stabilizing the system, whereas an increase in the constant immigration of prey populations tends to destabilize it. Stephano and Jung [35] used a modified Lotka-Volterra model, based on the work of Tahara et al. [30], which incorporated a small number of immigrants into the Lotka-Volterra framework and applied Holling type I, II and III functional responses. Their models therefore combined the effects of prey refuge with limited immigration. A refuge factor, considered with and without immigration, was used to numerically investigate the influence of prey refuge and immigration. The results showed that system stability depends on the type of functional response: the model remained stable under Holling type I and III responses but became unstable under Holling type II.

The Crowley-Martin functional response is considered a more general model than the Holling and DeAngelis functional responses, as it accounts for predator interference and offers greater flexibility in representing real-world interactions. Incorporating immigration and prey refuges can promote more stable and coexistent dynamics, particularly in endangered systems. Motivated by the research background described above, we introduce a more general predator-prey model using the Crowley-Martin functional response and incorporating prey refuge and immigration for both species to examine the dynamic behavior of this model, which may address some of the problems related to predator-prey interactions to real-world situations and provide a deeper understanding of ecological interactions.

The paper is organized as follows. Section 2 presents the proposed model, which incorporates the Crowley-Martin functional response along with the refuge of prey and the immigration of both species. Section 3 discusses the analytical aspects of the model, including existence, positivity, boundedness, and local and global stability. Section 4 provides numerical simulations that support and illustrate the theoretical findings. Finally, Section 5 concludes the paper.

## 2. The model

### 2.1 *Formulation of model*

In 2021, Alebraheem [31] used the Crowley-Martin functional response predator-prey model with oscillation prey immigration. More recently, Surendar et al. [36] employed the Crowley-Martin functional response in combination with a Holling type II numerical response to describe predator-prey interactions, incorporating prey refuge and intraspecific competition among predators. Let the densities of prey and predator be denoted by $x$ and $y$, respectively. In the absence of predators, prey species follow logistic growth. The Crowley-Martin functional response is used to describe both prey consumption and the resulting predator population growth. In addition, prey refuge and immigration of both species are included as key factors. Based on these considerations, we propose the following non-dimensional predator-prey model incorporating the Crowley-Martin functional response, prey refuge, and immigration of both species:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right)-\frac{\zeta(1-m)xy}{1+\gamma(1-m)x+dy+\gamma(1-m)dxy}+\eta_1,\\ \frac{dy}{dt} &= -\delta y+\frac{\nu\zeta(1-m)xy}{1+\gamma(1-m)x+dy+\gamma(1-m)dxy}+\eta_2,\end{aligned} \tag{1}$$

subject to

$$x(0)=x_0\geq 0,\quad y(0)=y_0\geq 0.$$

All initial conditions and parameters are assumed to have positive values and $m \in [0,\ 1)$, for biological relevance. The biological interpretations of the parameters are provided in Table 1.

**Table 1.** List of parameters with their biological meaning

| Symbol | The meaning |
|---|---|
| $r$ | Inherent growth rate of prey |
| $K$ | Carrying capacity of the system |
| $\zeta$ | Capture rate |
| $m$ | Refuge prey |
| $\gamma$ | Handling time |
| $\delta$ | Death rate of predator |
| $\nu$ | Conversion rate |
| $d$ | Interference of predator |
| $\eta_1$ | Prey immigrants size |
| $\eta_2$ | Predator immigrants size |

# 3. Analytical results

## 3.1 *The existence and positivity*

Our purpose is to verify that the solution of the model (1) exists and is positive.

**Theorem 3.1** The solution to model (1) exists, is unique, and is positive for all values of $t \geq 0$ in the region $\psi \times [0, \chi)$ where:

$$\psi = \left\{ (x, y) \in R_{+}^{2} : \max(|x|, |y|) \leq L \right\},$$

for $L$ is sufficiently large.

**Proof.** Consider a mapping $G(Z) = (G_1(Z), G_2(Z))$, where:

$$G_1(Z) = rx\left(1 - \frac{x}{K}\right) - \frac{\zeta(1-m)xy}{1+\gamma(1-m)x+dy+\gamma(1-m)dxy} + \eta_1,$$

$$G_2(Z) = -\delta y + \frac{v\zeta(1-m)xy}{1+\gamma(1-m)x+dy+\gamma(1-m)dxy} + \eta_2. \tag{2}$$

In the case of any $Z, \bar{Z} \in \psi$, it follows from (1) that:

$$||G(Z) - G(\bar{Z})||$$

$$= \left| G_1(Z) - G_1(\bar{Z})| + |G_2(Z) - G_2(\bar{Z}) \right|$$

$$= \left| rx\left(1 - \frac{x}{K}\right) - \frac{\zeta(1-m)xy}{1+\gamma(1-m)x+dy+\gamma(1-m)dxy} + \eta_1 - r\bar{x}\left(1 - \frac{\bar{x}}{K}\right) \right.$$

$$\left. + \frac{\zeta(1-m)\bar{x}\bar{y}}{1+\gamma(1-m)\bar{x}+d\bar{y}+\gamma(1-m)d\bar{x}\bar{y}} - \eta_1 \right|$$

$$\left| -\delta y + \frac{v\zeta(1-m)xy}{1+\gamma(1-m)x+dy+\gamma(1-m)dxy} + \eta_2 + \delta\bar{y} - \frac{v\zeta(1-m)\bar{x}\bar{y}}{1+\gamma(1-m)\bar{x}+d\bar{y}+\gamma(1-m)d\bar{x}\bar{y}} - \eta_2 \right|$$

$$= \left| rx\left(1 - \frac{x}{K}\right) - \frac{\zeta(1-m)xy}{1+\gamma(1-m)x+dy+\gamma(1-m)dxy} - r\bar{x}\left(1 - \frac{\bar{x}}{K}\right) \right.$$

$$\left. + \frac{\zeta(1-m)\bar{x}\bar{y}}{1+\gamma(1-m)\bar{x}+d\bar{y}+\gamma(1-m)d\bar{x}\bar{y}} \right|$$

$$\left| -\delta y + \frac{\nu\zeta(1-m)xy}{1+\gamma(1-m)x+dy+\gamma(1-m)dxy} + \delta\bar{y} - \frac{\nu\zeta(1-m)\bar{x}\bar{y}}{1+\gamma(1-m)\bar{x}+d\bar{y}+\gamma(1-m)d\bar{x}\bar{y}} \right|$$

$$\leq \left(r + \frac{2rL}{K} + \zeta(1-m)L\right)\left|x-\bar{x}\right| + \left(2\delta + \nu\zeta(1-m)\right)\left|y-\bar{y}\right|$$

$$\leq C\left|\left|Z-\bar{Z}\right|\right|$$

where: $C = \max\left(r + \dfrac{2rL}{K} + \zeta(1-m)L,\ 2\delta + \nu\zeta(1-m)\right)$. Thus, $G(Z)$ is locally Lipschitzian in $R_+^2$, and a solution of model (1) exists and is unique.

Model (1) can then be written as follows:

$$\frac{dx}{dt} = xA_1(x, y), \quad x(0) > 0,$$

$$\frac{dy}{dt} = yA_2(x, y), \quad y(0) > 0. \tag{3}$$

From model (3), we get

$$\frac{dx}{dt} = x(0)\ \exp\left[\int_0^t A_1(x(s), y(s))\right] \geq 0,$$

$$\frac{dy}{dt} = y(0)\ \exp\left[\int_0^t A_2(x(s), y(s))\right] \geq 0.$$

Since the solution is locally Lipschitzian in $R_+^2$, with initial values $x(0) = x_0 \geq 0$ and $y(0) = y_0 \geq 0$, the solution of the model (1) remains positive.

This proves the theorem. □

## 3.2 *The boundedness*

**Theorem 3.2** In system (1), all the solutions that start in $R_+^2$ are bounded in the region $\omega = \left\{(x, y) \in R_+^2 : 0 \leq Q \leq \dfrac{\alpha}{c} + \epsilon\right\}$, where $0 < c < \delta$.

**Proof.** For system (1), consider $x(t), y(t)$ to be any solution. Let $Q(t) \in C^1(R_+ \to R_+)$ be as $Q(t) = x(t) + \dfrac{1}{\nu}y(t) > 0$ and $c > 0$ be a constant.

The derivative of $Q$ with respect to time $(t)$ is

$$\frac{dQ}{dt} = \frac{dx}{dt} + \frac{1}{\nu}\frac{dy}{dt}. \tag{4}$$

$$\begin{aligned}\frac{dQ}{dt} + cQ = rx\left(1 - \frac{x}{K}\right) - \frac{\zeta(1-m)xy}{1+\gamma(1-m)x+dy+\gamma(1-m)dxy} + \eta_1 \\ - \frac{\delta}{\nu}y + \frac{\zeta(1-m)xy}{1+\gamma(1-m)x+dy+\gamma(1-m)dxy} + \frac{\eta_2}{\nu} + cx + \frac{cy}{\nu}.\end{aligned} \tag{5}$$

$$\frac{dQ}{dt} + cQ = rx - \frac{rx^2}{K} + \eta_1 - \frac{\delta}{\nu}y + \frac{\eta_2}{\nu} + cx + \frac{cy}{\nu}. \tag{6}$$

$$\frac{dQ}{dt} + cQ = rx - \frac{rx^2}{K} + cx + \eta_1 + \frac{\eta_2}{\nu} - \frac{1}{\nu}(\delta - c)y. \tag{7}$$

By selecting $0 < c < \delta$, then

$$\frac{dQ}{dt} + cQ \le rx - \frac{rx^2}{K} + cx + \eta_1 + \frac{\eta_2}{\nu}. \tag{8}$$

Consider $\eta_1 + \frac{\eta_2}{\nu} = \beta > 0$, then Eq. (8) can be assumed as follows:

$$\frac{dQ}{dt} + cQ \le rx - \frac{rx^2}{K} + cx + \beta. \tag{9}$$

Then $\max\left\{rx - \frac{rx^2}{K} + cx + \beta\right\} = \frac{c^2K + 2rcK + r^2K + 4r\beta}{4r} = \alpha$. Then Eq. (9) implies the following:

$$\frac{dQ}{dt} + cQ \le \alpha. \tag{10}$$

Thus,

$$Q(t) \le \frac{\alpha}{c} + Q(0)e^{-ct}.$$

$\lim\sup_{t\to\infty} Q \le \frac{\alpha}{c}$. So, $Q$ is bounded. As a result, all the solutions of (1) are contained within

$$\Phi = \left\{(x, y) \in R_+^2 : 0 \le Q \le \frac{\alpha}{c} + \epsilon\right\}. \tag{11}$$

Thus, the solution of system (1) is bounded. □

**Remark 3.2.1** The model solutions are biologically viable because of their existence, positivity, and boundedness properties.

## 3.3 *Equilibrium points and stability analysis*

In this section, we focus on analyzing and drawing conclusions about the conditions for the existence of different equilibrium solutions of model (1).

### 3.3.1 *Equilibrium points of the model (1) involving prey refuge and without immigration*

In this subsection, the model (1), which incorporates prey refuge but excludes immigration, has three equilibrium points as follows:

$E_0 = (0, 0)$: Extinction of both species.

$E_1 = (K, 0)$: Coexistence of prey and extinction of predator.

$E_2 = (\dot{x}, \dot{y})$: Coexistence of both species.

$E_0 = (0, 0)$ and $E_1 = (K, 0)$ exist unconditionally, while the equilibrium point $E_2 = (\dot{x}, \dot{y})$ is determined by identifying the positive roots of the following equations:

$$r\left(1-\frac{\dot{x}}{K}\right)-\frac{\zeta(1-m)\dot{y}}{1+\gamma(1-m)\dot{x}+d\dot{y}+\gamma(1-m)d\dot{x}\dot{y}}=0,$$

$$-\delta+\frac{\nu\zeta(1-m)\dot{x}}{1+\gamma(1-m)\dot{x}+d\dot{y}+\gamma(1-m)d\dot{x}\dot{y}}=0$$

Put

$$\Psi_1(\dot{x}, \dot{y})=r\left(1-\frac{\dot{x}}{K}\right)-\frac{\zeta(1-m)\dot{y}}{1+\gamma(1-m)\dot{x}+d\dot{y}+\gamma(1-m)d\dot{x}\dot{y}}=0. \tag{12}$$

Equation (12) shows the following:

(1) If $\dot{x}=0$, then $\dot{y}=\dfrac{r}{\zeta(1-m)-r}=\dot{y}_i$. $\dot{y}_i>0$ under the following condition:

$$\zeta(1-m)>r. \tag{13}$$

(2) If $\dot{y}=0$, then $\dot{x}=K=\dot{x}_i$. Consequently, without any condition, it has $\dot{x}_i>0$.

(3) $\dfrac{d\dot{y}}{d\dot{x}}=-\dfrac{\dfrac{\partial\Psi_1}{\partial\dot{x}}}{\dfrac{\partial\Psi_1}{\partial\dot{y}}}=\dfrac{\dfrac{r}{K}-\dfrac{\zeta(1-m)^2\gamma d\dot{y}(1+d\dot{y})}{(1+\gamma(1-m)\dot{x}+d\dot{y}+\gamma(1-m)d\dot{x}\dot{y})^2}}{\dfrac{\zeta(1-m)(1+\gamma(1-m)\dot{x})}{(1+\gamma(1-m)\dot{x}+d\dot{y}+\gamma(1-m)d\dot{x}\dot{y})^2}}$. Then $\dfrac{d\dot{y}}{d\dot{x}}<0$ is under the following condition:

$$r(1+\gamma(1-m)\dot{x}+d\dot{y}+\gamma(1-m)d\dot{x}\dot{y})^2<K\zeta(1-m)^2\gamma d\dot{y}(1+d\dot{y}). \tag{14}$$

The isocline (12) passes through the points $(\dot{x}_i, 0)$ and $(0, \dot{y}_i)$. Equation (12) shows that $\dot{y}$ decreases with $\dot{x}$ under conditions (14) and (15).

Put

$$\Psi_2(\dot{x},\, \dot{y}) = -\delta + \frac{\nu\zeta(1-m)\dot{x}}{1+\gamma(1-m)\dot{x}+d\dot{y}+\gamma(1-m)d\dot{x}\dot{y}} = 0. \tag{15}$$

According to Equation (15):

(1) If $\dot{y} = 0$, then $\dot{x} = \dfrac{\delta}{\nu\zeta(1-m)-\delta\gamma(1-m)} = \dot{x}_{ii}$. $\dot{x}_{ii} > 0$ under the following condition:

$$\nu\zeta > \delta\gamma. \tag{16}$$

(2) $\dfrac{d\dot{y}}{d\dot{x}} = -\dfrac{\dfrac{\partial\Psi_2}{\partial\dot{x}}}{\dfrac{\partial\Psi_2}{\partial\dot{y}}} = \dfrac{\dfrac{\nu\zeta(1-m)(1+d\dot{y})}{(1+\gamma(1-m)\dot{x}+d\dot{y}+\gamma(1-m)d\dot{x}\dot{y})^2}}{\dfrac{\nu\zeta(1-m)\dot{x}(d+d\gamma(1-m))\dot{x}}{(1+\gamma(1-m)\dot{x}+d\dot{y}+\gamma(1-m)d\dot{x}\dot{y})^2}} = \dfrac{1+d\dot{y}}{d\dot{x}(1+\gamma(1-m))\dot{x}} > 0$ always.

The isocline (15) has a positive slope and passes through $(\dot{x}_{ii},\, 0)$ when the condition (14) is met. Therefore, Equation (15) shows that $\dot{y}$ increases with $\dot{x}$. This shows that the isocline (15) represents an increasing function of $\dot{y}$.

According to the preceding analysis, both isoclines (12) and (15) cross at a unique point $(\dot{x},\, \dot{y})$ if:

$$\dot{x}_{ii} < \dot{x}_i. \tag{17}$$

### 3.3.2 *Equilibrium points of the model (1) involving prey refuge and immigration*

The two populations coexist; $E_2 = (\ddot{x} \neq 0,\, \ddot{y} \neq 0)$ is determined by identifying the positive roots of the following equations:

$$r\left(1-\frac{\ddot{x}}{K}\right) - \frac{\zeta(1-m)\ddot{y}}{1+\gamma(1-m)\ddot{x}+d\ddot{y}+\gamma(1-m)d\ddot{x}\ddot{y}} + \frac{\eta_1}{\ddot{x}} = 0,$$

$$-\delta + \frac{\nu\zeta(1-m)\ddot{x}}{1+\gamma(1-m)\ddot{x}+d\ddot{y}+\gamma(1-m)d\ddot{x}\ddot{y}} + \frac{\eta_2}{\ddot{y}} = 0$$

Let

$$\Psi_1(\ddot{x},\, \ddot{y}) = r\left(1-\frac{\ddot{x}}{K}\right) - \frac{\zeta(1-m)\ddot{y}}{1+\gamma(1-m)\ddot{x}+d\ddot{y}+\gamma(1-m)d\ddot{x}\ddot{y}} + \frac{\eta_1}{\ddot{x}} = 0. \tag{18}$$

Equation (22) shows the following:

(1) If $\ddot{y} = 0$, then $r\left(1-\dfrac{\ddot{x}}{K}\right) + \dfrac{\eta_1}{\ddot{x}} = 0$, which gives

$$\frac{r\ddot{x}^2}{K} - r\ddot{x} - \eta_1 = 0. \tag{19}$$

Eq. (23) shows two roots, one is positive and the other is negative. Therefore, the positive root is taken ($\ddot{x}_i > 0$).

(2) $\dfrac{d\ddot{y}}{d\ddot{x}} = -\dfrac{\dfrac{\partial \Psi_1}{\partial \ddot{x}}}{\dfrac{\partial \Psi_1}{\partial \ddot{y}}} = \dfrac{\dfrac{r}{K} - \dfrac{\zeta(1-m)^2\gamma d\ddot{y}(1+d\ddot{y})}{(1+\gamma(1-m)\ddot{x}+d\ddot{y}+\gamma(1-m)d\ddot{x}\ddot{y})^2} + \dfrac{\eta_1}{\ddot{x}^2}}{\dfrac{\zeta(1-m)(1+\gamma(1-m)\ddot{x})}{(1+\gamma(1-m)\ddot{x}+d\ddot{y}+\gamma(1-m)d\ddot{x}\ddot{y})^2}}$. Then $\dfrac{d\ddot{y}}{d\ddot{x}} < 0$ is under the following condition after simplification:

$$\frac{r}{K} + \frac{\eta_1}{\ddot{x}^2} < \frac{\zeta(1-m)^2\gamma d\ddot{y}}{(1+\gamma(1-m)\ddot{x}+d\ddot{y}+\gamma(1-m)d\ddot{x}\ddot{y})^2}. \tag{20}$$

Through the preceding analysis, the positive root of Eq. (13) and the condition (14) represent a decreasing function of $\dot{x}$.

$$\Psi_2(\ddot{x}, \ddot{y}) = -\delta + \frac{\nu\zeta(1-m)\ddot{x}}{1+\gamma(1-m)\ddot{x}+d\ddot{y}+\gamma(1-m)d\ddot{x}\ddot{y}} + \frac{\eta_2}{\ddot{y}} = 0. \tag{21}$$

According to Equation (25):

(1) If $\ddot{x} = 0$, then $\ddot{y} = \dfrac{\eta_2}{\delta}$.

(2)

$$\frac{d\ddot{y}}{d\ddot{x}} = -\frac{\dfrac{\partial \Psi_2}{\partial \ddot{x}}}{\dfrac{\partial \Psi_2}{\partial \ddot{y}}} = \frac{\dfrac{\nu\zeta(1-m)(1+d\ddot{y})}{(1+\gamma(1-m)\ddot{x}+d\ddot{y}+\gamma(1-m)d\ddot{x}\ddot{y})^2}}{\dfrac{\nu\zeta(1-m)\ddot{x}(d+d\gamma(1-m)\ddot{x})}{(1+\gamma(1-m)\ddot{x}+d\ddot{y}+\gamma(1-m)d\ddot{x}\ddot{y})^2} + \dfrac{\eta_2}{\ddot{y}^2}}$$

$$= \frac{\nu\zeta(1-m)(1+d\ddot{y})\ddot{y}^2}{\nu\zeta(1-m)\ddot{x}\ddot{y}(d+d\gamma\ddot{x}) + \eta_2(1+\gamma(1-m)\ddot{x}+d\ddot{y}+\gamma(1-m)d\ddot{x}\ddot{y})^2}.$$

Then $\dfrac{d\ddot{y}}{d\ddot{x}} > 0$ always.

This shows that the isocline (25) represents an increasing function of $\dot{y}$.

According to the preceding analysis, both isoclines (12) and (15) cross at a unique point under the positive root of Eq. (13) and the condition (14).

**Remark 3.2.2** According to the preceding analysis, model (1) with prey refuge but without immigration leads to the extinction of one or both species, whereas model (1) with prey refuge and immigration indicates that extinction does not occur.

**Theorem 3.3** Under the condition

$$r < \frac{2r\bar{x}}{K} + \frac{\zeta(1-m)\bar{y}(1+d\bar{y})}{(1+\gamma(1-m)\bar{x}+d\bar{y}+\gamma(1-m)d\bar{x}\bar{y})^2}, \tag{22}$$

and

$$\frac{\nu\zeta(1-m)\bar{x}(1+\gamma(1-m)\bar{x})}{(1+\gamma(1-m)\bar{x}+d\bar{y}+\gamma(1-m)d\bar{x}\bar{y})^2} < \delta. \tag{23}$$

The coexistence equilibrium point $E_I = (\bar{x}, \bar{y})$ is locally asymptotically stable.

**Proof.** At $E_I = (\bar{x}, \bar{y})$, the Jacobian matrix is given in the following form:

$$J = \begin{bmatrix} r - \dfrac{2r\bar{x}}{K} - \dfrac{\zeta(1-m)\bar{y}(1+d\bar{y})}{(1+\gamma(1-m)\bar{x}+d\bar{y}+\gamma(1-m)d\bar{x}\bar{y})^2} & -\dfrac{\zeta(1-m)\bar{x}(1+\gamma(1-m)\bar{x})}{(1+\gamma(1-m)\bar{x}+d\bar{y}+\gamma(1-m)d\bar{x}\bar{y})^2} \\ \dfrac{\nu\zeta(1-m)\bar{y}(1+d\bar{y})}{(1+\gamma(1-m)\bar{x}+d\bar{y}+\gamma(1-m)d\bar{x}\bar{y})^2} & -\delta + \dfrac{\nu\zeta(1-m)\bar{x}(1+\gamma(1-m)\bar{x})}{(1+\gamma(1-m)\bar{x}+d\bar{y}+\gamma(1-m)d\bar{x}\bar{y})^2} \end{bmatrix}. \tag{24}$$

where

$$J_{11} = r - \frac{2r\bar{x}}{K} - \frac{\zeta(1-m)\bar{y}(1+d\bar{y})}{(1+\gamma(1-m)\bar{x}+d\bar{y}+\gamma(1-m)d\bar{x}\bar{y})^2},$$

$$J_{12} = -\frac{\zeta(1-m)\bar{x}(1+\gamma(1-m)\bar{x})}{(1+\gamma(1-m)\bar{x}+d\bar{y}+\gamma(1-m)d\bar{x}\bar{y})^2},$$

$$J_{21} = \frac{\nu\zeta(1-m)\bar{y}(1+d\bar{y})}{(1+\gamma(1-m)\bar{x}+d\bar{y}+\gamma(1-m)d\bar{x}\bar{y})^2},$$

$$J_{22} = -\delta + \frac{\nu\zeta(1-m)\bar{x}(1+\gamma(1-m)\bar{x})}{(1+\gamma(1-m)\bar{x}+d\bar{y}+\gamma(1-m)d\bar{x}\bar{y})^2}.$$

The characteristic equation of $J(E_I)$ is given by

$$\lambda^2 - (\text{trace}(J(E_I)))\lambda + \det(J(E_I)) = 0, \tag{25}$$

where

$$\text{trace}(J(E_I)) = J_{11} + J_{22},$$

$$\det(J(E_I)) = J_{11}J_{22} - J_{12}J_{21}.$$

Consequently, the coexistence equilibrium point is locally asymptotically stable if and only if $\text{trace}(J(E_2)) < 0$ and $\det(J(E_2)) > 0$ are met when $J_{11}$ and $J_{22}$ are negative values, because $J_{12}$ is always a negative value and $J_{21}$ is a positive value, as a result, they are met under criteria (22) and (23). ☐

**Corollary 3.3.1** The system (1) is unstable if one of the conditions (22) and (23) is not met.

**Theorem 3.4** Under the following condition,

$$\frac{r}{K\bar{y}} + \frac{\nu\zeta(1-m)(d+\gamma(1-m)d\bar{x})}{(1+\gamma(1-m)\bar{x}+d\bar{y}+\gamma(1-m)d\bar{x}\bar{y})^2} + \frac{\eta_1}{\bar{y}\bar{x}^2} + \frac{\eta_2}{\bar{x}\bar{y}^2} < \frac{\zeta(1-m)(\gamma(1-m)+\gamma(1-m)d\bar{y})}{(1+\gamma(1-m)\bar{x}+d\bar{y}+\gamma(1-m)d\bar{x}\bar{y})^2}, \tag{26}$$

the coexistence equilibrium point $E_I = (\bar{x},\, \bar{y})$ is globally asymptotically stable.

**Proof.** Put

$$Z_1 = r\bar{x}\left(1 - \frac{\bar{x}}{K}\right) - \frac{\zeta(1-m)\bar{x}\bar{y}}{1+\gamma(1-m)\bar{x}+d\bar{y}+\gamma(1-m)d\bar{x}\bar{y}} + \eta_1,$$

$$Z_2 = -\delta\bar{y} + \frac{\nu\zeta(1-m)\bar{x}\bar{y}}{1+\gamma(1-m)\bar{x}+d\bar{y}+\gamma(1-m)d\bar{x}\bar{y}} + \eta_2.$$

Select the Dulac function in the following form:

$$F = \frac{1}{\bar{x}\bar{y}},$$

where $F$ is a continuously differentiable function in $R^2_+$.

Now,

$$\nabla(DZ_1, DZ_2) = \frac{\partial(FZ_1)}{\partial\bar{x}} + \frac{\partial(FZ_2)}{\partial\bar{y}}$$

$$= \frac{-r}{K\bar{y}} + \frac{\zeta(1-m)(\gamma(1-m)+\gamma(1-m)d\bar{y})}{(1+\gamma(1-m)\bar{x}+d\bar{y}+\gamma(1-m)d\bar{x}\bar{y})^2}$$

$$-\frac{\eta_1}{\bar{y}\bar{x}^2} - \frac{\nu\zeta(1-m)(d+\gamma(1-m)d\bar{)}}{(1+\gamma(1-m)\mp d\mp\gamma(1-m)d\bar{x}\bar{y})^2} - \frac{\eta_2}{\bar{x}\bar{y}^2}.$$

For $(x,\, y) \in (0,\, \infty) \times (0,\, \infty)$, $\nabla < 0$ if condition (26) is met. By Theorem 3.2 and the Bendixson-Dulac theorem, system (1) has no periodic orbits with condition (26). Thus, $E_2 = (\bar{x},\, \bar{y})$ is globally asymptotically stable. □

**Remark 3.4.1** Prey refuge and the immigration of both species are important factors influencing the uniqueness of the coexistence equilibrium point, the system's stability, and the existence of a limit cycle.

**Remark 3.4.2** Local stability conditions indicate that small perturbations that affect the ecosystem will lead it to return to stability, whereas global stability conditions imply that large perturbations in the system demonstrate its resilience for survival and stability.

**Remark 3.4.3** The stability conditions indicate that immigration plays a crucial role in maintaining the stability of the global system. Consequently, the long-term impact of immigration on system stability is evident, as exemplified by the rescue phenomenon [38, 39].

## 4. Numerical simulations

Our goal in this section is to validate the theoretical results and to provide a deeper explanation and interpretation of the findings. Numerical simulations are performed using Mathematica. We primarily investigate the effects of prey refuges and immigration on system stability. To illustrate the dynamics, we use phase portrait trajectories with zero-growth isoclines and time-series plots. The phase portrait trajectories with zero-growth isoclines depict the overall framework of the system's dynamics and the densities of prey and predators, while the time-series plots show how prey and predator populations vary over time. The initial conditions for all numerical simulations are specified as follows:

$$x_0 = 2.0,\ y_0 = 1.0.$$

As indicated by Theorems 3.3 and 3.4 and Corollary 3.3.1, the system exhibits two main types of dynamics equilibrium and cyclic; thus, two sets of parameter values are considered.

The first set represents stable dynamics and is given as follows:

$$r = 1.0,\ \zeta = 3.0,\ K = 5,\ \nu = 0.5,\ \delta = 0.5,\ \gamma = 1.0,\ d = 0.7. \tag{27}$$

The second set represents cyclic dynamics and is given as follows:

$$r = 1.0,\ \zeta = 3.5,\ K = 6,\ \nu = 0.75,\ \delta = 0.5,\ \gamma = 1.25,\ d = 0.7. \tag{28}$$

Assuming no prey refuges ($m = 0.0$) and no immigration ($\eta_1 = 0.0$ and $\eta_2 = 0.0$) for both species, Figure 1 shows that the dynamic behavior of model (1) leads to stable coexistence at the point (1.208, 0.916) after oscillations over a certain period. In Figure 2, with a low refuge rate ($m = 0.1$) and no immigration, the system remains stable around the point (1.556, 1.072), with a slight increase in prey and predator densities. When the refuge rate increases to a moderate value ($m = 0.5$) and there is still no immigration, Figure 3 shows that the species continue to coexist stably around the point (3.103, 1.177), with prey and predator densities higher than in the previous cases.

In Figure 4, when the refuge rate is high ($m = 0.9$) and there is no immigration for either species ($\eta_1 = 0.0$ and $\eta_2 = 0.0$), the predators go extinct, and the prey reach their carrying capacity. A high refuge rate in this scenario is therefore detrimental to coexistence. This outcome is also reported in [37].

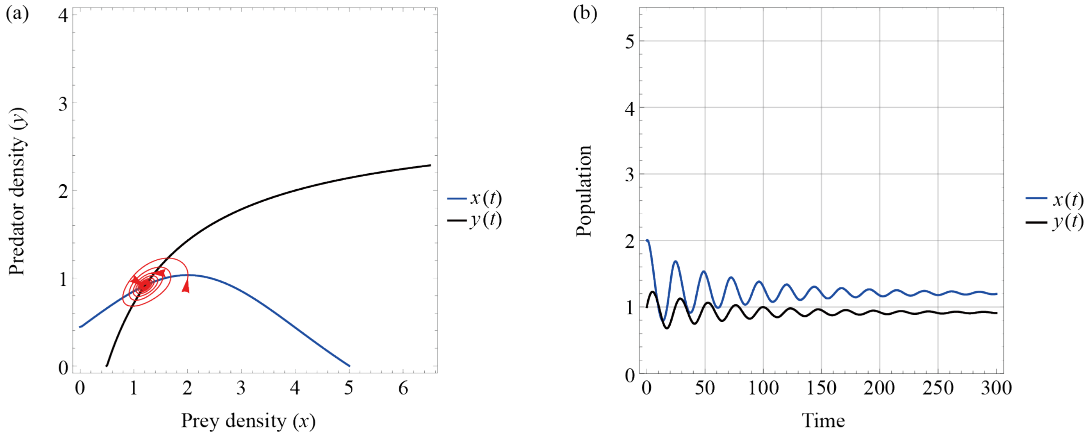

**Figure 1.** Dynamics of the model (1) with parameter values $r = 1.0$, $\zeta = 3.0$, $K = 5$, $\nu = 0.5$, $\delta = 0.5$, $\gamma = 1.0$, $d = 0.7$, $m = 0.0$, $\eta_1 = 0.0$, and $\eta_2 = 0.0$: (a) Zero-growth isoclines and phase portrait paths; (b) Time series

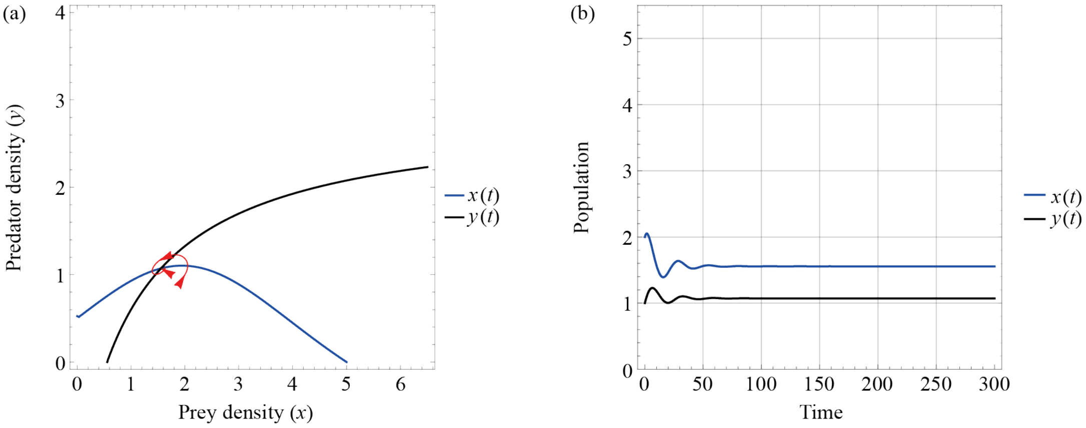

**Figure 2.** Dynamics of the model (1) with parameter values $r = 1.0$, $\zeta = 3.0$, $K = 5$, $v = 0.5$, $\delta = 0.5$, $\gamma = 1.0$, $d = 0.7$, $m = 0.1$, $\eta_1 = 0.0$, and $\eta_2 = 0.0$: (a) Zero-growth isoclines and phase portrait paths; (b) Time series

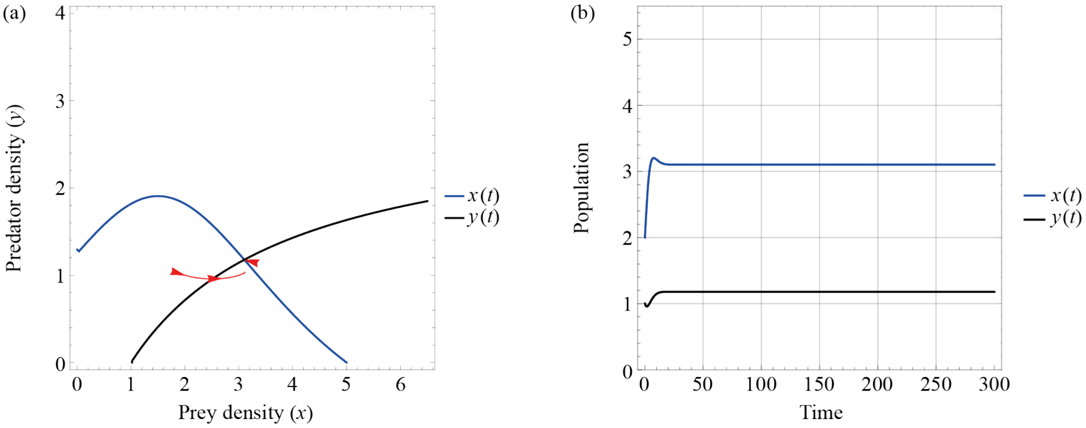

**Figure 3.** Dynamics of the model (1) with parameter values $r = 1.0$, $\zeta = 3.0$, $K = 5$, $v = 0.5$, $\delta = 0.5$, $\gamma = 1.0$, $d = 0.7$, $m = 0.5$, $\eta_1 = 0.0$, and $\eta_2 = 0.0$: (a) Zero-growth isoclines and phase portrait paths; (b) Time series

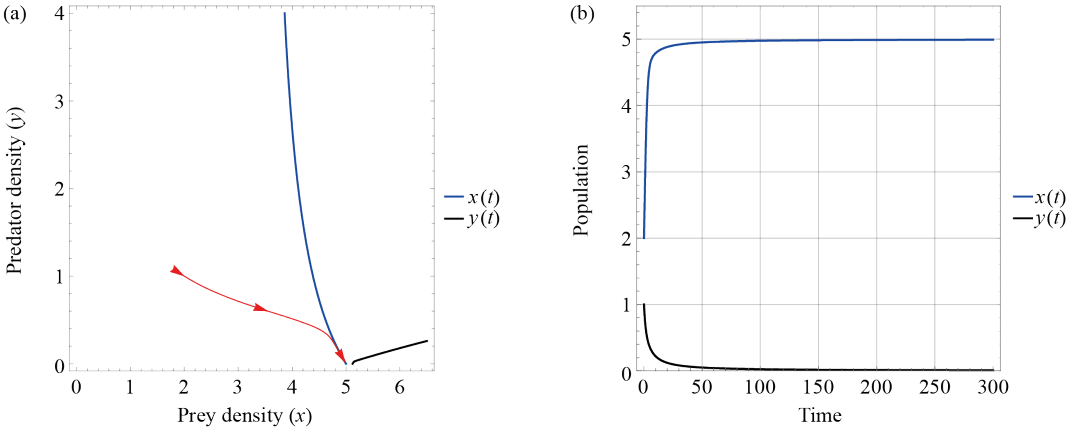

**Figure 4.** Dynamics of the model (1) with parameter values $r = 1.0$, $\zeta = 3.0$, $K = 5$, $v = 0.5$, $\delta = 0.5$, $\gamma = 1.0$, $d = 0.7$, $m = 0.9$, $\eta_1 = 0.0$, and $\eta_2 = 0.0$: (a) Zero-growth isoclines and phase portrait paths; (b) Time series

When the refuge rate is sufficiently high, coexistence may be disrupted. One important approach to mitigate this effect is to allow immigration into the environment, which can help preserve the ecosystem and biodiversity, as illustrated in Figure 5 for a high refuge rate ($m = 0.9$) with immigration.

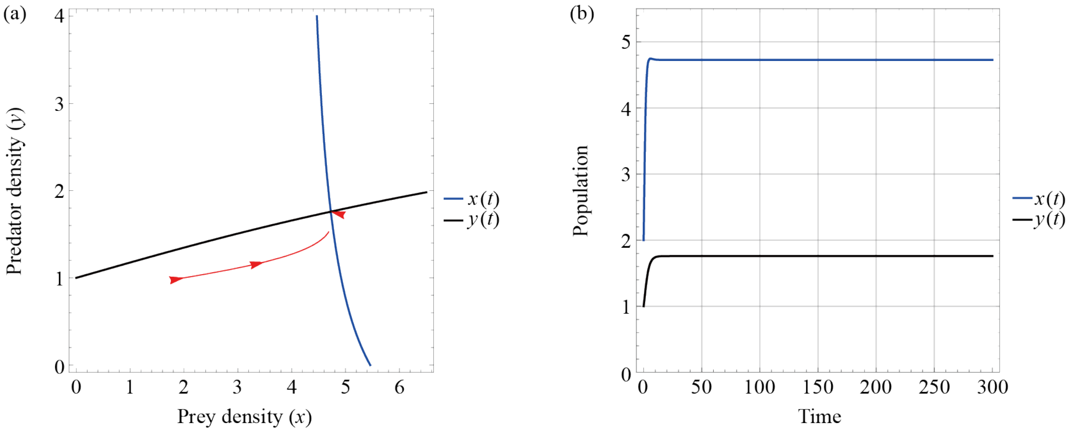

**Figure 5.** Dynamics of the model (1) with parameter values $r = 1.0$, $\zeta = 3.0$, $K = 5$, $\nu = 0.5$, $\delta = 0.5$, $\gamma = 1.0$, $d = 0.7$, $m = 0.9$, $\eta_1 = 0.5$, and $\eta_2 = 0.5$: (a) Zero-growth isoclines and phase portrait paths; (b) Time series

In Figure 6, which represents the bifurcation diagram with respect to the prey refuge parameter $m$, low or absent prey refuges show fluctuations that stabilize as the refuge rate increases. As the refuge rate becomes very high, the prey density approaches the carrying capacity, while the predator density declines toward extinction.

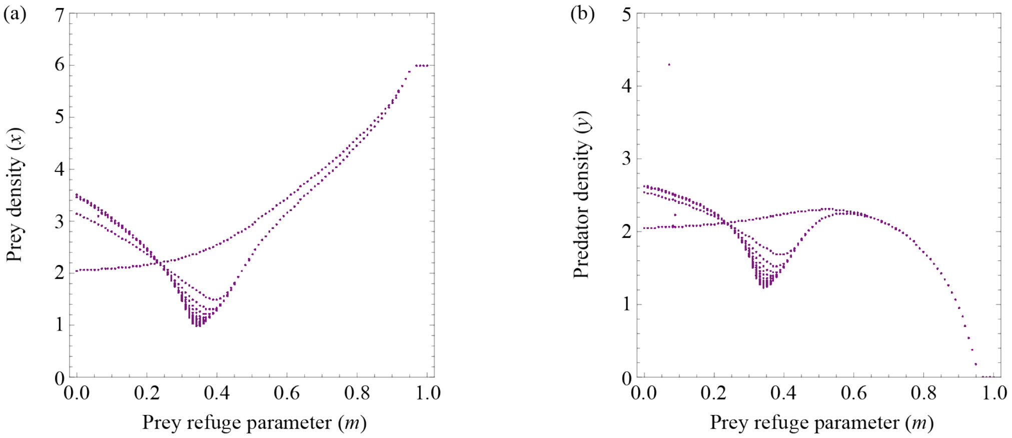

**Figure 6.** Bifurcation diagram of model (1) with respect to the prey refuge parameter $m$, with the other parameter values fixed as $r = 1.0$, $\zeta = 3.5$, $K = 6$, $\nu = 0.75$, $\delta = 0.5$, $\gamma = 1.25$, $d = 0.7$, $\eta_1 = 0.0$, and $\eta_2 = 0.0$: (a) Prey density ($x$); (b) Predator density ($y$)

**Remark 4.0.1** Ecological justifications for constant immigration include: the constant immigration of species can support endangered species, which can be explained ecologically by the rescue phenomenon [38, 39]. In addition, constant immigration improves the stability and survival of species, as explained in some references, for example [30–32].

Figure 7, which represents the second set of parameters, illustrates the cyclic dynamic behavior of the model (1) with no prey refuges ($m = 0.0$) and no immigration ($\eta_1 = 0.0$ and $\eta_2 = 0.0$). With a low refuge rate ($m = 0.1$) and no immigration for both species ($\eta_1 = 0.0$ and $\eta_2 = 0.0$), Figure 8 shows that the cycle becomes slightly smaller and shifts slightly away from the axes.

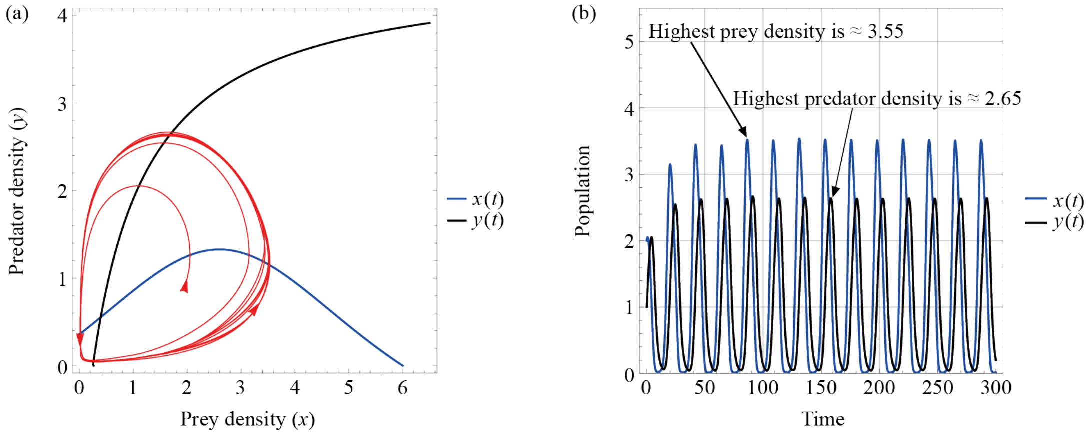

**Figure 7.** Dynamics of the model (1) with parameter values $r = 1.0$, $\zeta = 3.5$, $K = 6$, $\nu = 0.75$, $\delta = 0.5$, $\gamma = 1.25$, $d = 0.7$, $m = 0.0$, $\eta_1 = 0.0$, and $\eta_2 = 0.0$: (a) Zero-growth isoclines and phase portrait paths; (b) Time series

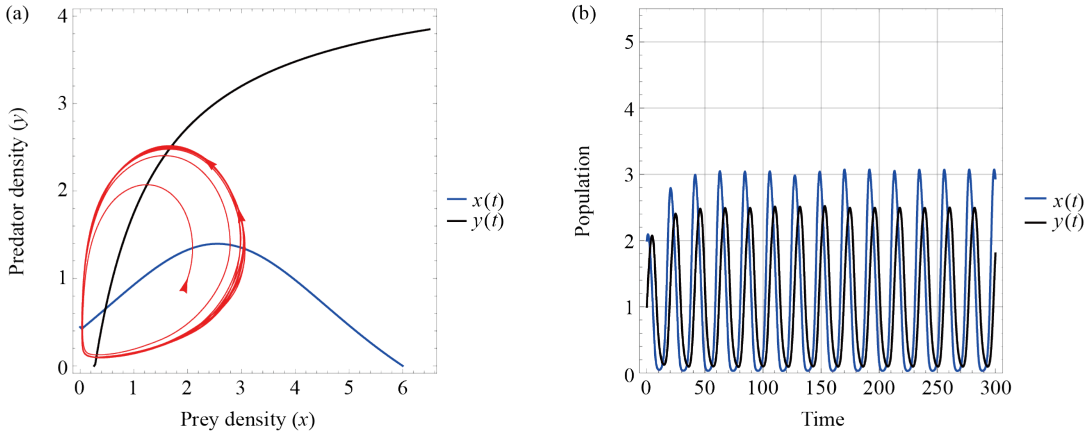

**Figure 8.** Dynamics of the model (1) with parameter values $r = 1.0$, $\zeta = 3.5$, $K = 6$, $\nu = 0.75$, $\delta = 0.5$, $\gamma = 1.25$, $d = 0.7$, $m = 0.1$, $\eta_1 = 0.0$, and $\eta_2 = 0.0$: (a) Zero-growth isoclines and phase portrait paths; (b) Time series

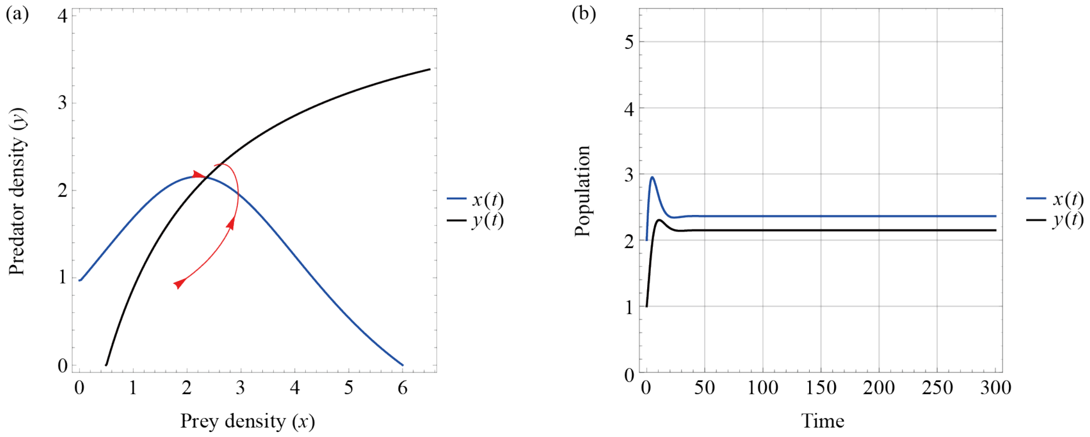

**Figure 9.** Dynamics of the model (1) with parameter values $r = 1.0$, $\zeta = 3.5$, $K = 6$, $\nu = 0.75$, $\delta = 0.5$, $\gamma = 1.25$, $d = 0.7$, $m = 0.5$, $\eta_1 = 0.0$, and $\eta_2 = 0.0$: (a) Nullclines and phase portrait paths; (b) Time series

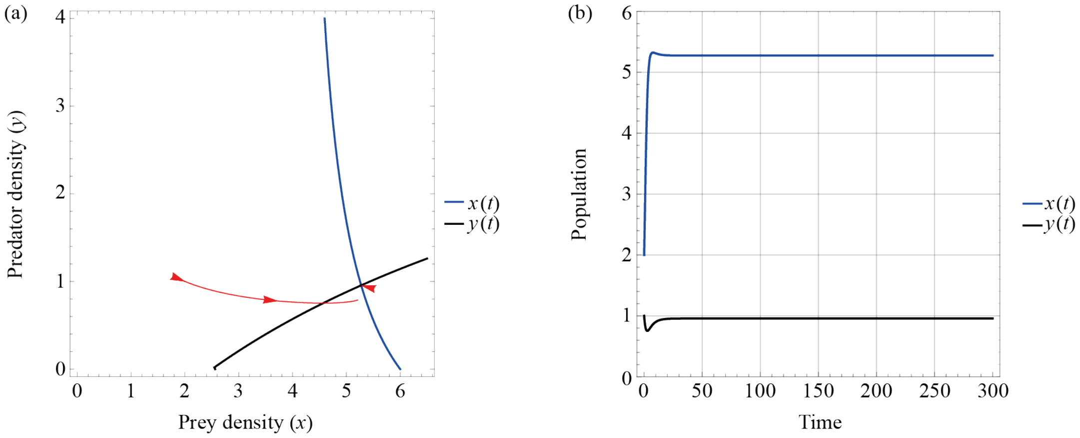

**Figure 10.** Dynamics of the model (1) with parameter values $r = 1.0$, $\zeta = 3.5$, $K = 6$, $\nu = 0.75$, $\delta = 0.5$, $\gamma = 1.25$, $d = 0.7$, $m = 0.9$, $\eta_1 = 0.0$, and $\eta_2 = 0.0$: (a) Zero-growth isoclines and phase portrait paths (b) Time series

When the refuge rate ($m = 0.5$) increases and there is no immigration for both species ($\eta_1 = 0.0$ and $\eta_2 = 0.0$), it is observed that there is a significant change in the system's dynamics from periodic coexistence to stability at point (2.361, 2.148), while the species densities remain below the maximum values observed in the cyclic dynamics (in Figure 7 the highest densities of prey is approximately 3.55 and that of predator is approximately 2.65), as shown in Figure 9. When the refuge rate is further increased to $m = 0.9$ with no immigration, Figure 10 shows that the dynamics remain stable at (5.275, 0.956): prey density rises, predator density declines, and the prey does not reach its carrying capacity due to the presence of predators. These results highlight a difference in the second set of parameters (cyclic dynamics) compared to the first set, where high refuge rates led to a substantial increase in prey density and predator extinction. In Figure 11, with a high refuge rate ($m = 0.9$) and immigration of both species ($\eta_1 = 0.5$ and $\eta_2 = 0.5$), the system remains stable at (5.407, 2.551), and the densities of both prey and predators increase.

Figures 12 and 13 present the bifurcation diagrams with respect to the prey and predator immigration parameters $\eta_1$ and $\eta_2$, respectively. Both figures indicate that the system coexists stably. In Figure 12, the densities of both prey and predator increase with increasing prey immigration parameter $\eta_1$. In contrast, Figure 13 shows that increasing the predator immigration parameter $\eta_2$ leads to higher predator density and lower prey density, which can be interpreted as the intensified predation pressure resulting from the larger predator population.

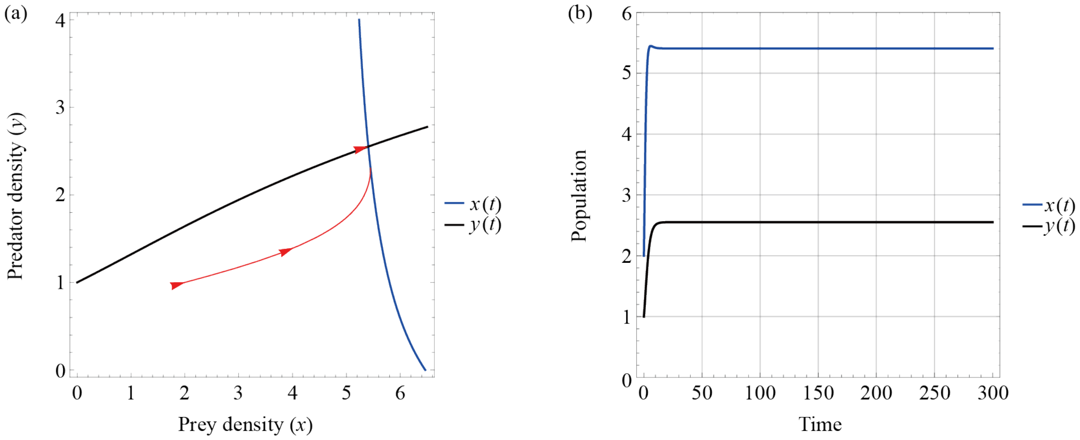

**Figure 11.** Dynamics of the model (1) with parameter values $r = 1.0$, $\zeta = 3.5$, $K = 6$, $\nu = 0.75$, $\delta = 0.5$, $\gamma = 1.25$, $d = 0.7$, $m = 0.9$, $\eta_1 = 0.5$, and $\eta_2 = 0.5$: (a) Zero-growth isoclines and phase portrait paths; (b) Time series

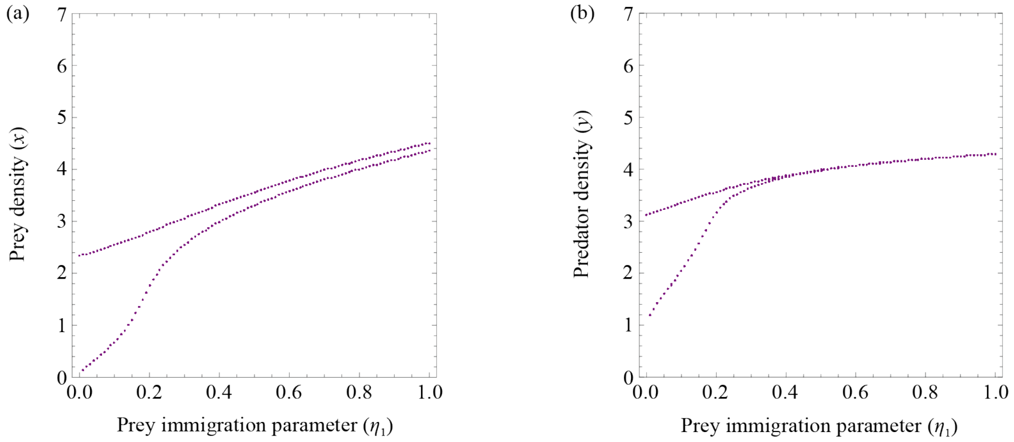

**Figure 12.** Bifurcation diagram of model (1) with respect to the prey immigration parameter $\eta_1$, with the other parameter values fixed as $r = 1.0$, $\zeta = 3.5$, $K = 6$, $\nu = 0.75$, $\delta = 0.5$, $\gamma = 1.25$, $d = 0.7$, $m = 0.5$, and $\eta_2 = 0.5$: (a) Prey density ($x$); (b) Predator density ($y$)

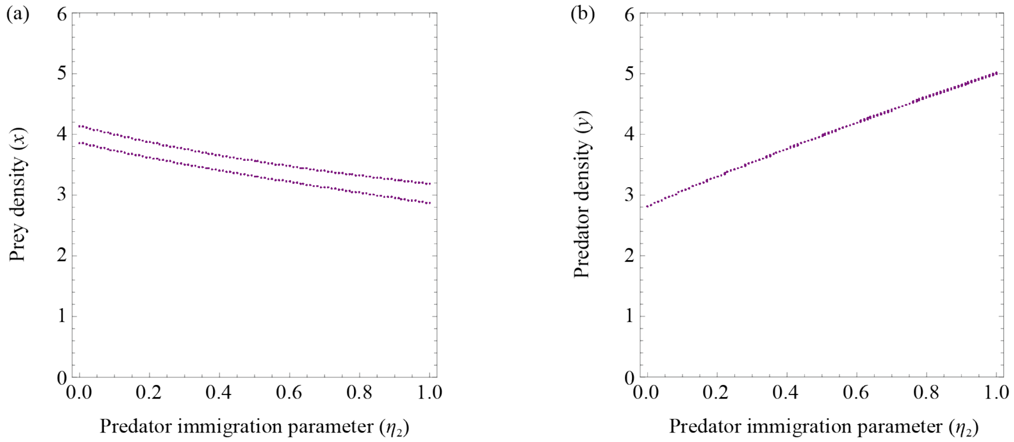

**Figure 13.** Bifurcation diagram of model (1) with respect to the predator immigration parameter $\eta_2$, with the other parameter values fixed as $r = 1.0$, $\zeta = 3.5$, $K = 6$, $\nu = 0.75$, $\delta = 0.5$, $\gamma = 1.25$, $d = 0.7$, $m = 0.5$ and $\eta_1 = 0.5$: (a) Prey density ($x$) (b) Predator density ($y$)

**Remark 4.0.2** Ecological justifications for the refuge of prey include: refuges can play an important role in biological pest control; for example, almond orchards may contain "hotspots" of high spider mite density. A hotspot is an area where predators cannot effectively control prey and is thus considered a refuge for the prey [40]. Prey refuges help stabilize prey-predator systems, reduce predation rates and predator densities, and prevent extinction due to predation [24, 26, 41].

## 5. Conclusions

Mathematical modeling plays a pivotal role in understanding many mathematical ecological problems. Predator-prey models have long been a central topic in ecological mathematics, and extensive theoretical as well as experimental research has addressed various aspects of these models, resolving many important questions. In this paper, we formulate a predator-prey model with a Crowley-Martin functional response, incorporating prey refuges and immigration for both species. We establish the existence, positivity, and boundedness of the model, thereby demonstrating its biological validity.

The equilibrium points of the model were first determined in the presence of prey refuges but without immigration, revealing three distinct equilibrium points. Next, when both prey refuges and immigration were incorporated, the model exhibited a single coexistence equilibrium, where both species persist without extinction due to the stabilizing effect of constant immigration. Previous research [37] has shown that high levels of prey refuge can negatively affect species survival, which is consistent with our model in the absence of immigration (see Figure 4). From this analysis, we conclude that constant immigration of both species plays a crucial role in supporting their long-term survival in the environment.

Mathematical analysis revealed that prey refuge and immigration play a pivotal role in determining the stability and survival of the system. Numerical simulations were employed to validate and illustrate the theoretical findings. The model was examined under two types of dynamic behavior: stable and cyclic species interactions. In the stable case, with or without immigration, prey density increases and predator density decreases as the prey refuge rate rises, owing to reduced predation pressure. However, when the prey refuge rate becomes sufficiently high, predator extinction occurs, and prey density approaches the carrying capacity of the system. In contrast, when immigration is incorporated even at high refuge rates, the system maintains stable coexistence.

In the case of coexistence with cyclic dynamics, the model shows that the presence of refuges and the absence of immigration lead to system stability as the rate of prey refuges increases to a certain extent. This outcome is consistent with several studies in the literature [24–28]. However, species densities remain below their upper limits under cyclic dynamics. When the rate of refuges increases significantly, prey densities increase while predator densities decline markedly. In contrast, when immigration is present for both species, their densities increase, thus enhancing species survival.

Prey refuges and immigration represent complementary ecological mechanisms that improve population persistence and promote system stability. Refuges provide prey species with protection against predation, thus reducing predation pressure, stabilizing predator-prey interactions, and preventing extinction events that may arise from uncontrolled predation. Immigration, on the other hand, reinforces local populations by replenishing numbers and mitigating demographic or environmental stochasticity, thus safeguarding species persistence. This dynamic underlies conservation strategies such as reintroducing endangered species and restocking declining populations to maintain biodiversity and ecological functioning. Collectively, these processes exemplify the rescue effect, demonstrating how spatial refuges and external demographic inputs act synergistically to buffer populations, stabilize systems, and sustain long-term ecosystem flexibility.

## Acknowledgement

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## Conflict of interest

The author declares that he has no conflict of interest.

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