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Research Article

Analysis of q-Fractional Differential Equations with Nonlinear q-Integral Conditions: Uniqueness and Nonuniqueness Solution in Banach Spaces

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Abstract: In this work, we discuss the solutions to the boundary value problem for fractional q-difference equations with nonlinear integral conditions:

$$^{c}D_{q,\gamma}\nu(\iota)-\lambda\nu(\iota)=\mathscr{J}\left(\iota,\nu(\iota),{}^{c}D_{q,\gamma}\nu(\iota)\right),\ \iota\in[0,\ell],\ 1<\gamma\leq 2, \tag{1}$$

with

$$\nu(0)-\nu'(0)=c_1\int_0^{\ell}\mathscr{G}_1(\tau,\nu(\tau))\,d_q\tau, \tag{2}$$

and

$$\nu(\ell)-\nu'(\ell)=c_2\int_0^{\ell}\mathscr{G}_2(\tau,\nu(\tau))\,d_q\tau. \tag{3}$$

By applying the fixed point theorems of Banach and Krasnoselskii, we establish the existence of solutions for the above problem (1)-(3). Some illustrative examples are given. This paper generalises some earlier results in the literature.

Keywords: caputo fractional q-difference equation, fixed point theorems, integral boundary conditions

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1. Introduction

Over the past years, fractional differential calculus has attracted the interest of several academics, due to its importance for mathematical modeling. Its applications can be observed in domains such as physics, engineering, biology,

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and finance, where standard calculus often fails, see [1–8] and the references therein. Regarding the qualitative properties of solutions to fractional differential equations, further details can be found in [1, 2, 9–11] and the references cited therein.

The evolution of quantum calculus (or q-calculus) is well-established, with comprehensive accounts of its fundamental concepts, results, and methods provided in [12–14]. In the early twentieth century, the quantum difference calculus itself became one of the most active areas within fractional differential equations, attracting considerable attention from researchers, see [15–17] and the references cited therein. In particular, important works have addressed initial and boundary value problems for both ordinary and fractional q-difference equations [18–24].

Recently, Allouch et al. [1] investigated the existence of solutions to the Boundary Value Problem (BVP) for fractional q-difference equations:

$$^{c}D_{q,\gamma}\nu(\iota) = \mathscr{J}(\iota, \nu(\iota)), \ \iota \in [0, \ell], 1 < \gamma \leq 2,$$

with nonlinear integral conditions

$$\nu(0) - \nu'(0) = \int_0^\ell \mathscr{G}_1(\tau, \nu(\tau))\, d\tau,$$

and

$$\nu(\ell) - \nu'(\ell) = \int_0^\ell \mathscr{G}_2(\tau, \nu(\tau))\, d\tau,$$

where $0 < q < 1$, the functions $\mathscr{G}_1, \mathscr{G}_2$: $[0, \ell] \times \mathscr{E} \to \mathscr{E}$, and $\mathscr{J}: [0, \ell] \times \mathscr{E} \to \mathscr{E}$ are continuous with $\mathscr{E}$ is a Banach space. $^{c}D_{q,\gamma}$ represents Caputo fractional q-difference derivative of order γ. The author's results are based on Mönch's fixed point theorem and the technique of measures of noncompactness. This work is considered one of the few on difference equations with integral boundary conditions on Banach space.

In this research, we investigate the existence of solutions to the BVP for fractional q-difference equations with nonlinear integral conditions:

$$^{c}D_{q,\gamma}\nu(\iota) - \lambda\nu(\iota) = \mathscr{J}\left(\iota, \nu(\iota), {}^{c}D_{q,\gamma}\nu(\iota)\right), \ \iota \in [0, \ell], 1 < \alpha \leq 2, \tag{4}$$

$$\nu(0) - \nu'(0) = c_1 \int_0^\ell \mathscr{G}_1(\tau, \nu(\tau))\, d_q\tau, \tag{5}$$

and

$$\nu(\ell) - \nu'(\ell) = c_2 \int_0^\ell \mathscr{G}_2(\tau, \nu(\tau))\, d_q\tau, \tag{6}$$

where $q \in (0, 1)$ and $\lambda, c_1, c_2 \in \mathbb{R}$, the functions $\mathscr{G}_1, \mathscr{G}_2$: $[0, \ell] \times \mathscr{E} \to \mathscr{E}$, and $\mathscr{J}: [0, \ell] \times \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ are continuous wich $\mathscr{E}$ is a reflexive Bnach space. $^{c}D_{q,\gamma}$ represents Caputo fractional q-difference derivative of order γ.

The structure of the article is as follows: Section 2 contains the important theorems and lemmas needed for the present study. Our results, based on fixed point theory, are presented in section 3. We also provide examples to support our results. This study improves upon the results, that have already been reported in [1].

2. Concepts and materials

Firstly, we start by some useful spaces. The Banach space $C([0, \ell], \mathscr{E})$ of all continuous functions ν: $[0, \ell] \to \mathscr{E}$, with the norm

$$\|\nu\|_\infty = \sup\{\|\nu(\iota)\|, \text{ for all } \iota \in [0, \ell]\},$$

where $\mathscr{E}$ is a separable Banach spaces. Let $L^1([0, \ell], \mathscr{E})$, the space of Lebesgue integrable functions φ: $[0, \ell] \to \mathscr{E}$ which are Bochner integrable, normed by

$$\|\varphi\|_{L^1} = \int_{[0, \ell]} \|\varphi(\iota)\| dt.$$

We also use the Banach space $\mathscr{C}_{q, \gamma}([0, \ell], \mathscr{E})$ defined by

$$\mathscr{C}_{q, \gamma}([0, \ell], \mathscr{E}) = \left\{\nu\colon \nu \in C([0, \ell], \mathscr{E}),\ {}^cD_{q, \gamma}\nu \in C([0, \ell], \mathscr{E})\right\},$$

equipped with the norm

$$\|\nu\|_q = \max\left\{\|\nu\|_\infty, \left\|{}^cD_{q, \gamma}\nu\right\|_\infty\right\}.$$

Let's go over some concepts of fractional q-calculus, see [12–14]. For $\alpha, \beta, \gamma \in \mathbb{R}$, let $q \in (0, 1)$

$$[\alpha]_q = \frac{q^\alpha - 1}{q - 1} = 1 + q + q^2 \ldots + q^{\alpha - 1}.$$

The q-analogue of the power function $(\alpha - \beta)^{(m)}$ with $m \in \mathbb{N}$ is

$$(\alpha - \beta)^{(0)} = 1,\ \ (\alpha - \beta)^{(m)} = \prod_{i=0}^{m-1}\left(\alpha - \beta q^i\right),\ \ \alpha, \beta \in \mathbb{R},\ \ m \in \mathbb{N}.$$

More generally,

$$(\alpha - \beta)^{(\gamma)} = \alpha^\gamma \prod_{i=0}^{\infty} \frac{\alpha - \beta q^i}{\alpha - \beta q^{\gamma + i}}.$$

Note that, if $\beta = 0$ then

$$\alpha^{(\gamma)} = \alpha^{\gamma}.$$

The q-gamma function is defined by:

$$\Gamma_q(\iota) = \frac{(1-q)^{(\iota-1)}}{(1-q)^{\iota-1}}, \ \iota \in \mathbb{R} \backslash \{0, -1, -2, ...\}, \ 0 < q < 1,$$

and satisfies

$$\Gamma_q(\iota+1) = [\iota]_q \Gamma_q(\iota).$$

The q-derivative of a function $\mathfrak{g}$: $[0, \ell] \to \mathscr{E}$ is defined by

$$D_q \mathfrak{g}(\iota) = \frac{d_q \mathfrak{g}(\iota)}{d_q \iota} = \frac{\mathfrak{g}(q\iota) - \mathfrak{g}(\iota)}{(q-1)\iota}, \ \iota \neq 0, \ D_q \mathfrak{g}(0) = \lim_{\iota \to 0} D_q \mathfrak{g}(\iota),$$

and q-derivatives of higher order by

$$D_{q,m} \mathfrak{g}(\iota) = \begin{cases} \mathfrak{g}(\iota), \text{ if } m = 0, \\ \\ D_q D_{q,m-1} \mathfrak{g}(\iota), \text{ if } m \in \mathbb{N}^*. \end{cases}$$

The q-integral of a function $\mathfrak{g}$ defined in the interval $[0, \beta]$ is given by

$$\int_0^{\iota} \mathfrak{g}(\tau) d_q \tau = \iota (1-q) \sum_{m=0}^{\infty} \mathfrak{g}(\iota q^n) q^n, \ 0 \leq |q| < 1, \ t \in [0, \beta].$$

If $\alpha \in [0, \beta]$ and $\mathfrak{g}$ defined in the interval $[0, \beta]$, its integral from α to β is defined by

$$\int_{\alpha}^{\beta} \mathfrak{g}(\tau) d_q \tau = \int_0^{\beta} \mathfrak{g}(\tau) d_q \tau - \int_0^{\alpha} \mathfrak{g}(\tau) d_q \tau.$$

Similarly, as done for derivatives, it can be defined an operator $\mathscr{I}_{q,n}$, namely,

$$(\mathscr{I}_{q,0} \mathfrak{g})(\iota) = \mathfrak{g}(\iota) \text{ and } (\mathscr{I}_{q,m} \mathfrak{g})(\iota) = \mathscr{I}_q(\mathscr{I}_{q,m-1} \mathfrak{g})(\iota), \ m \in \mathbb{N}.$$

The essential theorem of calculus relates to these operators $\mathscr{I}_q$ and D_q, i.e

$$D_q(\mathscr{I}_q\mathfrak{g})(\iota) = \mathfrak{g}(\iota),$$

and if $\mathfrak{g}$ is continuous at $\iota = 0$, then

$$\mathscr{I}_q(D_q\mathfrak{g})(\iota) = \mathfrak{g}(\iota) - \mathfrak{g}(0).$$

We recommend the reader to see [13] for more details as well as the fundamental properties of these operators.

Definition 1 Let $\gamma \geq 0$ and $\mathfrak{g}$ be a function defined on $[0, \ell]$. The fractional q-integral of the Riemann-Liouville type is

$$\mathscr{I}_{q,\gamma}\mathfrak{g}(\iota) = \begin{cases} \mathfrak{g}(\iota), \text{ if } \gamma = 0 \\ \\ \frac{1}{\Gamma_q(\gamma)} \int_0^\iota (\iota - q\tau)^{(\gamma-1)} \mathfrak{g}(\tau) d_q\tau, \text{ if } \gamma > 0 \end{cases}, \quad \iota \in [0, \ell].$$

Definition 2 The Riemann-Liouville fractional q-derivative of order $\gamma \geq 0$ is defined by

$$D_{q,\gamma}\mathfrak{g}(\iota) = \begin{cases} \mathfrak{g}(\iota), \text{ if } \gamma = 0 \\ \\ \left(D_{q,[\gamma]}\mathscr{I}_{q,[\gamma]-\gamma}\mathfrak{g}\right)(\iota), \text{ if } \gamma > 0 \end{cases}, \quad \iota \in [0, \ell].$$

The smallest value greater than or equal to γ is $[\gamma]$.

Definition 3 The Caputo fractional q-derivative of order $\gamma \geq 0$ is defined by

$${}^cD_{q,\gamma}\mathfrak{g}(\iota) = \begin{cases} \mathfrak{g}(\iota), \text{ if } \gamma = 0 \\ \\ \left(\mathscr{I}_{q,[\gamma]-\gamma}D_{q,[\gamma]}\mathfrak{g}\right)(\tau), \text{ if } \gamma > 0 \end{cases}, \quad \iota \in [0, \ell].$$

Lemma 1 ([17]) Let $\gamma \geq 0$. Then the following equality holds:

$$\left(\mathscr{I}^c_{q,\gamma}D_{q,\gamma}\mathfrak{g}\right)(\iota) = \mathfrak{g}(\iota) - \sum_{j=0}^{[\gamma]-1} \frac{\iota^j}{\Gamma_q(j+1)} \left(D_{q,\gamma}\mathfrak{g}\right)(0).$$

Lemma 2 Let $\gamma \in (1, 2]$ and $c_1, c_2 \in \mathbb{R}$ and $q \in (0, 1)$. For $\mathscr{M} \in C([0, \ell], \mathscr{E})$. Then the following system:

$${}^cD_{q,\gamma}\nu(\iota) = \mathscr{M}(\iota), \quad \iota \in [0, \ell], \tag{7}$$

with nonlinear integral conditions

$$\nu(0) - \nu'(0) = c_1 \int_0^{\ell} h_1(\tau)\, d_q\tau, \tag{8}$$

and

$$\nu(\ell) - \nu'(\ell) = c_2 \int_0^{\ell} h_2(\tau)\, d_q\tau, \tag{9}$$

has a solution that shown below

$$\nu(\iota) = \mathscr{K}(\iota) + \int_0^{\ell} G(\iota, \tau)\, \mathscr{M}(\tau)\, d_q\tau. \tag{10}$$

The function $\mathscr{K}(\iota)$ and $G(\iota, \tau)$ are given by:

$$\mathscr{K}(\iota) = \frac{c_1(\ell - \iota - 1)}{\ell} \int_0^{\ell} h_1(\tau)\, d_q\tau + \frac{c_2(\iota + 1)}{\ell} \int_0^{\ell} h_2(\tau)\, d_q\tau, \tag{11}$$

and

$$G(\iota, \tau) = \frac{1}{\Gamma_q(\gamma)} \begin{cases} (\iota - q\tau)^{(\gamma-1)} + \frac{\iota+1}{\ell}\phi(\tau),\ 0 < \tau < \iota < \ell, \\ \\ \frac{\iota+1}{\ell}\phi(\tau),\ 0 < \iota < \tau < \ell, \end{cases} \tag{12}$$

where

$$\phi(\tau) = (\gamma - 1)(\ell - q\tau)^{(\gamma-2)} - (\ell - q\tau)^{(\gamma-1)}, \tag{13}$$

which yields the following bound:

$$(\ell - \ell q)^{(\gamma-2)} - \ell^{(\gamma-1)} \leq \phi(\tau) \leq \ell^{(\gamma-2)}. \tag{14}$$

Proof. On the equation ${}^{c}D_{q,\gamma}\nu(\iota) = \mathscr{M}(\iota)$ we can apply the operator $\mathscr{I}_{q,\gamma}$ we find

$$\nu(\iota) = \frac{1}{\Gamma_q(\gamma)} \int_0^{\iota} (\iota - q\tau)^{(\gamma-1)} \mathscr{M}(\tau)\, d_q\tau + \iota\rho_1 + \rho_2,$$

where $\rho_1, \rho_2 \in \mathbb{R}$ are arbitrary constant. Through the boundary conditions that are given in (8)-(9), we get

$$\rho_2-\rho_1=c_1\int_0^{\ell}h_1(\tau)\,d_q\tau,$$

and

$$\frac{1}{\Gamma_q(\gamma)}\int_0^{\ell}(\ell-q\tau)^{(\gamma-1)}\mathscr{M}(\tau)\,d_q\tau+\ell\rho_1+\rho_2=\frac{\gamma-1}{\Gamma_q(\gamma)}\int_0^{\ell}(\ell-q\tau)^{(\gamma-2)}\mathscr{M}(\tau)\,d_q\tau+\rho_1+c_2\int_0^{\ell}h_2(\tau)\,d_q\tau.$$

With a simple calculation ρ_1 and ρ_2 are given by:

$$\begin{aligned}\rho_1=&\frac{1}{\ell\Gamma_q(\gamma)}\int_0^{\ell}\left((\gamma-1)(\ell-q\tau)^{(\gamma-2)}-(\ell-q\tau)^{(\gamma-1)}\right)\mathscr{M}(\tau)\,d_q\tau\\&+\frac{1}{\ell}\int_0^{\ell}(c_2h_2(\tau)-c_1h_1(\tau))\,d_q\tau,\end{aligned}$$

and

$$\begin{aligned}\rho_2=&\frac{1}{\ell\Gamma_q(\gamma)}\int_0^{\ell}\left((\gamma-1)(\ell-q\tau)^{(\gamma-2)}-(\ell-q\tau)^{(\gamma-1)}\right)\mathscr{M}(\tau)\,d_q\tau\\&+\frac{1}{\ell}\int_0^{\ell}(c_2h_2(\tau)+c_1(\ell-1)h_1(\tau))\,d_q\tau.\end{aligned}$$

Hence,

$$\begin{aligned}\nu(t)=&\frac{1}{\Gamma_q(\gamma)}\int_0^{\iota}(\iota-q\tau)^{(\gamma-1)}\mathscr{M}(\tau)\,d_q\tau\\&+\frac{\iota+1}{\ell\Gamma_q(\gamma)}\int_0^{\ell}\left((\gamma-1)(\ell-q\tau)^{(\gamma-2)}-(\ell-q\tau)^{(\gamma-1)}\right)\mathscr{M}(\tau)\,d_q\tau\\&+\frac{\iota+1}{\ell}c_2\int_0^{\ell}h_2(\tau)\,d_q\tau+\frac{(\ell-\iota-1)}{\ell}c_1\int_0^{\ell}h_1(\tau)\,d_q\tau\\=&\frac{1}{\Gamma_q(\gamma)}\left(\int_0^{\iota}\left((\iota-q\tau)^{(\gamma-1)}+\frac{\iota+1}{\ell}\left((\gamma-1)(\ell-q\tau)^{(\gamma-2)}-(\ell-q\tau)^{(\gamma-1)}\right)\right)\mathscr{M}(\tau)\,d_q\tau\right)\\&+\frac{\iota+1}{\ell\Gamma_q(\gamma)}\int_{\iota}^{\ell}\left((\gamma-1)(\ell-q\tau)^{(\gamma-2)}-(\ell-q\tau)^{(\gamma-1)}\right)\mathscr{M}(\tau)\,d_q\tau\end{aligned}$$

$$+\frac{\iota+1}{\ell}c_2\int_0^{\ell}h_2(\tau)\,d_q\tau+\frac{(\ell-\iota-1)}{\ell}c_1\int_0^{\ell}h_1(\tau)\,d_q\tau.$$

So

$$\nu(\iota)=\mathscr{K}(\iota)+\int_0^{\mathscr{T}}G(\iota,\,\tau)\mathscr{M}(\tau)\,d_q\tau,$$

where $\mathscr{K}(\iota)$, $G(\iota,\,\tau)$ are given by (11) and (12) respectively. □

Lemma 3 The Green function $G(\iota,\,\tau)$ has the following proporties:

$$\mathscr{B}\leq G(\iota,\,\tau)\leq\mathscr{A}, \tag{15}$$

where

$$\mathscr{A}=\frac{1}{\Gamma_q(\gamma)}\left(\ell^{(\gamma-1)}+\frac{(\ell+1)\,\ell^{(\gamma-2)}}{\ell}\right),$$

and

$$\mathscr{B}=-\frac{\ell+1}{\ell\Gamma_q(\gamma)}\ell^{(\gamma-1)}.$$

Proof. Let $\iota,\,\tau\in[0,\,\ell]$, we have, if $\tau<\iota$

$$\begin{aligned}G(\iota,\,\tau)=&\frac{1}{\Gamma_q(\gamma)}\left((\iota-q\tau)^{(\gamma-1)}+\frac{\iota+1}{\ell}\phi(\tau)\right)\\ \leq&\frac{1}{\Gamma_q(\gamma)}\left(\ell^{(\gamma-1)}+\frac{(\ell+1)\,\ell^{(\gamma-2)}}{\ell}\right)=\mathscr{A},\end{aligned}$$

if $\tau>\iota$

$$G(\iota,\,\tau)=\frac{\iota+1}{\ell\Gamma_q(\gamma)}\phi(\tau)\leq\frac{(\ell+1)\,\ell^{(\gamma-2)}}{\ell\Gamma_q(\gamma)}\leq\mathscr{A}.$$

Hence, $G(\iota,\,\tau)\leq\mathscr{A}$. On other hand, for all $\iota,\,\tau\in[0,\,\ell]$, we have, if $0<\tau<\iota<\ell$

$$G(\iota,\,\tau)=\frac{1}{\Gamma_q(\gamma)}\left((\iota-q\tau)^{(\gamma-1)}+\frac{\iota+1}{\ell}\phi(\tau)\right)$$

$$\geq \frac{1}{\Gamma_q(\gamma)}\left((\iota - q\tau)^{(\gamma-1)} + \frac{\iota+1}{\ell}\left(\ell^{(\gamma-2)}(1-q)^{(\gamma-2)} - \ell^{(\gamma-1)}\right)\right)$$

$$\geq -\frac{\ell+1}{\ell\Gamma_q(\gamma)}\ell^{(\gamma-1)} = \mathscr{B},$$

if $0 < \iota < \tau < \ell$

$$G(\iota, \tau) = \frac{\iota+1}{\ell\Gamma_q(\gamma)}\phi(\tau) \geq -\frac{\ell+1}{\ell\Gamma_q(\gamma)}\ell^{(\gamma-1)} = \mathscr{B}.$$

Then $G(\iota, \tau) \geq \mathscr{B}$. □

Let $\mathscr{E}$ be a Banach spaces. The operator Λ: $\mathscr{E} \to \mathscr{E}$ is called

i. k-Lipschitz if and only if there exists $k > 0$ such that

$$\|\Lambda v - \Lambda v^*\|_{\mathscr{E}} \leq k\|v - v^*\|_{\mathscr{E}}\text{. for all } v,\ v^* \in \mathscr{E}.$$

ii. A contraction if and only if it is k-Lipschitz with $k < 1$.

In addition, we introduce Krasnoselskii and Banach fixed point theorems (see [25]), which play a fundamental role in our analysis.

Theorem 1 (Banach fixed point theorem) If Λ: $\mathscr{E} \to \mathscr{E}$, is a contraction operator. Then Λ has a unique fixed point.

Theorem 2 (Krasnoselskii fixed point theorem). Let $\mathscr{D}$ be a closed convex nonempty subset of a Banach space $(\mathscr{E}, \|.\|)$. Suppose that Λ_1 and Λ_2 map $\mathscr{D}$ into $\mathscr{E}$ such that

(i) $v_1, v_2 \in \mathscr{D}$, implies $\Lambda_1 v_1 + \Lambda_2 v_2 \in \mathscr{D}$;

(ii) Λ_1 is a contraction mapping;

(iii) Λ_2 is completely continuous.

Then there exists $v \in \mathscr{D}$ with $v = \Lambda_1 v + \Lambda_2 v$.

Finally, we present the solution's definition related to the syatem (4)-(6).

Definition 4 A function $v \in \mathscr{C}_{q,\gamma}([0, \ell], \mathscr{E})$ is a solution of the system (4)-(6) if v satisfies the equation

$${}^{c}D_{q,\gamma}v(\iota) - \lambda v(\iota) = \mathscr{J}\left(\iota, v(\iota), {}^{c}D_{q,\gamma}v(\iota)\right),\ \text{ for all } \iota \in [0, \ell],$$

where $\gamma \in (1, 2]$ and $q \in (0, 1)$.

Also, the integral conditions

$$v(0) - v'(0) = c_1\int_0^{\ell}\mathscr{G}_1(\tau, v(\tau))\,d_q\tau,$$

and

$$v(\ell) - v'(\ell) = c_2\int_0^{\ell}\mathscr{G}_2(\tau, v(\tau))\,d_q\tau,$$

hold.

3. Main results

Let, for $r \in \mathbb{R}^+$, the set

$$\mathscr{C}_r = \left\{ \nu \in \mathscr{C}_{q,\gamma}([0, \ell], \mathscr{E}), \|\nu\|_q \leq r \right\}.$$

Clearly $\mathscr{C}_r$ is a closed bounded convex set in the Banach space $\mathscr{C}_{q,\gamma}([0, \ell], \mathscr{E})$. In addition, for the first result, we assume that $\mathscr{E}$ is reflexive Bnach space, and for all $\iota \in [0, \ell]$, there exist constants $\alpha \in \mathbb{R}$ such that

$$\| \mathscr{J}(\iota, \nu, \upsilon) \| \leq \alpha, \text{ for all } \nu, \upsilon \in \mathscr{E}, \tag{16}$$

$$\lambda \in \left\{ \|\nu\| \in]-1, 1[\text{ with } \|\nu\| < \frac{1}{\ell \max(\mathscr{A}, |\mathscr{B}|)} \right\}, \tag{17}$$

$$c_1 \int_0^\ell \mathscr{G}_1(\tau, \nu(\tau)) d_q\tau > c_2 \int_0^\ell \mathscr{G}_2(\tau, \nu(\tau)) d_q\tau, \tag{18}$$

and

$$\max\left(\frac{|\mathscr{K}(0)| + \ell\alpha}{1 - \ell \max(\mathscr{A}, |\mathscr{B}|)|\lambda|}, \frac{\alpha}{1 - |\lambda|} \right) \leq r. \tag{19}$$

Theorem 3 Suppose that (16)-(19) hold. If

$$|\lambda| \max\{\mathscr{T} \max(\mathscr{A}, |\mathscr{B}|), 1\} < 1,$$

then the problem (4)-(6) has at least one solution in $\mathscr{C}_{q,\gamma}([0, \ell], \mathscr{E})$ for all $\iota \in [0, \ell]$.

Proof. We consider the operator Λ: $\mathscr{C}_r \to \mathscr{C}_r$ as follow $\Lambda(\nu) = \Lambda_1(\nu) + \Lambda_2(\nu)$ where Λ_1, Λ_2 are defined by

$$(\Lambda_1\nu)(\iota) = \mathscr{K}(\iota) + \lambda \int_0^\ell G(\iota, \tau)\nu(\tau) d_q\tau,$$

and

$$(\Lambda_2\nu)(\iota) = \int_0^\ell G(\iota, \tau) \mathscr{J}\left(\tau, \nu(\tau), {}^cD_{q,\gamma}\nu(\tau)\right) d_q\tau.$$

We will prove that Λ satisfies the assumptions of Krasnoselskii's fixed point theorem.

Let's start by proving that $\Lambda_1 u + \Lambda_2 v \in \mathscr{C}_r$ for all $u, v \in \mathscr{C}_r$. As $\mathscr{K}(\iota)$ is decreassing due to (18) and $\mathscr{J}$ is continous, we get for all $\iota \in [0, \ell]$,

$$\begin{aligned}
\|(\Lambda_1 u)(\iota) + (\Lambda_2 v)(\iota)\| &= \left\| \mathscr{K}(\iota) + \int_0^\ell G(t, \tau)\left(\lambda v(\tau) + \mathscr{J}\left(\tau, v(\tau), {}^c D_{q,\gamma} v(\tau)\right)\right) d_q\tau \right\| \\
&\leq \|\mathscr{K}(\iota)\| + \int_0^\ell |G(t, \tau)| \left\|\lambda v(\tau) + \mathscr{J}\left(\tau, v(\tau), {}^c D_{q,\gamma} v(\tau)\right)\right\| d_q\tau \\
&\leq |\mathscr{K}(0)| + \max(\mathscr{A}, |\mathscr{B}|) \int_0^\ell \left(|\lambda| \|v(\tau)\| + \left\|\mathscr{J}\left(\tau, v(\tau), {}^c D_{q,\gamma} v(\tau)\right)\right\|\right) d_q\tau \\
&\leq |\mathscr{K}(0)| + \ell \max(\mathscr{A}, |\mathscr{B}|) |\lambda| r + \ell\alpha.
\end{aligned}$$

Then,

$$\|\Lambda_1 u + \Lambda_2 v\|_\infty \leq |\mathscr{K}(0)| + \ell\left(\max(\mathscr{A}, |\mathscr{B}|)|\lambda| r + \alpha\right).$$

On other hand,

$$\begin{aligned}
\left\|\left({}^c D_{q,\gamma}\Lambda_1 u\right)(\iota) + \left({}^c D_{q,\gamma}\Lambda_2 v\right)(\iota)\right\| &= \left\|\lambda u(\iota) + \mathscr{J}\left(\tau, v(\iota), {}^c D_{q,\gamma} v(\iota)\right)\right\| \\
&\leq |\lambda| \|u(\iota)\| + \left\|f\left(t, v(\iota), {}^c D_{q,\gamma} v(\iota)\right)\right\| \\
&\leq |\lambda| r + \alpha.
\end{aligned}$$

Thus,

$$\left\|{}^c D_{q,\gamma}\Lambda_1 u + {}^c D_{q,\gamma}\Lambda_2 v\right\|_\infty \leq |\lambda| r + \alpha.$$

By (19), we get

$$\|\Lambda_1 u + \Lambda_2 v\|_q \leq r.$$

Then $\Lambda_1 u + \Lambda_2 v \in \mathscr{C}_r$ for all $u, v \in \mathscr{C}_r$. Now, we shall prove that Λ_1 is contraction

$$(\Lambda_1 v)(t) = \mathscr{K}(\iota) + \lambda \int_0^\ell G(\iota, \tau) v(\tau) d_q\tau.$$

Let $u, v \in \mathscr{C}_r$, then, for all $\iota \in [0, \ell]$, we have

$$\|(\Lambda_1 u)(\iota) + (\Lambda_1 v)(\iota)\| \leq |\lambda| \int_0^\ell |G(t, s)| \|u(\iota) - v(\iota)\| d_q\tau$$

$$\leq |\lambda| \max(\mathscr{A}, |\mathscr{B}|) \ell \|u(t) - v(t)\|.$$

Then

$$\|\Lambda_1 u + \Lambda_1 v\| \leq |\lambda| \max(\mathscr{A}, |\mathscr{B}|) \ell \|u - v\|.$$

On other hand

$$\left\|\left({}^c D_{q,\gamma}\Lambda_1 u\right)(\iota) + \left({}^c D_{q,\gamma}\Lambda_1 v\right)(\iota)\right\| \leq |\lambda| \|u(\iota) - v(\iota)\|.$$

Then

$$\left\|{}^c D_{q,\gamma}\Lambda_1 u +{}^c D_{q,\gamma}\Lambda_1 v\right\|_\infty \leq |\lambda| \|u - v\|_\infty.$$

Employing (19), then

$$\|\Lambda_1 u + \Lambda_1 v\|_q \leq |\lambda| \max\{\ell \max(\mathscr{A}, |\mathscr{B}|), 1\} \|u - v\|.$$

For $l = |\lambda| \max\{\mathscr{T} \max(\mathscr{A}, |\mathscr{B}|), 1\}$, obviously, Λ_1 is a contractive operator if $l < 1$.

Now, we show that Λ_2 is completely continuous. First, we start by proving that Λ_2 is continuous. Let $\{v_n\} \in \mathscr{C}_r$ be a sequence such that $v_n \to v \in \mathscr{C}_r$ as $n \to \infty$, for each $\iota \in [0, \ell]$,

$$\|(\Lambda_2 v_n)(t) - (\Lambda_2 v)(t)\| \leq \int_0^\ell |G(\iota, \tau)| \left\|\mathscr{J}\left(\tau, v_n(\iota), {}^c D_{q,\gamma} v_n(\iota)\right) - \mathscr{J}\left(\tau, \nu(\iota), {}^c D_{q,\gamma}\nu(\tau)\right)\right\| d_q\tau.$$

Since $\mathscr{J}$ is continuous, then

$$\|\Lambda_2 v_n - \Lambda_2 v\|_\infty \to 0,\, n \to \infty.$$

On other hand,

$$\left\|\left({}^c D_{q,\gamma}\Lambda_2 v_n\right)(t) - \left({}^c D_{q,\gamma}\Lambda_2 v\right)(t)\right\| \leq \left\|\mathscr{J}\left(\iota, v_n(\iota), {}^c D_{q,\gamma} v_n(\iota)\right) - \mathscr{J}\left(\iota, \nu(\iota), {}^c D_{q,\gamma}\nu(\iota)\right)\right\|.$$

Therefore, by the continuity of $\mathscr{J}$ we get

$$\left\|{}^{c}D_{q,\,\gamma}\Lambda_2 v_n - {}^{c}D_{q,\,\gamma}\Lambda_2 v\right\|_{\infty} \to 0,\, n \to \infty.$$

We conclude that

$$\|\Lambda_2 v_n - \Lambda_2 v\|_q \to 0,\, n \to \infty.$$

The second step is to prove $\Lambda_2(\mathscr{C}_r)$ is bounded in $\mathscr{C}_{q,\,\gamma}([0,\,\ell]\,,\,\mathscr{E})$.We have for each $\Lambda_2 v$ where $v \in \mathscr{C}_r$

$$\|\Lambda_2 v(t)\| \leq \int_0^{\ell} |G(\iota,\,\tau)| \left\|\mathscr{J}\left(\tau,\,v(\tau),\,{}^{c}D_{q,\,\gamma}v(\tau)\right)\right\| d_q\tau$$

$$\leq \max(\mathscr{A},\,|\mathscr{B}|)\,\ell\alpha = c.$$

Then

$$\|\Lambda_2 v\|_{\infty} \leq \max(\mathscr{A},\,|\mathscr{B}|)\,\ell\alpha = c.$$

On other hand,

$$\left\|\left({}^{c}D_{q,\,\gamma}\Lambda_2 v\right)(\iota)\right\| \leq \left\|\mathscr{J}\left(t,\,v(\iota),\,{}^{c}D_{q,\,\gamma}v(\iota)\right)\right\|$$

So

$$\left\|{}^{c}D_{q,\,\gamma}\Lambda_2 v\right\|_{\infty} \leq \alpha.$$

Then $\|\Lambda_2 v\|_q \leq \max(c,\,\alpha) = \mathscr{R}$. For the third step, we shall prove that $\Lambda_2(\mathscr{C}_r)$ is equicontinuous set. Let $\iota_1,\,\iota_2 \in [0,\,\ell]$, with $\iota_1 < \iota_2$, for each $\upsilon = \Lambda_2(v)$, we have

$$\|\upsilon(\iota_2) - \upsilon(\iota_1)\| = \int_0^{\ell} |G(t_2,\,s) - G(\iota_1,\,s)| \left\|\mathscr{J}\left(\tau,\,v(\tau),\,{}^{c}D_{q,\,\gamma}v(\tau)\right)\right\| d_q\tau,$$

if $0 < s < \iota_1 < \iota_2 < \ell$

$$\|\upsilon(\iota_2) - \upsilon(\iota_1)\| \leq \frac{\alpha}{\Gamma_q(\gamma)} \int_0^{\ell} \left|(\iota_2 - q\tau)^{(\gamma-1)} - (\iota_1 - q\tau)^{(\gamma-1)} + \frac{\iota_2 - \iota_1}{\ell}\phi(\tau)\right| d_q\tau,$$

if $0 < \iota_1 < \iota_2 < s < \ell$

$$\|\upsilon(\iota_2)-\upsilon(\iota_1)\| = \frac{\alpha}{\Gamma_q(\gamma)}\int_0^{\ell}\frac{\iota_2-\iota_1}{\ell}\left|\phi(\tau)\right| d_q\tau.$$

So, if $\iota_2 \to \iota_1$ the right-hand side of the above equalitys tends to zero.

For the last step, we shall prove that for all $\iota \in [0, \ell]$ the set $\overline{\mathscr{C}_r(\iota)} = \overline{\{\upsilon(\iota),\ \upsilon \in \mathscr{C}_r\}}$ is a compact set in $\mathscr{E}$.

Let $(\upsilon_n)_{n\in\mathbb{N}} \in \mathscr{C}_r$ be a sequence, then for $\iota \in [0, \ell]$ fixed, $\|\upsilon_n(\iota)\| \le r,$ this gives us that $(\upsilon_n(\iota))_{n\in\mathbb{N}}$ is bounded. Since $\mathscr{E}$ is reflexive then there existe a weakly convergent subsequence such that $\upsilon_{n_k}(\iota) \rightharpoonup \upsilon(\iota)$, wich implies $\upsilon_{n_k}(\iota) \to \upsilon(\iota)$ uniformly in $\mathscr{E}$. Therefor we conclude that the set $\mathscr{C}_r(\iota)$ is relatively compact. By the Arzelà-Ascoli theorem we arrive at the conclusion that Λ_2 is completely continuous.

So, with the Theorem 2 there is an $u \in \mathscr{C}_r$ such that $\Lambda_1(\nu)+\Lambda_2(\nu) = \nu$. The problem (4)-(6) has a solution ν in $\mathscr{C}_r$. □

Now, we provide an example to demonstrate the outcomes of Theorem 3.

Example 1 Let $\mathscr{E} = L^2([0, 1])$ the Banach space of all real square integrable functions equipped with the norm

$$\|\nu\|_{L^2} = \left(\int_0^1 |\nu(\tau)|^2 d\tau\right)^{\frac{1}{2}}.$$

We consider the q-fractional differential equation with nonlinear integral conditions, given by:

$$^cD_{0.5,\,1.5}\nu(\iota) - 0.2\nu(\iota) = \iota\nu(\iota)^c D_{0.5,\,1.5}\nu(\iota),\ \ \iota \in [0, 1], \tag{20}$$

$$\nu(0)-\nu'(0) = c_1\int_0^1 \nu(\tau)\, d_q\tau, \tag{21}$$

and

$$\nu(1)-\nu'(1) = \int_0^1 \left(c_1\nu(\tau)-\nu^2(\tau)\right) d_q\tau, \tag{22}$$

where

$$q = 0.5,\ \ \gamma = 1.5,\ \ \ell = 1,\ \ \lambda = 0.2,\ \ c_1, c_2 > 0,$$

and

$$\mathscr{G}_1(\iota, \nu) = \nu, \mathscr{G}_2(\iota, \nu) = \frac{c_1\nu-\nu^2}{c_2} \text{ and } \mathscr{J}(\iota, \nu, \mu) = \iota\nu\mu.$$

Let

$$\mathscr{C}_{0.5,\,1.5}\left([0,\,1]\,,\,\mathscr{E}\right)=\left\{\nu\colon\,\nu\in C\left([0,\,1]\,,\,\mathscr{E}\right),\,{}^{c}D_{0.5,\,1.5}\nu\in C\left([0,\,1]\,,\,\mathscr{E}\right)\right\},$$

for $r\in\mathbb{R}_{+}^{*}$, we propose the set

$$\mathscr{C}_{r}=\left\{\nu\in\mathscr{C}_{0.5,\,1.5}\left([0,\,1]\,,\,\mathscr{E}\right),\,\|\nu\|_{q}\leq r\right\},$$

such that r satisfy $r\geq\frac{2+\alpha}{0.34854}$. By simple calculation, we get

$$\mathscr{A}=\frac{1}{\Gamma_{0.5}\left(1.5\right)}\left(1^{(\gamma-1)}+2^{(\gamma-2)}\right)\simeq 3.2573,$$

and

$$\mathscr{B}=-\frac{1}{\Gamma_{0.5}\left(1.5\right)}1^{(0.5)}\simeq -0.70711.$$

For each $\nu,\,\mu\in\mathscr{E}$ and $\iota\in[0,\,1]$, we have

$$\|\mathscr{J}\left(\iota,\,\nu,\,\mu\right)\|=\|\iota\nu\mu\|\leq\frac{\|\nu\|_{L^{2}}\|\mu\|_{L^{2}}}{\sqrt{3}}=\alpha<\infty,$$

then the condition (16) is satisfied. Also,

$$\lambda=0.2\in\left\{\|\nu\|\in\left]-1,\,1\right[\colon\,\|\nu\|<\frac{1}{\ell\max\left(\mathscr{A},\,|\mathscr{B}|\right)}=\frac{1}{3.2573}\simeq 0.307\right\},$$

so the condition (17) is verified. The condition (18) is verified easily. On the other hand

$$\max\left(\frac{|\mathscr{K}\left(0\right)|+\ell\alpha}{1-\ell\max\left(\mathscr{A},\,|\mathscr{B}|\right)|\lambda|},\,\frac{\alpha}{1-|\lambda|}\right)=\max\left(\frac{2+\alpha}{0.34854},\,\frac{\alpha}{0.8}\right)$$

$$=\frac{2+a}{0.34854}\leq r.$$

Then (19) is verified.

The quantity

$$|\lambda|\,\ell\max\left\{\max\left(\mathscr{A},\,|\mathscr{B}|\right),\,1\right\}=0.65146<1.$$

By Theorem 3, for all $\iota \in [0, \ell]$, the problem (20)-(22) has at least one solution in $\mathscr{C}_{0.5,\, 1.5}([0, 1], \mathscr{E})$.

Now interested in providing sufficient conditions to guarantee that the solution to problem (4)-(6) is unique. Here, Banach's fixed point theorem provided the foundation for our strategy. Let us assume that the function $\mathscr{J}$ fulfills the following hypothesis. For all $\iota \in [0, \ell]$, there exist two positives constants $p_1, p_2 \in \mathbb{R}$ such that:

$$\|\mathscr{J}(\iota, \nu_1, \nu_2) - \mathscr{J}(\iota, \upsilon_1, \upsilon_2)\| \leq p_1 \|\nu_1 - \upsilon_1\| + p_2 \|\nu_2 - \upsilon_2\|, \text{ for all } \nu_1, \nu_2, \upsilon_1, \upsilon_2 \in \mathscr{E}. \tag{23}$$

Theorem 4 Assume that (23) holds. If

$$\rho = \max\left(\ell \max\left(\mathscr{A}, |\mathscr{B}|\right)\left(|\lambda| + p_1 + p_2\right), |\lambda| + p_1 + p_2\right) < 1,$$

Then, for all $\iota \in [0, \ell]$, the problem (4)-(6) has a unique solution in $\mathscr{C}_{q,\gamma}([0, \ell], \mathscr{E})$.

We consider the operator

$$\Lambda: \mathscr{C}_{q,\gamma}([0, \ell], \mathscr{E}) \to \mathscr{C}_{q,\gamma}([0, \ell], \mathscr{E})$$

as follow

$$\Lambda(\nu)(\iota) = \mathscr{K}(\iota) + \int_0^\ell G(\iota, \tau)\left(\lambda \nu(\tau) + \mathscr{J}\left(\tau, \nu(\tau), {}^cD_{q,\gamma}\nu(\tau)\right)\right) d_q\tau.$$

Proof. We will prove that Λ satisfies the assumptions of Banach's fixed point theorem. Consider the two elements ν and υ in $\mathscr{C}_{q,\gamma}([0, \ell], \mathscr{E})$, then, for each $\iota \in [0, \ell]$, we have

$$\begin{aligned}
\|\Lambda(\nu)(\iota) - \Lambda(\upsilon)(\iota)\| = & \left\| \int_0^\ell G(\iota, \tau)\left(\lambda\left(\nu(\tau) - \upsilon(\tau)\right) + \mathscr{J}\left(\tau, \nu(\tau), {}^cD_{q,\gamma}\nu(\tau)\right)\right.\right. \\
& \left.\left. - \mathscr{J}\left(\tau, \upsilon(\tau), {}^cD_{q,\gamma}\upsilon(\tau)\right) d_q\tau\right) \right\| \\
& \leq \int_0^\ell |G(\iota, \tau)| \left(|\lambda| \|\nu(\iota) - \upsilon(\iota)\|\right. \\
& \left. + \left\| \mathscr{J}\left(\iota, \nu(\iota), {}^cD_{q,\gamma}\nu(\iota)\right) - \mathscr{J}\left(\iota, \upsilon(\iota), {}^cD_{q,\gamma}\upsilon(\iota)\right) \right\| \right) d_q\tau \\
& \leq \ell \max\left(\mathscr{A}, |\mathscr{B}|\right)\left(|\lambda| \|\nu(\iota) - \upsilon(\iota)\| + p_1 \|\nu(\iota) - \upsilon(\iota)\|\right. \\
& \left. + p_2 \left\| {}^cD_{q,\gamma}\nu(\iota) - {}^cD_{q,\gamma}\upsilon(\iota) \right\| \right) \\
& \leq \ell \max\left(\mathscr{A}, |\mathscr{B}|\right)\left(\left(|\lambda| + p_1\right) \|\nu(\iota) - \upsilon(\iota)\| + p_2 \left\| {}^cD_{q,\gamma}\nu(\iota) - {}^cD_{q,\gamma}\upsilon(\iota) \right\| \right).
\end{aligned}$$

Therfore,

$$\|\Lambda\nu - \Lambda\upsilon\|_{\infty} \leq \ell \max(\mathscr{A}, |\mathscr{B}|)(|\lambda| + p_1 + p_2)\|\nu - \upsilon\|_q.$$

On other hand,

$$\begin{aligned}
\left\|{}^cD_{q,\gamma}\Lambda(\nu)(\iota) - {}^cD_{q,\gamma}(\upsilon)(\iota)\right\| &= \left\|\lambda(\nu(\tau) - \upsilon(\tau)) + \mathscr{J}\left(\iota, \nu(\iota), {}^cD_{q,\gamma}\nu(\iota)\right) - \mathscr{J}\left(\iota, \upsilon(\iota), {}^cD_{q,\gamma}\upsilon(\iota)\right)\right\| \\
&\leq |\lambda|\|\nu(\iota) - \upsilon(\iota)\| + \left\|\mathscr{J}\left(\iota, \nu(\iota), {}^cD_{q,\gamma}\nu(\iota)\right) - \mathscr{J}\left(\iota, \upsilon(\iota), {}^cD_{q,\gamma}\upsilon(\iota)\right)\right\| \\
&\leq |\lambda|\|\nu(\iota) - \upsilon(\iota)\| + p_1\|\nu(\iota) - \upsilon(\iota)\| + p_2\left\|{}^cD_{q,\gamma}\nu(\iota) - {}^cD_{q,\gamma}\upsilon(\iota)\right\| \\
&\leq (|\lambda| + p_1)\|\nu(\iota) - \upsilon(\iota)\| + p_2\left\|{}^cD_{q,\gamma}\nu(\iota) - {}^cD_{q,\gamma}\upsilon(\iota)\right\|.
\end{aligned}$$

So,

$$\left\|{}^cD_{q,\gamma}\Lambda\nu - {}^cD_{q,\gamma}\Lambda\upsilon\right\|_{\infty} \leq (|\lambda| + p_1 + p_2)\|\nu - \upsilon\|_q.$$

Then

$$\begin{aligned}
\|\Lambda\nu - \Lambda\upsilon\|_q &\leq \max(\ell \max(\mathscr{A}, |\mathscr{B}|)(|\lambda| + p_1 + p_2), |\lambda| + p_1 + p_2)\|\nu - \upsilon\|_q \\
&= \rho\|\nu - \upsilon\|_q.
\end{aligned}$$

Since $\rho < 1$, then the operator Λ is a contraction. So, Λ has a unique fixed point $\nu \in \mathscr{C}_{q,\gamma}([0, \ell], \mathscr{E})$ according Banach's theorem which is the unique solution of the problem (4)-(6). □

Example 2 Consider the q-fractional differential equation with nonlinear integral conditions, given by:

$$\begin{aligned}
&{}^cD_{0.7, 7/5}\nu(\iota) - \lambda\nu(\iota) = \sin(\iota)\nu(\iota) + \cos(\iota){}^cD_{0.7, 7/5}\nu(\iota), \ \iota \in [0, 1], \\
&\nu(0) - \nu'(0) = c_1\int_0^{\ell} g_1(s, \nu(s))\, d_qs, \\
&\nu(\ell) - \nu'(\ell) = c_2\int_0^{\ell} g_2(s, \nu(s))\, d_qs,
\end{aligned} \tag{24}$$

where

$$q = 0.7,\ \gamma = \frac{7}{5},\ \ell = 1,\ |\lambda| < 0.15,\ \text{ and } \mathscr{J}(t,\ \nu,\ \upsilon) = \nu \sin(t) + \upsilon \cos(t),$$

and $c_1,\ c_2,\ g_1(\iota,\ \nu),\ g_2(\iota,\ \nu)$, are all chosen arbitrarily. The condition (23) is verified due to the following inequality:

$$\|\mathscr{J}(\iota,\ \nu_1,\ \nu_2) - \mathscr{J}(\iota,\ \upsilon_1,\ \upsilon_2)\| \leq \frac{1}{5}\|\nu_1 - \upsilon_1\| + \frac{1}{10}\|\nu_2 - \upsilon_2\|.$$

By simple calculation, we get

$$\mathscr{A} = \frac{1}{\Gamma_q(\gamma)}\left(1^{(\gamma-1)} + 2^{(\gamma-2)}\right) = \frac{1}{\Gamma_{0.7}\left(\frac{7}{5}\right)}\left(1^{\left(\frac{7}{5}-1\right)} + 2^{\left(\frac{7}{5}-2\right)}\right) \approx 1.8334,$$

$$\mathscr{B} = -\frac{\ell+1}{\ell\Gamma_q(\gamma)}\ell^{(\gamma-1)} = -\frac{1+1}{\Gamma_{0.7}\left(\frac{7}{5}\right)}1^{\left(\frac{7}{5}-1\right)} \approx -2.2092,$$

and

$$\begin{aligned}\rho =& \max\left(\ell \max\left(\mathscr{A},\ |\mathscr{B}|\right)\left(|\lambda| + p_1 + p_2\right),\ |\lambda| + p_1 + p_2\right)\\ =& \max\left(2.2092\left(|\lambda| + \frac{1}{5} + \frac{1}{10}\right),\ 0.15 + \frac{1}{5} + \frac{1}{10}\right)\\ =& \max(0.99414,\ 0.45)\\ =& 0.99414 < 1.\end{aligned}$$

All the requirements of Theorem 4 are satisfied. Then the problem (24) has a unique solution in $\mathscr{C}_{0.7,\,7/5}([0,\ \ell],\ \mathscr{E})$.

Conflict of interest

The authors declare no competing financial interest.

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