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Research Article

Generalized Nonlinear Rational Contractions via Coupled Fixed Points Under a Binary Relation with Application to Fractional Differential Equations

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Abstract: In this article, we use a binary relation to demonstrate the existence and uniqueness of fixed points and coupled fixed point results in the context of metric space under the new generalized $(\phi,\ \psi)$-rational contraction. Moreover, we employed the mixed $\mathscr{Q}$-monotone property of a mapping and utilize the same to investigate the existence and uniqueness of coupled fixed points endowed with a binary relation $\mathscr{Q}$. Additionally, we present an example to demonstrate our recently validated findings. Lastly, we provide an application to fractional differential equation.

Keywords: Coupled fixed point (Cfp), mixed monotone property, $(\phi,\ \psi)$-rational contraction, binary relation, fractional differential equation

MSC: 47H10, 54H25

1. Introduction and preliminaries

The Banach contraction theorem [1] is a cornerstone of mathematical analysis, renowned for its wide-ranging applications across numerous fields. It asserts that any self-mapping on a complete metric space admits a unique fixed point. This powerful result has been instrumental in advancing areas such as functional analysis, optimization, differential equations, and dynamical systems. Due to its depth and adaptability, the concept of Banach contractions has inspired extensive research, resulting in a rich body of literature on its applications and generalizations. In recent years, the extension of fixed-point theory to partially ordered metric space has marked a significant advancement, opening new avenues for applications across various branches of mathematics.

On early contribution in this area was made by Turinici [2], who investigated fixed point results in ordered metrizable uniform spaces, providing a foundation for subsequent research. Building on this, Ran and Reurings [3] applied fixed point theorems in partially ordered metric space to solve matrix equations, demonstrating the approach's effectiveness in linear algebra problems. Likewise, Nieto and Rodriguez-Lopez [4] employed fixed point results in partially ordered metric space to study partial differential equations with periodic boundary conditions, underlining their significance in mathematical physics. For the relevant work, see [5–11].

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In 2015, Alam and Imdad [12] made a noteworthy contribution by generalizing the Banach Contraction Principle (BCP) through the use of an amorphous binary relation (in short, relation) in place of a partial order. This advancement paved the way for further research, inspiring a variety of relation-theoretic results by numerous scholars [13–15]. Subsequent studies have primarily aimed at extending the BCP to incorporate more general contractive conditions, often involving one or more auxiliary functions. A prominent generalization was introduced by Dutta and Choudhury [16], which has since been expanded and refined by many researchers [13], reflecting the continuous development in this domain. Additionally, Chandok et al. [17] contributed to the field by establishing the existence and uniqueness of fixed points for a specific class of rational-type contractions within partially ordered metric space, further enriching the theoretical landscape of fixed point theory.

Opoitsev [18, 19] introduced coupled fixed points, which Guo et al. [20, 21] further explored. In metric fixed theory, the Coupled fixed point (Cfp) theorem has gained importance lately. Using the idea of mixed monotonic property, Bhaskar and Lakshmikanthan [22] obtained order theoretic existence and uniqueness of fixed point results for a pair of mapping from $\mathfrak{S} \times \mathfrak{S} \to \mathfrak{S}$ in partially ordered metric space after that in 2009, Lakshmikantham and Ćirić [23] introduced some Cfp results for nonlinear contraction.

In this article, we establish the existence and uniqueness of fixed point results in metric spaces using a relation under a newly generalized $(\phi,\ \psi)$-rational contraction. In section 4, we investigate the existence and uniqueness of coupled fixed point results by employing the mixed $\mathfrak{K}$-monotone property. An illustrative example is provided to support the validity of our main results. In section 5, we present an application to a fractional differential equation.

Throughout this manuscript, $\mathbb{N}_0$ stands for the set of whole numbers, $\mathbb{N}$ stands for the set of natural numbers & $\mathbb{R}$ signifies the set of real numbers.

Definition 1 [22] Consider the mapping $\daleth : \mathfrak{S} \times \mathfrak{S} \to \mathfrak{S}$, where $\mathfrak{S}$ is a non empty set. Then an element $(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})$ is called Cfp if

$$\daleth(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) = \tilde{\mathfrak{A}} \ \ \&\ \ \daleth(\tilde{\mathfrak{E}},\ \tilde{\mathfrak{A}}) = \tilde{\mathfrak{E}}.$$

A Cfp $(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})$ is called a strong Cfp if $\tilde{\mathfrak{A}} = \tilde{\mathfrak{E}}$.

Definition 2 [22, 24] Let $\mathscr{Q}$ be a relation on $\mathfrak{S}$. Then $\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}} \in \mathfrak{S}$,

(i) The inverse relation $\mathscr{Q}^{-1} = \{(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) \in \mathfrak{S}^2 : (\tilde{\mathfrak{E}},\ \tilde{\mathfrak{A}}) \in \mathscr{Q}\}$ & symmetric closure $\mathscr{Q}^s := \mathscr{Q} \cup \mathscr{Q}^{-1}$.

(ii) $\tilde{\mathfrak{A}}$ & $\tilde{\mathfrak{E}}$ are $\mathscr{Q}$-comparative if either $(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) \in \mathscr{Q}$ or $(\tilde{\mathfrak{E}},\ \tilde{\mathfrak{A}}) \in \mathscr{Q}$. This is denoted as $[\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}] \in \mathscr{Q}$.

(iii) A sequence $\tilde{\mathfrak{A}}_n \subset \mathfrak{S}$ is called $\mathscr{Q}$- preserving if $(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{A}}_{n+1}) \in \mathscr{Q}\ \forall\ n \in \mathbb{N}_0$.

Definition 3 [25] Let $(\mathfrak{S},\ \wp)$ be a metric space equipped with a relation $\mathscr{Q}$. We say that $(\mathfrak{S},\ \wp)$ is $\mathscr{Q}$-complete if every $\mathscr{Q}$-preserving Cauchy sequence $\{\tilde{\mathfrak{A}}_n\}\ \in\ \mathfrak{S}$, there is some $\tilde{\mathfrak{A}} \in \mathfrak{S}$ converges.

Definition 4 [12] Let $(\mathfrak{S},\ \wp)$ be a metric space. A relation $\mathscr{Q}$ defined on $\mathfrak{S}$ is called $\wp$-self-closed if whenever $\{\tilde{\mathfrak{A}}_n\}$ is an $\mathbb{R}$-preserving sequence and

$$\tilde{\mathfrak{A}}_n \xrightarrow{\wp} \tilde{\mathfrak{A}}$$

then $\exists$ a subsequence $\{\tilde{\mathfrak{A}}_{n_k}\}$ of $\{\tilde{\mathfrak{A}}_n\}$ with $[\tilde{\mathfrak{A}}_{n_k},\ \tilde{\mathfrak{A}}] \in \mathscr{Q}\ \forall\ k \in \mathbb{N}_0$.

Definition 5 [12] Let $\mathfrak{S} \neq \emptyset$ & $\daleth : \mathfrak{S} \to \mathfrak{S}$ be a self-mapping. A relation $\mathscr{Q}$ on $\mathfrak{S}$ is called $\daleth$-closed if $\forall\ \tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}} \in \mathfrak{S}$

$$(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) \in \mathscr{Q} \implies (\daleth\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{E}}) \in \mathscr{Q}.$$

Proposition 1 [12] Let $\mathfrak{S}$, $\daleth$ and $\mathscr{Q}$ be same as in Definition 5. $\mathscr{Q}^s$ must also be $\daleth$-closed if $\mathscr{Q}$ is $\daleth$-closed.

Definition 6 [26] Let $\mathfrak{S} \neq \emptyset$ & $\mathscr{Q}$ a relation on $\mathfrak{S}$. A subset Y of $\mathfrak{S}$ is called $\mathscr{Q}$-directed if for every $\tilde{\mathfrak{A}}$, $\tilde{\mathfrak{E}} \in Y$, $\exists$ $z \in \mathfrak{S}$ such that $(\tilde{\mathfrak{A}}, z) \in \mathscr{Q}$ and $(\tilde{\mathfrak{E}}, z) \in \mathscr{Q}$.

Definition 7 [27] Let $\mathscr{G} \in \mathbb{N}_0$, $\mathscr{G} \geq 2$, be a relation $\mathscr{Q}$ on a non empty set $\mathfrak{S}$ is called $\mathscr{G}$-transitive if for any $\tilde{\mathfrak{A}}_0$, $\tilde{\mathfrak{A}}_1$, $\tilde{\mathfrak{A}}_2, \ldots, \tilde{\mathfrak{A}}_{\mathscr{G}} \in \mathfrak{S}$

$$(\tilde{\mathfrak{A}}_{i-1}, \tilde{\mathfrak{A}}_i) \in R \text{ for each } i(1 \leq i \leq \mathscr{G}) \implies (\tilde{\mathfrak{A}}_0, \tilde{\mathfrak{A}}_{\mathscr{G}}) \in R.$$

Definition 8 [27] A relation $\mathscr{Q}$ on a nonempty set $\mathfrak{S}$ is called locally finitely transitive if for each denumerable subset E of $\mathfrak{S}$, $\exists\, \mathscr{G} = \mathscr{G}(E) \geq 2$, such that the restriction $R|_E$ is $\mathscr{G}$-transitive.

Definition 9 [28] Let $\mathfrak{S} \neq \emptyset$ and $\urcorner : \mathfrak{S} \to \mathfrak{S}$ be a self mapping. A relation $\mathscr{Q}$ on $\mathfrak{S}$ is called locally finitely $\urcorner$-transitive if for each denumerable subset E of $\urcorner(\mathfrak{S})$, $\exists\, \mathscr{G} = \mathscr{G}(E) \geq 2$, such that the restriction $\mathscr{Q}|_E$ is $\mathscr{G}$-transitive.

Proposition 2 [28] Let $\mathfrak{S} \neq \emptyset$, $\mathscr{Q}$ be a relation on $\mathfrak{S}$ & $\urcorner : \mathfrak{S} \to \mathfrak{S}$ be a self-mapping. Then

(i) $\mathscr{Q}$ is $\urcorner$-transitive $\iff \mathscr{Q}|_{\urcorner(\mathfrak{S})}$ is transitive,

(ii) $\mathscr{Q}$ is locally finitely $\urcorner$-transitive $\iff \mathscr{Q}|_{\urcorner(\mathfrak{S})}$ is locally finitely transitive,

(iii) $\mathscr{Q}$ is transitive $\to \mathscr{Q}$ is finitely transitive $\to \mathscr{Q}$ is locally transitive $\to \mathscr{Q}$ is locally finitely $\urcorner$-transitive,

(iv) $\mathscr{Q}$ is transitive $\to \mathscr{Q}$ is $\urcorner$-transitive $\to \mathscr{Q}$ is locally finitely $\urcorner$-transitive.

Lemma 1 [29] Let $(\mathfrak{S}, \wp)$ be a metric space and $\{\tilde{\mathfrak{A}}_n\}$ a sequence in $\mathfrak{S}$. If $\{\tilde{\mathfrak{A}}_n\}$ is not a Cauchy sequence, then $\exists$ $\varepsilon > 0$ & subsequences $\{\tilde{\mathfrak{A}}_{n_\varsigma}\}$ & $\{\tilde{\mathfrak{A}}_{m_\varsigma}\}$ of $\{\tilde{\mathfrak{A}}_n\}$ such that

(i) $\varsigma \leq m_\varsigma < n_\varsigma\ \forall\, \varsigma \in \mathbb{N}$,

(ii) $d(\tilde{\mathfrak{A}}_{m_\varsigma}, \tilde{\mathfrak{A}}_{n_\varsigma}) \geq \varepsilon$,

(iii) $d(\tilde{\mathfrak{A}}_{m_\varsigma}, \tilde{\mathfrak{A}}_{p_\varsigma}) \geq \varepsilon$, $\forall\, p_\varsigma \in \{m_{\varsigma+1}, m_{\varsigma+2}, \ldots, n_{\varsigma-2}, n_{\varsigma-1}\}$.

Additionally, if $\lim_{n\to\infty} \wp(\tilde{\mathfrak{A}}_n, \tilde{\mathfrak{A}}_{n+1}) = 0$, then

(iv) $\lim_{\varsigma\to\infty} \wp(\tilde{\mathfrak{A}}_{m_\varsigma}, \tilde{\mathfrak{A}}_{n_\varsigma+p}) = \varepsilon\ \forall\, p \in \mathbb{N}_0$.

Lemma 2 [27] Let $\mathfrak{S} \neq \emptyset$, $\mathscr{Q}$ a relation on $\mathfrak{S}$ and $\{\tilde{\mathfrak{A}}_n\}$ is a $\mathscr{Q}$-preserving sequence in $\mathfrak{S}$. If $\mathscr{Q}$ is a $\mathscr{G}$-transitive on the set $Y := \{\tilde{\mathfrak{A}}_n : n \in \mathbb{N}_0\}$ for some natural number $\mathscr{G} \geq 2$, then

$$(\tilde{\mathfrak{A}}_n, \tilde{\mathfrak{A}}_{n+1+r(\mathscr{G}-1)}) \in \mathscr{Q}, \quad \forall\ n,\ r \in \mathbb{N}_0.$$

Definition 10 Let $(\mathfrak{S}, \urcorner)$ as a partial ordered set. Then a mapping $\urcorner : \mathfrak{S} \times \mathfrak{S} \to \mathfrak{S}$ is said to admits a mixed monotone property if

$$\tilde{\mathfrak{A}}_1 \preceq \tilde{\mathfrak{A}}_2 \implies \urcorner(\tilde{\mathfrak{A}}_1, \tilde{\mathfrak{E}}) \preceq \urcorner(\tilde{\mathfrak{A}}_2, \tilde{\mathfrak{E}}) \qquad \forall\, \tilde{\mathfrak{A}}_1, \tilde{\mathfrak{A}}_2, \tilde{\mathfrak{E}} \in \mathfrak{S}$$

and

$$\tilde{\mathfrak{E}}_1 \preceq \tilde{\mathfrak{E}}_2 \implies \urcorner(\tilde{\mathfrak{A}}, \tilde{\mathfrak{E}}_2) \preceq \urcorner(\tilde{\mathfrak{A}}, \tilde{\mathfrak{E}}_1) \qquad \forall\, \tilde{\mathfrak{E}}_1, \tilde{\mathfrak{E}}_2, \tilde{\mathfrak{A}} \in \mathfrak{S}.$$

Definition 11 [30] Let $\mathfrak{S} \neq \emptyset$ be a set equipped with a relation $\mathscr{Q}$. Then a mapping $\urcorner : \mathfrak{S} \times \mathfrak{S} \to \mathfrak{S}$ is said to possess a mixed $\mathscr{Q}$-monotone property if

$$(\tilde{\mathfrak{A}}_1, \tilde{\mathfrak{A}}_2) \in \mathscr{Q} \implies (\urcorner(\tilde{\mathfrak{A}}_1, \tilde{\mathfrak{E}}), \urcorner(\tilde{\mathfrak{A}}_2, \tilde{\mathfrak{E}})) \in \mathscr{Q} \qquad \forall\, \tilde{\mathfrak{A}}_1, \tilde{\mathfrak{A}}_2, \tilde{\mathfrak{E}} \in \mathfrak{S}$$

and

$$(\tilde{\mathfrak{E}}_1,\ \tilde{\mathfrak{E}}_2) \in \mathscr{Q} \implies (\urcorner(\tilde{\mathfrak{E}}_1,\ \tilde{\mathfrak{A}}),\ \urcorner(\tilde{\mathfrak{E}}_2,\ \tilde{\mathfrak{A}})) \in \mathscr{Q} \qquad \forall\, \tilde{\mathfrak{E}}_1,\ \tilde{\mathfrak{E}}_2,\ \tilde{\mathfrak{A}} \in \mathfrak{S}.$$

Example 1 [30] Let $\mathfrak{S} = [1,\ 4]$ be a partial ordered set with usual order less than equal ($\leq$), and consider the function $\urcorner : \mathfrak{S} \times \mathfrak{S} \to \mathfrak{S}$ given by

$$\urcorner(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) = \begin{cases} 2 & \text{if } \tilde{\mathfrak{A}} \in \{2,\ 4\} \\ \\ 4 & \text{otherwise.} \end{cases}$$

Observe that $\urcorner(1,\ 2) \leq \urcorner(2,\ 2)$, even though $1 \leq 2$. Hence, $\mathfrak{S}$ does not satisfy mixed monotone property. Now, consider a relation $\mathscr{Q}$ on $\mathfrak{S}$ defined by

$$\mathscr{Q} = \{(2,\ 2),\ (4,\ 2),\ (2,\ 2),\ (4,\ 4)\}.$$

Then $\urcorner$ satisfy the assumptions of Definition 11. Hence $\urcorner$ satisfies mixed $\mathscr{Q}$-monotone property.

2. A new class of $(\Phi,\ \Psi)$-contraction

Let Φ denote the class of the functions $\phi : [0,\ \infty) \to [0,\ \infty)$ that satisfying the following assumptions:

Φ_1 : ϕ is right continuous;

Φ_2 : ϕ is monotonic increasing and $\phi(0) = 0$.

Let Ψ denote the class of the functions $\psi : [0,\ \infty) \to [0,\ \infty)$ that satisfying the following assumptions:

Ψ_1 : $\psi(\flat) > 0,\ \flat > 0$;

Ψ_2 : $\lim_{\flat \to r} \inf \psi(r) > 0, \forall\, r > 0$.

Remark 1 [13] Axiom Ψ_1 is equivalent to the following:

Ψ_1' : If $\exists\, \flat \in [0,\ \infty)$ such that $\psi(\flat) = 0$, then $\flat = 0$.

Proposition 3 [13] If $\exists$ a pair of auxiliary functions $\phi,\ \psi : [0,\ \infty) \to [0,\ \infty)$, which satisfies axioms Φ_2 and Ψ_1 such that $\forall\, \mathfrak{s} \in [0,\ \infty)$ and $\flat \in (0,\ \infty)$,

$$\phi(\mathfrak{s}) \leq \phi(\flat) - \psi(\flat),$$

then

$$\mathfrak{s} < \flat.$$

Proposition 4 Let $(\mathfrak{S},\ \wp)$ be a metric space and $\urcorner$ be a self mapping on $\mathfrak{S}$. If $\exists$ an auxiliary functions $\phi,\ \psi : [0,\ \infty) \to [0,\ \infty)$, which satisfies axioms Φ_2 and Ψ_1 respectively, such that $\urcorner$ is a $(\phi,\ \psi)$-contraction of Alam et al. [13], then $\urcorner$ is contractive and hence is continuous.

Given a relation $\mathscr{Q}$ and a self-mapping $\urcorner$ on a non-empty set $\mathfrak{S}$, we adopt the following notations:

(i) $F(\urcorner) :=$ the set of all fixed points of $\urcorner$.
(ii)

$$\mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) = \max\left\{\frac{\wp(\tilde{\mathfrak{A}},\ \urcorner\tilde{\mathfrak{A}})[1+\wp(\tilde{\mathfrak{E}},\ \urcorner\tilde{\mathfrak{E}})]}{1+\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})},\ \frac{\wp(\tilde{\mathfrak{E}},\ \urcorner\tilde{\mathfrak{E}})\wp(\tilde{\mathfrak{A}},\ \urcorner\tilde{\mathfrak{A}})}{1+\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})},\ \frac{\wp(\tilde{\mathfrak{A}},\ \urcorner\tilde{\mathfrak{E}})+\wp(\tilde{\mathfrak{E}},\ \urcorner\tilde{\mathfrak{A}})}{2},\ \wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})\right\}.$$

(iii)

$$\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) = \max\left\{\frac{\wp(\tilde{\mathfrak{A}},\ \urcorner\tilde{\mathfrak{A}})[1+\wp(\tilde{\mathfrak{E}},\ \urcorner\tilde{\mathfrak{E}})]}{1+\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})},\ \wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})\right\}.$$

Remark 2 Observe that $\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) \leq \mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})$ $(\forall\ \tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}} \in \mathfrak{S})$.

Now, we define the new generalized $(\phi,\ \psi)$-rational contraction as follows:

Definition 12 Let $(\mathfrak{S},\ \wp)$ be complete metric space and $\urcorner : \mathfrak{S} \to \mathfrak{S}$ be a self-mapping, then $\urcorner$ is said to satisfy new generalized $(\phi,\ \psi)$-rational contraction if for $\phi \in \Phi$ & $\psi \in \Psi$ such that

$$\phi(d(\urcorner\tilde{\mathfrak{A}},\ \urcorner\tilde{\mathfrak{E}})) \leq \phi(\mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})) - \psi(\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}))\ \ \forall\ \tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}} \in \mathfrak{S}\ \text{ with }\ (\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) \in \mathbb{R}.$$

Due to the symmetry of $\wp$, the following conclusion is immediate.

Proposition 5 Let $(\mathfrak{S},\ \wp)$ be a metric space equipped with a relation $\mathscr{Q}$, a mapping $\urcorner : \mathfrak{S} \to \mathfrak{S}$ and an auxiliary function $\phi \in \Phi$ & $\psi \in \Psi$. Then the following contractive assumptions are equivalent:

(a) $\phi(\wp(\urcorner\tilde{\mathfrak{A}},\ \urcorner\tilde{\mathfrak{E}})) \leq \phi(\mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})) - \psi(\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}))\ \ \forall\ \tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}} \in \mathfrak{S}\ \text{ with }\ (\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) \in \mathscr{Q}$.

(b) $\phi(\wp(\urcorner\tilde{\mathfrak{A}},\ \urcorner\tilde{\mathfrak{E}})) \leq \phi(\mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})) - \psi(\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}))\ \ \forall\ \tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}} \in \mathfrak{S}\ \text{ with }\ [\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}] \in \mathscr{Q}$.

3. Main result

Theorem 1 Let $(\mathfrak{S},\ \wp)$ be a metric space endowed with a binary relation $\mathscr{Q}$ on $\mathfrak{S}$ and let $\urcorner$: $\mathfrak{S} \to \mathfrak{S}$ be a mapping. Assume that the following conditions hold:

(C_i) $(\mathfrak{S},\ \wp)$ is $\mathscr{Q}$-complete,

(C_{ii}) $\exists\ \tilde{\mathfrak{A}}_0$ such that $(\tilde{\mathfrak{A}}_0,\ \urcorner\tilde{\mathfrak{A}}_0) \in \mathscr{Q}$,

(C_{iii}) R is $\urcorner$-closed and locally finitely $\urcorner$-transitive,

(C_{iv}) either $\urcorner$ is $\mathscr{Q}$-continuous or $\mathscr{Q}$ is $\wp$-self-closed,

(C_v) $\urcorner$ satisfy the new generalized $(\phi,\ \psi)$-rational contraction.

Then $\urcorner$ possesses a fixed point.

Proof. We shall from the results into four parts.

Step I: By condition (C_{ii}), $\exists\ \tilde{\mathfrak{A}}_0 \in\mathfrak{S}$ such that $(\tilde{\mathfrak{A}}_0,\ \urcorner\tilde{\mathfrak{A}}_0) \in \mathscr{Q}$. Now we define the sequence of Picard iterates

$$\tilde{\mathfrak{A}}_{n+1} = \urcorner^n\tilde{\mathfrak{A}}_0 = \urcorner\tilde{\mathfrak{A}}_n. \tag{1}$$

If $\urcorner\tilde{\mathfrak{A}}_0 = \tilde{\mathfrak{A}}_0$ then nothing to prove.

If $\urcorner\tilde{\mathfrak{A}}_0 \neq \tilde{\mathfrak{A}}_0$ then by the condition (C_{iii}), we get

$$(\daleth\tilde{\mathfrak{A}}_0,\ \daleth^2\tilde{\mathfrak{A}}_0),\ (\daleth^2\tilde{\mathfrak{A}}_0,\ \daleth^3\tilde{\mathfrak{A}}_0),\ (\daleth^3\tilde{\mathfrak{A}}_0,\ \daleth^2\tilde{\mathfrak{A}}_0),\ \ldots,\ (\daleth^n\tilde{\mathfrak{A}}_0,\ \daleth^{n+1}\tilde{\mathfrak{A}}_0),\ \ldots,\ \in \mathscr{Q}.$$

As $(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{A}}_{n+1}) \in \mathscr{Q}\ \forall\ n \in \mathbb{N}_0$, i.e., $\{\tilde{\mathfrak{A}}_n\}$ is $\mathscr{Q}$-preserving sequence.

Denote $\wp_n := \wp(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{A}}_{n+1})$. If $\exists\ n_0 \in \mathbb{N}_0$ so that $\wp_{n_0} = 0$, then by (1), we conclude that $\tilde{\mathfrak{A}}_{n_0} = \tilde{\mathfrak{A}}_{n_0+1} = \daleth(\tilde{\mathfrak{A}}_{n_0})$ such that $\tilde{\mathfrak{A}}_{n_0}$ is a fixed point of $\daleth$.

Step II: Let $\wp_n > 0\ \ \forall\ \ n \in \mathbb{N}_0$. Applying condition (C_v), we get

$$\phi(\wp(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{A}}_{n+1})) = \phi(\wp(\daleth\tilde{\mathfrak{A}}_{n-1},\ \daleth\tilde{\mathfrak{A}}_n)) \leq \phi(\mathscr{V}(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)) - \psi(\mathscr{W}(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)) \tag{2}$$

where

$$\mathscr{V}(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n) = \max\left\{\frac{\wp(\tilde{\mathfrak{A}}_n,\ \daleth\tilde{\mathfrak{A}}_n)[1+\wp(\tilde{\mathfrak{A}}_{n-1},\ \daleth\tilde{\mathfrak{A}}_{n-1})]}{1+\wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)},\ \frac{\wp(\tilde{\mathfrak{A}}_{n-1},\ \daleth\tilde{\mathfrak{A}}_{n-1})\wp(\tilde{\mathfrak{A}}_n,\ \daleth\tilde{\mathfrak{A}}_n)}{1+\wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)},\right.$$

$$\left.\frac{\wp(\tilde{\mathfrak{A}}_{n-1},\ \daleth\tilde{\mathfrak{A}}_n)+\wp(\tilde{\mathfrak{A}}_n,\ \daleth\tilde{\mathfrak{A}}_{n-1})}{2},\ \wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)\right\}.$$

$$= \max\left\{\wp(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{A}}_{n+1}),\ \frac{\wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)\wp(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{A}}_{n+1})}{1+\wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)},\ \frac{\wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_{n+1})+\wp(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{A}}_n)}{2},\ \wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)\right\}$$

$$= \max\left\{\wp(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{A}}_{n+1}),\ \frac{\wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)+\wp(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{A}}_{n+1})}{2},\ \wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)\right\}$$

$$= \max\left\{\wp(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{A}}_{n+1}),\ \wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)\right\}.$$

Similarly,

$$\mathscr{W}(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n) = \max\left\{\frac{\wp(\tilde{\mathfrak{A}}_n,\ \daleth\tilde{\mathfrak{A}}_n)[1+\wp(\tilde{\mathfrak{A}}_{n-1}),\ \daleth\tilde{\mathfrak{A}}_{n-1})]}{1+\wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)},\ \wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)\right\}$$

$$= \max\left\{\wp(\tilde{\mathfrak{A}}_{n+1},\ \tilde{\mathfrak{A}}_n),\ \wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)\right\}.$$

Therefore equation (2) can be rewritten as

$$
\begin{aligned}
\phi(\wp_n) &= \phi(\wp(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{A}}_{n+1})) \\
&\leq \phi(\max(\wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n),\ \wp(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{A}}_{n+1}))) - \psi(\max(\wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n),\ \wp(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{A}}_{n+1}))) \qquad (3) \\
&\leq \phi(\max(\wp_{n-1},\ \wp_n)) - \psi(\max(\wp_{n-1},\ \wp_n)).
\end{aligned}
$$

Now if $\wp_n > \wp_{n-1}$ then by (3), we get

$$
\phi(\wp_n) \leq \phi(\wp_n) - \psi(\wp_n).
$$

By properties of Φ, we get $\psi(\wp_n) \leq 0$, which is a contradiction to condition Ψ_2.
Hence we have $\wp_n < \wp_{n-1}$ which amount to say that $\wp_n$ is decreasing sequence. Then equation (3) yields that

$$
\phi(\wp_n) \leq \phi(\wp_{n-1}) - \psi(\wp_{n-1}). \qquad (4)
$$

In light of Proposition 4 and (4) gives rise

$$
\wp_n < \wp_{n-1} \ \ \forall n \in \mathbb{N}_0,
$$

This implies that $\{\wp_n\}$ is a decreasing sequence of positive real numbers. Since it is bounded below by 0, there is an element $\kappa \geq 0$ such that

$$
\lim_{n\to\infty} \wp_n = \kappa. \qquad (5)
$$

We now claim that $\kappa = 0$. Suppose, to the contrary that $\kappa > 0$. Taking upper limit in equation (4), we obtain

$$
\begin{aligned}
\lim_{n\to\infty}\sup\ \phi(\wp_n) &\leq \lim_{n\to\infty}\sup\ \phi(\wp_{n-1}) + \lim_{n\to\infty}\sup\ [-\psi(\wp_{n-1})] \\
&\leq \lim_{n\to\infty}\sup\ \phi(\wp_{n-1}) - \lim_{n\to\infty}\inf\ \psi(\wp_{n-1}).
\end{aligned} \qquad (6)
$$

Using (5) and (6) reduces to

$$
\phi(\kappa) \leq \phi(\kappa) - \lim_{n\to\infty}\inf \psi(\wp_{n-1}),
$$

implying thereby

$$\lim_{\wp_n \to \kappa} \inf \psi(\wp_{n-1}) = \lim_{n\to\infty} \inf \psi(\wp_{n-1}) \le 0,$$

which contradicts the property of Ψ_2. Therefore, one has

$$\lim_{n\to\infty} \wp_n = \lim_{n\to\infty} \wp(\tilde{\mathfrak{A}}_n,\, \tilde{\mathfrak{A}}_{n+1}) = 0. \tag{7}$$

Step III: We now aim to show that $\{\tilde{\mathfrak{A}}_n\}$ to be Cauchy sequence. Assume, for contradiction, that $\{\tilde{\mathfrak{A}}_n\}$ is not a Cauchy sequence. Therefore, by Lemma 1, $\exists\ \varepsilon > 0$ & subsequences $\{\tilde{\mathfrak{A}}_{n_\varsigma}\}$ & $\{\tilde{\mathfrak{A}}_{m_\varsigma}\}$ of $\{\tilde{\mathfrak{A}}_n\}$ such that $\varsigma \le m_\varsigma < n_\varsigma$, $d(\tilde{\mathfrak{A}}_{m_\varsigma},\, \tilde{\mathfrak{A}}_{n_\varsigma}) \ge \varepsilon$ and $d(\tilde{\mathfrak{A}}_{m_\varsigma},\, \tilde{\mathfrak{A}}_{p_\varsigma}) < \varepsilon$, where $p_\varsigma \in \{m_\varsigma + 1,\, m_\varsigma + 2,\, \ldots,\, n_\varsigma - 2,\, n_\varsigma - 1\}$. Further, using (6) and Lemma 1, we have

$$\lim_{\varsigma\to\infty} d(\tilde{\mathfrak{A}}_{m_\varsigma},\, \tilde{\mathfrak{A}}_{n_\varsigma + p}) = \varepsilon \ \ \forall p \in \mathbb{N}_0. \tag{8}$$

As $\{x_n\}$ is $\mathscr{Q}$-preserving and $\{\tilde{\mathfrak{A}}_n\} \subset \urcorner(\mathfrak{S})$ (by assumption (1)) and hence the range $E := \{\tilde{\mathfrak{A}}_n : n \in \mathbb{N}_0\}$ (of the sequence $\{\tilde{\mathfrak{A}}_n\}$), is a denumerable subset of $\urcorner(\mathfrak{S})$. By locally finitely $\urcorner$-transitivity of $\mathscr{Q}$, $\exists$ a natural number $\mathscr{G} = \mathscr{G}(E) \ge 2$, such that the restriction $\mathscr{Q}|_E$ is $\mathscr{G}$-transitive.

As $m_\varsigma < n_\varsigma$ & $\mathscr{G} - 1 > 0$, using Division Algorithm one has

$$n_\varsigma - m_\varsigma = (\mathscr{G} - 1)(\alpha_\varsigma - 1) + (\mathscr{G} - \beta_\varsigma)$$

$$\alpha_\varsigma - 1 \ge 0,\ 0 \le \mathscr{G} - \beta_\varsigma < \mathscr{G} - 1$$

$$\iff \begin{cases} n_\varsigma + \beta_\varsigma = m_\varsigma + 1 + (\mathscr{G} - 1)\alpha_\varsigma \\ \alpha_\varsigma \ge 1,\ 1 < \beta_\varsigma \le \mathscr{G}. \end{cases}$$

Here above α_ς & β_ς are natural numbers, where β_ς take + ve integral value in interval $(1,\, \mathscr{G}]$. Hence, we can choose subsequences $\{\tilde{\mathfrak{A}}_{n_\varsigma}\}$ & $\{\tilde{\mathfrak{A}}_{m_\varsigma}\}$ of $\{\tilde{\mathfrak{A}}_n\}$ satisfying (8) such that β_ς remains constant say β, which is independent of k. Write

$$m'_\varsigma = n_\varsigma + \beta = m_\varsigma + 1 + (\mathscr{G} - 1)\alpha_\varsigma, \tag{9}$$

where $\beta (1 < \beta \le \mathscr{G})$ is constant. Owing to (8) & (9), we get

$$\lim_{n\to\infty} \wp(\tilde{\mathfrak{A}}_{m_\varsigma},\, \tilde{\mathfrak{A}}_{m'_\varsigma}) = \lim_{n\to\infty} \wp(\tilde{\mathfrak{A}}_{m_\varsigma},\, \tilde{\mathfrak{A}}_{n_\varsigma} + \beta) = \varepsilon. \tag{10}$$

By the triangular inequality, one obtains

$$\wp(\tilde{\mathfrak{A}}_{m_\varsigma+1}, \tilde{\mathfrak{A}}_{m'_\varsigma+1}) \leq \wp(\tilde{\mathfrak{A}}_{m_\varsigma+1}, \tilde{\mathfrak{A}}_{m_\varsigma}) + \wp(\tilde{\mathfrak{A}}_{m_\varsigma}, \tilde{\mathfrak{A}}_{m'_\varsigma}) + \wp(\tilde{\mathfrak{A}}_{m'_\varsigma}, \tilde{\mathfrak{A}}_{m'_\varsigma+1}),$$

and

$$\wp(\tilde{\mathfrak{A}}_{m_\varsigma}, \tilde{\mathfrak{A}}_{m'_\varsigma}) \leq \wp(\tilde{\mathfrak{A}}_{m_\varsigma}, \tilde{\mathfrak{A}}_{m_\varsigma+1}) + \wp(\tilde{\mathfrak{A}}_{m_\varsigma+1}, \tilde{\mathfrak{A}}_{m'_\varsigma+1}) + \wp(\tilde{\mathfrak{A}}_{m'_\varsigma+1}, \tilde{\mathfrak{A}}_{m'_\varsigma}),$$

therefore, we have

$$\begin{aligned} \wp(\tilde{\mathfrak{A}}_{m_\varsigma}, \tilde{\mathfrak{A}}_{m'_\varsigma}) - \wp(\tilde{\mathfrak{A}}_{m_\varsigma}, \tilde{\mathfrak{A}}_{m_\varsigma+1}) - \wp(\tilde{\mathfrak{A}}_{m'_\varsigma+1}, \tilde{\mathfrak{A}}_{m'_\varsigma}) &\leq \wp(\tilde{\mathfrak{A}}_{m_\varsigma+1}, \tilde{\mathfrak{A}}_{m'_\varsigma+1}) \\ &\leq \wp(\tilde{\mathfrak{A}}_{m_\varsigma+1}, \tilde{\mathfrak{A}}_{m_\varsigma}) + \wp(\tilde{\mathfrak{A}}_{m_\varsigma}, \tilde{\mathfrak{A}}_{m'_\varsigma}) + \wp(\tilde{\mathfrak{A}}_{m'_\varsigma}, \tilde{\mathfrak{A}}_{m'_\varsigma+1}) \end{aligned}$$

which on letting $\varsigma \to \infty$ and using (7) and (10), gives rise

$$\lim_{n\to\infty} \wp(\tilde{\mathfrak{A}}_{m_\varsigma+1}, \tilde{\mathfrak{A}}_{m'_\varsigma+1}) = \varepsilon. \tag{11}$$

In light of (9) & Lemma 2, we have $\wp(\tilde{\mathfrak{A}}_{m_\varsigma}, \tilde{\mathfrak{A}}_{m'_\varsigma}) \in \mathscr{Q}$. By (1) and condition (C_v), we have

$$\begin{aligned} \phi(\wp(\tilde{\mathfrak{A}}_{m_\varsigma+1}, \tilde{\mathfrak{A}}_{m'_\varsigma+1})) &= \phi(\wp(\urcorner\tilde{\mathfrak{A}}_{m_\varsigma}, \urcorner\tilde{\mathfrak{A}}_{m'_\varsigma})) \\ &\leq \phi(\mathscr{V}(\tilde{\mathfrak{A}}_{m_\varsigma}, \tilde{\mathfrak{A}}_{m'_\varsigma})) - \psi(\mathscr{W}(\tilde{\mathfrak{A}}_{m_\varsigma}, \tilde{\mathfrak{A}}_{m'_\varsigma})). \end{aligned}$$

Let $\mathscr{V}_\varsigma = \mathscr{V}(\tilde{\mathfrak{A}}_{m_\varsigma}, \tilde{\mathfrak{A}}_{m'_\varsigma})$ and $\mathscr{W}_\varsigma = \mathscr{W}(\tilde{\mathfrak{A}}_{m_\varsigma}, \tilde{\mathfrak{A}}_{m'_\varsigma})$ then

$$\phi(\wp(\tilde{\mathfrak{A}}_{m_\varsigma+1}, \tilde{\mathfrak{A}}_{m'_\varsigma+1})) \leq \phi(\mathscr{V}_\varsigma) - \psi(\mathscr{W}_\varsigma). \tag{12}$$

So,

$$\begin{aligned} \mathscr{V}_\varsigma = \mathscr{V}(\tilde{\mathfrak{A}}_{m_\varsigma}, \tilde{\mathfrak{A}}_{m'_\varsigma}) = \max\Bigg\{ &\frac{\wp(\tilde{\mathfrak{A}}_{m'_\varsigma}, \urcorner\tilde{\mathfrak{A}}_{m'_\varsigma})[1+\wp(\tilde{\mathfrak{A}}_{m_\varsigma}, \urcorner\tilde{\mathfrak{A}}_{m_\varsigma})]}{1+\wp(\tilde{\mathfrak{A}}_{m_\varsigma}, \tilde{\mathfrak{A}}_{m'_\varsigma})}, \frac{d(\tilde{\mathfrak{A}}_{m_\varsigma}, \urcorner\tilde{\mathfrak{A}}_{m_\varsigma})\wp(\tilde{\mathfrak{A}}_{m'_\varsigma}, \urcorner\tilde{\mathfrak{A}}_{m'_\varsigma})}{[1+\wp(\tilde{\mathfrak{A}}_{m_\varsigma}, \tilde{\mathfrak{A}}_{m'_\varsigma})]}, \\ &\frac{\wp(\tilde{\mathfrak{A}}_{m_\varsigma}, \urcorner\tilde{\mathfrak{A}}_{m'_\varsigma}) + \wp(\tilde{\mathfrak{A}}_{m'_\varsigma}, \urcorner\tilde{\mathfrak{A}}_{m_\varsigma})}{2}, \wp(\tilde{\mathfrak{A}}_{m_\varsigma}, \tilde{\mathfrak{A}}_{m'_\varsigma}) \Bigg\} \end{aligned}$$

$$= \max\left\{\frac{\wp(\tilde{\mathfrak{A}}_{m'_\varsigma},\ \tilde{\mathfrak{A}}_{m'_\varsigma+1})[1+\wp(\tilde{\mathfrak{A}}_{m_\varsigma},\ \tilde{\mathfrak{A}}_{m_\varsigma+1})]}{1+\wp(\tilde{\mathfrak{A}}_{m_\varsigma},\ \tilde{\mathfrak{A}}_{m'_\varsigma})},\ \frac{\wp(\tilde{\mathfrak{A}}_{m_\varsigma},\ \tilde{\mathfrak{A}}_{m_\varsigma+1})\wp(\tilde{\mathfrak{A}}_{m'_\varsigma},\ \tilde{\mathfrak{A}}_{m'_\varsigma+1})}{[1+\wp(\tilde{\mathfrak{A}}_{m_\varsigma},\ \tilde{\mathfrak{A}}_{m'_\varsigma})]},\right.$$

$$\left.\frac{d(\tilde{\mathfrak{A}}_{m_\varsigma},\ \tilde{\mathfrak{A}}_{m'_\varsigma+1})+\wp(\tilde{\mathfrak{A}}_{m'_\varsigma},\ \tilde{\mathfrak{A}}_{m_\varsigma+1})}{2},\ \wp(\tilde{\mathfrak{A}}_{m_\varsigma},\ \tilde{\mathfrak{A}}_{m'_\varsigma})\right\}.$$

Then taking limit $\varsigma \to \infty$ and using (7), (10), (11) we get

$$\mathscr{V}_\varsigma = \max\ \{0,\ 0,\ \varepsilon,\ \varepsilon\} = \varepsilon. \tag{13}$$

Analogously,

$$\mathscr{W}_\varsigma = \mathscr{W}(\tilde{\mathfrak{A}}_{m_\varsigma},\ \tilde{\mathfrak{A}}_{m'_\varsigma}) = \max\left\{\frac{\wp(\tilde{\mathfrak{A}}_{m'_\varsigma},\ \daleth\tilde{\mathfrak{A}}_{m'_\varsigma})[1+\wp(\tilde{\mathfrak{A}}_{m_\varsigma},\ \daleth\tilde{\mathfrak{A}}_{m_\varsigma})]}{1+\wp(\tilde{\mathfrak{A}}_{m_\varsigma},\ \tilde{\mathfrak{A}}_{m'_\varsigma})},\ \wp(\tilde{\mathfrak{A}}_{m_\varsigma},\ \tilde{\mathfrak{A}}_{m'_\varsigma})\right\}$$

$$= \max\left\{\frac{\wp(\tilde{\mathfrak{A}}_{m'_\varsigma},\ \tilde{\mathfrak{A}}_{m'_\varsigma+1})[1+\wp(\tilde{\mathfrak{A}}_{m_\varsigma},\ \tilde{\mathfrak{A}}_{m_\varsigma+1})]}{1+\wp(\tilde{\mathfrak{A}}_{m_\varsigma},\ \tilde{\mathfrak{A}}_{m'_\varsigma})},\ \wp(\tilde{\mathfrak{A}}_{m_\varsigma},\ \tilde{\mathfrak{A}}_{m'_\varsigma})\right\}.$$

Then taking limit $\varsigma \to \infty$ and using (7), (10), (11) we get

$$\mathscr{W}_\varsigma = \max\ \{0,\ \varepsilon\} = \varepsilon. \tag{14}$$

Taking upper limit in (11), one gets

$$\lim_{\varsigma\to\infty}\sup\phi(\wp(\tilde{\mathfrak{A}}_{m_\varsigma+1},\ \tilde{\mathfrak{A}}_{m'_\varsigma+1})) \le \lim_{\varsigma\to\infty}\sup\phi(\mathscr{V}_\varsigma) + \lim_{\varsigma\to\infty}\sup[-\psi(\mathscr{W}_\varsigma)],$$

which on using (10), (11), (13) & (14) becomes

$$\phi(\varepsilon) \le \phi(\varepsilon) - \lim_{\varsigma\to\infty}\inf\psi(\varepsilon),$$

thereby yielding

$$\lim_{\varsigma\to\infty}\inf\psi(\varepsilon) = \lim_{\varsigma\to\infty}\inf\psi(\varepsilon) \le 0,$$

which contradicts to the property Ψ_2. It follows that $\{\tilde{\mathfrak{A}}_n\}$ is an $\mathscr{Q}$-preserving Cauchy in $\mathfrak{S}$. As $(\daleth,\ \tilde{\mathfrak{A}})$ is complete metric space then there exists $\tilde{\mathfrak{A}} \in \daleth$ such that the sequence $\{\tilde{\mathfrak{A}}_n\}$ converges to $\tilde{\mathfrak{A}}$.

Step IV: Now, we shall use condition $C_{iv})$. Firstly suppose that $\daleth$ is $\mathscr{Q}$-continuous. Then $\tilde{\mathfrak{A}}_{n+1} = \daleth\tilde{\mathfrak{A}}_n \xrightarrow{\wp} \daleth\tilde{\mathfrak{A}}$. But by uniqueness of limits, we have $\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}}) = 0$. Hence, $\tilde{\mathfrak{A}}$ is a fixed point of $\daleth$.

Alternately, if $\mathscr{Q}$ is $\wp$-self closed. As $\{\tilde{\mathfrak{A}}_n\}$ is an $\mathscr{Q}$-preserving sequence in $\mathfrak{S}$ and $\tilde{\mathfrak{A}}_n \xrightarrow{\wp} \tilde{\mathfrak{A}}$, there is a subsequence $\{\tilde{\mathfrak{A}}_{n_\varsigma}\}$ of $\{\tilde{\mathfrak{A}}_n\}$ with $[\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{A}}] \in \mathscr{Q}\ \forall\ \varsigma \in \mathbb{N}_0$. In view of condition (v) and Proposition 5, we have

$$
\begin{aligned}
\phi(\wp(\tilde{\mathfrak{A}}_{n_\varsigma+1},\ \daleth\tilde{\mathfrak{A}})) &= \phi(\wp(\daleth\tilde{\mathfrak{A}}_{n_\varsigma},\ \daleth\tilde{\mathfrak{A}})) \\
&\leq \phi(\mathscr{V}(\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{E}})) - \psi(\mathscr{W}(\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{A}})).
\end{aligned} \tag{15}
$$

Let $\mathscr{V}_{n_\varsigma} = \mathscr{V}(\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{A}})$ and $\mathscr{W}_{n_\varsigma} = \mathscr{W}(\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{A}})$ then

$$
\phi(\wp(\tilde{\mathfrak{A}}_{n_\varsigma+1},\ \daleth\tilde{\mathfrak{A}})) = \phi(\mathscr{V}_{n_\varsigma}) - \psi(\mathscr{W}_{n_\varsigma}). \tag{16}
$$

So,

$$
\begin{aligned}
\mathscr{V}_{n_\varsigma} = \mathscr{V}(\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{A}}) = \max\Bigg\{ & \frac{\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}})[1+\wp(\tilde{\mathfrak{A}}_{n_\varsigma},\ \daleth\tilde{\mathfrak{A}}_{n_\varsigma})]}{1+\wp(\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{A}})},\ \frac{\wp(\tilde{\mathfrak{A}}_{n_\varsigma},\ \daleth\tilde{\mathfrak{A}}_{n_\varsigma})\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}})}{[1+\wp(\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{A}})]}, \\
& \frac{\wp(\tilde{\mathfrak{A}}_{n_\varsigma},\ \daleth\tilde{\mathfrak{A}})+d(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}}_{n_\varsigma})}{2},\ \wp(\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{A}})\Bigg\} \\
= \max\Bigg\{ & \frac{\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}})[1+\wp(\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{A}}_{n_\varsigma}+1)]}{1+\wp(\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{A}})},\ \frac{\wp(\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{A}}_{n_\varsigma}+1)\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}})}{[1+\wp(\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{A}})]}, \\
& \frac{\wp(\tilde{\mathfrak{A}}_{n_\varsigma},\ \daleth\tilde{\mathfrak{A}})+d(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{A}}_{n_\varsigma}+1)}{2},\ \wp(\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{A}})\Bigg\}.
\end{aligned}
$$

Taking limit $\varsigma \to \infty$ and using $\tilde{\mathfrak{A}}_{n_\varsigma} \xrightarrow{\wp} \tilde{\mathfrak{A}}$, we obtain

$$
\lim_{\varsigma\to\infty} \mathscr{V}_{n_\varsigma} = \max\left\{\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}}),\ \wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}}),\ \frac{\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}})}{2},\ 0\right\} = \wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}}). \tag{17}
$$

Similarly,

$$
\mathscr{W}_{n_\varsigma} = \mathscr{W}(\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{A}}) = \max\left\{\frac{\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}})[1+\wp(\tilde{\mathfrak{A}}_{n_\varsigma},\ \daleth\tilde{\mathfrak{A}}_{n_\varsigma})]}{1+\wp(\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{A}})},\ \wp(\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{A}})\right\}
$$

$$= \max\left\{\frac{\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}})[1+\wp(\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{A}}_{n_\varsigma}+1)]}{1+\wp(\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{A}})},\ \wp(\tilde{\mathfrak{A}}_{n_\varsigma},\ \tilde{\mathfrak{A}})\right\}.$$

Taking limit $\varsigma \to \infty$ and using $\tilde{\mathfrak{A}}_{n_\varsigma} \xrightarrow{\wp} \tilde{\mathfrak{A}}$, we obtain

$$\lim_{\varsigma\to\infty} \mathscr{W}_{n_\varsigma} = \max\left\{\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}}),\ 0\right\} = \wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}}). \tag{18}$$

Taking upper limit in equation (16), we get

$$\lim_{\varsigma\to\infty} \sup \phi d(\tilde{\mathfrak{A}}_{n_\varsigma},\ \daleth\tilde{\mathfrak{A}}) \le \lim_{\varsigma\to\infty} \sup \phi(\mathscr{V}_{n_\varsigma}) + \lim_{\varsigma\to\infty} \sup[-\psi(\mathscr{W}_{n_\varsigma})],$$

which on using (17), (18) becomes

$$\phi(\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}})) \le \phi(\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}}) - \lim_{\varsigma\to\infty} \inf \psi(\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}})),$$

yielding thereby

$$\lim_{\varsigma\to\infty} \inf \psi(\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}})) = \lim_{\varsigma\to\infty} \inf \psi(\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}})) \le 0,$$

which contradicts to the property Ψ_2. Hence, $\tilde{\mathfrak{A}} = \daleth\tilde{\mathfrak{A}}$, that is $\tilde{\mathfrak{A}}$ is a fixed point of $\daleth$. □

Theorem 2 In addition to Theorem 1, If $\daleth(\mathfrak{S})$ is $\mathscr{Q}^s$-directed, then $\daleth$ admits unique fixed point.

Proof. Let $\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}$ be two fixed points, i.e., $\daleth\tilde{\mathfrak{A}} = \tilde{\mathfrak{A}}\ \ \&\ \ \daleth\tilde{\mathfrak{E}} = \tilde{\mathfrak{E}}$ then two cases arise.

Case I: If $(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) \in \mathscr{Q}$ then

$$\phi(\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})) = \phi(\wp(\daleth\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{E}})) \le \phi(\mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})) - \psi(\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})). \tag{19}$$

So, then

$$\mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) = \max\left\{\frac{\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}})[1+d(\tilde{\mathfrak{E}},\ \daleth\tilde{\mathfrak{E}})]}{1+\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})},\ \frac{d(\tilde{\mathfrak{E}},\ \daleth\tilde{\mathfrak{E}})\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}})}{1+\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})},\ \frac{\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{E}})+\wp(\tilde{\mathfrak{E}},\ \daleth\tilde{\mathfrak{A}})}{2},\ \wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})\right\}$$

$$= \max\left\{\frac{\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{A}})[1+\wp(\tilde{\mathfrak{E}},\ \tilde{\mathfrak{E}})]}{1+\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})},\ \frac{d(\tilde{\mathfrak{E}},\ \tilde{\mathfrak{E}})\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{A}})}{1+\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})},\ \frac{d(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})+\wp(\tilde{\mathfrak{E}},\ \tilde{\mathfrak{A}})}{2},\ \wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})\right\}$$

$$= \max\left\{0,\ 0,\ \wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}),\ \wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})\right\} = \wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}).$$

Similarly, we obtain $\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) = \wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})$. Then from (19) can be written as

$$\phi(\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})) = \phi(\wp(\daleth\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{E}})) \leq \phi(\mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})) - \psi(\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}))$$

$$\leq \phi(\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})) - \psi(\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})),$$

yielding there by $\psi(\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})) < 0$, which is a contradiction to the condition Ψ_2. Hence $\tilde{\mathfrak{A}} = \tilde{\mathfrak{E}}$.

Case II: If $(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) \notin \mathscr{Q}$ then by $\daleth(\mathfrak{S})$ is $\mathscr{Q}^s$-directed then $\exists\, z \in \mathfrak{S}$ such that $(\tilde{\mathfrak{A}},\ z) \in \mathscr{Q}$ and $(z,\ \tilde{\mathfrak{E}}) \in \mathscr{Q}$. Since $\mathscr{Q}$ is $\daleth$-closed $\daleth^n z$ will be related to $\daleth^n\tilde{\mathfrak{A}}$ i.e., $(\daleth^n z,\ \daleth^n\tilde{\mathfrak{A}} = \tilde{\mathfrak{A}}) \in \mathscr{Q}$ for any $n \in \mathbb{N}_0$. Then by condition (C_v) of Theorem 1, for any $n \in \mathbb{N}_0$, we have

$$\begin{aligned} \phi(d(\daleth^n z,\ \tilde{\mathfrak{A}})) &= \phi(d(\daleth^n z,\ \daleth^n\tilde{\mathfrak{A}})) \\ &\leq \phi(\mathscr{V}(\daleth^{n-1}\tilde{\mathfrak{A}},\ \daleth^{n-1}\tilde{\mathfrak{A}})) - \psi(\mathscr{W}(\daleth^{n-1}\tilde{\mathfrak{A}},\ \daleth^{n-1}\tilde{\mathfrak{A}})). \end{aligned} \tag{20}$$

Now,

$$\mathscr{V}(\daleth^{n-1}z,\ \daleth^{n-1}\tilde{\mathfrak{A}}) = \max\left\{ \frac{\wp(\daleth^{n-1}\tilde{\mathfrak{A}},\ \daleth\daleth^{n-1}\tilde{\mathfrak{A}})[1+\wp(\daleth^{n-1}z,\ \daleth\daleth^{n-1}z)]}{1+\wp(\daleth^{n-1}z,\ \daleth^{n-1}\tilde{\mathfrak{A}})},\ \frac{\wp(\daleth^{n-1}z,\ \daleth\daleth^{n-1}z)[\wp(\daleth^{n-1}\tilde{\mathfrak{A}},\ \daleth\daleth^{n-1}\tilde{\mathfrak{A}})]}{1+\wp(\daleth^{n-1}z,\ \daleth^{n-1}\tilde{\mathfrak{A}})} \right.$$

$$\left. \frac{\wp(\daleth^{n-1}z,\ \daleth\daleth^{n-1}\tilde{\mathfrak{A}}) + \wp(\daleth^{n-1}\tilde{\mathfrak{A}},\ \daleth\daleth^{n-1}z)}{2},\ \wp(\daleth^{n-1}z,\ \daleth^{n-1}\tilde{\mathfrak{A}}) \right\}$$

$$= \max\left\{ \frac{\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{A}})[1+\wp(\daleth^{n-1}z,\ \daleth\daleth^{n-1}z)]}{1+\wp(\daleth^{n-1}z,\ \tilde{\mathfrak{A}})},\ \frac{d(\daleth^{n-1}z,\ \daleth^n z)[\wp(\daleth^{n-1}\tilde{\mathfrak{A}},\ \daleth^n\tilde{\mathfrak{A}})]}{1+d(\daleth^{n-1}z,\ \tilde{\mathfrak{A}})}, \right.$$

$$\left. \frac{\wp(\daleth^{n-1}z,\ \tilde{\mathfrak{A}}) + \wp(\tilde{\mathfrak{A}},\ \daleth^n z)}{2},\ \wp(\daleth^{n-1}z,\ \tilde{\mathfrak{A}}), \right\} = \wp(\daleth^{n-1}z,\ \tilde{\mathfrak{A}}).$$

Similarly, for $\mathscr{W}(\daleth^{n-1}z,\ \daleth^{n-1}\tilde{\mathfrak{A}}) = \wp(\daleth^{n-1}z,\ \tilde{\mathfrak{A}})$. Then (20) reduces to

$$\phi(\wp(\daleth^{n-1}z,\ \tilde{\mathfrak{A}})) \leq \phi(\wp(\daleth^{n-1}z,\ \tilde{\mathfrak{A}})) - \psi(\wp(\daleth^{n-1}z,\ \tilde{\mathfrak{A}})). \tag{21}$$

Using Proposition 4 one has

$$\wp(\daleth^n z,\ \tilde{\mathfrak{A}}) < \wp(\daleth^{n-1}z,\ \tilde{\mathfrak{A}}).$$

Which amount to say that $\{\wp(\daleth^n z,\ \tilde{\mathfrak{A}})\}$ is a decreasing of positive real numbers, which is bounded below by 0, $\exists\ \varsigma \geq 0$ such that

$$\wp(\daleth^n z,\ \tilde{\mathfrak{A}}) = \varsigma.$$

Our claim is that $\varsigma = 0$, for that let on contrary that $\varsigma > 0$ then taking upper limit to the equation (21) we get

$$\limsup_{n\to 0} \phi(\wp(\daleth^n z,\ \tilde{\mathfrak{A}})) \le \limsup_{n\to 0} \phi(\wp(\daleth^{n-1} z,\ \tilde{\mathfrak{A}})) + \limsup_{n\to 0}[-\psi(\wp(\daleth^{n-1} z,\ \tilde{\mathfrak{A}}))]$$

$$\le \limsup_{n\to 0} \phi(\wp(\daleth^{n-1} z,\ \tilde{\mathfrak{A}})) - \liminf_{n\to 0} \psi(\wp(\daleth^{n-1} z,\ \tilde{\mathfrak{A}})),$$

by right continuity of ϕ one gets

$$\phi(\varsigma) \le \phi(\varsigma) - \liminf_{n\to\infty} \psi(\varsigma),$$

which amounts to say that $\lim_{n\to\infty} \inf \psi(\varsigma) \le 0$, which is contradiction to the condition Ψ_2. Hence, $\lim_{n\to\infty}\{\wp(\daleth^n z,\ \tilde{\mathfrak{A}})\} = 0$ means $\lim_{n\to\infty} \daleth^n z = \tilde{\mathfrak{A}}$. Analogously, we can proved that $\lim_{n\to\infty} \daleth^n z = \tilde{\mathfrak{E}}$. Then by unicity of limit we have $\tilde{\mathfrak{A}} = \tilde{\mathfrak{E}}$. Hence, $\daleth$ admits unique fixed point. □

Corollary 1 If we replace the contractive condition of Theorem 1 and the hypothesis Theorem 2, are also satisfied. Also assume that:

$$\phi(\wp(\daleth\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{E}})) \le \phi(\mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})) - \psi(\mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})),$$

where

$$\mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) = \max\left\{\frac{\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}})[1+\wp(\tilde{\mathfrak{E}},\ \daleth\tilde{\mathfrak{E}})]}{1+\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})},\ \frac{\wp(\tilde{\mathfrak{E}},\ \daleth\tilde{\mathfrak{E}})\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}})}{1+\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})},\ \frac{\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{E}})+\wp(\tilde{\mathfrak{E}},\ \daleth\tilde{\mathfrak{A}})}{2},\ \wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})\right\},$$

for all $\phi \in \Phi$ & $\psi \in \Psi$. Then $\daleth$ has unique fixed point.

Corollary 2 If we replace the contractive condition of Theorem 1 and the hypothesis Theorem 2, are also satisfied. Also assume that:

$$\phi(\wp(\daleth\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{E}})) \le \phi(\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})) - \psi(\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})),$$

where

$$\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) = \max\left\{\frac{\wp(\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{A}})[1+\wp(\tilde{\mathfrak{E}},\ \daleth\tilde{\mathfrak{E}})]}{1+\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})},\ \wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})\right\},$$

for all $\phi \in \Phi$ & $\psi \in \Psi$. Then $\daleth$ has unique fixed point.

Taking ϕ as identity mapping and $\psi(t) = (1-k)t$ for $k \in (0,\ 1)$. We deduce the several versions of our newly proved results in the context of metric, universal relation and rational expression.

Remark 3 Under the setting of $\wp = p$ (i.e., partial metric), $\mathscr{Q} = \mathfrak{S} \times \mathfrak{S}$ in Theorem 1 then our results deduce to Corollary 2.4 of Kumar et al. [31].

Remark 4 Under the setting of $\wp = p$ (i.e., partial metric), $\mathscr{Q} = \mathfrak{S} \times \mathfrak{S}$ and replacing $\mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})$ by $\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})$ in Theorem 1 then our results deduce to corollary 2.5 of Kumar et al. [31].

Remark 5 Under universal relation $(i.e.,\ \mathscr{Q} = \mathfrak{S}^2)$, Theorems 1 and 2 reduce to the following fixed point theorem of Alam et al. [12].

Remark 6 (Error bounds and convergence rate) Let $(\mathfrak{S},\ \wp)$ be a complete metric space and $\daleth : \mathfrak{S} \to \mathfrak{S}$ satisfy the new generalized $(\phi,\ \psi)$-rational contraction. Assume that $p \in \mathfrak{S}$ is the unique fixed point of $\daleth$ and there exists $c \in (0,\ 1)$ such that

$$\phi(\mathscr{V}(\tilde{\mathfrak{A}},\ p)) \leq \phi(\wp(\tilde{\mathfrak{A}},\ p)), \quad \psi(\mathscr{W}(\tilde{\mathfrak{A}},\ p)) \geq c\ \phi(\wp(\tilde{\mathfrak{A}},\ p)), \qquad \forall\, \tilde{\mathfrak{A}} \in \mathfrak{S}.$$

Then the Picard iterates $\tilde{\mathfrak{A}}_{n+1} = \daleth\tilde{\mathfrak{A}}_n$ satisfy

$$\phi(\wp(\tilde{\mathfrak{A}}_n,\ p)) \leq (1-c)^n \phi(\wp(\tilde{\mathfrak{A}}_0,\ p)), \quad \text{hence} \quad \wp(\tilde{\mathfrak{A}}_n,\ p) \leq \phi^{-1}\big((1-c)^n \phi(\wp(\tilde{\mathfrak{A}}_0,\ p))\big).$$

Moreover,

$$\wp(\tilde{\mathfrak{A}}_n,\ p) \leq \sum_{k=n}^{\infty} \phi^{-1}\big((1-c)^k \phi(\wp(\tilde{\mathfrak{A}}_1,\ \tilde{\mathfrak{A}}_0))\big),$$

which yields explicit a posterior error bounds. In particular:

- If $\phi(t) = t$, then $\wp(\tilde{\mathfrak{A}}_n,\ p) \leq (1-c)^n \wp(\tilde{\mathfrak{A}}_0,\ p)$ and $\wp(\tilde{\mathfrak{A}}_n,\ p) \leq \dfrac{(1-c)^n}{c}\ d(\tilde{\mathfrak{A}}_1,\ \tilde{\mathfrak{A}}_0)$.
- If $\phi(t) = t^\alpha$ $(\alpha > 0)$, then $\wp(\tilde{\mathfrak{A}}_n,\ p) \leq (1-c)^{n/\alpha} \wp(\tilde{\mathfrak{A}}_0,\ p)$ and $\wp(\tilde{\mathfrak{A}}_n,\ p) \leq \dfrac{(1-c)^{n/\alpha}}{1-(1-c)^{1/\alpha}}\ \wp(\tilde{\mathfrak{A}}_1,\ \tilde{\mathfrak{A}}_0)$.

4. Coupled fixed point results with mixed R-monotone property

Definition 13 Let $(\mathfrak{S},\ \wp)$ be complete metric space and $\daleth : \mathfrak{S} \to \mathfrak{S}$ be a self mapping, then $\daleth$ is said to satisfy new generalized $(\phi,\ \psi)$-rational contraction if for $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$\phi(\wp(\daleth(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}),\ \daleth(\upsilon,\ \omega))) \leq \phi(\mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}},\ \upsilon,\ \omega)) - \psi(\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}},\ \upsilon,\ \omega))\ \forall\, \tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}},\ \upsilon,\ \omega \in \mathfrak{S} \text{ with } (\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}},\ \upsilon,\ \omega) \in R.$$

where,

$$\mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}},\ \upsilon,\ \omega) = \max\left\{\frac{\wp(\upsilon,\ \urcorner(\upsilon,\ \omega))[1+\wp(\tilde{\mathfrak{A}},\ \urcorner(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}))]}{1+\wp(\tilde{\mathfrak{A}},\ \upsilon)},\ \frac{\wp(\tilde{\mathfrak{A}},\ \urcorner(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})),\ \wp(\upsilon,\ \urcorner(\upsilon,\ \omega))}{1+d(\upsilon,\ \omega)},\right.$$

$$\left.\frac{\wp(\tilde{\mathfrak{A}},\ \urcorner(\upsilon,\ \omega))+\wp(\upsilon,\ \urcorner(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}))}{2},\ \wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})\right\}$$

$$\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}},\ \upsilon,\ \omega) = \max\left\{\frac{\wp(\upsilon,\ \urcorner(\upsilon,\ \omega))[1+\wp(\tilde{\mathfrak{A}},\ \urcorner(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}))]}{1+\wp(\tilde{\mathfrak{A}},\ \upsilon)},\ \wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})\right\}.$$

Theorem 3 Let $(\mathfrak{S},\ \wp)$ be a complete metric space endowed with binary relation $\mathscr{Q}$ on $\mathfrak{S}$ and let $\urcorner : \mathfrak{S}\times\mathfrak{S}\to\mathfrak{S}$ a mapping. Assume that the following conditions hold:
(C_i) $\exists\ \tilde{\mathfrak{A}}_0,\ \tilde{\mathfrak{E}}_0 \in \mathfrak{S}$ such that $(\tilde{\mathfrak{A}}_0,\ \urcorner(\tilde{\mathfrak{A}}_0,\ \tilde{\mathfrak{E}}_0)) \in \mathscr{Q}$ and $(\urcorner(\tilde{\mathfrak{E}}_0,\ \tilde{\mathfrak{A}}_0),\ \tilde{\mathfrak{E}}_0) \in \mathscr{Q}$,
(C_{ii}) $\urcorner$ satisfies mixed $\mathscr{Q}$-monotone property,
(C_{iii}) $\mathscr{Q}$ is $\urcorner$-closed,
(C_{iv}) either $\urcorner$ is continuous or $\mathscr{Q}$ is $\wp$-self closed,
(C_v) $\urcorner$ satisfy the new generalized $(\phi,\ \psi)$-rational contraction.
Proof. By condition (C_i) ensures the availability of $\tilde{\mathfrak{A}}_0,\ \tilde{\mathfrak{E}}_0 \in \mathfrak{S}$ such that

$$(\tilde{\mathfrak{A}}_0,\ \urcorner(\tilde{\mathfrak{A}}_0,\ \tilde{\mathfrak{E}}_0)) \in \mathscr{Q}\ \ \&\ \ (\urcorner(\tilde{\mathfrak{E}}_0,\ \tilde{\mathfrak{A}}_0),\ \tilde{\mathfrak{E}}_0) \in \mathscr{Q}. \tag{22}$$

Assume $\urcorner(\tilde{\mathfrak{A}}_0,\ \tilde{\mathfrak{E}}_0) = \tilde{\mathfrak{A}}_1$ & $\urcorner(\tilde{\mathfrak{E}}_0,\ \tilde{\mathfrak{A}}_0) = \tilde{\mathfrak{E}}_1$. Then we can have $\tilde{\mathfrak{A}}_2,\ \tilde{\mathfrak{E}}_2 \in \mathfrak{S}$ such that $\urcorner(\tilde{\mathfrak{A}}_1,\ \tilde{\mathfrak{E}}_1) = \tilde{\mathfrak{A}}_2$ and $\urcorner(\tilde{\mathfrak{E}}_1,\ \tilde{\mathfrak{A}}_1) = \tilde{\mathfrak{E}}_2$. We denote

$$\urcorner^2(\tilde{\mathfrak{A}}_0,\ \tilde{\mathfrak{E}}_0) = \urcorner(\urcorner(\tilde{\mathfrak{A}}_0,\ \tilde{\mathfrak{E}}_0),\ \urcorner(\tilde{\mathfrak{E}}_0,\ \tilde{\mathfrak{A}}_0)) = \urcorner(\tilde{\mathfrak{A}}_1,\ \tilde{\mathfrak{E}}_1) = \tilde{\mathfrak{A}}_2$$

$$\urcorner^2(\tilde{\mathfrak{E}}_0,\ \tilde{\mathfrak{A}}_0) = \urcorner(\urcorner(\tilde{\mathfrak{E}}_0,\ \tilde{\mathfrak{A}}_0),\ \urcorner(\tilde{\mathfrak{A}}_0,\ \tilde{\mathfrak{E}}_0)) = \urcorner(\tilde{\mathfrak{E}}_1,\ \tilde{\mathfrak{A}}_1) = \tilde{\mathfrak{E}}_2.$$

Following the same procedure, one can inductively construct sequences $\{\tilde{\mathfrak{A}}_n\}$ and $\{\tilde{\mathfrak{E}}_n\}$ such that

$$\tilde{\mathfrak{A}}_n = \urcorner^n(\tilde{\mathfrak{A}}_0,\ \tilde{\mathfrak{E}}_0) \quad \text{and} \quad \tilde{\mathfrak{E}}_n = \urcorner^n(\tilde{\mathfrak{E}}_0,\ \tilde{\mathfrak{A}}_0). \tag{23}$$

Now, we will show that

$$(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{A}}_{n+1}) \in R \quad \text{and} \quad (\tilde{\mathfrak{E}}_n,\ \tilde{\mathfrak{E}}_{n+1}) \in \mathscr{Q}\ \forall\ n\ \in\ \mathbb{N}_0. \tag{24}$$

Now, mathematical induction is to be used to prove this fact.
From equation (22), $(\tilde{\mathfrak{A}}_0,\ \tilde{\mathfrak{A}}_1) \in \mathscr{Q}$ and $(\tilde{\mathfrak{E}}_0,\ \tilde{\mathfrak{E}}_1) \in \mathscr{Q}$. So, equation (24) is true for $n = 0$. Suppose that for $t > 0$,

$$(\tilde{\mathfrak{A}}_t,\ \tilde{\mathfrak{A}}_{t+1}) \in R \quad \text{and} \quad (\tilde{\mathfrak{E}}_t,\ \tilde{\mathfrak{E}}_{t+1}) \in \mathscr{Q}.$$

We now confirm that (24) also true for $n = t+1$. Since $(\tilde{\mathfrak{A}}_t,\ \tilde{\mathfrak{A}}_{t+1}) \in \mathscr{Q}$, applying assumption (C_{ii}) we obtain,

$$(\daleth(\tilde{\mathfrak{A}}_t,\ \tilde{\mathfrak{E}}_t)),\ \daleth(\tilde{\mathfrak{A}}_{t+1},\ \tilde{\mathfrak{E}}_{t+1}) \in \mathscr{Q}$$

$$\Longrightarrow (\daleth(\daleth^t(\tilde{\mathfrak{A}}_0,\ \tilde{\mathfrak{E}}_0),\ \daleth^t(\tilde{\mathfrak{E}}_0,\ \tilde{\mathfrak{A}}_0)),\ \daleth(\daleth^{t+1}(\tilde{\mathfrak{A}}_0,\ \tilde{\mathfrak{E}}_0),\ \daleth^{t+1}(\tilde{\mathfrak{E}}_0,\ \tilde{\mathfrak{A}}_0))) \in \mathscr{Q}$$

$$\Longrightarrow (\daleth^{t+1}(\tilde{\mathfrak{A}}_0,\ \tilde{\mathfrak{E}}_0),\ \daleth^{t+2}(\tilde{\mathfrak{A}}_0,\ \tilde{\mathfrak{E}}_0)) \in \mathscr{Q}$$

$$\Longrightarrow (\tilde{\mathfrak{A}}_{t+1},\ \tilde{\mathfrak{A}}_{t+2}) \in \mathscr{Q}.$$

Similarly, we can show that $(\tilde{\mathfrak{E}}_{t+2},\ \tilde{\mathfrak{E}}_{t+1}) \in \mathscr{Q}$. Hence (24) holds $\forall\ n \in \mathbb{N}_0$. Now, choosing $\tilde{\mathfrak{A}} = \tilde{\mathfrak{A}}_n$, $\upsilon = \tilde{\mathfrak{A}}_{n-1}$, $\tilde{\mathfrak{E}} = \tilde{\mathfrak{E}}_n$, $\omega = \tilde{\mathfrak{E}}_{n-1}$ in condition (C_v)

$$\begin{aligned} \phi(\wp(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{A}}_{n+1})) &= \phi\left(\wp(\daleth(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{E}}_{n-1}),\ \daleth(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{E}}_n)\right) \\ &\leq \phi(\mathscr{V}(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{E}}_{n-1},\ \tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{E}}_n)) - \psi(\mathscr{W}(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{E}}_{n-1},\ \tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{E}}_n)), \end{aligned} \tag{25}$$

where,

$$\begin{aligned} \mathscr{V}(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{E}}_{n-1},\ \tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{E}}_n) = \max\Bigg\{ & \frac{\wp(\tilde{\mathfrak{A}}_n,\ \daleth(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{E}}_n))[1+\wp(\tilde{\mathfrak{A}}_{n-1},\ \daleth(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{E}}_{n-1}))]}{1+\wp(\tilde{\mathfrak{A}}_{n-1,\ \tilde{\mathfrak{A}}_n})}, \\ & \frac{\wp(\tilde{\mathfrak{A}}_{n-1},\ \daleth(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{E}}_{n-1}))[\wp(\tilde{\mathfrak{E}}_{n-1},\ \daleth(\tilde{\mathfrak{E}}_{n-1},\ \tilde{\mathfrak{A}}_{n-1}))]}{1+\wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)}, \\ & \frac{\wp(\tilde{\mathfrak{A}}_{n-1},\ \daleth(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{E}}_n))+[\wp(\tilde{\mathfrak{A}}_n,\ \daleth(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{E}}_{n-1}))}{2},\ \wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)\Bigg\}. \end{aligned}$$

$$\begin{aligned} = \max\Bigg\{ & \frac{\wp(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{A}}_n))[1+\wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_{n-1})]}{1+\wp(\tilde{\mathfrak{A}}_{n-1,\ \tilde{\mathfrak{A}}_n})},\ \frac{\wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_{n-1})[d(\tilde{\mathfrak{E}}_{n-1},\ \tilde{\mathfrak{E}}_{n-1})]}{1+\wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)}, \\ & \frac{\wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)+\wp(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{A}}_{n-1})}{2},\ \wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)\Bigg\} \end{aligned} \tag{26}$$

$$= \max\left\{0,\ 0,\ \wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n),\ \wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)\right\}$$

$$= \wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n).$$

$$
\begin{aligned}
\mathscr{W}(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{E}}_{n-1},\ \tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{E}}_n) &= \max\left\{\frac{\wp(\tilde{\mathfrak{A}}_n,\ \daleth(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{E}}_n))[1+\wp(\tilde{\mathfrak{A}}_{n-1},\ \daleth(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{E}}_{n-1}))]}{1+\wp(\tilde{\mathfrak{A}}_{n-1,\ \tilde{\mathfrak{A}}_n})},\ \wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)\right\} \\
&= \max\left\{\frac{\wp(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{A}}_n)[1+\wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_{n-1})]}{1+\wp(\tilde{\mathfrak{A}}_{n-1,\ \tilde{\mathfrak{A}}_n})},\ \wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)\right\} \\
&= \max\left\{0,\ \wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)\right\} \\
&= \wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n).
\end{aligned} \tag{27}
$$

Therefore, from (25), (26) and (27)

$$
\begin{aligned}
\phi(\wp(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{A}}_{n+1})) &= \phi\left(\wp(\daleth(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{E}}_{n-1}),\ \daleth(\tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{E}}_n)\right) \\
&\le \phi(\mathscr{V}(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{E}}_{n-1},\ \tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{E}}_n)) - \psi(\mathscr{W}(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{E}}_{n-1},\ \tilde{\mathfrak{A}}_n,\ \tilde{\mathfrak{E}}_n))) \\
&\le \phi(\wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)) - \psi(\wp(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{A}}_n)).
\end{aligned} \tag{28}
$$

By applying the conclusion of the Theorem 1 and we deduce that $\daleth$ admits a fixed point.
Hence, $\exists\ \tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}} \in \mathfrak{S}$ such that $\tilde{\mathfrak{A}}_n \to \tilde{\mathfrak{A}}$ as $n \to \infty$ and $\tilde{\mathfrak{E}}_n \to \tilde{\mathfrak{E}}$ as $n \to \infty$, the continuity of $\daleth$ implies that

$$
\tilde{\mathfrak{A}} = \lim_{n\to\infty} \tilde{\mathfrak{A}}_n = \lim_{n\to\infty} \daleth(\tilde{\mathfrak{A}}_{n-1},\ \tilde{\mathfrak{E}}_{n-1}) = \daleth\big(\lim_{n\to\infty} \tilde{\mathfrak{A}}_{n-1},\ \lim_{n\to\infty} \tilde{\mathfrak{E}}_{n-1}\big) = \daleth(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}).
$$

$$
\tilde{\mathfrak{E}} = \lim_{n\to\infty} \tilde{\mathfrak{E}}_n = \lim_{n\to\infty} \daleth(\tilde{\mathfrak{E}}_{n-1},\ \tilde{\mathfrak{A}}_{n-1}) = \daleth\big(\lim_{n\to\infty} \tilde{\mathfrak{E}}_{n-1},\ \lim_{n\to\infty} \tilde{\mathfrak{A}}_{n-1}\big) = \daleth(\tilde{\mathfrak{E}},\ \tilde{\mathfrak{A}}).
$$

Thus $\tilde{\mathfrak{A}} = \daleth(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})$ and $\tilde{\mathfrak{E}} = \daleth(\tilde{\mathfrak{E}},\ \tilde{\mathfrak{A}})$. Hence, $\daleth$ has Cfp. □

Theorem 4 In addition of Theorem 3, if $\daleth(\mathfrak{S}^2)$ is $\mathscr{Q}^s$-directed, then $\daleth$ has a unique Cfp.

Proof. From Theorem 3, the set of Cfp of $\daleth$ is nonempty. Let $(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})$ & $(\tilde{\mathfrak{A}}^*,\ \tilde{\mathfrak{E}}^*)$ are two Cfps of $\daleth$. Then $\tilde{\mathfrak{A}} = \daleth(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})$; $\tilde{\mathfrak{E}} = \daleth(\tilde{\mathfrak{E}},\ \tilde{\mathfrak{A}})$ and $\tilde{\mathfrak{A}}^* = \daleth(\tilde{\mathfrak{A}}^*,\ \tilde{\mathfrak{E}}^*)$; $\tilde{\mathfrak{E}}^* = \daleth(\tilde{\mathfrak{E}}^*,\ \tilde{\mathfrak{A}}^*)$. We aim to show that $\tilde{\mathfrak{A}} = \tilde{\mathfrak{A}}^*$ and $\tilde{\mathfrak{E}} = \tilde{\mathfrak{E}}^*$. By $\daleth(\mathfrak{S}^2)$ is $\mathscr{Q}^s$-directed, there exists $\sigma \in \mathfrak{S}$ and $\mathfrak{q} \in \mathfrak{S}$ such that $(\tilde{\mathfrak{A}},\ p) \in \mathscr{Q}$; $(\tilde{\mathfrak{A}}^*,\ \sigma) \in \mathscr{Q}$ and $(\tilde{\mathfrak{E}},\ \mathfrak{q}) \in \mathscr{Q}$; $(\tilde{\mathfrak{E}}^*,\ \mathfrak{q}) \in \mathscr{Q}$. Put $\sigma_0 = \sigma$ and $\mathfrak{q}_0 = \mathfrak{q}$. Then $(\tilde{\mathfrak{A}},\ \sigma_0) \in \mathscr{Q}$ and $(\mathfrak{q}_0,\ \tilde{\mathfrak{E}}) \in \mathscr{Q}$. Let $\sigma_1 = \daleth(\sigma_0,\ \mathfrak{q}_0)$ and $\mathfrak{q}_1 = \daleth(\mathfrak{q}_0,\ \sigma_0)$. Following the same inductive approach as in Theorem 1, we define sequences $\{\sigma_n\}$ & $\{\mathfrak{q}_n\}$ such that

$$
\sigma_{n+1} = \daleth(\sigma_n,\ \mathfrak{q}_n)\ \ \&\ \ \mathfrak{q}_{n+1} = \daleth(\mathfrak{q}_n,\ \sigma_n)\ \ \forall\ \ n \ge 0. \tag{29}
$$

As $\daleth$ is $\mathscr{Q}$-preserving, we obtain

$$
(\sigma_n,\ \sigma_{n+1}) \in \mathscr{Q}\ \ \&\ \ (\mathfrak{q}_n,\ \mathfrak{q}_{n+1}) \in \mathscr{Q}\ \ \forall\ \ n \ge 0. \tag{30}
$$

By following similar arguments as in proof of Theorem 3, one can show that $\{\sigma_n\}$ and $\{\mathfrak{q}_n\}$ are both Cauchy sequence in $\mathfrak{S}$ & $\exists\ \varsigma,\ \rho \in \mathfrak{S}$ such that

$$\lim_{n\to\infty} \sigma_n = \varsigma \ \ \& \ \ \lim_{n\to\infty} \mathfrak{q}_n = \rho. \tag{31}$$

We assert that

$$(\tilde{\mathfrak{A}},\ \sigma_n) \in \mathscr{Q} \ \ \& \ \ (\mathfrak{q}_n,\ \tilde{\mathfrak{E}}) \in \mathscr{Q}, \quad \forall\ \ n \geq 0. \tag{32}$$

We now employ mathematical induction to prove this fact. By condition (C_v) and (32) we have, $\forall\ n \geq 0$

$$\begin{aligned}\phi(\wp(\tilde{\mathfrak{A}},\ \sigma_n)) &= \phi\left(\wp(\daleth(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}),\ \daleth(\sigma_{n-1},\ \mathfrak{q}_{n-1})\right)\\ &\leq \phi(\mathscr{V}(\sigma_{n-1},\ \mathfrak{q}_{n-1},\ \tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})) - \psi(\mathscr{W}(\sigma_{n-1},\ \mathfrak{q}_{n-1},\ \tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})),\end{aligned} \tag{33}$$

where,

$$\begin{aligned}\mathscr{V}(\sigma_{n-1},\ \mathfrak{q}_{n-1},\ \tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) = \ \max\Bigg\{ &\frac{\wp(\tilde{\mathfrak{A}},\ \daleth(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}))[1+\wp(\sigma_{n-1},\ \daleth(\sigma_{n-1},\ \mathfrak{q}_{n-1}))]}{1+\wp(\sigma_{n-1},\ \tilde{\mathfrak{A}})},\\ &\frac{\wp(\sigma_{n-1},\ \daleth(\sigma_{n-1},\ \mathfrak{q}_{n-1}))[\wp(\mathfrak{q}_{n-1},\ \daleth(\mathfrak{q}_{n-1},\ \sigma_{n-1}))]}{1+\wp(\sigma_{n-1,\ \tilde{\mathfrak{A}}})},\\ &\frac{\wp(\sigma_{n-1},\ \daleth(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}))+[\wp(\tilde{\mathfrak{A}},\ \daleth(\sigma_{n-1},\ \mathfrak{q}_{n-1}))}{2},\ \wp(\sigma_{n-1},\ \tilde{\mathfrak{A}})\Bigg\},\\ = \ \max\Bigg\{ &\frac{\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{A}}))[1+\wp(\sigma_{n-1},\ \sigma_{n-1})]}{1+\wp(\sigma_{n-1,\ \tilde{\mathfrak{A}}})},\ \frac{\wp(\sigma_{n-1},\ \sigma_{n-1})[\wp(\mathfrak{q}_{n-1},\ \mathfrak{q}_{n-1})]}{1+\wp(\sigma_{n-1,\ \tilde{\mathfrak{A}}})},\\ &\frac{\wp(\sigma_{n-1},\ \tilde{\mathfrak{A}})+\wp(\tilde{\mathfrak{A}},\ \sigma_{n-1})}{2},\ \wp(\sigma_{n-1},\ \tilde{\mathfrak{A}})\Bigg\}\\ = \ &\wp(\sigma_{n-1},\ \tilde{\mathfrak{A}})\end{aligned} \tag{34}$$

$$\begin{aligned}\mathscr{W}(\sigma_{n-1},\ \mathfrak{q}_{n-1},\ \tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) &= \max\left\{\frac{\wp(\tilde{\mathfrak{A}},\ \daleth(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}))[1+\wp(\sigma_{n-1},\ \daleth(\sigma_{n-1},\ \mathfrak{q}_{n-1}))]}{1+\wp(\sigma_{n-1,\ \tilde{\mathfrak{A}}})},\ \wp(\sigma_{n-1},\ \tilde{\mathfrak{A}})\right\}\\ &= \max\left\{\frac{\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{A}})[1+\wp(\sigma_{n-1},\ \sigma_{n-1})]}{1+\wp(\sigma_{n-1,\ \tilde{\mathfrak{A}}})},\ \wp(\sigma_{n-1},\ \tilde{\mathfrak{A}})\right\} = \wp(\sigma_{n-1},\ \tilde{\mathfrak{A}}).\end{aligned} \tag{35}$$

Therefore, from (33), (34) and (35)

$$\phi(\wp(\tilde{\mathfrak{A}},\ \sigma_n)) = \phi\left(\wp(\daleth(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}),\ \daleth(\sigma_{n-1},\ \mathfrak{q}_{n-1})\right)$$

$$\leq \phi(\mathscr{V}(\sigma_{n-1},\ \mathfrak{q}_{n-1},\ \tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})) - \psi(\mathscr{W}(\sigma_{n-1},\ \mathfrak{q}_{n-1},\ \tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})) \tag{36}$$

$$= \phi(\wp(\sigma_{n-1},\ \tilde{\mathfrak{A}})) - \psi(\wp(\sigma_{n-1},\ \tilde{\mathfrak{A}})).$$

By properties of Φ, we get $\psi(\wp(\sigma_{n-1},\ \tilde{\mathfrak{A}})) \leq 0$, which is a contradiction to condition Ψ_2.

Hence, we have $\wp(\sigma_n,\ \tilde{\mathfrak{A}}) < \wp(\sigma_{n-1},\ \tilde{\mathfrak{A}})$ which amount to say that $\wp(\sigma_n,\ \tilde{\mathfrak{A}})$ is decreasing sequence. Then equation (36) yields that

$$\phi(\wp(\sigma_n,\ \tilde{\mathfrak{A}})) \leq \phi(\wp(\sigma_{n-1},\ \tilde{\mathfrak{A}})) - \psi(\wp(\sigma_{n-1},\ \tilde{\mathfrak{A}})). \tag{37}$$

This implies that the sequence $\{\wp(\tilde{\mathfrak{A}},\ \sigma_n)\}$ is a monotone decreasing and bounded below, $\exists\ \varsigma \geq 0$ such that

$$\lim_{n\to 0} \wp(\tilde{\mathfrak{A}},\ \sigma_n) = \varsigma.$$

Now, we show that $\varsigma = 0$, let $\varsigma \neq 0$ then taking upper limit to the equation (36) we get

$$\limsup_{n\to\infty} \phi(\wp(\tilde{\mathfrak{A}},\ \sigma_n)) \leq \limsup_{n\to\infty} \phi(\wp(\tilde{\mathfrak{A}},\ \sigma_{n-1})) + \limsup_{n\to\infty}[-\psi(\wp(\tilde{\mathfrak{A}},\ \sigma_{n-1}))]$$

$$\leq \limsup_{n\to\infty} \phi(\wp(\tilde{\mathfrak{A}},\ \sigma_{n-1})) - \liminf_{n\to\infty}[\psi(\wp(\tilde{\mathfrak{A}},\ \sigma_{n-1}))],$$

by right continuity of ϕ we get

$$\phi(\varsigma) \leq \phi(\varsigma) - \liminf_{n\to\infty} \psi(\varsigma)$$

which amounts to say that $\lim_{n\to\infty}\inf\psi(\varsigma) \leq 0$, which is contradiction to the condition Ψ_2. Hence, $\lim_{n\to\infty}\{\wp(\tilde{\mathfrak{A}},\ \sigma_n)\} = 0$ means $\lim_{n\to\infty}\sigma_n = \tilde{\mathfrak{A}}$. Analogously, we can proved that $\lim_{n\to\infty}\sigma_n = \tilde{\mathfrak{A}}^*$. Then by unicity of limit we have $\tilde{\mathfrak{A}} = \tilde{\mathfrak{A}}^*$. Similarly, we can show that $\tilde{\mathfrak{E}} = \tilde{\mathfrak{E}}^*$. Hence, $\daleth$ has a unique Cfp.

Corollary 3 If we replace the contractive condition of Theorem 3 and the hypothesis Theorem 4, are also satisfied. Also assume that:

$$\phi(\wp(\daleth(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}),\ \daleth(\upsilon,\ \omega))) \leq \phi(\mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}},\ \upsilon,\ \omega)) - \psi(\mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}},\ \upsilon,\ \omega))\ \ \forall\ \tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}},\ \upsilon,\ \omega \in \mathfrak{S}\ \text{ with }\ (\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}},\ \upsilon,\ \omega) \in \mathbb{R}.$$

where,

$$\mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}},\ \upsilon,\ \omega)=\max\left\{\frac{\wp(\upsilon,\ \daleth(\upsilon,\ \omega))[1+\wp(\tilde{\mathfrak{A}},\ \daleth(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}))]}{1+\wp(\tilde{\mathfrak{A}},\ \upsilon)},\ \frac{\wp(\tilde{\mathfrak{A}},\ \daleth(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})),\ \wp(\upsilon,\ \daleth(\upsilon,\ \omega))}{1+d(\upsilon,\ \omega)},\right.$$

$$\left.\frac{\wp(\tilde{\mathfrak{A}},\ \daleth(\upsilon,\ \omega))+\wp(\upsilon,\ \daleth(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}))}{2},\ \wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})\right\},$$

for all $\phi\in\Phi$ and $\psi\in\Psi$. Then $\daleth$ has unique fixed point.

Corollary 4 If we replace the contractive condition of Theorem 3 and the hypothesis Theorem 4, are also satisfied. Also assume that:

$$\phi(\wp(\daleth(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}),\ \daleth(\upsilon,\ \omega)))\leq\phi(\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}},\ \upsilon,\ \omega))-\psi(\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}},\ \upsilon,\ \omega))\ \forall\ \tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}},\ \upsilon,\ \omega\in\mathfrak{S}\quad\text{with}\quad(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}},\ \upsilon,\ \omega)\in\mathbb{R}.$$

where,

$$\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}},\ \upsilon,\ \omega)=\max\left\{\frac{\wp(\upsilon,\ \daleth(\upsilon,\ \omega))[1+\wp(\tilde{\mathfrak{A}},\ \daleth(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}))]}{1+\wp(\tilde{\mathfrak{A}},\ \upsilon)},\ \wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})\right\},$$

for all $\phi\in\Phi$ and $\psi\in\Psi$. Then $\daleth$ has unique fixed point. □

5. Application to a fractional differential equation

Now we are going to find the solution of a fractional differential equation Boundary Value Problem (BVP) by means of fixed point theorem. Let us consider the problem

$$D^{\alpha}_{0_+}\mathfrak{g}(\tilde{\mathfrak{A}})=\lambda f(\tilde{\mathfrak{A}},\ \mathfrak{g}(\tilde{\mathfrak{A}})),\ 0<\tilde{\mathfrak{A}}<1 \tag{38}$$

$$\mathfrak{g}(0)=\mathfrak{g}(1)=\mathfrak{g}'(0)=\mathfrak{g}'(1)=0, \tag{39}$$

where $3<\alpha\leq 4$, $\lambda\in\mathbb{R}^+$, $f:[0,\ 1]\times\mathbb{R}\to\mathbb{R}$ is continuous and $D^{\alpha}_{0_+}$ denotes the standard Riemann-Liouville fractional derivative. Define a relation $\mathbb{R}$ on $\mathfrak{S}$ as

$$uRv\iff\mathfrak{g}(\tilde{\mathfrak{A}})\leq v(\tilde{\mathfrak{A}})\ \forall\ \mathfrak{g},\ v\in C[0,\ 1]\quad\text{and}\quad\tilde{\mathfrak{A}}\in[0,\ 1].$$

In 2009, Xu et al. [32] transform the above BVP in to an integral equation, which as follows:

Lemma 3 BVP (38), (39) is analogous to the integral equation

$$\mathfrak{g}(\tilde{\mathfrak{A}})=\lambda\int_0^1K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}}))d\tilde{\mathfrak{E}}, \tag{40}$$

where

$$K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) = \begin{cases} \dfrac{(\tilde{\mathfrak{A}}-\tilde{\mathfrak{E}})^{\alpha-1}+(1-\tilde{\mathfrak{E}})^{\alpha-2}\tilde{\mathfrak{A}}^{\alpha-2}[(\tilde{\mathfrak{E}}-\tilde{\mathfrak{A}})\alpha-2(1-\tilde{\mathfrak{A}})\tilde{\mathfrak{E}}]}{\Gamma(\alpha)}, & \text{if } 0 \le \tilde{\mathfrak{E}} \le \tilde{\mathfrak{A}} \le 1 \\[2ex] \dfrac{(1-\tilde{\mathfrak{E}})^{\alpha-2}\tilde{\mathfrak{A}}^{\alpha-2}[(\tilde{\mathfrak{E}}-\tilde{\mathfrak{A}})\alpha-2(1-\tilde{\mathfrak{A}})\tilde{\mathfrak{E}}]}{\Gamma(\alpha)}, & \text{if } 0 \le \tilde{\mathfrak{A}} \le \tilde{\mathfrak{E}} \le 1. \end{cases}$$

To reduce the problem into more simpler one, Xu et al. [32] introduced another lemma to bound the value of the kernel of the integral equation (40) as follows:

Lemma 4 For all $\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}} \in (0,\ 1)$, we have $A_1 \le K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) \le A_0$, where $A_1 = \dfrac{(\alpha-2)\tilde{\mathfrak{E}}^2(1-\tilde{\mathfrak{E}})^{\alpha-2}\tilde{\mathfrak{A}}^{\alpha-2}(1-\tilde{\mathfrak{A}})^2}{\Gamma(\alpha)}$, $A_0 = \dfrac{\tilde{\mathfrak{A}}^{\alpha-2}(1-\tilde{\mathfrak{A}})^{\alpha-2}}{\Gamma(\alpha)}B_0$ and $B_0 = \max\{\alpha-1,\ (\alpha-2)^2\}$.

Now for existence and uniqueness of the solution of BVP we have following theorem.

Theorem 5 Consider the BVP (38), (39) and assume that any of the following conditions hold:

(i) $\exists$ a real number β with $0 \le \beta \le 1$, such that

$$|f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}})) - f(\tilde{\mathfrak{E}},\ v(\tilde{\mathfrak{E}}))| \le \beta|f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}}))| - \beta \sup_{0\le\tilde{\mathfrak{A}}\le 1}|\mathfrak{g}(\tilde{\mathfrak{A}})|,$$

for all real valued continuous functions $\mathfrak{g}(\tilde{\mathfrak{E}}),\ v(\tilde{\mathfrak{E}})$ defined on $[0,\ 1]$ and $\lambda A_1 \ge 1$.

(ii) $\exists$ a real number β with $0 \le \beta < 1$, such that

$$|f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}})) - f(\tilde{\mathfrak{E}},\ v(\tilde{\mathfrak{E}}))| \le \beta|\mathfrak{g}(\tilde{\mathfrak{E}}) - v(\tilde{\mathfrak{E}})|$$

for all real valued continuous functions $\mathfrak{g}(\tilde{\mathfrak{E}}),\ v(\tilde{\mathfrak{E}})$ defined on $[0,\ 1]$ and $\lambda A_0 \le 1$.

Then, the problem has a unique solution in $C[0,\ 1]$.

Proof. We know that $C[0,\ 1]$ with d the supremum metric is complete metric space. Define a selfmap $\urcorner$ on $C[0,\ 1]$ by

$$(\urcorner\mathfrak{g})(\tilde{\mathfrak{A}}) = \lambda\int_0^1 K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}}))d\tilde{\mathfrak{E}},$$

for $\mathfrak{g} \in C[0,\ 1]$ and $\tilde{\mathfrak{A}} \in [0,\ 1]$. Then the fixed point of $\urcorner$ is the solution of the BVP (38) and (39).

Now we will show that all the hypothesis of Theorem 1 and contractivity condition of Corollary 4 are satisfied.

To prove that the relation $\mathscr{Q}$ is $\urcorner$-closed, take $\mathfrak{g},\ v \in \mathfrak{S}$ such that $\mathfrak{g}\mathscr{Q}v$, i.e., By the assumption, we have $u_0 = 0 \in C[0,\ 1]$ such that

$$0 = \mathfrak{g}(\tilde{\mathfrak{A}}) = \lambda\int_0^1 K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}}))d\tilde{\mathfrak{E}},$$

$$\le \urcorner\mathfrak{g}_0 = 0,$$

this implies that $\mathfrak{g}_0\mathscr{Q}\urcorner\mathfrak{g}_0$, implying thereby $\mathfrak{S}(\urcorner,\ \mathscr{Q})$ is nonempty.

Now we prove that the relation $\mathscr{Q}$ to be $\daleth$-closed, choose $\mathfrak{g},\ v \in C[0,\ 1]$ such that $\mathfrak{g}\mathscr{Q}v$, then for $v(t) \geq \mathfrak{g}(t)$

$$K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}})) \leq K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})f(\tilde{\mathfrak{E}},\ v(\tilde{\mathfrak{E}}))$$

$$\int_0^1 K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}}))d\tilde{\mathfrak{E}} \leq \int_0^1 K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})f(\tilde{\mathfrak{E}},\ v(\tilde{\mathfrak{E}}))d\tilde{\mathfrak{E}}$$

$$\daleth\mathfrak{g}(\tilde{\mathfrak{A}}) \leq \daleth v(\tilde{\mathfrak{A}}),$$

implies that $(\daleth u,\ \daleth v) \in \mathscr{Q}$. Hence, $\mathscr{Q}$ is $\daleth$-closed. Firstly, assume that conditions (i) holds. Then for $v(t) \geq u(t)$ we have

$$\lambda K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})|f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}})) - f(\tilde{\mathfrak{E}},\ v(\tilde{\mathfrak{E}}))| \leq \beta\lambda K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})|f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}}))| - \beta\lambda K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) \sup_{0\leq\tilde{\mathfrak{A}}\leq 1} |\mathfrak{g}(\tilde{\mathfrak{A}})|.$$

So

$$\begin{aligned}\int_0^1 \lambda K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})|f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}})) - f(\tilde{\mathfrak{E}},\ v(\tilde{\mathfrak{E}}))| \leq \int_0^1 \beta\lambda K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})|f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}}))| \\ - \int_0^1 \beta\lambda K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) \sup_{0\leq\tilde{\mathfrak{A}}\leq 1} |\mathfrak{g}(\tilde{\mathfrak{A}})| d\tilde{\mathfrak{E}}.\end{aligned}$$

$$\begin{aligned}\implies \int_0^1 |\lambda K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}})) - \lambda K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})f(\tilde{\mathfrak{E}},\ v(\tilde{\mathfrak{E}}))| &\leq \int_0^1 \beta\lambda K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})|f(\tilde{\mathfrak{E}},\ u(\tilde{\mathfrak{E}}))| \\ &\quad - \int_0^1 \beta\lambda K A_1 \sup_{0\leq\tilde{\mathfrak{A}}\leq 1} |\mathfrak{g}(\tilde{\mathfrak{A}})| \wp\tilde{\mathfrak{E}}. \\ &\leq \int_0^1 \beta\lambda K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})|f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}}))| \wp\tilde{\mathfrak{E}} - \beta\lambda K A_1 |\mathfrak{g}(\tilde{\mathfrak{A}})| \\ &\leq \int_0^1 \beta\lambda K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})|f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}}))| \wp\tilde{\mathfrak{E}} - \beta|\mathfrak{g}(\tilde{\mathfrak{A}}) \\ &\leq \beta \left| \int_0^1 \lambda K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}})) \wp\tilde{\mathfrak{E}} - \mathfrak{g}(\tilde{\mathfrak{A}}) \right|.\end{aligned}$$

Therefore, for any $\mathfrak{g},\ v \in C[0,\ 1]$ and $\tilde{\mathfrak{A}} \in [0,\ 1]$, we have

$$\left(|(\daleth\mathfrak{g})(\tilde{\mathfrak{A}}) - (\daleth v)(\tilde{\mathfrak{E}})|\right) = \lambda\left|\int_0^1 K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}}))\wp\tilde{\mathfrak{E}} - \int_0^1 K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})f(\tilde{\mathfrak{E}},\ v(\tilde{\mathfrak{E}}))\wp\tilde{\mathfrak{E}}\right|$$

$$\leq \lambda\int_0^1 K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})|f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}})) - f(\tilde{\mathfrak{E}},\ v(\tilde{\mathfrak{E}}))|\wp\tilde{\mathfrak{E}}$$

$$\leq \beta\left|\int_0^1 \beta\lambda K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}}))\wp\tilde{\mathfrak{E}} - \mathfrak{g}(\tilde{\mathfrak{A}})\right|$$

$$= \beta|(\daleth\mathfrak{g})(\tilde{\mathfrak{A}}) - \mathfrak{g}(\tilde{\mathfrak{A}})|.$$

So,

$$\wp(\daleth\mathfrak{g},\ \daleth v) \leq \beta\wp(\daleth\mathfrak{g},\ \mathfrak{g}) \leq \mathscr{V}(\mathfrak{g},\ v).$$

Similarly,

$$\wp(\daleth\mathfrak{g},\ \daleth v) \leq \beta\wp(\daleth\mathfrak{g},\ \mathfrak{g}) \leq \mathscr{W}(\mathfrak{g},\ v).$$

Now, if we choose the condition (ii), then

$$\left(|(\daleth\mathfrak{g})(\tilde{\mathfrak{A}}) - (\daleth v)(\tilde{\mathfrak{A}})|\right) = \lambda\left|\int_0^1 K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}}))d\tilde{\mathfrak{E}} - \int_0^1 K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})f(\tilde{\mathfrak{E}},\ v(\tilde{\mathfrak{E}}))d\tilde{\mathfrak{E}}\right|$$

$$\leq \lambda\int_0^1 K(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})|f(\tilde{\mathfrak{E}},\ \mathfrak{g}(\tilde{\mathfrak{E}})) - f(\tilde{\mathfrak{E}},\ v(\tilde{\mathfrak{E}}))|d\tilde{\mathfrak{E}}$$

$$\leq \lambda\beta A_0\int_0^1 |\mathfrak{g}(\tilde{\mathfrak{E}}) - \mathfrak{g}(\tilde{\mathfrak{A}})|$$

$$= \lambda\beta A_0\wp(\mathfrak{g},\ v)$$

$$\leq \beta\wp(\mathfrak{g},\ v).$$

Therefore,

$$\wp(\daleth\mathfrak{g},\ \daleth v) \leq \beta\wp(\mathfrak{g},\ v) \leq \mathscr{V}(\mathfrak{g},\ v).$$

Similarly,

$$\wp(\daleth\mathfrak{g},\ \daleth v) \le \beta\wp(\mathfrak{g},\ v) \le \mathscr{W}(\mathfrak{g},\ v).$$

Then for $\mathfrak{g},\ v \in C[0,\ 1]$ and the assumption of (i) and (ii) we have

$$\phi(\wp(\daleth\mathfrak{g},\ \daleth v)) \le \phi(\mathscr{V}(\wp(\mathfrak{g},\ v))) - \psi(\mathscr{W}(\wp(\mathfrak{g},\ v))). \tag{41}$$

Using Remark 2, we have

$$\phi(\wp(\daleth\mathfrak{g},\ \daleth v)) \le \phi(\mathscr{V}(\wp(\mathfrak{g},\ v))) - \psi(\mathscr{W}(\wp(\mathfrak{g},\ v))). \tag{42}$$

For $\phi(t) = t$ and $\psi(t) = (1-\beta)t$, equation (26) holds true. Then we can say that $\daleth$ satisfy the contractive condition of the Theorem 1.

For $\mathscr{Q}$ to be $\wp$-self closed, assume $\{\mathfrak{g}_n\}$ a $\mathscr{Q}$- preserving Cauchy sequence converging to $\mathfrak{g} \in C[0,\ 1]$. As $\{\mathfrak{g}_n\}$ is $\mathscr{Q}$-preserving, we have

$$\mathfrak{g}_0(\tilde{\mathfrak{A}}) \le \mathfrak{g}_1(\tilde{\mathfrak{A}}) \le \mathfrak{g}_2(\tilde{\mathfrak{A}}) \le \cdots \le \mathfrak{g}_n(\tilde{\mathfrak{A}}) \le \mathfrak{g}_{n+1}(\tilde{\mathfrak{A}}) \le \cdots \le \mathfrak{g}(\tilde{\mathfrak{A}}) \qquad \tilde{\mathfrak{A}} \in [0,\ 1],$$

then we get $\mathfrak{g}_n\mathscr{Q}\mathfrak{g}\ \forall n \in \mathbb{N}$. Therefore, $\mathscr{Q}$ is $\wp$- self closed.

Hence, we verify that all assumptions of Theorem 1 are satisfied. Hence $\daleth$ possess fixed point, which amounts to say that the integral equation (40) has a solution. Finally we can say that the fractional differential equations (38) and (39) have a solution in $C[0,\ 1]$. □

Now we can give an example in support of Theorem 1.

Example 2 Consider $\mathfrak{S} = [0,\ 1]$ is a metric space with usual metric $\wp$. Suppose a relation $\mathscr{Q} = \left\{(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) \in \mathfrak{S} : \tilde{\mathfrak{E}} \ge \tilde{\mathfrak{A}} \ge \frac{2}{3}\right\}$ and mapping $\daleth : \mathfrak{S} \to \mathfrak{S}$ defined by

$$\daleth(\tilde{\mathfrak{A}}) = \begin{cases} 0 & \text{if } \tilde{\mathfrak{A}} \in \left[0,\ \frac{2}{3}\right) \\ \frac{\tilde{\mathfrak{A}}}{6}, & \text{if } \tilde{\mathfrak{A}} \in \left[\frac{2}{3},\ 1\right]. \end{cases}$$

It is clear that $(\mathfrak{S},\ d)$ is $\mathscr{Q}$-complete, $\mathscr{Q}$ is $\daleth$-closed & $\daleth$ is $\mathscr{Q}$-continuous function. Now, $\forall\ (\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) \in \mathscr{Q}$. One can easily verify that $\frac{1}{6}d(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) \le \mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})$ and $\frac{1}{6}d(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) \le \mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})$, then we have

$$\wp(\daleth\tilde{\mathfrak{A}},\ \daleth\tilde{\mathfrak{E}}) = \frac{1}{6}|\tilde{\mathfrak{A}} - \tilde{\mathfrak{E}}|$$

$$= \frac{1}{6}d(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) - \frac{\left[\frac{1}{6}\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})\right]}{10}$$

$$\leq \mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) - \frac{[\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})]}{10}$$

$$\phi(\wp(T\tilde{\mathfrak{A}},\ T\tilde{\mathfrak{E}})) \leq \phi(\mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})) - \psi(\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})),$$

which implies that $\urcorner$ satisfies the assumption (v) of Theorem 1 for $\phi(t) = t$ and $\psi(t) = \frac{[t]}{10}$. Consequently all the hypothesis of Theorem 1 are satisfied and hence $\urcorner$ has fixed point namely $\tilde{\mathfrak{A}} = 0$. Further the fixed point is also unique.

Now, we demonstrate the convergence of the Picard iteration scheme and the existence and uniqueness of the fixed point.

Let $\tilde{\mathfrak{A}}_n$ be the Picard sequence defined by $\tilde{\mathfrak{A}}_{n+1} = \urcorner(\tilde{\mathfrak{A}}_n)$, for any $\tilde{\mathfrak{A}}_0 \in [0,\ 1]$.

If $\tilde{\mathfrak{A}}_0 \in \left[0,\ \frac{2}{3}\right)$, then $\urcorner(x_0) = 0$, and hence $\tilde{\mathfrak{A}}_1 = 0$ and $\tilde{\mathfrak{A}}_n = 0$ for all $n \geq 1$.

If $\tilde{\mathfrak{A}}_0 \in \left[\frac{2}{3},\ 1\right]$, then $\tilde{\mathfrak{A}}_1 = \frac{\tilde{\mathfrak{A}}_0}{6} \leq \frac{1}{6} < \frac{2}{3}$, so again $\tilde{\mathfrak{A}}_2 = 0$ and $\tilde{\mathfrak{A}}_n = 0$ for all $n \geq 2$.

Thus, for any initial point $\tilde{\mathfrak{A}}_0 \in [0,\ 1]$, the sequence $\tilde{\mathfrak{A}}_n$ converges to 0 in at most two iterations. Therefore, 0 is a fixed point of $\urcorner$.

To prove uniqueness, let $\tilde{\mathfrak{A}}$ be any fixed point of $\urcorner$, i.e., $\urcorner(\tilde{\mathfrak{A}}) = \tilde{\mathfrak{A}}$. If $\tilde{\mathfrak{A}} \in \left[0,\ \frac{2}{3}\right)$, then $\urcorner(\tilde{\mathfrak{A}}) = 0$, implying $\tilde{\mathfrak{A}} = 0$. If $\tilde{\mathfrak{A}} \in \left[\frac{2}{3},\ 1\right]$, then $\tilde{\mathfrak{A}} = \tilde{\mathfrak{A}}/6$, which also gives $\tilde{\mathfrak{A}} = 0$. Hence, the fixed point is unique.

Example 3 Consider $\mathfrak{S} = (-1,\ 1]$ is a metric space with usual metric $\wp$. Suppose a relation $\mathscr{Q} = \{(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) \in \mathfrak{S} : \tilde{\mathfrak{A}} > \tilde{\mathfrak{E}} \geq 0\}$ and mapping $\urcorner : \mathfrak{S} \to \mathfrak{S}$ defined by

$$\urcorner(\tilde{\mathfrak{A}}) = \begin{cases} \tilde{\mathfrak{A}} + 1, & \text{if } -1 < \tilde{\mathfrak{A}} < 0, \\ \frac{\tilde{\mathfrak{A}}}{4}, & \text{if } 0 \leq \tilde{\mathfrak{A}} \leq 1. \end{cases}$$

It is clear that $(\mathfrak{S},\ d)$ is $\mathscr{Q}$-complete, $\mathscr{Q}$ is $\urcorner$-closed & $\urcorner$ is $\mathscr{Q}$-continuous but not continuous function. Now, $\forall$ $(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) \in \mathscr{Q}$. One can easily verify that $\frac{1}{4}d(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) \leq \mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})$ and $\frac{1}{4}d(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) \leq \mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})$, then we have

$$\wp(\urcorner\tilde{\mathfrak{A}},\ \urcorner\tilde{\mathfrak{E}}) = \frac{1}{4}|\tilde{\mathfrak{A}} - \tilde{\mathfrak{E}}|$$

$$= \frac{1}{4}d(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) - \frac{\left[\frac{1}{4}\wp(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})\right]}{10}$$

$$\leq \mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}}) - \frac{[\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})]}{10}$$

$$\phi(\wp(T\tilde{\mathfrak{A}},\ T\tilde{\mathfrak{E}})) \leq \phi(\mathscr{V}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})) - \psi(\mathscr{W}(\tilde{\mathfrak{A}},\ \tilde{\mathfrak{E}})),$$

which implies that T satisfies the assumption (v) of Theorem 1 for $\phi(t) = t$ and $\psi(t) = \frac{[t]}{10}$. Consequently all the hypothesis of Theorem 1 are satisfied and hence $\urcorner$ has fixed point namely $\tilde{\mathfrak{A}} = 0$. Further the fixed point is also unique.

Notice that assumption (v) of Theorem 1 does not hold for the whole space (for example take $\tilde{\mathfrak{A}} = \frac{-1}{2}$ and $\tilde{\mathfrak{E}} = \frac{3}{5}$). Therefore, this example cannot be solved by the existing results, which establishes the importance of our result.

6. Conclusion

In this paper, we established existence and uniqueness results for fixed points and coupled fixed points under a binary relation via generalized $(\phi,\ \psi)$-rational contraction. we also established some coupled fixed point results for mappings with mixed $\mathscr{Q}$-monotone property using relation-theoretic approach. An illustrative example and an application to fractional differential equations validated our findings. Theoretically, the proposed results enrich fixed point theory by introducing a broader class of contractions that generalize several existing theorems and provide a unified framework for studying nonlinear mappings under binary relations. Practically, these results can be applied to the analysis of nonlinear functional and fractional differential equations, offering effective tools for modeling and solving complex problems in applied sciences and engineering. Future research may extend these results to broader settings such as Symmetric space, cone metric space, partial or dislocated metric spaces and integral inclusions, as well as explore applications in nonlinear systems and computational methods.

Conflict of interest

The author declares no competing financial interest.

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