Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

UNIVERSAL WISER PUBLISHER

Research Article

New Constructions of Landsberg Non-Berwaldian (α, β)-Metrics

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Received: 20 July 2025; **Revised:** 9 September 2025; **Accepted:** 19 September 2025

Abstract: In this paper, we present new constructions of non-regular Finsler metrics of (α, β)-type that satisfy the Landsberg condition without being Berwaldian. These examples contribute to the ongoing investigation of the Landsberg-Berwald problem by expanding the class of known non-Berwaldian Landsberg spaces. The results offer deeper insight into the geometric structure and richness of such metrics within the framework of Finsler geometry.

Keywords: Landsberg metrics, Berwald metrics, (α, β)-metrics

MSC: 53C60, 53B40, 58B20

1. Introduction

The Landsberg Problem is one of the central and most intriguing open problems in Finsler geometry, revolving around the relationship between two important classes of Finsler spaces: Landsberg spaces and Berwald spaces. In Finsler geometry, the metric depends not only on the position $x \in M$ on a manifold M, but also on the direction $y \in T_xM$ in the tangent space. This direction-dependence introduces rich geometric structures, one of which is the Cartan tensor C_{ijk}, measuring the deviation of the metric from being Riemannian. The Landsberg tensor L_{ijk} captures how the Cartan tensor changes along geodesics and is a key quantity for studying the geometry of Finsler spaces.

A Landsberg space is defined as a Finsler manifold for which the Landsberg tensor vanishes identically: $L_{ijk} = 0$. This means that, along geodesics, the Cartan tensor is parallel in a certain sense. On the other hand, a Berwald space is one where the Chern (or Berwald) connection coefficients are independent of the directional variable y, meaning that the parallel translation is linear and the geodesics behave much like those in a Riemannian setting. Every Berwald space is also a Landsberg space because the Cartan tensor is constant along geodesics in a Berwald space, which automatically implies $L_{ijk} = 0$.

Geometrically, in a Berwald manifold (M, F), and for any piecewise smooth curve $c(t)$ connecting two points $p, q \in M$, the Berwald parallel translation P_c is linear isometry between (T_pM, F_p) and (T_qM, F_q). This is equivalent to that the geodesic spray of F is quadratic. But, in a Landsberg manifold the parallel translation P_c along c preserves the induced Riemannian metrics on the slit tangent spaces, that is, $P_c : (T_pM\backslash\{0\}, g_p) \longrightarrow (T_qM\backslash\{0\}, g_q)$ is an isometry. This is equivalent to the horizontal covariant derivative of the metric tensor of F with respect to Berwald connection vanishes.

This question-whether every regular Landsberg space must be Berwaldian-is famously known as the Landsberg unicorn problem, first posed several decades ago. Despite numerous efforts and partial results, a definitive counterexample

DOI: https://doi.org/10.37256/cm.6620257914

or a general proof has remained elusive. Notably, all known regular Landsberg metrics have been shown to be Berwaldian, leading to the conjecture that non-Berwaldian Landsberg spaces (the so-called "unicorns") may not exist. For details and surveys on the problem, we refer, for example, to the two surveys [1, 2].

It is well known that every Berwald space is necessarily Landsberg. However, whether there exist regular Landsberg spaces that are not Berwald remains a central unresolved problem in Finsler geometry. In [3], the author employed the inverse problem to analyze explicit examples of non-Berwaldian Landsberg metrics of $(\alpha,\ \beta)$-types. By studying the geodesic spray and applying suitable deformations, the paper produced new and simpler examples of non-Berwaldian Landsberg metrics. This method also yields a more accessible formulation of a class of metrics previously discovered by [4, 5]. For more explicit examples in the class of spherically symmetric metrics see [6, 7].

In a recent development, Heefer et al. [8] constructed a new family of exact vacuum solutions to Pfeifer and Wohlfarth's field equations in Finsler gravity, building upon one of the solutions previously obtained in [3]. Their construction is a compelling analogy to classical cosmological models.

In this note, we build upon of the results obtained in [3]. We manage to construct new Riemannian metric α and a one form β that fulfill the requirements of an $(\alpha,\ \beta)$-type to be a non-regular Landsberg Berwaldian spaces. Namely, α and β are given by:

$$\alpha = \sqrt{(y^1)^2 + c\, e^{2x^1} \varphi(\hat{y})}, \qquad \beta = y^1$$

where c is an arbitrary constant and φ is a quadratic function in $\hat{y}$ ($\hat{y}$ stands for the variables $y^2,\ \ldots,\ y^n$).

By constructing explicit examples of non-regular $(\alpha,\ \beta)$-metrics that are Landsberg but not Berwaldian, our work contributes new evidence and tools to this longstanding problem, offering insights that may influence both the theoretical classification of Finsler spaces and their application, for instance, in physics.

We conclude this work with several remarks. One of them concerns the potential use of conformal transformations to obtain further solutions and examples of non-regular Landsberg metrics that are not Berwaldian. Additionally, we correct few typographical errors in [3].

2. Notations and preliminaries

Let M be an n-dimensional smooth manifold. The coordinates on M are denoted by x^i and corresponding fibre coordinates y^i. The tangent bundle of M is denoted by TM while the slit tangent bundle is denoted by $\mathscr{T}M$. On the tangent bundle TM, the vector 1-form J defined locally by $J = \frac{\partial}{\partial y^i} \otimes dx^i$ is known as the natural almost tangent structure. The vertical (Liouville) vector field is $\mathscr{C} = y^i \frac{\partial}{\partial y^i}$.

Let $X \in \mathfrak{X}(M)$. We denote by $\mathscr{L}_X$ the Lie derivative with respect to X, and by i_X the interior product with X. For $f \in C^\infty(M)$, df denotes the differential of f.

A skew-symmetric $C^\infty(TM)$-linear map

$$L : \left(\mathfrak{X}(M)\right)^\ell \longrightarrow \mathfrak{X}(M) \tag{1}$$

is called a vector ℓ-form on M.

Each vector ℓ-form L determines two graded derivations of the Grassmann algebra of M, denoted by i_L and d_L, defined as follows:

$$i_L f = 0, \quad i_L df = df \circ L, \qquad (f \in C^\infty(M)), \tag{2}$$

$$d_L := [i_L,\ d] = i_L \circ d - (-1)^{\ell-1}\ d \circ i_L. \tag{3}$$

A vector field $S \in \mathfrak{X}(\mathscr{T}M)$ is called a *spray* if it satisfies $JS = \mathscr{C}$ and $[\mathscr{C},\ S] = S$. Locally, any spray can be represented as

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}, \tag{4}$$

where the functions $G^i = G^i(x,\ y)$ are homogeneous of degree two in y and are referred to as the *spray coefficients*.

We denote partial derivatives by $\partial_i := \dfrac{\partial}{\partial x^i}$ and $\dot{\partial}_i := \dfrac{\partial}{\partial y^i}$.

Definition 1 A *Finsler manifold* is a pair $(M,\ F)$, where M is a differentiable manifold and $F : TM \to \mathbb{R}$ is a function satisfying:

(a) F is smooth and positive on $\mathscr{T}M$ and vanishes only at the zero section,

(b) F is positively homogeneous of degree one in y,

(c) The metric tensor $g_{ij} = \dot{\partial}_i\dot{\partial}_j E$, where $E := \dfrac{1}{2}F^2$, is non-degenerate on $\mathscr{T}M$.

Such a pair $(M,\ F)$ is referred to as a *regular Finsler space*. If F fails to be smooth or defined in certain directions, it is termed a *non-regular* Finsler function. Sometimes $(M,\ F)$ is called conic Finsler manifold when F fulfills the above conditions on a conic subset of $\mathscr{T}M$.

The non-degeneracy of the 2-form $dd_J E$ implies the existence of a unique spray S on TM satisfying the Euler-Lagrange equation:

$$i_S dd_J E = -dE. \tag{5}$$

This spray is known as the *geodesic spray* of F.

For a Finsler metric $F(x,\ y)$, the geodesic spray coefficients are given by:

$$G^i = \frac{1}{4} g^{ih} \left(y^r \partial_r \dot{\partial}_h F^2 - \partial_h F^2 \right). \tag{6}$$

The associated nonlinear connection N^i_j and Berwald connection coefficients G^i_{jk} are given by:

$$N^i_j := \dot{\partial}_j G^i, \quad G^i_{jk} := \dot{\partial}_k N^i_j, \tag{7}$$

and the Berwald and Landsberg tensors are defined, respectively, by:

$$G^i_{jkh} := \dot{\partial}_h G^i_{jk}, \quad L_{jkh} := -\frac{1}{2} F \ell_i G^i_{jkh}, \quad \text{where } \ell_i := \dot{\partial}_i F. \tag{8}$$

Definition 2 A Finsler manifold $(M,\ F)$ is called *Berwald* if its Berwald tensor G^i_{jkh} vanishes identically. Moreover, $(M,\ F)$ is called *Landsberg* if its Landsberg tensor L_{jkh} vanishes identically.

3. Landsberg metrics of $(\alpha,\ \beta)$-types

The geodesic spray of an $(\alpha,\ \beta)$-metric $F=\alpha\phi(s)$, where $s:=\dfrac{\beta}{\alpha}$ is given by [5]

$$G^i=G^i_\alpha+\alpha Q s^i{}_0+\Theta\{-2\alpha Q s_0+r_{00}\}\left\{\frac{y^i}{\alpha}+\frac{Q'}{Q-sQ'}b^i\right\}, \tag{9}$$

where

$$r_{ij}=\frac{1}{2}(b_{i|j}+b_{j|i}),\quad s_{ij}=\frac{1}{2}(b_{i|j}-b_{j|i}), \tag{10}$$

$$r_{00}:=r_{ij}y^iy^j,\quad s_{i0}:=s_{ij}y^j,\quad s^i_j:=s_{hj}a^{ih},\quad s_j:=s_{ij}b^i,\quad b^i:=b_ja^{ij} \tag{11}$$

$$Q(s):=\frac{\phi'}{\phi-s\phi'}, \tag{12}$$

$$\Theta(s):=\frac{Q-sQ'}{2(1+sQ+(b^2-s^2)Q')}, \tag{13}$$

where $|$ denotes the covariant derivative with respect to the Levi-Civita connection of α, and Q' (resp. ϕ') presents the derivative of Q (respectively, ϕ) with respect to s.

Shen [5] characterized all $(\alpha,\ \beta)$-metrics of Landsberg type which are not Berwaldian. Precisely, he obtained the following class:

$$F=\alpha\phi(s),\quad \phi(s)=c_4\exp\left(\int_0^s\frac{c_1\sqrt{1-(t/b_o)^2}+c_3t}{1+t\left(c_1\sqrt{1-(t/b_o)^2}+c_3t\right)}dt\right), \tag{14}$$

where

$$b=b_0,\quad b_{i|j}=b_{j|i},\quad b_{i|j}=k(a_{ij}-b_ib_j), \tag{15}$$

where k is a non zero scalar function on M.

These metrics form a class of non-regular Landsberg metrics that are not Berwaldian; they become Berwaldian if and only if $k=0$.

The coefficients of the geodesic spray of the class (14) are given by

$$G^i=G^i_\alpha+\frac{c_1k\sqrt{\alpha^2-(\beta/b_0)^2}}{2(1+c_3b_0^2)}\left\{b_0^2y^i-\beta b^i+\frac{c_3k}{c_1}\sqrt{\alpha^2-(\beta/b_0)^2}b^i\right\}. \tag{16}$$

3.1 *Special Riemannian metric*

In [3], the author introduced the following choice for α and β

$$\alpha = f(x^1)\sqrt{(y^1)^2 + \varphi(\hat{y})}, \quad \beta = f(x^1)y^1, \tag{17}$$

where $f(x^1)$ is smooth function on the base manifold M, and φ is a quadratic function in the variables $\hat{y} = (y^2, \ldots, y^n)$. The function φ must be chosen such that the metric tensor associated with α is non-degenerate. Specifically, φ can be expressed as

$$\varphi = c_{\lambda\mu} y^\lambda y^\mu, \tag{18}$$

where $c_{\lambda\mu}$ is a symmetric, non-singular $(n-1)\times(n-1)$ matrix of arbitrary constants. Here, the Greek indices $\mu, \nu, \ldots$ are used to range over the values $2, 3, \ldots, n$.

In this note, we introduce another choice for the sake of adding or obtaining new solutions to the unicorn's problem i.e. new examples of non-regular Landsberg spaces that are not Berwaldian. That is, we have

$$\alpha = \sqrt{(y^1)^2 + f(x^1)\varphi(\hat{y})}, \quad \beta = y^1, \tag{19}$$

In this case, the components a_{ij} and the inverse a^{ij} of the metric α are given respectively by

$$a_{11} = 1, \quad a_{1\mu} = 0, \quad a_{\lambda\mu} = f(x^1)c_{\lambda\mu}; \quad a^{11} = 1, \quad a^{1\mu} = 0, \quad a^{\lambda\mu} = \frac{1}{f(x^1)}c^{\lambda\mu}, \tag{20}$$

where $(c^{\lambda\mu})$ is the inverse matrix of $(c_{\lambda\mu})$.

Using (6), we can find the geodesic spray G^i_α of α, as follows

$$G^1_\alpha = -\frac{1}{4}f'(x^1)\varphi(\hat{y}), \quad G^\mu_\alpha = \frac{1}{2}\frac{f'(x^1)}{f(x^1)}y^1 y^\mu. \tag{21}$$

The coefficients $\gamma^h_{ij} := \dot{\partial}_i\dot{\partial}_j G^h_\alpha$ of Levi-Civita connection of α are given by

$$\gamma^1_{11} = 0, \quad \gamma^1_{\lambda\mu} = -\frac{1}{2}f'(x^1)c_{\lambda\mu}, \quad \gamma^\mu_{1\lambda} = \frac{f'(x^1)}{2f(x^1)}\delta^\mu_\lambda, \quad \gamma^1_{1\mu} = \gamma^\mu_{11} = \gamma^\mu_{\lambda\nu} = 0. \tag{22}$$

Proposition 1 With respect $\alpha = \sqrt{(y^1)^2 + f(x^1)\varphi(\hat{y})}$, the one form $\beta = b_i y^i = y^1$ has the following properties:
(i) $\|\beta\|_\alpha = 1$.
(ii) β is closed that is, $s_{ij} = 0$.
(iii) $b_{i|j} = a_{ij} - b_i b_j$ if and only if $f(x^1) = c\, e^{2x^1}$.
Proof. (i) We have $b_1 = 1$ and $b_\mu = 0$. Now, one can see that

$$b^1 = 1, \quad b^\mu = 0. \tag{23}$$

Hence, $\|\beta\|_\alpha = b^2 = b_i b^i = 1$.

(ii) The proof follows by making use of the facts that b_i are constants and γ^h_{ij} are symmetric in the lower indices.

(iii) We have $b_{i|j} = \partial_j b_i - b_h \gamma^h_{ij} = -\gamma^1_{ij}$ which reads

$$b_{1|\mu} = 0, \quad b_{\mu|1} = 0, \quad b_{\lambda|\mu} = \frac{1}{2} f'(x^1) c_{\lambda\mu}. \tag{24}$$

from which we get $b_{i|j} = a_{ij} - b_i b_j$ if and only if

$$\frac{1}{2} f'(x^1) = f(x^1). \tag{25}$$

The above Ordinary Differential Equation (ODE) has the solution $f(x^1) = c\, e^{2x^1}$. □

4. New non-regular solutions for the unicorn's problem

From now on, we fix the following choice of α and β as follows:

$$\alpha = \sqrt{(y^1)^2 + c\, e^{2x^1} \varphi(\hat{y})}, \quad \beta = y^1 \tag{26}$$

where c is a positive constant and $\varphi(\hat{y})$ is given by (18).

Proposition 2 Let $F = \alpha\phi(s)$ be an (α, β) metric with special choice (26) of α and β. Then the components of Berwald curvature of F are $G^1_{ijk} = 0$, $G^h_{1jk} = 0$, and

$$\begin{aligned} G^\lambda_{\nu\mu\rho} = \frac{c_1 \sqrt{c\, e^{2x^1}}}{2(1+c_3)} \Bigg\{ & \frac{1}{\sqrt{\varphi}} (c_{\mu\rho} \delta^\lambda_\nu + c_{\mu\nu} \delta^\lambda_\rho + c_{\nu\rho} \delta^\lambda_\mu) \\ & - \frac{1}{\varphi^{3/2}} (c_{\mu\gamma} c_{\rho\sigma} \delta^\lambda_\nu y^\gamma y^\sigma + c_{\mu\gamma} c_{\nu\sigma} \delta^\lambda_\rho y^\gamma y^\sigma + c_{\nu\gamma} c_{\rho\sigma} \delta^\lambda_\mu y^\gamma y^\sigma) \\ & - \frac{1}{\varphi^{3/2}} (c_{\mu\nu} c_{\rho\sigma} y^\sigma + c_{\mu\rho} c_{\nu\sigma} y^\sigma + c_{\rho\nu} c_{\mu\sigma} y^\sigma) y^\lambda + \frac{3}{\varphi^{5/2}} c_{\nu\gamma} c_{\mu\sigma} c_{\rho\xi} y^\gamma y^\sigma y^\xi y^\lambda \Bigg\}. \end{aligned} \tag{27}$$

Proof. The geodesic spray coefficients G^i are given with the help of the equations (16) and (21) as follows

$$G^1 = -\frac{c_3 c\, e^{2x^1} \varphi}{2(1+c_3)}, \quad G^\mu = P y^\mu, \quad P = \left(y^1 + \frac{c_1}{2(1+c_3)} \sqrt{c\, e^{2x^1} \varphi} \right). \tag{28}$$

Now, by taking the derivative of G^i with respect the direction y^j, then we obtain the coefficients of the non linear connection N^i_j of F:

$$N^1_1 = 0, \quad N^1_\mu = -\frac{c_3 c\, e^{2x^1}}{1+c_3} c_{\mu\nu} y^\nu, \quad N^\mu_1 = y^\mu,$$

$$N^\lambda_\nu = \left(y^1 + \frac{c_1}{2(1+c_3)} \sqrt{c\, e^{2x^1} \varphi(\hat{y})} \right) \delta^\lambda_\nu + \frac{c_1 \sqrt{c\, e^{2x^1}}}{2(1+c_3)\sqrt{\varphi}} c_{\nu\sigma} y^\sigma y^\lambda.$$

Also, the coefficients $G^h_{ij} = \dot{\partial}_j N^h_i$ of F are calculated as follows:

$$G^1_{1i} = 0, \quad G^1_{\mu\nu} = -\frac{c\, e^{2x^1}}{a^2} c_{\mu\nu}, \quad G^\lambda_{1\nu} = \delta^\lambda_\nu, \quad G^\lambda_{11} = 0$$

$$G^\lambda_{\nu\mu} = \frac{\sqrt{c\, e^{2x^1}}}{a} \left(\frac{c_{\mu\rho} y^\rho \delta^\lambda_\nu + c_{\nu\rho} y^\rho \delta^\lambda_\mu + c_{\mu\nu} y^\lambda}{\sqrt{\varphi}} - \frac{c_{\mu\rho} c_{\nu\gamma} y^\rho y^\gamma y^\lambda}{\varphi^{3/2}} \right).$$

Using the fact that $G^h_{ijk} = \dot{\partial}_k G^h_{ij}$ we have

$$G^1_{ijk} = 0, \quad G^h_{1jk} = 0. \tag{29}$$

Now, the only non zero components are $G^\lambda_{\mu\nu\gamma}$. These components are given by

$$G^\lambda_{\nu\mu} = \frac{c_1 \sqrt{c\, e^{2x^1}}}{2(1+c_3)} \left(\frac{c_{\mu\rho} y^\rho \delta^\lambda_\nu + c_{\nu\rho} y^\rho \delta^\lambda_\mu + c_{\mu\nu} y^\lambda}{\sqrt{\varphi}} - \frac{c_{\mu\rho} c_{\nu\gamma} y^\rho y^\gamma y^\lambda}{\varphi^{3/2}} \right).$$

Hence the components $G^\lambda_{\nu\mu\rho}$ are obtained by the derivative of $G^\lambda_{\nu\mu}$ with respect to y^i. □

The following theorem introduces the first class of Landsberg metrics which are not Berwaldian.

Theorem 1 Let (M, F) be an n-dimensional non-regular Finsler manifold where $n \geq 3$, $a \neq 0$, and

$$F = \left(a y^1 + \sqrt{c\, e^{2x^1} \varphi(\hat{y})} \right) e^{\frac{a y^1}{a y^1 + \sqrt{c\, e^{2x^1} \varphi(\hat{y})}}}. \tag{30}$$

Then, (M, F) is a Landsberg manifold which is not Berwaldian. Moreover, the geodesic spray of F is given by

$$G^1 = -\frac{c}{2a^2} e^{2x^1} \varphi(\hat{y}), \quad G^\mu = P y^\mu, \quad P = \left(y^1 + \frac{1}{a} \sqrt{c\, e^{2x^1} \varphi(\hat{y})} \right), \tag{31}$$

where $\varphi(\hat{y})$ is given by (18).

Proof. The class

$$F=\left(a\beta+\sqrt{\alpha^2-\beta^2}\right)e^{\frac{a\beta}{a\beta+\sqrt{\alpha^2-\beta^2}}},\tag{32}$$

with the choice (17), is discussed in [3]. So, the constants c_1 and c_3 in (16) (The spray coefficients of a Landsberg metric of $(\alpha,\ \beta)$ metric type) are given by

$$c_1=2a,\quad c_3=a^2-1.\tag{33}$$

Now, by (28), the geodesic spray coefficients G^i are given by

$$G^1=-\frac{c}{2a^2}e^{2x^1}\varphi(\hat{y}),\quad G^\mu=Py^\mu,\quad P=\left(y^1+\frac{1}{a}\sqrt{ce^{2x^1}\varphi(\hat{y})}\right).\tag{34}$$

Since, by Proposition 2, $G^1_{ijk}=0$ and $G^h_{1jk}=0$, then to prove that $(M,\ F)$ is a Landsberg manifold, it is enough to show that $L_{\nu\mu\rho}=-\frac{1}{2}FG^\lambda_{\nu\mu\rho}\ell_\mu=0$. So, we have

$$\ell_\mu=\frac{\sqrt{ce^{2x^1}}}{ay^1+\sqrt{ce^{2x^1}\varphi}}e^{\frac{ay^1}{ay^1+\sqrt{ce^{2x^1}\varphi}}}c_{\mu\delta}y^\delta,\tag{35}$$

Now, we have

$$\begin{aligned}L_{\nu\mu\rho}=&-\frac{1}{2}FG^\lambda_{\nu\mu\rho}\ell_\mu\\
=&-\frac{ce^{2x^1}e^{\frac{2ay^1}{ay^1+\sqrt{ce^{2x^1}\varphi}}}}{2a}\left\{\frac{1}{\sqrt{\varphi}}(c_{\mu\rho}c_{\nu\delta}+c_{\mu\nu}c_{\rho\delta}+c_{\nu\rho}c_{\mu\delta})\right.\\
&-\frac{1}{\varphi^{3/2}}(c_{\mu\gamma}c_{\rho\sigma}c_{\nu\delta}y^\gamma y^\sigma+c_{\mu\gamma}c_{\nu\sigma}c_{\rho\delta}y^\gamma y^\sigma+c_{\nu\gamma}c_{\rho\sigma}c_{\mu\delta}y^\gamma y^\sigma)\\
&\left.-\frac{1}{\varphi^{3/2}}(c_{\mu\nu}c_{\rho\sigma}y^\sigma+c_{\mu\rho}c_{\nu\sigma}y^\sigma+c_{\rho\nu}c_{\mu\sigma}y^\sigma)c_{\lambda\delta}y^\lambda+\frac{3}{\varphi^{5/2}}c_{\nu\gamma}c_{\mu\sigma}c_{\rho\xi}c_{\lambda\delta}y^\gamma y^\sigma y^\xi y^\lambda\right\}y^\delta\\
=&-\frac{ce^{2x^1}e^{\frac{2ay^1}{ay^1+\sqrt{ce^{2x^1}\varphi}}}}{2a}\left\{\frac{1}{\sqrt{\varphi}}(c_{\mu\rho}c_{\nu\delta}+c_{\mu\nu}c_{\rho\delta}+c_{\nu\rho}c_{\mu\delta})y^\delta\right.\\
&-\frac{1}{\varphi^{3/2}}(c_{\mu\gamma}c_{\rho\sigma}c_{\nu\delta}y^\gamma y^\sigma+c_{\mu\gamma}c_{\nu\sigma}c_{\rho\delta}y^\gamma y^\sigma+c_{\nu\gamma}c_{\rho\sigma}c_{\mu\delta}y^\gamma y^\sigma)y^\delta\end{aligned}$$

$$-\frac{1}{\sqrt{\varphi}}(c_{\mu\nu}c_{\rho\sigma}y^{\sigma}+c_{\mu\rho}c_{\nu\sigma}y^{\sigma}+c_{\rho\nu}c_{\mu\sigma}y^{\sigma})+\frac{3}{\varphi^{3/2}}c_{\nu\gamma}c_{\mu\sigma}c_{\rho\xi}y^{\gamma}y^{\sigma}y^{\xi}\Big\}$$

$$=0.$$

Keeping in mind that $\delta^{\mu}_{\mu}=n-1$, the mean Berwald curvature $E_{ij}:=G^{k}_{ijk}$ is given by $E_{1j}=0$, and

$$E_{\nu\mu}=G^{\lambda}_{\nu\mu\lambda}$$

$$=-\frac{c\,e^{2x^1}\,e^{\frac{2ay^1}{ay^1+\sqrt{c\,e^{2x^1}\varphi}}}}{2a}\Big\{\frac{n+1}{\sqrt{\varphi}}c_{\mu\nu}-\frac{n+1}{\varphi^{3/2}}c_{\mu\gamma}c_{\nu\sigma}y^{\gamma}y^{\sigma}) \tag{36}$$

$$-\frac{1}{\varphi^{3/2}}(\varphi c_{\mu\nu}+2c_{\mu\lambda}c_{\nu\sigma}y^{\lambda}y^{\sigma})+\frac{3}{\varphi^{3/2}}c_{\nu\gamma}c_{\mu\sigma}y^{\gamma}y^{\sigma}\Big\}.$$

Now, $c^{\nu\mu}E_{\nu\mu}$ can be calculated by

$$c^{\nu\mu}E_{\nu\mu}=-\frac{c\,e^{2x^1}\,e^{\frac{2ay^1}{ay^1+\sqrt{c\,e^{2x^1}}}}}{2a}\Big\{(n^2-1)-(n+1)-(n+1)+3\Big\}=-n(n-2)\frac{c\,e^{2x^1}\,e^{\frac{2ay^1}{ay^1+\sqrt{c\,e^{2x^1}}}}}{2a}.$$

Which is zero only in dimension $n=2$. Since $n\geq 3$, then F is not Berwaldian. That is, the proof is completed. □

The next theorem introduces a second class of solutions. Its proof follows a similar approach to that used in Theorem 1. Moreover, for the geodesic spray calculations, we employ the parameter choices $c_1=2a$ and $c_3=a^2-2$ into (28).

Theorem 2 Let $(M,\ F)$ be an n-dimensional non-regular Finsler manifold where $n\geq 3,\ a\neq 0,\ \pm 1$ and

$$F=\left((a+1)y^1+\sqrt{c\,e^{2x^1}\varphi(\hat{y})}\right)^{(1+a)/2}\left((a-1)y^1+\sqrt{c\,e^{2x^1}\varphi(\hat{y})}\right)^{(1-a)/2} \tag{37}$$

Then $(M,\ F)$ is a non-regular Landsberg metric which is not Berwaldian. In addition, the geodesic spray of F is given by

$$G^1=-\frac{1}{2(a^2-1)}c\,e^{2x^1}\varphi(\hat{y}),\qquad G^{\mu}=Py^{\mu},\qquad P=\left(y^1+\frac{a}{a^2-1}\sqrt{c\,e^{2x^1}\varphi(\hat{y})}\right), \tag{38}$$

where $\varphi(\hat{y})$ is given by (18).

The following theorem presents a third class of solutions. The proof proceeds analogously to that of Theorem 1, using the specific parameter values $c_1=\dfrac{3a}{2}$ and $c_3=\dfrac{a^2-2}{2}$.

Theorem 3 Let $(M,\ F)$ be an n-dimensional non-regular Finsler manifold where $n\geq 3,\ a\neq 0$ and

$$F = ay^1 + \frac{c\,e^{2x^1}\varphi(\hat{y})}{ay^1 + 2\sqrt{c\,e^{2x^1}\varphi(\hat{y})}}. \tag{39}$$

Then $(M,\ F)$ is a non-regular Landsberg non-Berwaldian metric. Moreover, the geodesic spray of F is given by

$$G^1 = -\frac{1}{a^2}c\,e^{2x^1}\varphi(\hat{y}), \quad G^\mu = Py^\mu, \quad P = \left(y^1 + \frac{3}{2a}\sqrt{c\,e^{2x^1}\varphi(\hat{y})}\right)y^\mu, \tag{40}$$

where $\varphi(\hat{y})$ is given by (18).

The following theorem presents the fourth and last class of solutions discussed in this work. The proof proceeds analogously to that of Theorem 1, but for the general values for the constants $c_1 = p$ and $c_3 = q$.

Theorem 4 Let $(M,\ F)$ be an n-dimensional non-regular Finsler manifold where $n \geq 3$, and

$$F = \sqrt{\alpha^2 + p\beta\sqrt{\alpha^2 - \beta^2} + q\beta^2}\, e^{\frac{p}{\sqrt{p^2-4q-4}} \operatorname{arctanh}\left(\frac{p\beta + 2\sqrt{\alpha^2-\beta^2}}{\beta\sqrt{p^2-4q-4}}\right)}, \tag{41}$$

where α and β are given by the special choice (26), and p, q are arbitrary constants. Then $(M,\ F)$ is a non-regular Landsberg non-Berwaldian metric. Additionally, the geodesic spray is given by

$$G^1 = -\frac{1}{2(1+q)}c\,e^{2x^1}\varphi(\hat{y}), \quad G^\mu = Py^\mu, \quad P = \left(y^1 + \frac{p}{2(1+q)}\sqrt{c\,e^{2x^1}\varphi(\hat{y})}\right), \tag{42}$$

where $\varphi(\hat{y})$ is given by (18).

Proof. One can see that the geodesic spray coefficients G^i, by (28), are given as follows

$$G^1 = -\frac{c\,e^{2x^1}\varphi}{2(1+q)}, \quad G^\mu = Py^\mu, \quad P = \left(y^1 + \frac{p}{2(1+q)}\sqrt{c\,e^{2x^1}\varphi}\right). \tag{43}$$

Moreover, the components $G^\lambda_{\nu\mu\rho}$ are given by

$$\begin{aligned} G^\lambda_{\nu\mu\rho} = \frac{p\sqrt{ce^{2x^1}}}{2(1+q)}\Bigg\{ & \frac{1}{\sqrt{\varphi}}(c_{\mu\rho}\delta^\lambda_\nu + c_{\mu\nu}\delta^\lambda_\rho + c_{\nu\rho}\delta^\lambda_\mu) \\ & - \frac{1}{\varphi^{3/2}}(c_{\mu\gamma}c_{\rho\sigma}\delta^\lambda_\nu y^\gamma y^\sigma + c_{\mu\gamma}c_{\nu\sigma}\delta^\lambda_\rho y^\gamma y^\sigma + c_{\nu\gamma}c_{\rho\sigma}\delta^\lambda_\mu y^\gamma y^\sigma) \\ & - \frac{1}{\varphi^{3/2}}(c_{\mu\nu}c_{\rho\sigma}y^\sigma + c_{\mu\rho}c_{\nu\sigma}y^\sigma + c_{\rho\nu}c_{\mu\sigma}y^\sigma)y^\lambda + \frac{3}{\varphi^{5/2}}c_{\nu\gamma}c_{\mu\sigma}c_{\rho\xi}y^\gamma y^\sigma y^\xi y^\lambda \Bigg\}. \end{aligned}$$

That is, the proof can be proceeded by following the argument as in the proof of Theorem 1. □

5. Concluding remarks

We end this work by the following remarks:

• For the sake of obtaining more examples of Landsberg metrics that are not Berwaldian, we focus our attention to the conformal transformation. By [9], a Landsberg metric F remains Landsberg under a conformal transformation $e^{\sigma(x)}F$ if and only the σ T-condition is satisfied, that is $\sigma_h T^h_{ijk} = 0$, where T^h_{ijk} is the T-tensor of F and $\sigma_h := \dfrac{\partial \sigma}{\partial x^h}$. Let us check the σ T-condition for the general class of solutions obtained here in this work. For the Finsler function $F = \alpha\phi(s)$ given in Theorem 4, the function $\phi(s)$ is given by

$$\phi(s) = \sqrt{1 + ps\sqrt{1-s^2} + qs^2}\, e^{\frac{p}{\sqrt{p^2-4q-4}} \operatorname{arctanh}\left(\frac{ps+2\sqrt{1-s^2}}{s\sqrt{p^2-4q-4}}\right)}. \tag{44}$$

Then, by (12), the function Q is calculated as follows

$$Q = p\sqrt{1-s^2} + qs. \tag{45}$$

Now, by [10, Proof of Theorem 4.2], F satisfy the σ T-condition. Moreover, by [10, Theorem 3.3 (c)] the factor σ is given by an arbitrary function $f(x^1)$. That is, [9, Corollary 2.9] the conformal transformation of all classes obtained in this work by a factor $f(x^1)$ remains non-regular Landsberg and non-Berwaldian metrics.

• Some typos in [3] to be fixed and should be mentioned here:

(a) In [3, Theorem 4.7], there a term $f(x^1)$ is missing in the definition of the function P. The correct expression for P should be:

$$P = \left(y^1 + \frac{a}{a^2-1}\frac{\sqrt{\alpha^2-\beta^2}}{f(x^1)}\right)\frac{f'(x^1)}{f(x^1)}. \tag{46}$$

(b) Also, In [3, Theorem 4.10] there a term $f(x^1)$ is missing in the definition of the function P. The correct expression for P should be:

$$P = \left(y^1 + \frac{3}{2a}\frac{\sqrt{\alpha^2-\beta^2}}{f(x^1)}\right)\frac{f'(x^1)}{f(x^1)}.$$

(c) The object s_i is defined as $s_i = s_{ij}b^j$ in [3, Eq. (2.5)], but the correct definition is $s_j := s_{ij}b^i$.

(d) In [3, below Eq. (2.10)], the Riemannian metric α should be written in the form:

$$\alpha = \sqrt{e^{-2x^1}(y^1)^2 + ((y^2)^2 + (y^3)^2)}. \tag{47}$$

(e) We consider the general case mentioned in Remark 4.12. For simplicity, we replace c_1 (resp. c_2) with $\dfrac{c_1}{b^2}$ $\left(\text{resp. } \dfrac{c_2}{b^2}\right)$, and thus obtain

$$\frac{c_2 s + c_1\sqrt{b^2-s^2}}{b^2+c_2 s^2+c_1 s\sqrt{b^2-s^2}} = \frac{1}{2}\frac{2c_2 s\sqrt{b^2-s^2}-2c_1 s^2+c_1 b^2}{\sqrt{b^2-s^2}\left(b^2+c_2 s^2+c_1 s\sqrt{b^2-s^2}\right)} \tag{48}$$

$$+\frac{1}{2}\frac{c_1 b^2}{\sqrt{b^2-s^2}\left(b^2+c_2 s^2+c_2 s\sqrt{b^2-s^2}\right)} \tag{49}$$

Then, we have

$$\phi(s) = \exp\left(\int \frac{c_2 s + c_1\sqrt{b^2-s^2}}{b^2+c_2 s^2+c_1 s\sqrt{b^2-s^2}} ds\right)$$

$$= \sqrt{b^2+c_2 s^2+c_1 s\sqrt{b^2-s^2}}\, e^{\frac{c_1}{\sqrt{c_1^2-4c_1-4}} \operatorname{arctanh} \frac{c_1 s+2\sqrt{b^2-s^2}}{s\sqrt{c_1^2-4c_2-4}}}. \tag{50}$$

Acknowledgement

The author gratefully acknowledge the referees for their constructive comments and valuable suggestions, which have improved the quality of this paper.

Conflict of interest

The author declares no competing financial interests.

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