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Research Article

Collocation Method by Exponential Functions for Solving the System of Differential Equations and Error Bound

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Abstract: This paper introduces three novel numerical approaches for solving linear matrix differential equations. Leveraging exponential functions and collocation points, these methods transform the original problem into a system of algebraic equations through matrix operations. The first method employs negative exponential functions, the second adopts positive exponential functions, and the third combines both into extended exponential functions. Error analysis is provided, and an error problem formulated via the residual function is solved using the proposed techniques to estimate errors. Numerical examples demonstrate the effectiveness of the methods and the accuracy of the error estimation.

Keywords**:** exponential function, Linear Differential Matrix Equations (LDME), collocation points

MSC: 34A30, 41A10

1. Introduction

First-order linear matrix differential equations of the form:

$$W'(x) = A(x)W(x) + C(x), \quad W(x_0) = W_0, \tag{1}$$

arise in control theory (Lyapunov/Sylvester equations), quantum dynamics (master equations), and parametric model reduction. Their numerical solution demands:

- Structure preservation: Maintaining solution properties (e.g., symmetry, positivity)
- Dimensional scalability: Handling $n \times p$ matrix solutions efficiently

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• Stiffness robustness: Resolving wide spectral ranges in $A(x)$.

The first-order linear matrix differential equations play an important are widely used in Chemistry [1], Physics [2], and Engineering [3]. Some researchers have studied on development and implementation of the numerical methods to solve these equations, via minimal, regular, and excessive space extension-based universalization [4], bernoulli polynomials for solving Linear Differential Matrix Equations (LDME) [5], discretization method [6], higher-order matrix splines [7] and Krylov method [8]. One type of these methods in literature consists of the collocation method and the different collocation methods have been used for solving various differential equations in the literature; in [9] using the rational Chebyshev functions for nonlinear differential equations, in [10] using the bernstein polynomials for fractional riccati type differential equations, in [11] using the radial basis function for Rosenau-KdV-RLW equation, in [12] using the Bernstein polynomials for differential Lyapunov and Sylvester matrix equations, in [13] using Chebyshev spectral collocation method for a system of nonlinear Volterra integral equations, in [14] using Jacobi spectral collocation method for solving multi-dimensional nonlinear fractional sub-diffusion equations, in [15] using haar wavelet collocation method for solving one-dimensional fractional boundary value problems, in [16] using the piecewise polynomials for terminal value problems of tempered fractional differential equations, in [17] using the general form of Bernstein polynomials for ψ-fractional differential equations, in [18] using the shifted Legendre polynomials for multiterm variable-order fractional differential equations, in [19] using the Mittag-Leffler function for fractional Riccati differential equations, in [20] using the Pell-Lucas polynomials for the parabolic-type partial integro-differential equations, in [21] using the Bell function for linear fractional differential equations, in [22] using the Bell polynomials for nonlinear Fredholm-Volterra integral equations and in [23] using the Bell polynomials for the fractional optimal control problems and using the exponential function in [24–26].

Exponential functions or exponential functions have various application areas. For example, many optical and quantum electronics [27], Human Immunodeficiency Virus (HIV) infection model of CD4+ T-cells [28], describing biological species living together [29], a nonlinear phenomena model [30], various statistical studies [31], the analysis of the control problem [32], the study of spectral synthesis [33, 34] and the problem regarding mean-periodic functions [35]. In this work, we implemented the exponential function and two methods: the extended exponential function and positive exponential function methods.

The computational cost of the collocation methods changes depending on the choice of the collocation points and functions. In this paper, we developed a collocation to find numerical solutions in three different exponential bases of the first-order linear matrix differential Eq. (2). The numerical solutions will be computed in three forms of different using the negative, positive, and extended exponential functions. Also, we will determine which of them works better by comparing the results.

In this paper, we consider the first-order linear matrix differential equations:

$$\begin{cases} W'(x) = A(x)W(x) + C(x), \quad x \in [x_0,\, x_f], \\ W(x_0) = W_0, \end{cases} \tag{2}$$

where $W(x)$ is an unknown matrix function of size $n \times p$, the matrices $A(x) \in \mathbb{R}^{n\times n}$, $C(x),\ W_0 \in \mathbb{R}^{n\times p}$ are all given. For foundational details on such equations, see [1].

The aim of this paper is to apply the exponential collocation method in three different forms of the exponential functions to obtain sets of approximate solutions of the first-order linear matrix differential equations (2). There have been strong contributions that used the exponential method to solve many problems in various fields of science, and we would like to mention: Our purpose in this study is to present three new methods (negative Exponential Matrix (EM) method, positive EM method, and extended EM method), which are based on the exponential basis (14) and the collocation points, to obtain the approximate solution of the problem (2) [36]. Let $A = [a_{ij}] \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{l\times k}$. Their Kronecker product is the matrix $A \otimes B \in \mathbb{R}^{nl\times mk}$, defined by for each successive blocks of size $l \times k$, we have

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{pmatrix}.$$

Example For $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 5 \\ 6 & 7 \end{pmatrix}$, the Kronecker product $A \otimes B$ is:

$$A \otimes B = \begin{pmatrix} 1 \cdot \begin{pmatrix} 0 & 5 \\ 6 & 7 \end{pmatrix} & 2 \cdot \begin{pmatrix} 0 & 5 \\ 6 & 7 \end{pmatrix} \\ 3 \cdot \begin{pmatrix} 0 & 5 \\ 6 & 7 \end{pmatrix} & 4 \cdot \begin{pmatrix} 0 & 5 \\ 6 & 7 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 5 & 0 & 10 \\ 6 & 7 & 12 & 14 \\ 0 & 15 & 0 & 20 \\ 18 & 21 & 24 & 28 \end{pmatrix}.$$

We shall note that the Kronecker product gives the following important relation:

$$\operatorname{vec}(AXB) = (B^T \otimes A)\operatorname{vec}(X), \tag{3}$$

where the vector $\operatorname{vec}(X)$ is defined as

$$\operatorname{vec}(X) = [X_{11}, X_{21}, \ldots, X_{n1}, \ldots, X_{1s}, X_{2s}, \ldots, X_{ns}]^T \in \mathbb{R}^{ns}. \tag{4}$$

We can organize the rest of this paper as follows. In Section 2, we first review some definitions and properties of exponential polynomials and approximation of functions and derivatives for matrix exponential functions and approximation of functions that are used throughout this paper. In Section 3, we explain how three exponential collocation methods for solving problem (2). In Section 4, we present the planning method and iterative algorithm. In Section 5, we study the error analysis and estimate the error. Finally, Section 6 is devoted to numerical applications and comparisons to demonstrate the accuracy of three numerical schemes for solving Eq. (2). Section 7 gives the conclusions of the paper.

2. Derivative for matrix exponential function and approximation of functions

In this section, we want to determine an explicit formula for the derivative matrix exponential function, and we will outline some of the basic definitions and properties of the exponential functions and approximation of functions.

2.1 *Matrix of negative exponential bases*

The set of negative exponential functions is defined by

$$\{1,\ e^{-x},\ e^{-2x},\ e^{-3x},\ \ldots\}. \tag{5}$$

Yuzbasi et al. [37] have solved generalized pantograph equations with variable coefficients by using the collocation-matrix method, which is based on the negative exponential basis (5).

Let $N \in \mathbb{N}$. The space of negative exponential is defined by:

$$\mathbb{H}^{Neg} := \text{span}\{1,\ e^{-x},\ e^{-2x},\ e^{-3x},\ ...,\ e^{-Nx}\},\ \ 0 \le x < \infty. \tag{6}$$

Let $N \in \mathbb{N}$. The matrix of negative exponential bases is defined by:

$$E_{Neg,\ N}(x) := \begin{pmatrix} 1 \\ e^{-x} \\ \vdots \\ e^{-Nx} \end{pmatrix}. \tag{7}$$

Let the matrix $E_{Neg,\ N}(x)$ defined in Eq. (7). Then,

$$E'_{Neg,\ N}(x) = D_{Neg,\ N} E_{Neg,\ N}(x), \tag{8}$$

where $D_{Neg,\ N}$ is the matrix in dimensional $(N+1) \times (N+1)$ and it is defined by:

$$D_{Neg,\ N} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & -1 & \ddots & 0 \\ \vdots & 0 & \ddots & 0 \\ \vdots & \cdots & 0 & -N \end{bmatrix}. \tag{9}$$

2.2 *Matrix of positive exponential bases*

Let $N \in \mathbb{N}$. The space of the positive exponential is defined by:

$$\mathbb{H}^{Pos} := \text{span}\{1,\ e^{x},\ e^{2x},\ e^{3x},\ ...,\ e^{Nx}\},\ \ 0 \le x < \infty. \tag{10}$$

Let $N \in \mathbb{N}$. The matrix of the positive exponential is defined by:

$$E_{Pos,\ N}(x) := \begin{pmatrix} 1 \\ e^{x} \\ \vdots \\ e^{Nx} \end{pmatrix}. \tag{11}$$

Let the matrix $E_{Pos,\ N}(x)$ be defined in Eq. (11). Then,

$$E'_{Pos,\ N}(x) = D_{Pos,\ N} E_{Pos,\ N}(x), \tag{12}$$

where $D_{Pos,\,N}$, which is the matrix in dimension $(N+1)\times(N+1)$ is defined by

$$D_{Pos,\,N} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & 0 & \ddots & 0 \\ \vdots & \cdots & 0 & N \end{bmatrix}. \tag{13}$$

2.3 *Matrix of the extended exponential bases*

Let $N \in \mathbb{N}$. The space of the extended exponential is defined by:

$$\mathbb{H}^{Ext} := \text{span}\{1,\, e^{-x},\, e^{x},\, e^{-2x},\, e^{2x},\, e^{-3x},\, e^{3x},\, \ldots,\, e^{Nx},\, e^{-Nx}\},\ \ 0 \le x < \infty. \tag{14}$$

Let $N \in \mathbb{N}$. The matrix of the extended exponential bases is defined by:

$$E_{Ext,\,N}(x) = \begin{pmatrix} 1 \\ e^{-x} \\ \vdots \\ e^{-Nx} \\ e^{x} \\ \vdots \\ e^{Nx} \end{pmatrix}. \tag{15}$$

Theorem 1 Let the matrix $E_{Ext,\,N}(x)$ be defined in Eq. (15). Then,

$$E'_{Ext,\,N}(x) = D_{Ext,\,N} E_{Ext,\,N}(x), \tag{16}$$

where $D_{Ext,\,N}$ is the matrix in dimensional $(2N+1)\times(2N+1)$ and it is defined by:

$$D_{Ext,\,N} = \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & -1 & 0 & \ddots & \ddots & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & 0 & -N & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & N \end{bmatrix}. \tag{17}$$

Proof. We multiply the matrix $D_{Ext,\,N}$ by the matrix $E_{Ext,\,N}(x)$ from the right side, and thus we have the matrix $E'_{Ext,\,N}(x)$. □

2.4 *Approximation of functions*

Let the space $\mathbb{H}^k$, where k is Neg, Pos, or Ext, be the finite-dimensional vector space $\mathbb{H}^k \subset L^2[x_0,\ x_f]$ is a complete subspace of $L^2[x_0,\ x_f]$ (space of the measurable functions defined on the interval $[x_0,\ x_f]$). Then, for every $f \in L^2[x_0,\ x_f]$, there exists a unique $g \in \mathbb{H}^k$ such that

$$\|f-g\|_2 \leq \|f-y\|_2, \quad \forall y \in \mathbb{H}^k,$$

the function g is called the best approximation to f out of $\mathbb{H}^k$. As $g \in \mathbb{H}^{Neg}$, we can get that

$$f(x) \approx g(x) = \sum_{i=0}^{N} \mathrm{l}_i e^{-ix} = \mathbb{L}E_{Neg,\ N}(x), \tag{18}$$

where $\mathbb{L} = [\mathrm{l}_0,\ \mathrm{l}_1,\ \ldots,\ \mathrm{l}_N] \in \mathbb{R}^{1\times(N+1)}$. As $g \in \mathbb{H}^{Pos}$, we have

$$f(x) \approx g(x) = \sum_{i=0}^{N} \mathrm{l}_i e^{ix} = \mathbb{L}E_{Pos,\ N}(x), \tag{19}$$

where $\mathbb{L} = [\mathrm{l}_0,\ \mathrm{l}_1,\ \ldots,\ \mathrm{l}_N] \in \mathbb{R}^{1\times(N+1)}$. As $g \in \mathbb{H}^{Ext}$, we write

$$f(x) \approx g(x) = \sum_{i=-N}^{N} \mathrm{l}_i e^{ix} = \mathbb{L}E_{Ext,\ N}(x), \tag{20}$$

where $\mathbb{L} = [\mathrm{l}_0,\ \mathrm{l}_1,\ \ldots,\ \mathrm{l}_N,\ \mathrm{l}_{-1},\ \mathrm{l}_{-2},\ \ldots,\ \mathrm{l}_{-N}] \in \mathbb{R}^{1\times(2N+1)}$, and the coefficient l_i obtained by collocation method.

3. Exponential collocation method

In this section, we propose three main approaches to solve the first-order linear matrix differential equations: The negative EM method, the positive EM method, and the extended EM method created by the following numerical methods Neg, Pos and Ext, respectively. First all, we suppose k is Neg, Pos, or Ext. Let $W(x) = [W_{ij}(x)]_{n\times p}$ solution of Eq. (2). Consequently, from (18), (19) or (20), we have

$$W_{ij}(x) = \mathbb{L}_{ij}E_{k,\ N}(x),\ \ i = 1,\ \ldots,\ n,\ \ j = 1,\ \ldots,\ p. \tag{21}$$

with the unknown row matrices $\mathbb{L}_{ij} \in \mathbb{R}^{1\times(\widetilde{m}+1)}$ where $\widetilde{m} = 2N$ for Ext and $\widetilde{m} = N$ for Neg and Pos. We can write

$$
\begin{aligned}
W(x) &= \begin{pmatrix} W_{11}(x) & \dots & W_{1p}(x) \\ \vdots & \ddots & \vdots \\ W_{n1}(x) & \dots & W_{np}(x) \end{pmatrix} \\
&\simeq \begin{pmatrix} \mathbb{L}_{11}E_{k,\,N}(x) & \dots & \mathbb{L}_{1p}E_{k,\,N}(x) \\ \vdots & \ddots & \vdots \\ \mathbb{L}_{n1}E_{k,\,N}(x) & \dots & \mathbb{L}_{np}E_{k,\,N}(x) \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{L}_{11} & \dots & \mathbb{L}_{1p} \\ \vdots & \ddots & \vdots \\ \mathbb{L}_{n1} & \dots & \mathbb{L}_{np} \end{pmatrix} \begin{pmatrix} E_{k,\,N}(x) & \dots & 0_{(\widetilde{m}+1)\times 1} \\ \vdots & \ddots & \vdots \\ 0_{(\widetilde{m}+1)\times 1} & \dots & E_{k,\,N}(x) \end{pmatrix} \\
&= \mathbb{C}\left(I_p \otimes E_{k,\,N}(x)\right), \\
&:= W_N(x),
\end{aligned} \tag{22}
$$

where $\mathbb{C} = \begin{pmatrix} \mathbb{L}_{11} & \dots & \mathbb{L}_{1p} \\ \vdots & \ddots & \vdots \\ \mathbb{L}_{n1} & \dots & \mathbb{L}_{np} \end{pmatrix} \in \mathbb{R}^{n\times p(\widetilde{m}+1)}$. Using equation (8), (12) or (16), we have

$$
W'(x) \simeq \mathbb{C}\left(I_p \otimes D_{k,\,N}E_{k,\,N}(x)\right). \tag{23}
$$

By (22) and (23) in equation (2), we derive

$$
\mathbb{C}\left(I_p \otimes D_{k,\,N}E_{k,\,N}(x)\right) = A(x)\mathbb{C}\left(I_p \otimes E_{k,\,N}(x)\right) + C(x) + R_N(x), \tag{24}
$$

where $R_N(x)$ is residual.

3.1 *Choice of collocation points*

The collocation points η_i are chosen as the Chebyshev-Gauss-Lobatto nodes in $[x_0,\, x_f]$ are defined by:

$$
\eta_i = \frac{x_f - x_0}{2}\left(\cos\frac{(2i-1)\pi}{2N} + 1\right) + x_0, \quad i = 1,\, \dots,\, N. \tag{25}
$$

This non-uniform distribution is preferred over equidistant points for three key reasons:

1. Spectral Accuracy: Chebyshev points minimize Runge's phenomenon, enabling exponential convergence for smooth solutions [38]. For analytic solutions (as in Examples 1-4), they typically achieve machine precision with fewer points than equidistant grids.

2. Stability: The clustering of points near the interval endpoints (x_0 and x_f) provides better resolution of boundary layer effects that may occur in matrix differential equations [39]. This is particularly advantageous when $A(x)$ has stiff components.

3. Optimal Interpolation: Chebyshev points are roots of orthogonal polynomials, minimizing the Lebesgue constant and ensuring numerically stable interpolation [40]. This property is crucial when approximating matrix-valued functions through exponential bases.

From condition

$$R_N(\eta_i) = 0_{n\times p}, \ \ 1 \le i \le N,$$

the equations (24) and (25), we obtain the coupled matrix equations

$$\mathbb{C}\mathscr{C}_i = \mathscr{D}_i\mathbb{C}\mathscr{E}_i + \mathscr{G}_i, \quad i = 1,\ 2,\ \ldots,\ N,$$

where

$$\mathscr{C}_i = I_p \otimes D_{k,\ N}E_{k,\ N}(\eta_i), \ \ \mathscr{D}_i = A(\eta_i), \ \ \mathscr{E}_i = I_{pk,\ N}(\eta_i),$$

and

$$\mathscr{G}_i = C(\eta_i).$$

Since from the initial condition we set $\mathbb{C}\left(I_p \otimes E_{k,\ N}(x_0)\right) = W(x_0)$ and define $\eta_0 = x_0$,

$$\begin{cases} \mathscr{C}_0 = 0_{p(\widetilde{m}+1)\times p}, \\ \mathscr{D}_0 = I_n, \\ \mathscr{E}_0 = I_p \otimes E_{k,\ N}(x_0), \\ \mathscr{G}_0 = -W_0. \end{cases}$$

Therefore

$$\mathbb{C}\mathscr{C}_i - \mathscr{D}_i\mathbb{C}\mathscr{E}_i = \mathscr{G}_i, \tag{26}$$

for $i = 0,\ 1,\ \ldots,\ N$. Using the Kronecker products relation (3) and the operator vec. The coupled matrix equations (26) are transformed into the linear system:

$$\mathbb{A}x = \mathbb{B}, \tag{27}$$

where

$$\mathbb{A} = \begin{bmatrix} \mathscr{A}_0 \\ \mathscr{A}_1 \\ \vdots \\ \mathscr{A}_N \end{bmatrix} \text{ and } \mathbb{B} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_N \end{bmatrix},$$

with the $\mathscr{A}_i$ and b_i are:

$$\begin{cases} b_i = \operatorname{vec}(\mathscr{G}_i) \in \mathbb{R}^{np\times 1}, \\ \mathscr{A}_i = \mathscr{H}_i^T \otimes I_n - \mathscr{E}_i^T \otimes \mathscr{D}_i \in \mathbb{R}^{np\times np(\widetilde{m}+1)}, \\ x = \operatorname{vec}(\mathbb{C}) \in \mathbb{R}^{np(\widetilde{m}+1)\times 1}. \end{cases}$$

A solution to (27) exists and is unique when: For the collocation system (27) with $N \geq \widetilde{m}+1$:

1. If $\operatorname{rank}(\mathbb{A}) = np(\widetilde{m}+1)$, there exists a unique solution.
2. If $\operatorname{rank}(\mathbb{A}) < np(\widetilde{m}+1)$, solutions exist iff $\mathbb{B} \in \operatorname{range}(\mathbb{A})$.

The first statement follows from the rank-nullity theorem. For the second, the Fredholm alternative guarantees existence when $\mathbb{B}$ is orthogonal to $\ker(\mathbb{A}^T)$. In practice, the exponential basis guarantees $\operatorname{rank}(\mathbb{A}) = np(\widetilde{m}+1)$ for distinct collocation points.

4. Implementation method

In this section, we plan a method for solving the first-order linear matrix differential equations (2). We employ the step-by-step method for solving (2) for $[x_0,\ x_f]$. To do so, we choose $s \neq 0$, begining with $x_0 := x_0$, $\mathbb{Z}_0 := W(x_0)$ and using the collocation points $x_i = x_0 + is, i = 1,\ 2,\ 3,\ \ldots$. On each subinterval $[x_i,\ x_{i+1})$. For $i = 0,\ 1,\ \ldots,\ \left[\dfrac{x_f - x_0}{s}\right] - 1$, by solving the following first-order linear matrix differential equations:

$$\begin{cases} \mathbb{Z}'(x) = A(x)\mathbb{Z}(x) + C(x), \quad x_i \leq t < x_{i+1}, \\ \mathbb{Z}(x_i) = \mathbb{Z}_i. \end{cases}$$

Hence, we approximate $W(x)$ by $\mathbb{Z}(x)$. Then, to find the approximate solution $\mathbb{Z}(x)$ of $W(x)$ on the next subinterval, we set $\mathbb{Z}_{i+1} = \mathbb{Z}(x_{i+1})$.

We summarize the above method to solve the first-order linear matrix differential equations for interval $[a,\ b]$ to long $h \in [0,\ 1]$ in the following algorithm.

Algorithm 1 The exponential collocation method for the first-order linear matrix differential equations $(a,\ h,\ \mathbb{Z}_0,\ m)$

Inputs: $A(x)$ an $n \times n$ matrix, $\mathbb{Z}_0$ an $n \times p$ matrix, $C(x)$ an $n \times p$ matrix and a.

1. Choose an integer m.
2. $b = a + h$
3. Compute $\mathbb{A}$ and $\mathbb{B}$.
4. Compute x_i solve the linear system equation (27).
5. $\mathbb{C}_i = \operatorname{vec}^{-1}(x_i)$.
6. $\mathbb{Z}(x) = \mathbb{C}_i \left(I_p \otimes B_{k,\ N}(x)\right)$.
7. End

4.1 *Computational complexity analysis*

The computational cost of Algorithm 4 depends critically on the basis size parameter N. We analyze each step's complexity using the Landau notation $\mathscr{O}(\cdot)$:

The complexity derives from:

1. The matrix $\mathbb{A}$ has Nnp rows and $np(\widetilde{m}+1)$ columns, where $\widetilde{m}=N$ (negative/positive bases) or $2N$ (extended basis). Its construction involves:

• $\mathscr{O}(N)$ evaluations of $E_{k,\,N}(\eta_i)$ and $D_{k,\,N}E_{k,\,N}(\eta_i)$,

• $\mathscr{O}(N^2p^2n^2)$ operations for Kronecker products in $\mathscr{A}_i$.

2. Solving the linear system (27) has complexity:

$$\mathscr{O}(\min\{r^2c,\, rc^2\}), \quad r=Nnp,\; c=np(\widetilde{m}+1),$$

with standard dense solvers. For $N>\widetilde{m}+1$, iterative methods with preconditioning reduce this to $\mathscr{O}(N^2p^2n^2(\widetilde{m}+1)^2)$.

3. Error estimation requires solving a smaller system of size $\mathscr{O}(Nnp\times npM)$, where $M=N+1$, giving $\mathscr{O}(N^2p^2n^2)$.

5. Error analysis

In this section, we analyse the error $\mathscr{E}_N(x)=W(x)-W_N(x)$. Our approximation solutions $W_N(x)$. Here, $W(x)$ and $W_N(x)$ represent the sets of exact solutions and our approximation solutions of the first-order linear matrix differential equations (2). This section includes the upper bound of error and the estimated error.

5.1 *Error estimation and correction procedure*

In this section, the error of the solution is estimated and also the solutions are improved with the aid of this error estimation. $\mathscr{E}_N(x)=W(x)-W_N(x)$ shows the actual error function.

Theorem 1 Let $W(x)$ and $W_N(x)$ be the exact solution and the approximate solution of (2). The error $\mathscr{E}_N(x)$ satisfies the problem

$$\begin{cases} \mathscr{E}_N'(x)=A(x)\mathscr{E}_N(x)-R_N(x), \\ \mathscr{E}_N(x_0)=W_0-W_N(x_0). \end{cases} \tag{28}$$

Proof. From (24), we can write

$$R_N(x)=W_N'(x)-A(x)W_N(x)-C(x),$$

and from (2), we know $W'(x)=A(x)W(x)+C(x)$. Hence we get

$$\begin{aligned}\mathscr{E}_N'(x)&=W'(x)-W_N'(x)\\ &=A(x)W(x)+C(x)-A(x)W_N(x)-C(x)-R_N(x)\\ &=A(x)(W(x)-W_N(x))-R_N(x)\end{aligned}$$

$$= A(x)\mathscr{E}_N(x) - R_N(x).$$

For $t = x_0$, we have $\mathscr{E}_N(x_0) = W(x_0) - W_N(x_0) = W_0 - W_N(x_0)$. Thus, the proof is finished. □

Corollary 1 We assume that the solution of the error problem (28) for truncated limited M by the presented method is $\mathscr{E}_{N,\, M}(x)$. Then the solution $W_N(x)$ can be improved by summing $\mathscr{E}_{N,\, M}(x)$. Hence, we have an improved solution as $W_{N,\, M}(x) = W_N(x) + \mathscr{E}_{N,\, M}(x)$.

When the exact solution of Eq. (2) is not available, we can evaluate the precision and quality of our computed solution by using the solution of the error equation. These metrics are used to assess how well our computed solution approximates the exact solution.

5.2 *Upper boundary of error*

In [41], the unique solution of the Eq. (28) is defined by

$$\mathscr{E}_N(x) = \Phi_A(x,\, x_0)\,\mathscr{E}_0 + \int_{x_0}^{x} \Phi_A(x,\, \tau) R_N(\tau) d\tau,$$

where the transition matrix $\Phi_A(x,\, x_0) = e^{\int_{x_0}^{x} A(s)ds}$. Let $A_0 = \left[\max_{\tau \in [x_0,\, x]} |A_{ij}(\tau)|\right]$ and the 2-logarithmic norm of the matrix A_0 is $\mu_2(A_0) = \dfrac{\lambda_{max}(A_0 + A_0^T)}{2}$ with $\mu_2(A_0) \neq 0$. Then, the following inequality is obtained:

$$\|\mathscr{E}_N(x)\|_2 \leq e^{(x-x_0)\mu_2(A_0)} \|\mathscr{E}_0\|_2 + \sigma_N(x) \frac{e^{(x-x_0)\mu_2(A_0)} - 1}{\mu_2(A_0)}. \tag{29}$$

where $\sigma_N(x) = \max_{\tau \in [x_0,\, x]} \|R_N(\tau)\|_2$.

Proof. We have

$$\begin{aligned}
\|\mathscr{E}_N(x)\|_2 &= \|\Phi_A(x,\, x_0)\,\mathscr{E}_0 + \int_{x_0}^{x} \Phi_A(x,\, \tau) R_N(\tau) d\tau\|_2 \\
&\leq \|\Phi_A(x,\, x_0)\,\mathscr{E}_0\|_2 + \|\int_{x_0}^{x} \Phi_A(x,\, \tau) R_N(\tau) d\tau\|_2 \\
&\leq \|e^{\int_{x_0}^{x} A(s)ds}\mathscr{E}_0\|_2 + \|\int_{x_0}^{x} e^{\int_{x_0}^{\tau} A(s)ds} R_N(\tau) d\tau\|_2 \\
&\leq \|e^{\int_{x_0}^{x} A(s)ds}\|_2 \|\mathscr{E}_0\|_2 + \int_{x_0}^{x} \|e^{\int_{\tau}^{x} A(s)ds}\|_2 \|R_N(\tau) d\tau\|_2 \\
&\leq \|e^{\int_{x_0}^{x} A_0 ds}\|_2 \|\mathscr{E}_0\|_2 + \int_{x_0}^{x} \|e^{\int_{\tau}^{x} A_0 ds}\|_2 \|R_N(\tau) d\tau\|_2 \\
&\leq \|e^{(x-x_0)A_0}\|_2 \|\mathscr{E}_0\|_2 + \int_{x_0}^{x} \|e^{(x-\tau)A_0}\|_2 \|R_N(\tau) d\tau\|_2,
\end{aligned}$$

as $\|e^{xA_0}\|_2 \leq e^{\mu_2(A_0)x}$, so

$$\|\mathscr{E}_N(x)\|_2^{(x-x_0)\mu_2(A_0)}\|\mathscr{E}_0\|_2 + \int_{x_0}^{x} e^{(x-\tau)\mu_2(A_0)}\|R_N(\tau)d\tau\|_2$$

$$\leq e^{(x-x_0)\mu_2(A_0)}\|\mathscr{E}_0\|_2 + \max_{\tau\in[x_0,\ x]}\|R_N(\tau)\|_2 \int_{x_0}^{x} e^{(x-\tau)\mu_2(A)}d\tau$$

$$\leq e^{(x-x_0)\mu_2(A_0)}\|\mathscr{E}_0\|_2 + \max_{\tau\in[x_0,\ x]}\|R_N(\tau)\|_2 \frac{e^{(x-x_0)\mu_2(A)}-1}{\mu_2(A)}.$$

□

5.3 *Convergence and stability*

5.3.1 *Convergence conditions*

The exponential collocation methods converge under the following sufficient conditions:

Theorem 3 [Convergence] For the matrix ODE (2) with $A(x)$ Lipschitz continuous and $C(x) \in C^N[x_0,\ x_f]$, the approximate solution $W_N(x)$ satisfies:

$$\|W - W_N\|_\infty \leq \frac{C_A C_L^N h^{\min(N+1,\ p)}}{N!} + \mathscr{O}(\kappa(\mathbb{A})\varepsilon_{\text{mach}}),$$

where:

- $C_A = \sup_x \|A^{(N)}(x)\|$
- C_L: Lipschitz constant of $A(x)$
- h: Subinterval length
- p: Order of exact solution smoothness.

Proof. Combine the truncation error of the exponential series (Lemma 1) with the stability bound (Proposition 5) via Gronwall's inequality. □

5.3.2 *Numerical stability*

The methods exhibit conditional stability governed by:

Theorem 4 [Stability Criterion] Let $\mu_2(A(x))$ be the logarithmic 2-norm of $A(x)$. For step size h and basis order N, the scheme is stable when:

$$h\mu_2(A) \leq \frac{2N+1}{N^2}, \quad \text{(Negative/Positive Bases)}$$

$$h\mu_2(A) \leq \frac{1}{2N}, \quad \text{(Extended Basis)}$$

The extended basis requires stricter stability due to bidirectional exponential terms, but gains accuracy for oscillatory solutions (Example 3).

For stiff systems ($\mu_2(A) \gg 1$), the subdomain approach in Algorithm 1 with $h \sim \mu_2(A)^{-1}$ maintains both stability and accuracy.

6. Numerical experiments

In this section, we give the numerical applications of the negative EM method, positive EM method, and extended EM method (Algorithm 4) presented in this paper. Also, we compare our results with the results of other methods in [7, 42–44]. All the experiments were performed on a laptop with an Intel Core i3 processor and 4 GB of RAM, using software MATLAB 2020 b.

Example 1 Firstly, we consider the first-order linear matrix differential equations [7, 44]:

$$\begin{cases} W'(x) = A(x)W(x) + C(x), \quad x \in [0,\ 1], \\ W_0 = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}, \end{cases} \tag{30}$$

where

$$A(x) = \begin{pmatrix} 1 & -1 \\ 1 & e^x \end{pmatrix} \text{ and } C(x) = \begin{pmatrix} -3e^{-x} - 1 & 2 - 2te^{-x} \\ -3e^{-x} - 2 & 1 - 2\cosh(x) \end{pmatrix}, \tag{31}$$

with the exact solution

$$W(x) = \begin{pmatrix} 2e^{-x} + 1 & e^{-x} - 1 \\ e^{-x} & -1 \end{pmatrix}. \tag{32}$$

The obtained results of the negative EM method, positive EM method, and extended EM method are given in Table 1. This table shows that our method gives better results methods in [7, 43, 44]. In Figure 1, the approximate solution and exact solution of Example 1 for $N = 4$ is compared. Table 1 shows that the negative EM method is superior to the positive EM method and the extended EM method because the solution contains elements used in the negative EM method.

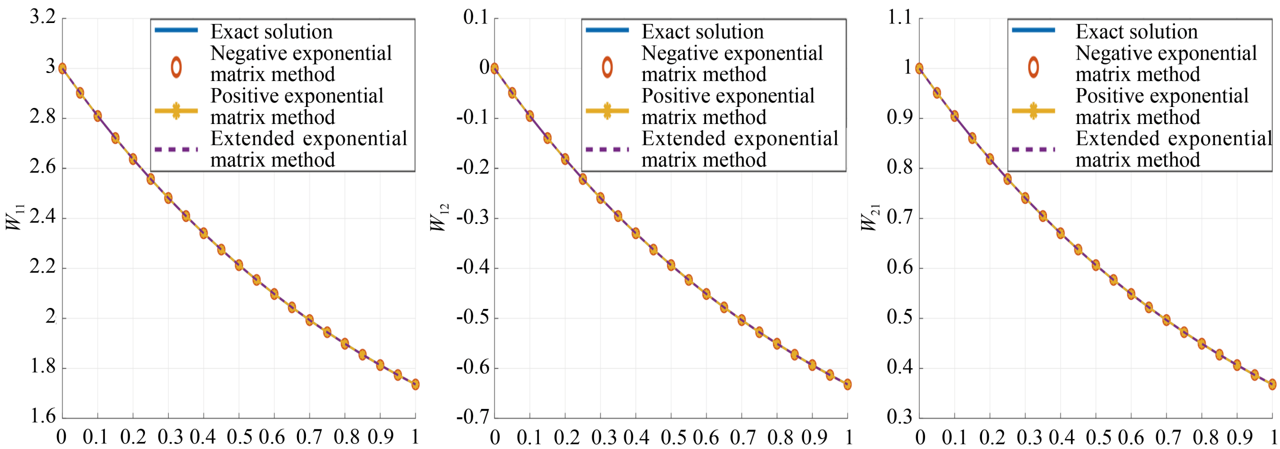


Figure 1. The approximate solutions for $N = 4$ and exact solution of Example 1

Table 1. Comparison of the methods in [7, 43, 44] and the proposed methods for (34)

Interval	$N=4$					
	Method [7]	Method [44]	Method [43]	Negative EM method	Positive EM method	Extended EM method
[0, 0.1]	5.06×10^{-8}	6.28×10^{-10}	5.52×10^{-11}	4.32×10^{-15}	7.67×10^{-8}	5.75×10^{-8}
[0.1, 0.2]	1.02×10^{-7}	1.22×10^{-9}	4.79×10^{-8}	4.21×10^{-15}	1.50×10^{-7}	1.28×10^{-7}
[0.2, 0.3]	1.54×10^{-7}	1.78×10^{-9}	8.25×10^{-8}	4.55×10^{-15}	2.20×10^{-7}	1.32×10^{-7}
[0.3, 0.4]	2.10×10^{-7}	2.98×10^{-9}	6.87×10^{-9}	5.55×10^{-15}	2.20×10^{-7}	1.32×10^{-7}
[0.4, 0.5]	2.70×10^{-7}	2.73×10^{-9}	7.16×10^{-8}	5.55×10^{-15}	2.86×10^{-7}	2.01×10^{-7}
[0.5, 0.6]	3.38×10^{-7}	3.09×10^{-9}	3.71×10^{-6}	5.66×10^{-15}	3.45×10^{-7}	3.32×10^{-7}
[0.6, 0.7]	4.19×10^{-7}	3.31×10^{-9}	4.36×10^{-5}	8.43×10^{-15}	3.95×10^{-7}	4.45×10^{-7}
[0.7, 0.8]	5.21×10^{-7}	3.41×10^{-9}	2.62×10^{-5}	9.10×10^{-15}	5.14×10^{-7}	5.47×10^{-7}
[0.8, 0.9]	6.59×10^{-7}	3.26×10^{-9}	1.07×10^{-4}	1.08×10^{-14}	6.98×10^{-7}	6.66×10^{-7}
[0.9, 1]	8.51×10^{-7}	2.80×10^{-9}	3.41×10^{-4}	1.09×10^{-14}	9.46×10^{-7}	8.42×10^{-7}
Interval	$N=5$					
	Method [7]	Method [44]	Method [43]	Negative EM method	Positive EM method	Extended EM method
[0, 0.1]	6.75×10^{-10}	2.79×10^{-12}	1.31×10^{-12}	5.66×10^{-15}	1.33×10^{-10}	1.89×10^{-10}
[0.1, 0.2]	1.36×10^{-9}	5.45×10^{-12}	6.57×10^{-10}	2.88×10^{-15}	2.74×10^{-10}	4.99×10^{-10}
[0.2, 0.3]	2.06×10^{-9}	7.94×10^{-12}	1.29×10^{-9}	1.88×10^{-15}	4.23×10^{-10}	4.99×10^{-10}
[0.3, 0.4]	2.80×10^{-9}	1.02×10^{-11}	6.60×10^{-11}	3.21×10^{-15}	4.23×10^{-10}	3.74×10^{-10}
[0.4, 0.5]	3.60×10^{-9}	1.22×10^{-11}	9.71×10^{-10}	4.44×10^{-15}	5.82×10^{-10}	3.69×10^{-10}
[0.5, 0.6]	4.50×10^{-9}	1.39×10^{-11}	1.30×10^{-7}	4.77×10^{-15}	7.51×10^{-10}	3.61×10^{-10}
[0.6, 0.7]	5.57×10^{-9}	1.50×10^{-11}	2.79×10^{-6}	5.88×10^{-15}	9.30×10^{-10}	3.31×10^{-10}
[0.7, 0.8]	6.93×10^{-9}	1.55×10^{-11}	2.66×10^{-5}	6.10×10^{-15}	1.12×10^{-9}	3.14×10^{-10}
[0.8, 0.9]	8.75×10^{-9}	1.50×10^{-11}	1.57×10^{-4}	7.38×10^{-15}	1.32×10^{-9}	3.84×10^{-10}
[0.9, 1]	1.13×10^{-8}	1.32×10^{-11}	6.83×10^{-4}	1.02×10^{-14}	1.52×10^{-9}	5.65×10^{-10}

Example 2 Secondly, let solve the first-order linear matrix differential equation [44]:

$$\begin{cases} W'(x) = A(x)W(x) + C(x), \quad x \in [0,\ 1], \\ W_0 = \begin{pmatrix} \frac{1}{8} & 0 \\ 1 & \frac{1}{8} \end{pmatrix}, \end{cases} \tag{33}$$

where

$$A(x) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \quad C(x) = \begin{pmatrix} c_{11}(x) & 0 \\ -1 & c_{22}(x) \end{pmatrix},$$

$c_{11}(x)$ and $c_{22}(x)$ are defined as:

$$c_{11}(x)=\begin{cases} -\dfrac{(1-2x)^2}{8}(2x^2-6-x),\ x\geq\dfrac{1}{2}, \\ \dfrac{(1-2x)^2}{8}(2x^2-6-x),\ x<\dfrac{1}{2}, \end{cases}$$

$$c_{22}(x)=\begin{cases} -\dfrac{(1-2x)^2}{8}((2x-7)\cos(x)+(2x-1)\sin(x)),\ x\geq\dfrac{1}{2}, \\ \dfrac{(1-2x)^2}{8}((2x-7)\cos(x)+(2x-1)\sin(x)),\ x<\dfrac{1}{2}, \end{cases}$$

and also the exact solution

$$W(x)=\begin{pmatrix} w_{11}(x) & 0 \\ 1 & w_{22}(x) \end{pmatrix}, \tag{34}$$

in which

$$w_{11}(x)=\begin{cases} \left(x-\dfrac{1}{2}\right)^3,\ x\geq\dfrac{1}{2}, \\ \left(\dfrac{1}{2}-x\right)^3,\ x<\dfrac{1}{2}, \end{cases}$$

and

$$w_{22}(x)=\begin{cases} \left(x-\dfrac{1}{2}\right)^3\cos(x),\ x\geq\dfrac{1}{2}, \\ \left(\dfrac{1}{2}-x\right)^3\cos(x),\ x<\dfrac{1}{2}. \end{cases}$$

In Figure 2, we compare the approximate solutions with the exact solutions in the interval $[0,\ 1]$. In Table 2 and Figure 3, 4, 5, the results of our methods (with negative, positive, and extended bases) and the method in [44] are compared for $N=4$ and $N=5$. From these comparisons, it is observed that our results are good. Also, it is seen from Figure 4 that the estimation error function is almost the actual error function.

Table 2. Comparison of the absolute errors of our methods and the method[44] for problem (37)

Interval	$N = 4$			
	Method [44]	Negative EM method	Positive EM method	Extended EM method
[0, 0.1]	7.98×10^{-9}	3.10×10^{-8}	7.87×10^{-8}	9.45×10^{-9}
[0.1, 0.2]	1.28×10^{-8}	6.55×10^{-8}	1.51×10^{-7}	8.89×10^{-8}
[0.2, 0.3]	1.46×10^{-8}	1.00×10^{-7}	2.18×10^{-7}	1.33×10^{-7}
[0.3, 0.4]	1.76×10^{-8}	1.00×10^{-7}	2.18×10^{-7}	1.33×10^{-7}
[0.4, 0.5]	1.65×10^{-8}	1.32×10^{-7}	2.83×10^{-7}	2.18×10^{-7}
[0.5, 0.6]	1.70×10^{-8}	1.66×10^{-7}	3.46×10^{-7}	3.22×10^{-7}
[0.6, 0.7]	3.31×10^{-8}	1.76×10^{-7}	3.22×10^{-7}	3.67×10^{-7}
[0.7, 0.8]	9.40×10^{-8}	2.16×10^{-7}	3.07×10^{-7}	4.28×10^{-7}
[0.8, 0.9]	7.81×10^{-8}	2.80×10^{-7}	2.99×10^{-7}	5.01×10^{-7}
[0.9, 1]	1.06×10^{-7}	3.75×10^{-7}	3.00×10^{-7}	5.82×10^{-7}
Interval	$N = 5$			
	Method [44]	Negative EM method	Positive EM method	Extended EM method
[0, 0.1]	1.51×10^{-10}	1.17×10^{-10}	1.93×10^{-10}	1.64×10^{-10}
[0.1, 0.2]	3.73×10^{-10}	2.33×10^{-10}	3.63×10^{-10}	3.70×10^{-10}
[0.2, 0.3]	5.90×10^{-10}	3.33×10^{-10}	5.15×10^{-10}	4.60×10^{-10}
[0.3, 0.4]	8.24×10^{-10}	3.33×10^{-10}	5.15×10^{-10}	4.60×10^{-10}
[0.4, 0.5]	1.07×10^{-9}	4.04×10^{-10}	6.55×10^{-10}	4.98×10^{-10}
[0.5, 0.6]	1.23×10^{-9}	5.22×10^{-10}	7.86×10^{-10}	5.33×10^{-10}
[0.6, 0.7]	1.19×10^{-9}	5.56×10^{-10}	7.47×10^{-10}	6.04×10^{-10}
[0.7, 0.8]	1.16×10^{-9}	7.70×10^{-10}	7.30×10^{-10}	6.95×10^{-10}
[0.8, 0.9]	1.16×10^{-9}	1.09×10^{-9}	7.34×10^{-10}	8.05×10^{-10}
[0.9, 1]	1.18×10^{-9}	1.53×10^{-9}	7.58×10^{-10}	9.25×10^{-10}

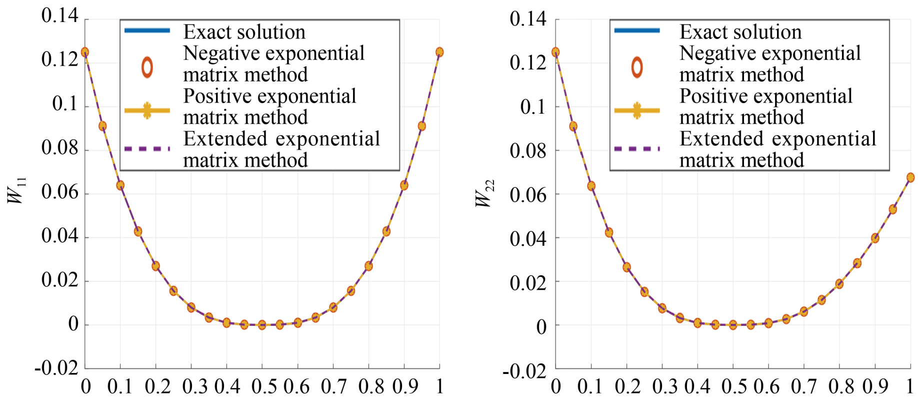


Figure 2. Comparisons of approximate solutions for $N = 4$ and exact solutions of the problem (33)

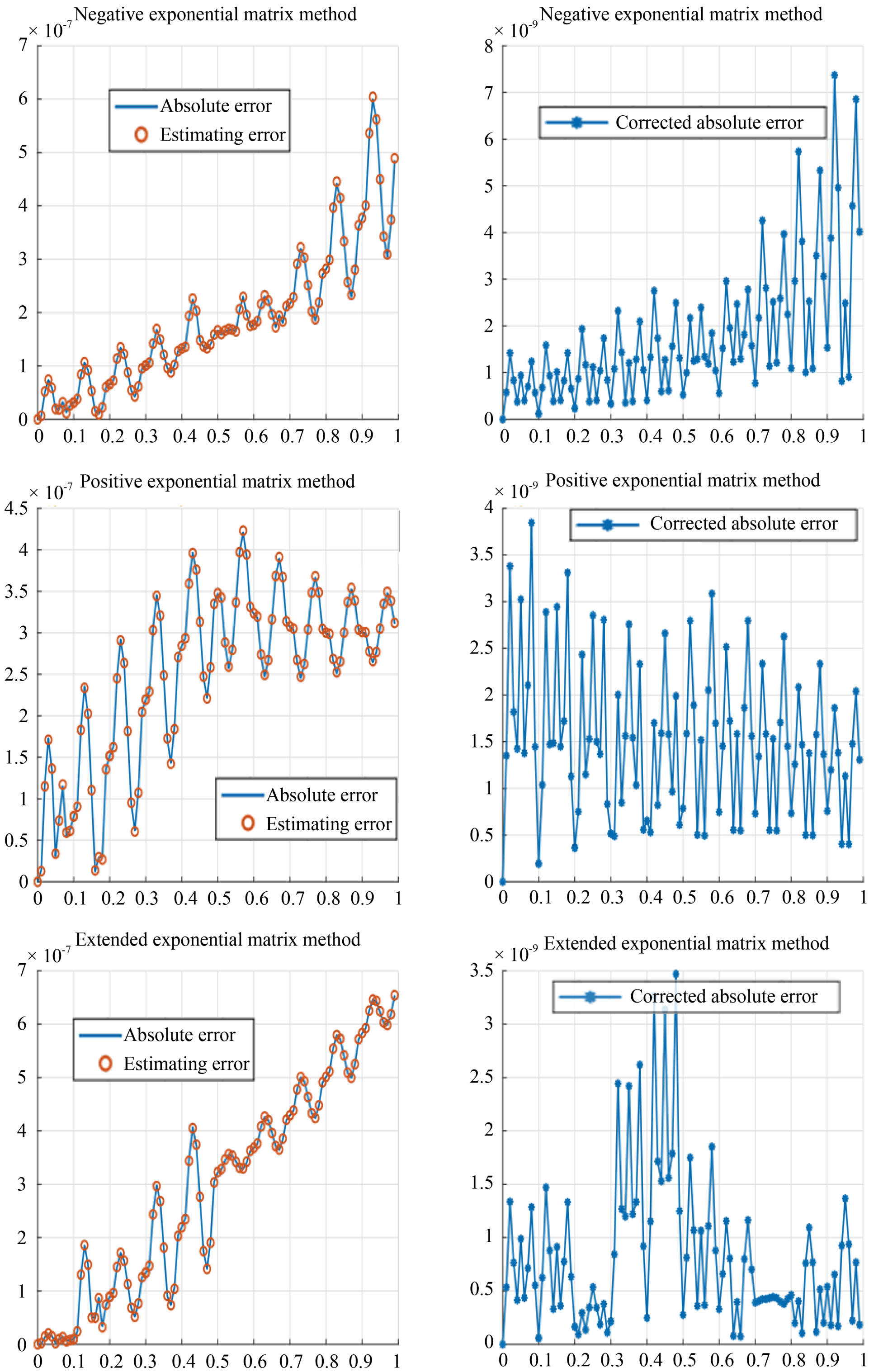


Figure 3. Comparison of the error functions (actual, estimation and corrected) in separate graphs for $N = 4$ and $M = N + 1$ of Example 2

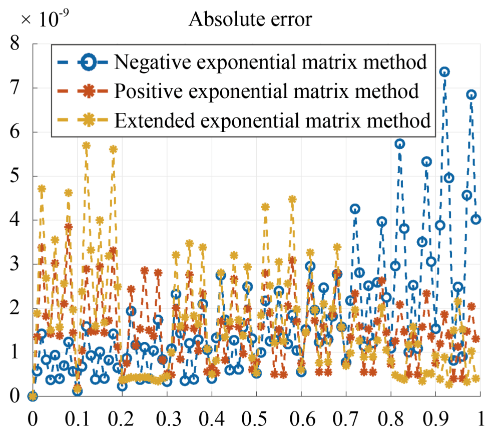
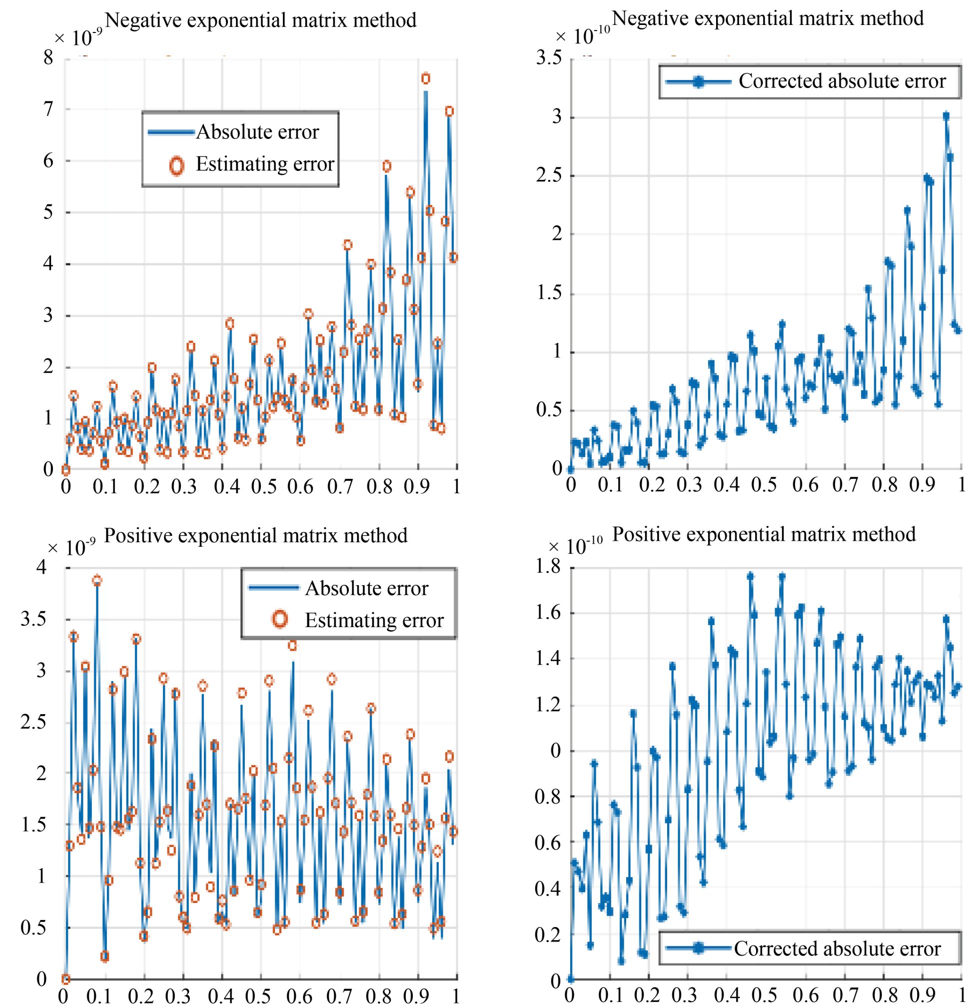


Figure 4. Comparison of the actual error functions (for negative, positive and extended bases) in single graph for $N = 5$ of Example 2



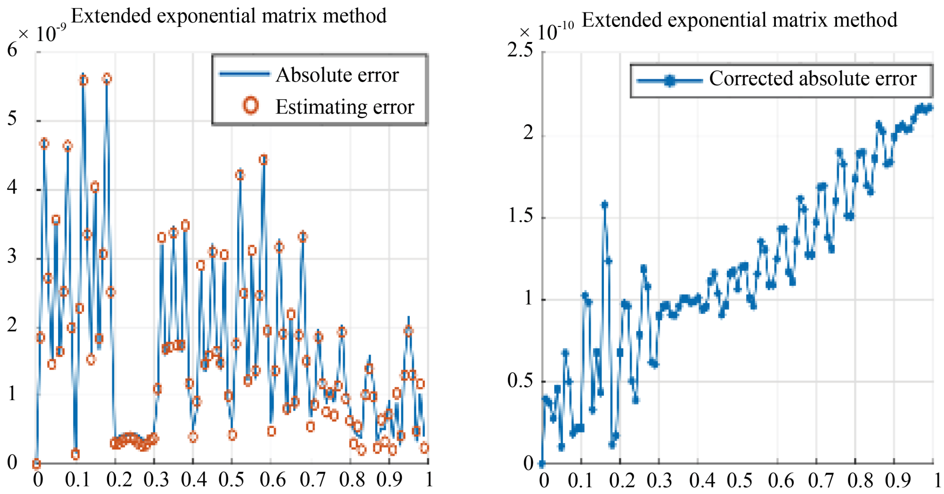


Figure 5. Comparison of the error functions (actual, estimation and corrected) in separate graph for $N = 5$ and $M = N + 1$ of Example 2

In Figure 6, we plot the absolute error (actual, estimation and bounded) in separate graphs for $N = 4$ of Example 3 where $A_0 := \left[\max_{\tau \in [0,\ 1]} |A_{ij}(\tau)|\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

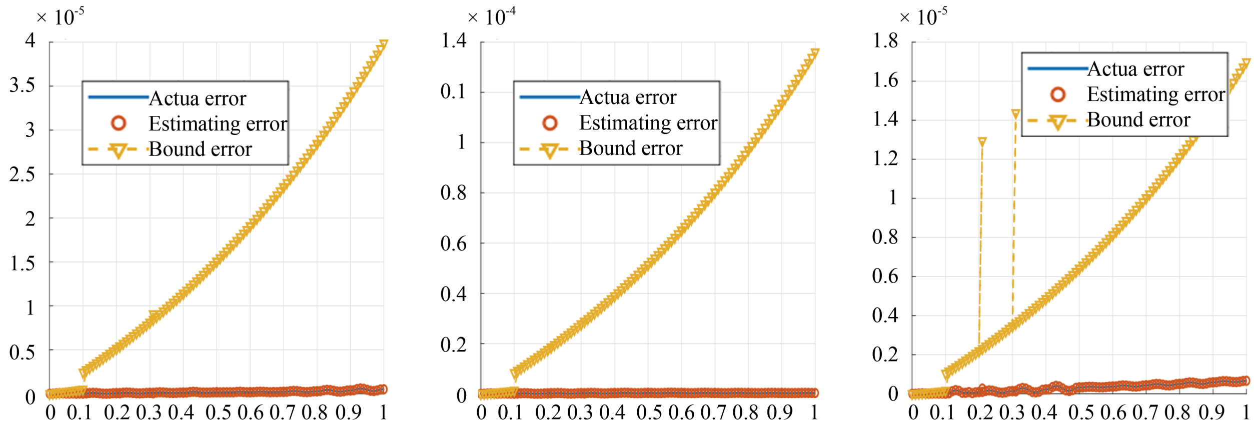


Figure 6. The absolutes error (actual, estimation and bound) in separate graphs for $N = 4$ and $M = N + 1$ of Example 2

In Figure 7, we compare the estimation error function is almost the actual error function for $N = 5$ and $M = N + 1$ of Example 2.

Example 3 Now let's consider the problem [7, 43, 44]:

$$\begin{cases} W'(x) = A(x)W(x), \quad x \in [0,\ 1], \\ W_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{cases} \tag{35}$$

where

$$A(x) = \frac{1}{x^3 - x - 1} \begin{pmatrix} 2x^2 - 1 & x^2 - 2x - 1 \\ -x - 1 & x^3 + x^2 - x - 1 \end{pmatrix}, \tag{36}$$

with the exact solution

$$W(x) = \begin{pmatrix} e^x \\ te^x \end{pmatrix}. \tag{37}$$

In Table 3, the comparison between the methods in [7, 44] and our methods for $N = 4$, $N = 5$ is presented. In Figure 7, the approximate solution for $N = 4$ is compared with the exact solution. Figure 8 and 9 show the comparisons of the error functions (actual, estimation, and corrected) of our methods (with positive, negative, and extended bases) in separate graphs for $N = 4$ and $N = 5$, respectively. Figure 10 and 11 show the comparisons of the actual error functions of our methods (with positive, negative, and extended bases) in the same graphics for $N = 4$ and $N = 5$, respectively. Also, it is seen from Figure 11 that the estimation error function is almost the same as the actual error function.

Table 3. Comparison of the methods in [7, 44] and our methods for the problem (39)

Interval	$N = 4$					
	Method [7]	Method [44]	Method [43]	Negative EM method	Positive EM method	Extended EM method
[0, 0.1]	1.14×10^{-7}	1.75×10^{-9}	2.19×10^{-10}	7.09×10^{-8}	1.44×10^{-9}	1.46×10^{-8}
[0.1, 0.2]	2.62×10^{-7}	3.97×10^{-9}	1.61×10^{-7}	1.64×10^{-7}	3.21×10^{-9}	2.91×10^{-8}
[0.2, 0.3]	4.51×10^{-7}	6.70×10^{-9}	3.45×10^{-7}	2.84×10^{-7}	5.36×10^{-9}	6.11×10^{-8}
[0.3, 0.4]	6.89×10^{-7}	1.01×10^{-8}	3.61×10^{-8}	2.84×10^{-7}	5.36×10^{-9}	6.11×10^{-8}
[0.4, 0.5]	9.89×10^{-7}	1.40×10^{-8}	4.02×10^{-8}	4.37×10^{-7}	7.95×10^{-9}	9.70×10^{-8}
[0.5, 0.6]	1.36×10^{-6}	1.90×10^{-8}	2.65×10^{-5}	6.31×10^{-7}	1.10×10^{-8}	1.33×10^{-7}
[0.6, 0.7]	1.82×10^{-6}	2.50×10^{-8}	3.98×10^{-5}	8.71×10^{-7}	1.47×10^{-8}	1.56×10^{-7}
[0.7, 0.8]	2.37×10^{-6}	3.30×10^{-8}	2.99×10^{-5}	1.16×10^{-6}	1.91×10^{-8}	1.66×10^{-7}
[0.8, 0.9]	3.05×10^{-6}	4.10×10^{-8}	1.52×10^{-5}	1.52×10^{-6}	2.43×10^{-8}	1.69×10^{-7}
[0.9, 1]	3.86×10^{-6}	5.20×10^{-8}	6.04×10^{-5}	1.96×10^{-6}	3.03×10^{-8}	1.76×10^{-7}
Interval	$N = 5$					
	Method [7]	Method [44]	Method [43]	Negative EM method	Positive EM method	Extended EM method
[0, 0.1]	1.80×10^{-9}	9.56×10^{-12}	4.37×10^{-12}	3.15×10^{-10}	1.49×10^{-12}	5.92×10^{-11}
[0.1, 0.2]	4.09×10^{-9}	2.15×10^{-11}	2.65×10^{-9}	7.33×10^{-10}	3.19×10^{-12}	7.82×10^{-11}
[0.2, 0.3]	7.00×10^{-9}	3.63×10^{-11}	6.48×10^{-9}	1.27×10^{-9}	5.14×10^{-12}	7.82×10^{-11}
[0.3, 0.4]	1.07×10^{-8}	5.45×10^{-11}	3.97×10^{-10}	1.27×10^{-9}	5.14×10^{-12}	5.25×10^{-11}
[0.4, 0.5]	1.53×10^{-8}	7.68×10^{-11}	7.49×10^{-9}	1.96×10^{-9}	7.37×10^{-12}	3.91×10^{-11}
[0.5, 0.6]	2.10×10^{-8}	1.04×10^{-10}	1.09×10^{-7}	2.84×10^{-9}	9.94×10^{-12}	5.07×10^{-11}
[0.6, 0.7]	2.80×10^{-8}	1.36×10^{-10}	2.98×10^{-6}	3.93×10^{-9}	1.28×10^{-11}	6.13×10^{-11}
[0.7, 0.8]	3.65×10^{-8}	1.75×10^{-10}	3.55×10^{-6}	5.27×10^{-9}	1.62×10^{-11}	7.24×10^{-11}
[0.8, 0.9]	4.67×10^{-8}	2.22×10^{-10}	2.62×10^{-6}	6.90×10^{-9}	2.00×10^{-11}	1.23×10^{-10}
[0.9, 1]	5.90×10^{-8}	2.76×10^{-10}	1.41×10^{-6}	8.87×10^{-9}	2.43×10^{-11}	1.27×10^{-10}

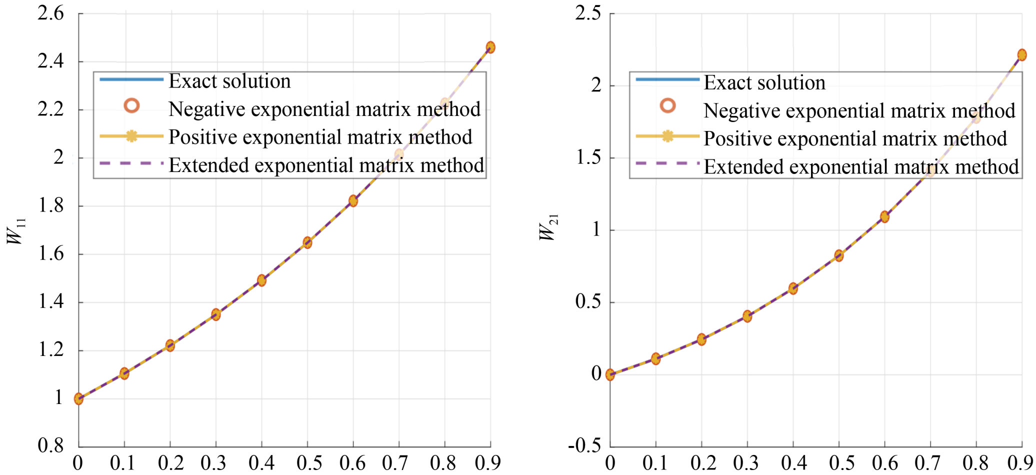
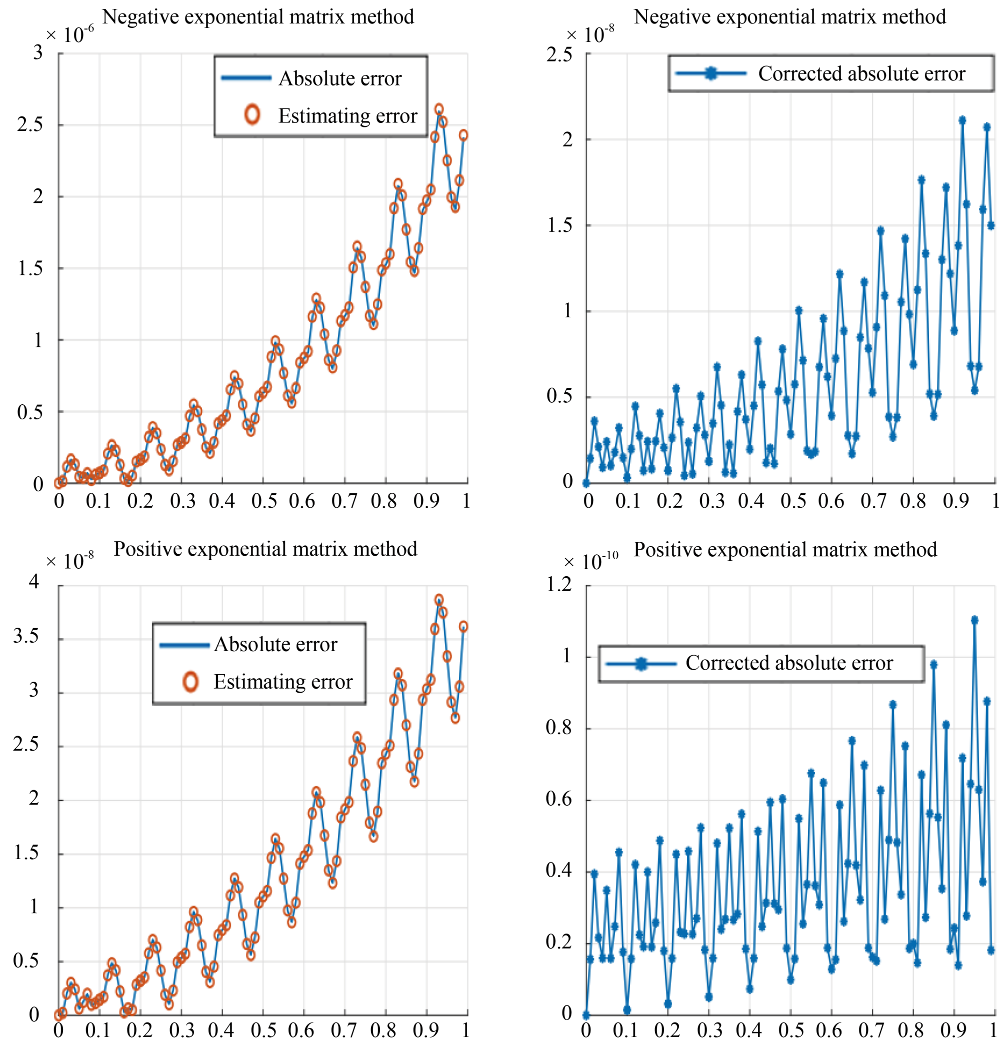


Figure 7. Comparison of the approximate solutions and exact solution for $N = 4$ of Example 3



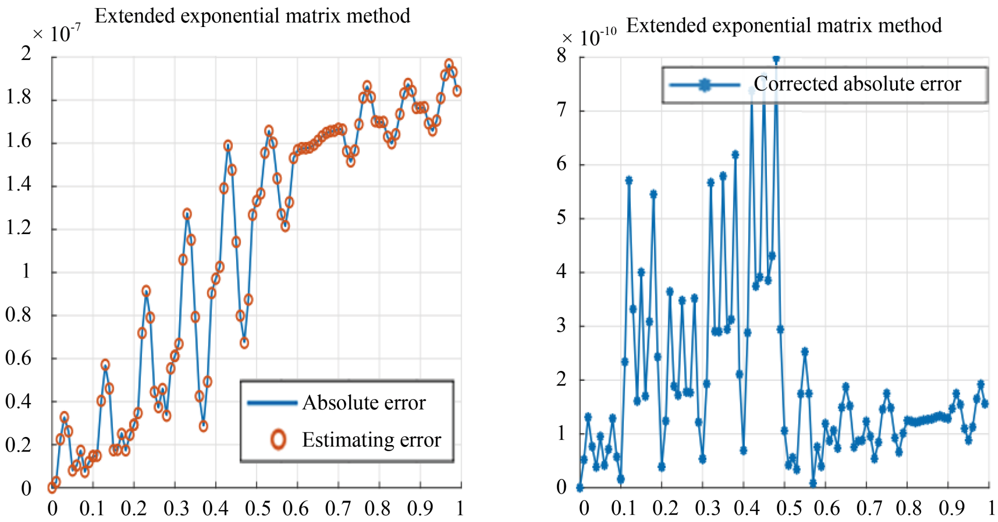
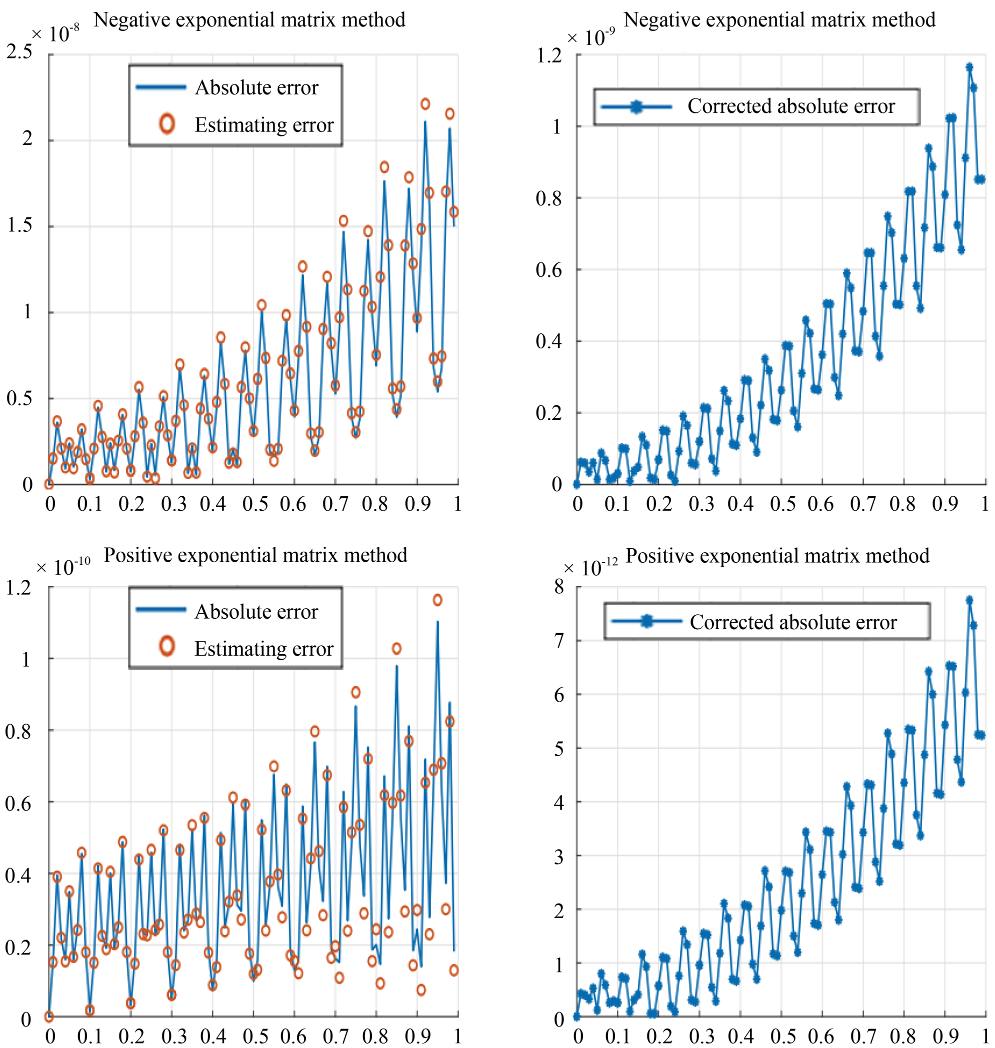


Figure 8. Comparison of the error functions (actual, estimation, and corrected) in separate graphs for $N = 4$ and $M = N+1$ of Example 3



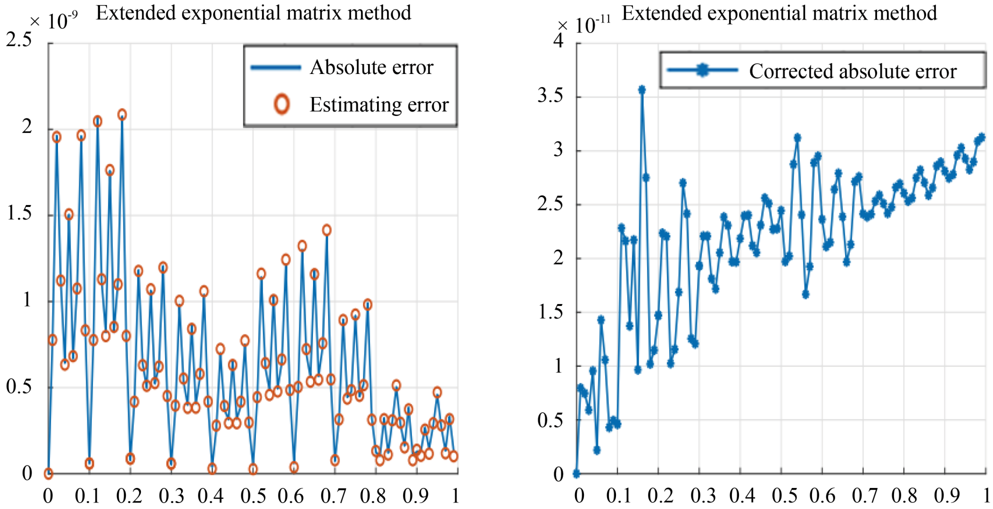


Figure 9. Comparison of the error functions (actual, estimation and corrected) in separate graphs for $N = 5$ and $M = N + 1$ of Example 3

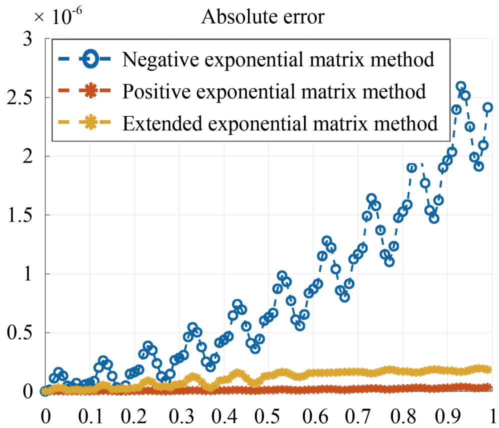


Figure 10. Comparison of the actual error functions (for negative, positive and extended bases) in single graph for $N = 4$ of Example 3

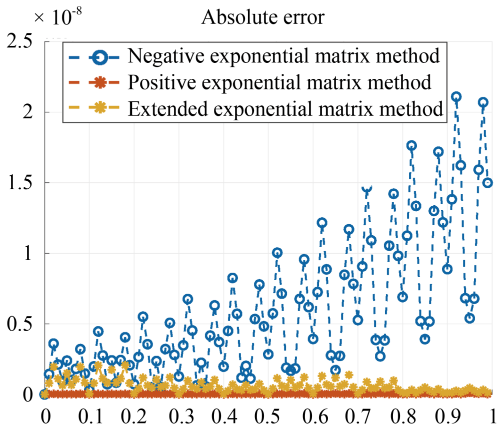


Figure 11. Comparison of the actual error functions (for negative, positive, and extended bases) in single chart for $N = 5$ of Example 3

In Figure 12, we plot the absolute error (actual, estimation, and bounded) in separate graphs for $N = 4$ of Example 3 where $A_0 := \left[\max_{\tau\in[0,\ 1]} |A_{ij}(\tau)| \right] = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

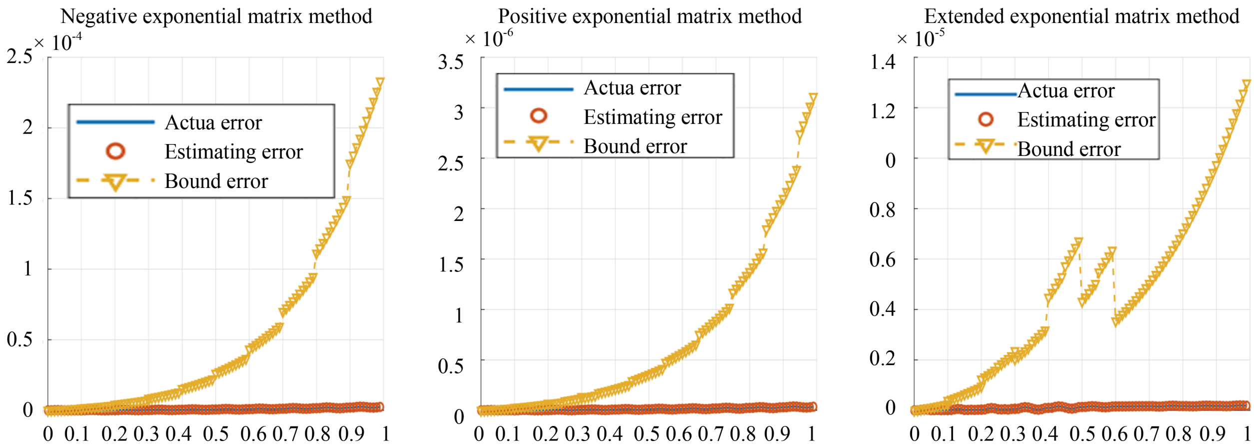


Figure 12. The absolute error (actual, estimation, and bounded) in separate graphs for $N = 4$ and $M = N+1$ of Example 3

Lastly, we solve the problem of the first-order linear matrix differential equation given by [42, 43]:

$$\begin{cases} W'(x) = A(x)W(x) + C(x), \quad x \in [0,\ 1], \\ W_0 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}, \end{cases} \tag{38}$$

where

$$A(x) = \begin{pmatrix} -1-x & 0 & -1+e^x+x \\ e^x & -x & 1 \\ 0 & -1 & e^x \end{pmatrix}, \tag{39}$$

and

$$C(x) = \begin{pmatrix} -(-1-x)(1+x)-(-1+e^x+x)x+1 & -(-1-x)(e^x+x)+e^x+1 \\ -x-e^x(1+t) & -e^x(e^x+x)+x(x^2+5x-1)+5+2x \\ 1-xe^x & -1+x(5+x) \end{pmatrix},$$

with the exact solution

$$W(x) = \begin{pmatrix} 1+x & e^x+x \\ 0 & -1+5x+x^2 \\ x & 0 \end{pmatrix}. \tag{40}$$

As in previous examples, we present the comparisons between the approximate solutions and the exact solutions, and also we compare the errors of our methods (with negative, positive, and extended bases) both among themselves and with the errors of other methods [42, 43] for $N = 4$ and $N = 5$. These comparisons can be from Table 4 and Figure 13, 14, 15. As we see, the solution calculated by the proposed method is in good agreement with the exact solution. Moreover, we note that the proposed method is better than the one found in [42, 43]. It is seen that the proposed three methods are more effective than other methods, and also our methods with positive bases are better than our methods with negative and extended bases in this example. The estimation error function is close to the actual error function. In addition, it is observed that the approximate solutions obtained with the error correction technique much better results give better results.

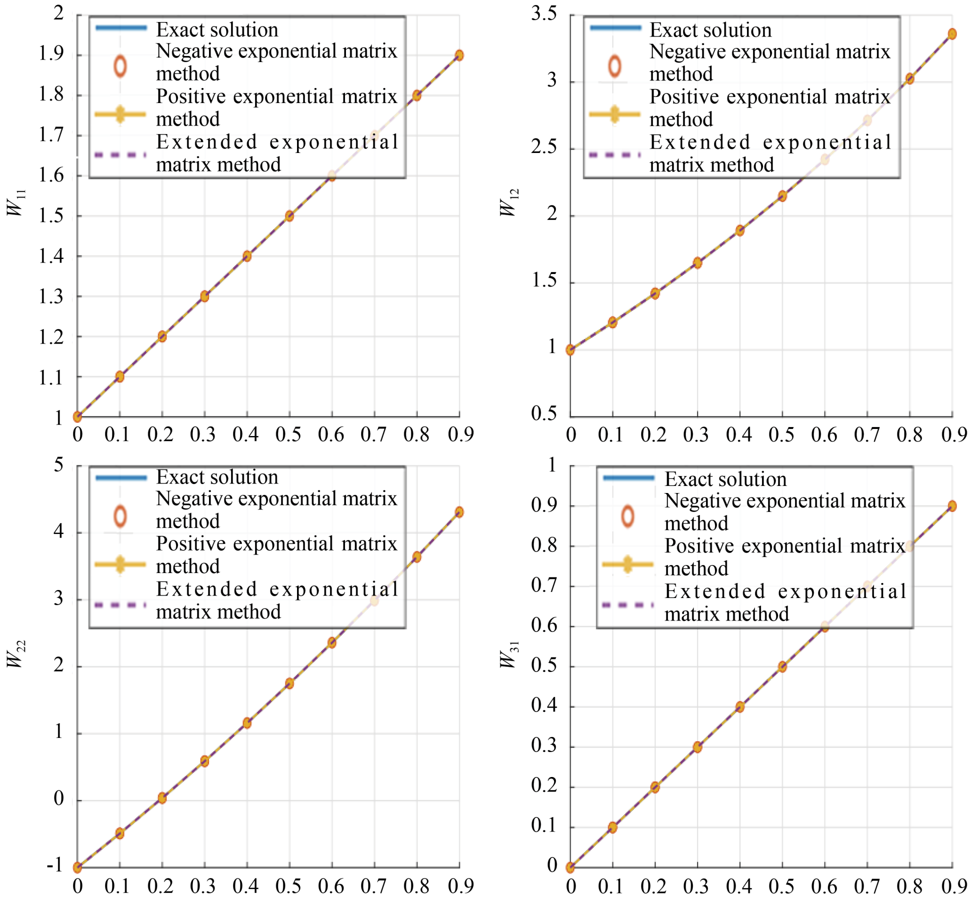
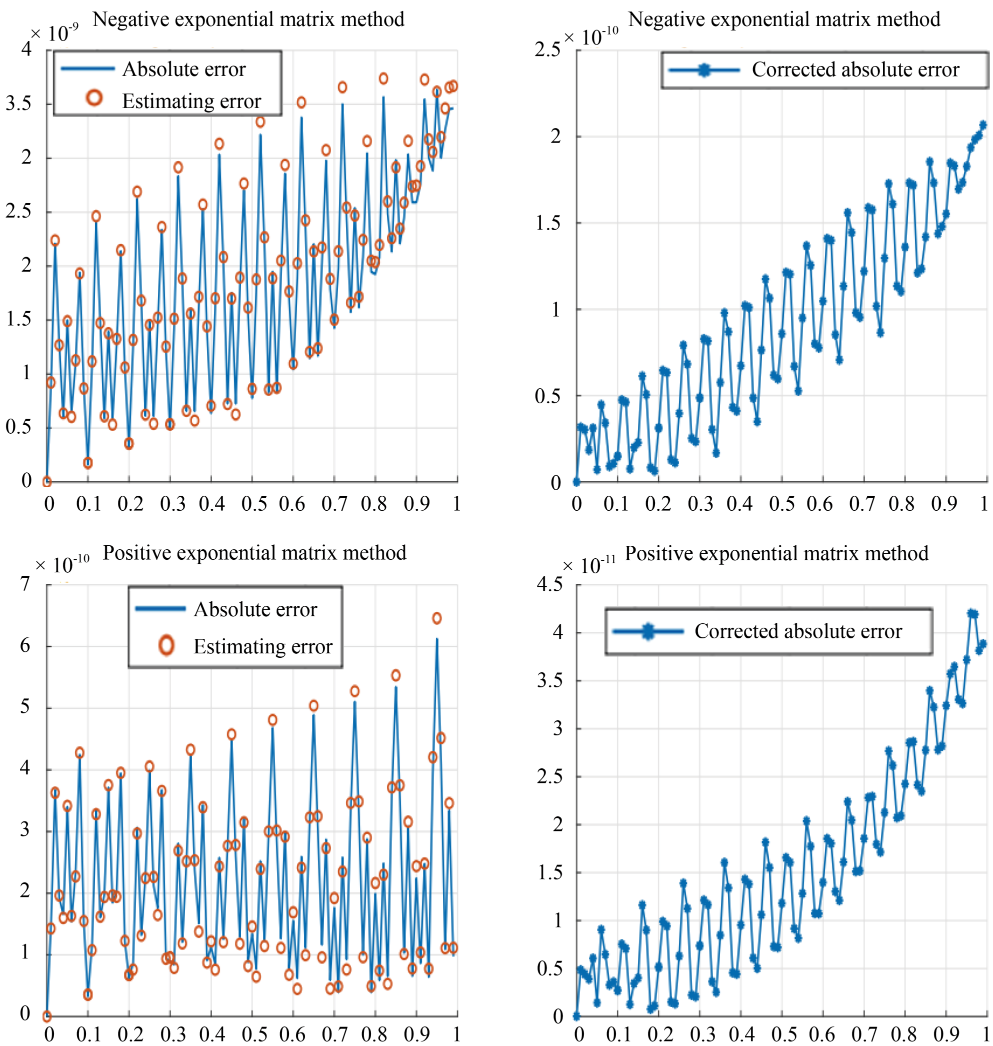


Figure 13. The approximate solutions and the exact solution for $N = 5$ of Example 4

Table 4. Comparison of the methods in [42] and the proposed methods for the problem (42)

Interval	$N = 5$			
	Method [42]	Negative EM method	Positive EM method	Extended EM method
[0, 0.1]	1.39×10^{-6}	1.61×10^{-10}	3.27×10^{-11}	2.00×10^{-10}
[0.1, 0.2]	1.39×10^{-6}	3.24×10^{-10}	6.21×10^{-11}	3.02×10^{-10}
[0.2, 0.3]	1.43×10^{-6}	4.85×10^{-10}	8.85×10^{-11}	3.81×10^{-10}
[0.3, 0.4]	1.43×10^{-6}	4.85×10^{-10}	8.86×10^{-11}	3.81×10^{-10}
[0.4, 0.5]	1.49×10^{-6}	6.38×10^{-10}	1.12×10^{-10}	4.18×10^{-10}
[0.5, 0.6]	1.49×10^{-6}	7.76×10^{-10}	1.34×10^{-10}	4.79×10^{-10}
[0.6, 0.7]	1.57×10^{-6}	1.04×10^{-9}	1.55×10^{-10}	7.20×10^{-10}
[0.7, 0.8]	1.57×10^{-6}	1.42×10^{-9}	1.76×10^{-10}	9.00×10^{-10}
[0.8, 0.9]	1.65×10^{-6}	1.92×10^{-9}	1.99×10^{-10}	8.54×10^{-10}
[0.9, 1]	1.65×10^{-6}	2.59×10^{-9}	2.24×10^{-10}	7.53×10^{-10}



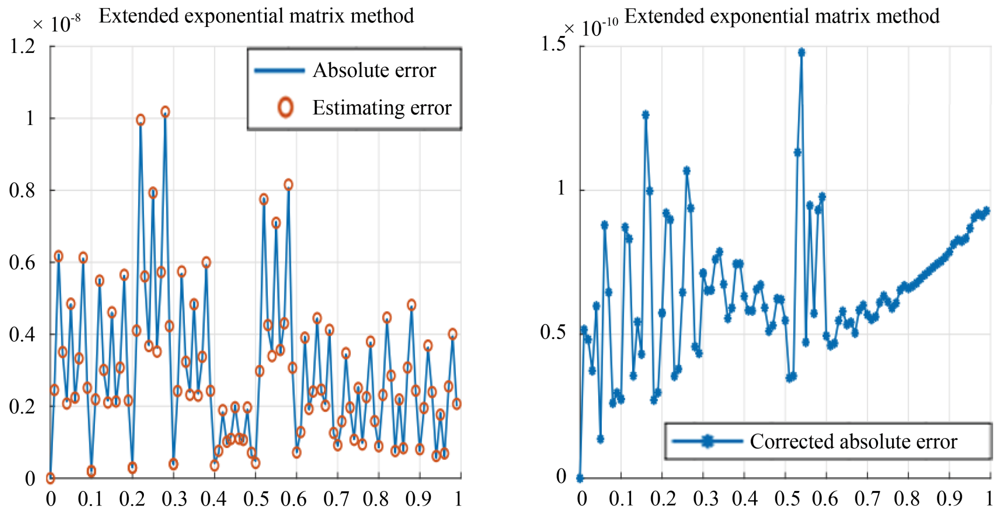


Figure 14. Comparisons of errors (actual, estimation, and corrected) of our methods (negative, positive and extended bases) for $N = 5$ and $M = N+1$ of Example 4

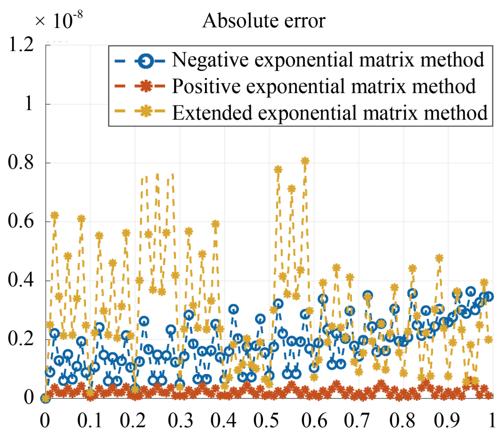


Figure 15. Comparison of the actual error functions of our methods (negative, positive, and extended bases) for $N = 5$ of Example 4

In Table 5, we report the CPU running times in seconds for the four examples $h = 0.1$.

Table 5. Report CPU running times in seconds for the four examples $h = 0.1$

Example	$N = 4$		
	Negative EM method	Positive EM method	Extended EM method
1	0.243940	0.152903	0.104998
2	0.247771	0.154041	0.102792
3	0.268362	0.090476	0.109939
4	0.457437	0.397607	0.240870

In Figure 16, we plot the absolutes error (actual, estimation and bounded) in separate graphs for $N = 4$ of Example 4 where $A_0 := \left[\max_{\tau\in[0,\ 1]} |A_{ij}(\tau)|\right] = \begin{pmatrix} -1 & 0 & 2.7183 \\ 2.7183 & 0 & 1 \\ 0 & -1 & 2.7183 \end{pmatrix}$.

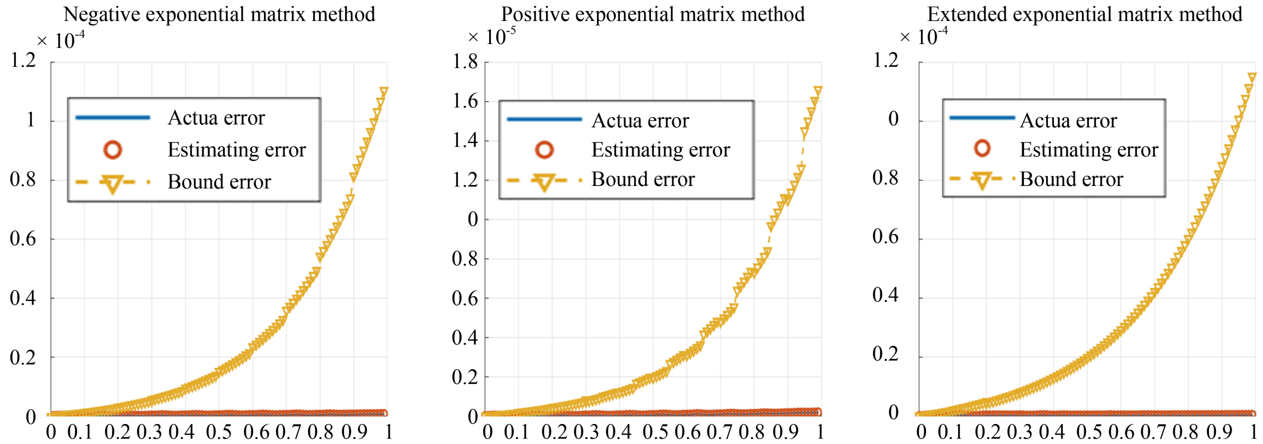


Figure 16. The absolute error (actual, estimation, and bounded) in separate graphs for $N = 4$ and $M = N+1$ of Example 4

7. Conclusion

In this paper, an exponential collocation method is designed separately for the negative, positive, and extended exponential bases to solve first-order linear matrix differential equations numerically. Although the results obtained for each of these bases are good and close to each other, it has been observed that the results differ between these bases used, depending on the problem. For example, it is seen that our method with negative bases gives good results in Example 2, and one with positive bases in Example 3 and Example 4 gives better results. Each example shows that the estimated error functions are very close to the actual error functions. This feature is important because it can be used to test the reliability of the result when the exact solution of the problem is unknown. In comparison with other methods, the presented approaches for each base give good results compared to other methods.

Data availability

Data sharing not applicable to this article as no data sets were generated or ana lysed during the current study.

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Conflict of interest

The authors declare that they have no conflict of interest.

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