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Research Article

A Novel Iterative Method for the Split Common Fixed Point Problem with Application to Variational Inequalities

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Abstract: We extend the notion of asymptotically nonexpansive mapping to the more general class, namely, e-enriched asymptotically nonexpansive mappings. It is shown, with an example, that the class of e-enriched asymptotically nonexpansive mappings is more general than the class of asymptotically nonexpansive mappings. Certain weak and strong convergence theorems are then proved for the iterative approximation of split common fixed point problem involving the class of (e, ϑ)-enriched strictly quasi-pseudocontractive mappings and the class of e-enriched asymptotically nonexpansive mappings in the domain of two Banach spaces. Furthermore, a significant result for the hierarchical variational inequality problem is obtained as a consequence of our main result.

***Keywords*:** fixed point, enriched nonlinear mapping, mapping, Hilbert space

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1. Introduction

The fundamental nonlinear problems of applied mathematics reduce to determining solutions of nonlinear functional equations (e.g. nonlinear integral equations, boundary value problems for nonlinear ordinary or partial differential equations, the existence of periodic solutions of nonlinear partial differential equations). These problems can be formulated in terms of finding the fixed points of a given nonlinear mapping on an infinite dimensional function space X into itself. Fixed point theorems give the conditions under which certain equation involving mappings (single or multivalued) have solutions. Recently, the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. Specifically, fixed point methods have been applied in such diverse fields as biology, chemistry, economics, engineering, game theory, physics, rocks waves in gases, the movement of viscous fluids, chemical reactions, steady-state tempreture distribution, neutron transport theory, variational inequalities, economics theories, epidemics, complementary problems, optimal control, heat radiation, nonlinear oscillation in biology, and optimization theory; see [1, 2] for more information.

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Let C be a nonempty closed convex subset of a real Banach E, and $E^\star$ be a dual space of E. For $\rho > 1$, an operator $J_\rho : E \longrightarrow 2^{E^\star}$ is given by

$$J_\rho(\xi) = \{\xi^\star \in E^\star : \langle \xi,\ \xi^\star \rangle = \|\xi\|^\rho \text{ and } \|\xi^\star\| = \|\xi\|^{\rho-1},\ \forall \xi \in E\}, \tag{1}$$

is called generalized duality map on E, where $\langle \cdot,\ \cdot \rangle$ is referred to as generalized duality pairing. If $\rho = 2$, we get the normalized duality mapping, which is often designated with J. Current literature is abound with the fact (see, for instance, [3]) that $J_\rho(\xi) = \|\xi\|^{\rho-2} J(\xi)$ if $\xi \neq 0$. The J_ρ becomes single-valued (which we shall designate with j_ρ throughout the paper) for the case where $E^\star$ is strictly convex and an identity in Hilbert spaces (H).

Let $G,\ \eth : C \longrightarrow C$ be two nonlinear mappings. The following designations will be used in this paper:

(a) $F(G),\ F(\eth)$ will designate the set of fixed points of G and $\eth$, respectively;

(b) $\mathbb{R},\ \mathbb{N}$ will designate the sets of real and natural numbers, respectively;

(c) $UCBS$ will designate uniformly convex Banach space;

(d) $USBS$ will designate uniformly smooth Banach space and if $\{\xi_n\}_{n=1}^\infty$ is a sequence in E, then;

(e) $\xi_n \to \xi^\star$ and $\xi_n \rightharpoonup \xi^\star$ will designate strong and weak convergence of $\{\xi_n\}_{n=1}^\infty$ to a point $\xi^\star \in E$ as $n \to \infty$, respectively.

Definition 1.1 Let C be as described above. A mapping $G : C \longrightarrow C$ is known as $(e,\ \vartheta)$-Enriched Strictly Pseudocontractive (e-ESPMs, for short) (see [4, 5]) if for all $\xi,\ \hbar \in C$, there exist $e \in [0,\ +\infty)$ and $j(\xi - \hbar) \in J(\xi - \hbar)$ such that

$$\langle e(\xi - \hbar) + G\xi - G\hbar,\ j((e+1)(\xi - \hbar)) \rangle \leq (e+1)^2 \|\xi - \hbar\|^2 - \vartheta \|\xi - \hbar - (G\xi - G\hbar)\|^2, \tag{2}$$

where $\vartheta = \dfrac{1}{2}(1 - \varpi)$ for some $\varpi \in [0,\ 1)$.

Remark 1.1 Some nonlinear mappings that could be recovered from (2) when either e or ϑ assumes the value 0 or 1 are as follows: if $e = 0$ in (2), a class of operators studied in [6] emerges while $\vartheta = 0$ reduces (2) to a class of e-Enriched Nonexpansive Mappings (e-ENM), see [7, 8]. Hence, the class of (e, ϑ)-enriched strict pseudocontractions contains the classes e-ENM and ϑ-strictly pseudocontractive mappings. For some studies pertaining to other generalizations of nonexpansive mappings and their relationship with other nonlinear operators, see [9–11] and the references that go with them.

Remark 1.2 Observe that if $\vartheta = 1$ in (2), then we have an e-Enriched Pseudocontraction (e-EPM, for short). Therefore, the class of $(e,\ \vartheta)$-ESPMs is a subclass of the class of e-EPMs; see, for instance, Example 2.5 and Example 2.6 in [12] for more details. Also, while the class of $(e,\ \vartheta)$-enriched strictly pseudocontractions inherits their Lipschitz property through their definition (see Proposition 3.3 in [13]), the class of e-enriched pseudocontractions is generally not continuous. It is worth noting that the class (e, ϑ)-ESPM was initially examined in [14] as a super class of the class of a ϑ-Strict Pseudocontraction (ϑ-SPM) (note that G is said to be ϑ-strictly pseudocontractive mapping if $\|G\xi - G\hbar\|^2 \leq \|\xi - \hbar\|^2 + \vartheta \|\xi - \hbar - (G\xi - G\hbar)\|^2$, for all $\xi,\ \hbar \in C$. For more information on ϑ-SPM, their generalizations and properties, see [6]. In [14] Berinde showed that if C is a bounded, closed and convex subset of a Hilbert space and $G : C \longrightarrow C$ is a $(e,\ \vartheta)$-ESPM, then G has a fixed point.

Setting $e = \dfrac{1}{\sigma} - 1$, for $\sigma \in (0,\ 1]$, in (2), we obtain, using Proposition 2.1 (3) (see below), that

$$\langle G_\sigma \xi - G_\sigma \hbar,\ j(\xi - \hbar) \rangle \leq \|\xi - \hbar\|^2 - \vartheta \|\xi - \hbar - (G_\sigma \xi - G_\sigma \hbar)\|^2, \tag{3}$$

where $G_\sigma = (1-\sigma)I + \sigma G$ is a ϑ-strict pseudocontraction. If we represent the identity map with I, then (3) can equivalently be written as

$$\langle (I-G_\sigma)\xi - (I-G_\sigma)\hbar,\ j(\xi-\hbar)\rangle \geq \vartheta\|\xi-\hbar-(G_\sigma\xi - G_\sigma\hbar)\|^2. \tag{4}$$

In a real Hilbert space, (3) becomes

$$\|G_\sigma\xi - G_\sigma\hbar\|^2 \leq \|\xi-\hbar\|^2 + \varpi\|\xi-\hbar-(G_\sigma\xi - G_\sigma\hbar)\|^2, \tag{5}$$

where G_σ is as described in inequality (3). If we set $\varpi = 1$ in (5), then a pseudocontraction ensues.

On the other hand, the split feasibility problem (Small Form Pluggable (SFP), for short), first considered in [15]) for the case of a finite dimensional Hilbert space, has been widely used in computer tomography, signal processing, image restoration, optimization and intensity-modulated radiation therapy. This problem basically seeks to

$$\text{find} \quad \xi^\star \in C \quad \text{such that} \quad A\xi^\star \in Q, \tag{6}$$

where $A : H_1 \longrightarrow H_2$ is a bounded linear operator, C and Q are nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively.

Notice that a point $\xi^\star \in C$ solves problem (6) provided it solves the fixed point equation:

$$\xi^\star = P_C(I - \gamma A^\star(I-P_Q)A)\xi^\star, \tag{7}$$

where P_C and P_Q are orthogonal projections of H_1 and H_2 onto C and Q, $A^\star$ is the corresponding adjoint operator of A, and γ is a positive real number. Aside from finite-dimensional spaces, some remarkable results have recently been announced in the infinite-dimensional Hilbert space. For this and some related results, see [16] and the references therein.

As a worthy generalization of problem (6), the split common fixed point problem (Split Common Fixed Point Problem (SCFPP), for short) was introduced lately. Let $\eth$ and G be selfmaps of C and Q, respectively. The SCFPP for the maps $\eth$ and G is an inverse problem whose focus is to find an element $\xi^\star$ in the fixed point set $F(\eth)$ such that the image under a bounded linear operator A is a member of another fixed point set $F(G)$. To be precise, the SCFPP for the aforementioned maps seeks to search for $\xi^\star \in C$ that guarantees

$$\xi^\star \in F(\eth) \quad \text{and} \quad A\xi^\star \in F(G), \tag{8}$$

For the rest of this paper, the set of solution set of a SCFPP will be designated by

$$\Gamma = \{\xi^\star \in F(\eth) :\ A\xi^\star \in F(G)\}. \tag{9}$$

Problem (8) was initiated by Moudafi [17] in the domain of a real Hilbert space. Subsequently, a lot of techniques have been deployed to solve the SFP and SCFPP as evident in [15, 16, 18–20]. Aside from this, Cui et al. [21] investigated the SCFPP of ϑ-quasi-strict pseudocontraction in two Hilbert space domains; in [22, 23], Takahashi et al. obtained remarkable convergence results for the SFP and Split Common Null Point Problem (SCNPF) in the setup of Banach spaces, respectively. Under appropriate conditions, they presented weak and strong convergence results of hybrid and Halpern-type sequences, respectively. Shortly, Tang et al. [24] obtained a remarkable conclusion in respect of the SCFPP for ϑ-quasi strictly pseudocontractive and asymptotically nonexpansive mappings in the setting of two real Banach spaces. Motivated and inspired by this study and the research going on in the domain of split feasibility problems and SCFPP, the following question becomes necessary:

Question 1.1 Is it possible to enrich the maps considered by Tang [24] so as to obtain results that include the ones announced in [24] as special cases?

In the spirit of oneness (particularly at zero) and considering the setback introduced by Picard's technique in relation to approximating fixed points of nonexpansive mappings, the enriched nonexpansive mapping was introduced in the setup of a real Hilbert space. This big feat, credited to Berinde [7], ushered in a new research direction in history. Following this remarkable achievement was the birth of Φ_T-enriched Lipschitzian mapping. Saleem et al. [5] demonstrated that this class of mappings includes the class of enriched contraction, enriched nonexpansive and Lipschitzian mappings as special cases. Also, it is known that the class of enriched nonlinear mappings, having been proved to be more general than the class of mappings from which they inherited their family name, provides richer bases in applications. Motivated by this and following the results reported in [24], in the present paper, a hybrid method involving $(e,\ \vartheta)$-Enriched Quasi Strictly Pseudocontractive Mappings ($(e,\ \vartheta)$-EQSPM) and $(e,\ \{\varpi_r\}_{r=1}^{\infty})$-enriched asymptotically nonexpansive mappings ($(e,\ \{\varpi_r\}_{r=1}^{\infty})$-European Association of Nuclear Medicine (EANM)) with $e \in [0,\ \infty)$ is examined for the SCFPP. To compare with the results obtained in Tang et al. [24], we considered operators that encompass the ones studied in [24]. Also, the techniques we employed are different from the ones used in [24], which, under standard conditions, ensured that weak and strong convergence results are obtained in real Banach spaces. As an application, we considered an algorithm for the Hierarchical Variational Inequality Problem (HVIP) in real Banach spaces. Our results generalize and improve the results of Censor [15], Bryne [16], Moudafi [18], Takahashi [22], and Takahashi et al. [23].

2. Preliminaries

Throughout this section, $B(E)$ and $S(E)$ will denote a unit ball and a unit sphere, respectively.

A uniform convexity of a Banach space E means that, for any given $\varepsilon > 0$, there is a $\delta > 0$ such that for all $\xi,\ \hbar \in E,\ \|\xi\| \leq 1,\ \|\hbar\| \leq 1,\ \|\xi - \hbar\|\varepsilon$, the inequality

$$\|\xi + \hbar\| \leq 2(1-\delta), \tag{10}$$

is fulfilled. The function $\delta_E : [0,\ 2] \longrightarrow [0,\ 1]$ is specified by

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|\xi + \| : \xi,\ \|\xi\|,\ \|\| \leq 1,\ \|\xi - \| \geq \varepsilon\right\}.$$

is called the modulus of convexity of E.

A uniform smoothness of a Banach space E means that, for any given $\varepsilon > 0$, there is a $\delta > 0$ such that for all $\xi,\ \hbar \in E,\ \|\xi\| = 1,\ \|\hbar\| \leq \delta$, the inequality

$$2^{-1}(\|\xi + \hbar\| + \|\xi - \hbar\| - 1 \leq \varepsilon\|\hbar\|), \tag{11}$$

is satisfied. The function $\rho_E : [0,\ \infty) \longrightarrow [0,\ \infty)$ defined by

$$\rho_E(\bar{\tau}) = \sup\left\{\frac{1}{2}(\|\xi+\hbar\|+\|\xi-\hbar\|)-1 : \xi \in S(E),\ \|\hbar\| \leq \bar{\tau}\right\},$$

is referred to as the modulus of smoothness of E. The moduli of convexity and smoothness are the basic quantitative characteristics of a Banach space that describe its geometric properties. The space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon > 0$, and it is uniformly smooth if and only if $\lim_{\bar{\tau}\to 0}\frac{\rho_E(\bar{\tau})}{\bar{\tau}} = 0$. For $\alpha > 1$, a normed space E is called α-uniformly smooth if there is a constant $c > 0$ such that $\rho_E(\tau) = c\tau^{\alpha}$. It is well known that L_p, ℓ_p, and Sobolev spaces $W_p^m(\Omega)$, for $1 < p < \infty$, are all p-uniformly smooth, and the following estimate remains valid

$$\rho_{L_p}(\tau) = \rho_{\ell_p}(\tau) = \rho_{W_p^m(\Omega)}(\tau) = \begin{cases} (1-\tau^p)^{\frac{1}{p}} - 1 \leq \dfrac{1}{p}\tau^p & 1 < p < 2; \\ \\ \dfrac{p-1}{2}\tau^2 + o(\tau^2) \leq \dfrac{p-1}{2}\tau^2, & p \geq 2, \end{cases} \tag{12}$$

where $\tau \geq 0$ (see [25]). From (12), it is clear that if $1 < p < 2$, then L_p, ℓ_p, or $W_p^m(\Omega)$ is not 2-uniformly smooth.

Consider now $E = L_p$, ℓ_p, or $W_p^m(\Omega)$ for $p \in [2,\ \infty)$. From (12), these spaces are 2-uniformly smooth. In addition, for these spaces, the following inequality remains valid:

$$\|\xi+\hbar\|^2 \leq \|\xi\|^2 + 2\langle \hbar,\ j(\xi)\rangle + (p-1)\|\hbar\|^2, \quad \forall \xi,\ \hbar \in E, \tag{13}$$

and $(p-1)$ is the best smoothness constant (see, for example, [26]). Comparing inequality (13) and the following inequality in [24]:

$$\|\xi+\hbar\|^2 \leq \|\xi\|^2 + 2\langle \hbar,\ j(\xi)\rangle + 2\|k\hbar\|^2, \quad \forall \xi,\ \hbar \in E, \tag{14}$$

we obtain that $2k^2 = (p-1)$, so that $k = \frac{\sqrt{p-1}}{\sqrt{2}}$. The condition that $k = \frac{1}{\sqrt{2}}$ therefore implies that $p \leq 2$. But if $p < 2$, then, from (12), E is not 2-uniformly smooth. Consequently, the only possibility is that $p = 2$.

The following properties of the functions $\delta_E(\varepsilon)$ and $\rho_E(\bar{\tau})$ are important to keep in mind through out the paper:

1. $\delta_E(\varepsilon)$ is defined on the interval $[0,\ 2]$, continuous and increasing on this interval, $\delta_E(0) = 0$,
2. $0 < \delta_E(\varepsilon) < 1$ if $0 < \varepsilon < 2$,
3. $\rho_E(\bar{\tau})$ is defined on the interval $[0,\ \infty)$, and is convex, continuous and increasing on this interval, with $\rho_E(0) = 0$,
4. the function $g_E(\varepsilon) = \varepsilon^{-1}\delta_E(\varepsilon)$ is continuous and non-decreasing on the interval $[0,\ 2]$, with $g_E(0) = 0$,
5. the function $h_E(\varepsilon) = \varepsilon^{-1}\rho_E$ is continuous and non-decreasing on the interval $[0,\ \infty)$, with $h_E(0) = 0$,
6. $\varepsilon^2\delta_E \geq (4L)^{-1}\eta^2\delta_E(\varepsilon)$ if $\delta_E(\varepsilon \geq \varepsilon > 0)$ and $\tau^2\rho_E(\sigma) \leq \sigma^2\rho_E(\tau)$ if $\sigma \geq \tau > 0$. Here, $1 < L < 1.7$ is the Figiel constant.

Lemma 2.1 [24] Let E be a UCBS. Given any positive number c, we can find a strictly increasing and continuous function $g : [0,\ \infty) \longrightarrow [0,\ \infty)$, with $g(0) = 0$, verifying the identity

$$\|t\xi + (1-t)\hbar\|^2 = t\|\xi\|^2 + (1-t)\|\hbar\|^2 - t(1-t)g(\|\xi-\hbar\|), \quad \forall \xi,\ \hbar \in E,$$

with $\|\xi\| \leq c$ and $\|\hbar\| \leq c$, $t \in [0,\ 1]$.

Lemma 2.2 [24] Let E be a 2-USBS with $k > 0$ as the best smoothness constant. Then,

$$\|\xi + \hbar\|^2 \leq \|\xi\|^2 + 2\langle \xi,\ J(\hbar)\rangle + 2\|k\hbar\|^2, \tag{15}$$

for all $\xi,\ \hbar \in E$.

Definition 2.1 [4] Let E be a Banach space, $\emptyset \neq C \subset E$ and $G : C \longrightarrow C$ be a mapping with $F(G) \neq \emptyset$. Then

1. G is referred to as being demiclosed at $\xi^\star$ if whenever $\{\xi_n\}_{n\geq 1}$ is a sequence in C such that $\xi_n \rightharpoonup \wp^\star \in C$ and $\{\xi_n - G\xi_n\}_{n=1}^{\infty} \to 0$, then $G\xi^\star = \wp^\star$. Further, G is referred to as being semicompact if whenever $\{\xi_n\}_{n\geq 1}$ is a bounded sequence in C such that $\{\xi_n - G\xi_n\}_{n\geq 1} \to 0$, $\{\xi_n\}_{n\geq 1}$ has a subsequence which converges strongly.

2. E is endowed with the Opial property when a sequence $\{\xi_n\}_{n=1}^{\infty}$ in E assures $\xi_n \to \wp$ as $n \to \infty$ and

$$\liminf_{n\to+\infty}\|\xi_n - \xi\| < \liminf_{n\to+\infty}\|\xi_n - \hbar\|, \tag{16}$$

for any $\hbar \in E$ with $\hbar \neq \xi$.

Theorem 2.1 [27] Let C be a nonempty, closed, and convex subset of a UCBS E. Suppose a mapping $G : C \longrightarrow E$ is nonexpansive. Then, $(I - G)$ is demiclosed at zero.

Proposition 2.1 [28] Let E be a real Banach space. For $1 < \rho < \infty$, J_ρ is characterized as follows:

1. $J_\rho(\xi) \neq \emptyset$ for all $\xi \in E$ and $D(J_\rho)$ (:the dormain of J_ρ) = E;
2. $J_\rho(\xi) = \|\xi\|^{\rho-1} J_2(\xi),\ \forall \xi \in E(\wp \neq 0)$;
3. $J_\rho(\kappa\xi) = \kappa^{\rho-1} J_\rho(\xi),\ \kappa \in [0,\ \infty)$;
4. $J_\rho(-\xi) = -J_\rho(\xi)$;
5. J_ρ is bounded.
6. E is USBS (equivalently, $E^\star$ is UCBS) provided J_ρ is single-valued and uniformly continuous on any bounded subset of E.

Lemma 2.3 [24] Let $\{\bar{a}_n\}_{n=1}^{\infty}$, $\{\tau_n\}_{n=1}^{\infty}$ and $\{\bar{b}_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers which ensure that $\sum_{n=1}^{\infty}\tau_n < \infty$, $\sum_{n=1}^{\infty}\bar{b}_n < \infty$ and

$$\bar{a_{n+1}} \leq (1 + \tau_n)\bar{a}_n + \bar{b}_n, \quad n \geq 1.$$

Then, $\lim_{n\to\infty}\bar{a}_n$ exists.

3. Main results

In this section, we first introduce the notion of an $(e,\ \{\varpi_n\}_{n=1}^{\infty})$-enriched asymptotically nonexpansive mapping ($(e,\ \{\varpi_n\}_{n=1}^{\infty})$-EANM, for short) in the Banach space domain and then prove our convergence results.

Definition 3.1 Let C be a nonempty, closed, and convex subset of a real Banach space E. An operator $G : C \longrightarrow C$ is referred to as $(e,\ \{\varpi_n\}_{n=1}^{\infty})$-EANM if there exist $e \in [0,\ \infty)$ and a sequence $\{\varpi_n\}_{n=1}^{\infty}$ of positive integers with $\varpi_n \to 1$ $(n \to \infty)$ such that, for all $\xi,\ \psi \in C$ and for all $n \geq 1$, the inequality

$$\|e(\xi - \psi) + G^n\xi - \Im^n\psi\| \leq (e+1)\varpi_n\|\xi - \psi\|, \tag{17}$$

holds.

Now, by setting $e = \frac{1}{\sigma} - 1$, for some $\sigma \in (0,\ 1]$, we obtain from (17) that

$$\begin{aligned}
&\|e(\xi-\psi)+G^n\xi-G^n\psi\| \le (e+1)\varpi_n\|\xi-\psi\| \\
&\Leftrightarrow \left\|\left(\frac{1}{\sigma}-1\right)(\xi-\psi)+G^n\xi-G^n\psi\| \le \frac{1}{\sigma}\varpi_n\|\xi-\psi\right\| \\
&\Leftrightarrow \frac{1}{\sigma}\|(1-\sigma)(\xi-\psi)+\sigma G^n\xi-\sigma G^n\psi\| \le \frac{1}{\sigma}\varpi_n\|\xi-\psi\| \\
&\Leftrightarrow \|(1-\sigma)\xi+\sigma G^n\xi-[(1-\sigma)\psi+\sigma G^n\psi]\| \le \varpi_n\|\xi-\psi\|.
\end{aligned} \tag{18}$$

Thus, G is a modified $(\sigma,\ \{\varpi_n\}_{n=1}^{\infty})$-enriched asymptotically nonexpansive mapping ($(\sigma,\ \{\varpi_n\}_{n=1}^{\infty})$-MEANM) whenever G is $(e,\ \{\varpi_n\}_{n=1}^{\infty})$)-EANM. Observe that if $e = 0$ in (17), we obtained an important class of asymptotically nonexpansive mappings studied by Goebel et al. in [29]. Also, if we set $G_\sigma = (1-\sigma)I + \sigma G$ in inequality (18), then the modified $(\sigma,\ \{\varpi_n\}_{n=1}^{\infty})$-enriched asymptotically nonexpansive mapping reduces to the class of mappings examined in [29].

Definition 3.2 Consider E and C as stated in Definition 3.1. A selfmap G of C is known as $(e,\ L)$-enriched uniformly Lipschitizian if there exist $e \in [0,\ \infty)$ and a positive real number L ensuring

$$\|e(\xi-\psi)+G^n\xi-G^n\psi\| \le (e+1)L\|\xi-\psi\|, \quad \forall\xi,\ \psi \in C,\ \forall n \ge 1, \tag{19}$$

holds.

Example 3.1 Every asymptotically nonexpansive mapping G verifying (17) is automatically $(e,\ \{\varpi_n\}_{n=1}^{\infty})$-EANM with $e = 0$.

Example 3.2 Every e-enriched nonexpansive mapping G verifying (17) is automatically $(e,\ \{\varpi_n\}_{n=1}^{\infty})$-EANM for each $n \in N$ with $\varpi_n = 1$.

Example 3.3 Any $(\sigma,\ \{\varpi_n\}_{n=1}^{\infty})$-EANM G validating (18) is a $\left(\frac{1}{e+1},\ \{\varpi_n\}_{n=1}^{\infty}\right)$-EANM validating (17).

Example 3.4 Let $E = \ell^\infty$ and $G : \ell^\infty \longrightarrow \ell^\infty$ be a mapping defined by

$$G(\xi_1,\ \xi_2,\ \cdots) = \frac{1}{\sqrt{2}}(\xi_1,\ \xi_2,\ \cdots), \quad \forall\xi \in \ell^\infty,$$

where $\xi = (\xi_1,\ \xi_2,\ \cdots)$. Then, G is an $(e,\ \{\varpi_n\}_{n=1}^{\infty})$-enriched asymptotically nonexpansive.

To see this, let $\xi \in \ell^\infty$ and observe that for $n = 2,\ 3,\ 4,\ \cdots,\ n-1,\ n$, we have

$$G\xi = \frac{1}{\sqrt{2}}\xi \tag{20}$$

$$G^2\xi = G(G\xi) = \frac{1}{\sqrt{2}}G(\xi) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\xi\right) = \frac{1}{\sqrt{2^2}}\xi = \frac{1}{2}\xi$$

$$G^3\xi = G^2(G\xi) = \frac{1}{\sqrt{2^2}}G\xi = \frac{1}{\sqrt{2^2}}\left(\frac{1}{\sqrt{2}}\xi\right) = \frac{1}{\sqrt{2^3}}\xi = \frac{1}{2\sqrt{2}}\xi$$

$$G^4\xi = G^3(G\xi) = \frac{1}{\sqrt{2^3}}G\xi = \frac{1}{\sqrt{2^3}}\left(\frac{1}{\sqrt{2}}\xi\right) = \frac{1}{\sqrt{2^4}}\xi = \frac{1}{4}\xi$$

$$\vdots$$

$$G^{n-1}\xi = G^{n-2}(G\xi) = \frac{1}{\sqrt{2^{n-2}}}G\xi = \frac{1}{\sqrt{2^{n-2}}}\left(\frac{1}{\sqrt{2}}\xi\right) = \frac{1}{\sqrt{2^{n-1}}}\xi$$

$$G^{n}\xi = G^{n-1}(G\xi) = \frac{1}{\sqrt{2^{n-1}}}G\xi = \frac{1}{\sqrt{2^{n-1}}}\left(\frac{1}{\sqrt{2}}\xi\right) = \frac{1}{\sqrt{2^{n}}}\xi.$$

Thus,

$$G^n(\xi_1,\ \xi_2,\ \cdots) = \begin{cases} \frac{1}{2^{2n-2}}(\xi_1,\ \xi_2,\ \cdots), & \text{for} \quad n \quad \text{even} \\ \frac{1}{\sqrt{2^{2n-1}}}(\xi_1,\ \xi_2,\ \cdots), & \text{for} \quad n \quad \text{odd}. \end{cases}$$

Clearly, $F(G) = 0$. Now, for any $e \in [0,\ \infty)$ (with n even) and $\xi \in R$, we obtain

$$\begin{aligned} \|e(\xi-\psi)+G^n\xi-G^n\psi\| &= \|e(\xi-0)+G^n\xi-0\| \\ &= \left(e+\frac{4}{2^{2n}}\right)\|\xi-0\| \\ &< \left(e+\frac{4}{2^{2n}}\right)\left(1+\frac{4}{2^{2n}}\right)\|\xi-0\| \\ &\leq (e+1)\left(1+\frac{4}{2^{2n}}\right)\|\xi-0\| \\ &= (e+1)\varpi_n\|\xi-0\|, \end{aligned}$$

where $\varpi_n = \left(1+\frac{4}{2^{2n}}\right)$. Observe that $\varpi_n \to 1$ as $n \to \infty$.

Also, for any $e \in [0,\ \infty)$ (with n odd) and $\xi \in \ell^\infty$, we obtain

$$
\begin{aligned}
\|e(\xi-\psi)+G^n\xi-G^n\psi\| &= \|e(\xi-0)+G^n\xi-0\| \\
&= \left(e+\frac{\sqrt{2}}{\sqrt{2^{2n}}}\right)\|\xi-0\| \\
&< \left(e+\frac{\sqrt{2}}{\sqrt{2^{2n}}}\right)\left(1+\frac{\sqrt{2}}{\sqrt{2^{2n}}}\right)\|\xi-0\| \\
&< (e+1)\left(1+\frac{\sqrt{2}}{\sqrt{2^{2n}}}\right)\|\xi-0\| \\
&= (e+1)\varpi_n\|\xi-0\|,
\end{aligned}
$$

where $\varpi_n = \left(1+\frac{\sqrt{2}}{\sqrt{2^{2n}}}\right)$. Observe that $\varpi_n \to 1$ as $n \to \infty$. It is not difficult to see that in the two cases, G satisfies (17). Hence, G is an $(e,\ \{\varpi_n\}_{n=1}^{\infty})$-enriched asymptotically nonexpansive.

Definition 3.3 [30] The mapping $G : E \longrightarrow E$ is said to satisfy condition (C) provided

$$
\frac{1}{2}\|\omega-G\omega\| \le \|\omega-\xi\| \Rightarrow \|G\omega-G\xi\| \le \|\omega-\xi\|, \quad \forall\omega,\ \xi \in E. \tag{21}
$$

Aside from failing to satisfy the continuity property, the nonexpansiveness condition of this class of operators does not involve all points in the domain of definition.

Example 3.5 Let $\mathbb{R}$ be the set of real numbers with the usual norm $\|.\| = |.|$, $E = (\mathbb{R},\ |.|)$, $[0,\ 1] = C \subset E$ and $G : \mathbb{R} \longrightarrow \mathbb{R}$ be a mapping defined by

$$
G\xi = \begin{cases} \frac{1}{2}\xi, & \text{for} \quad \xi \neq 1 \\ 0 & \text{for} \quad \xi = 1. \end{cases}
$$

Note that $F(G) = 0$. G is not Suzuki generalised nonexpansive. Indeed, take $\xi = 1$ and $\psi = \frac{3}{4}$. Then,

$$
\frac{1}{2}|\xi-G\xi| = \frac{1}{2}|1-0| > \frac{1}{4} = \left|1-\frac{3}{4}\right| = |\xi-\psi|.
$$

Thus, G is not covered by condition (C) and therefore cannot be Suzuki generalised nonexpansive (or nonexpansive). Notice that for n even,

$$G\xi = \begin{cases} \frac{1}{2^{2n-2}}\xi, & \text{if} \quad \xi \neq 1 \\ \\ 0, & \text{if} \quad \xi = 1. \end{cases}$$

It can easily be shown, following the same argument as in Example 3.4 (considering $\xi \neq 1$), that G is an $(e, \{\varpi_n\}_{n=1}^{\infty})$-enriched asymptotically nonexpansive mapping.

The examples below show that the class of $(e, \{\varpi_n\}_{n=1}^{\infty})$-enriched asymptotically nonexpansive mappings properly includes the classes of asymptotically nonexpansive and e-enriched nonexpansive mappings.

Example 3.6 Consider $E = \mathbb{R},\ C = \left[-\frac{1}{\tau},\ 1\right]$ with $\tau \in (0,\ 2)$. Define the selfmap G on C as

$$G\xi = \begin{cases} -\tau\xi & \text{if} \quad -\frac{1}{\tau} \leq \xi \leq 0 \\ \\ -\frac{1}{\tau}\xi, & \text{if} \quad 0 \leq \xi \leq 1. \end{cases}$$

Then, G is an $(e, \{\varpi_n\}_{n=1}^{\infty})$-EANM with $\varpi_n = 1$ but not an asymptotically nonexpansive mapping.

Indeed, noticing $F(G) = \{0\}$ and $G^2 = I$, with I designating an identity operator, it is evident that $G^{2n-1} = G\ \forall n \geq 1$. Since for every $(\xi^\star,\ \xi) \in F(G) \times C$, the following conditions:

$$\text{(i)} \quad \|e(\xi - \xi^\star) + G^{2n}\xi - G^{2n}\xi^\star\| \leq \|e\xi + G^{2n}\xi\| = (e+1)\|\xi - \xi^\star\|,$$

$$\text{(ii)} \quad \|e(\xi - \xi^\star) + G^{2n-1}\xi - G^{2n-1}\xi^\star\|^2 \leq \|e\xi + G^{2n-1}\xi\|^2 = (e-\tau)^2\|\xi - \xi^\star\|^2 < (e+1)^2\|\xi - \xi^\star\|^2,$$

for $-\frac{1}{\tau} \leq \xi \leq 0$ and for $0 \leq \xi \leq 1$, the inequality

$$\text{(iii)} \quad \|e(\xi - \xi^\star) + G^{2n-1}\xi - G^{2n-1}\xi^\star\|^2 \leq \|e\xi + G^{2n-1}\xi\|^2 = \left(e - \frac{1}{\tau}\right)^2 \|\xi - \xi^\star\|^2 < (e+1)^2\|\xi - \xi^\star\|^2,$$

hold, it follows from (17) that G is an $(e, \{\varpi_n\}_{n=1}^{\infty})$-EANM with $\varpi_n = 1$. But, if $-\frac{1}{\tau} \leq \xi \leq 0$, since $\tau > 1$, we obtain

$$\|G^{2n-1}\xi - G^{2n-1}\xi^\star\|^2 \leq \|G^{2n-1}\xi\|^2 = \|G\xi\|^2 = \tau^2\|\xi - \xi^\star\|^2 > \|\xi - \xi^\star\|^2,$$

which disqualifies G from being asymptotically nonexpansive.

Example 3.7 Let $E = \ell^\infty$ and $C = \{\xi \in \ell^\infty : \|\xi\|_\infty \leq 1\}$. Define the mapping $G : C \longrightarrow C$ as follows:

$$G\xi = (0,\ \xi_1^2,\ \xi_2^2,\ \cdots),\ \xi = (\xi_1,\ \xi_2,\ \cdots) \in C.$$

Then,

$$G^n\xi = (0,\ \xi_1^{2n},\ \xi_2^{2n},\ \cdots),\ \xi = (\xi_1,\ \xi_2,\ \cdots) \in C,$$

for $n \geq 1$. Clearly, $F(G) = \{\bar{0}\}$, where $\bar{0} = (0,\ 0,\ \cdots)$. Now, since

$$\begin{aligned}
\|e(\xi - \xi^\star) + G^n\xi - G^n\xi^\star\|_\infty &= \|e(\xi_1,\ \xi_2,\ \cdots) + (0,\ \xi_1^{2n},\ \xi_2^{2n},\ \cdots)\|_\infty \\
&\leq \|e(\xi_1,\ \xi_2,\ \cdots) + (0,\ \xi_1,\ \xi_2,\ \cdots)\|_\infty \\
&= \|((e+1)\xi_1,\ (\eta+1)\xi_2,\ \cdots)\|_\infty \\
&= (e+1)\|(\xi_1,\ \xi_2,\ \cdots)\|_\infty \\
&= (e+1)\|\xi - \xi^\star\|_\infty,
\end{aligned}$$

it follows from (17) that G is an $(e,\ \{\varpi_n\}_{n=1}^\infty)$-EANM for $\varpi_n = 1$. But, by setting $\xi = \left(\frac{3}{4},\ \frac{3}{4},\ \cdots\right),\ \psi = \left(\frac{1}{2},\ \frac{1}{2},\ \cdots\right) \in C$, we get

$$\begin{aligned}
\|e(\xi - \psi) + G\xi - G\psi\|_\infty &= \left\| e\left(\frac{1}{4},\ \frac{1}{4},\ \cdots\right) + \left(0,\ \frac{5}{16},\ \frac{5}{16},\ \cdots\right) \right\|_\infty \\
&= \left\| \left(\frac{e}{4} + \frac{5}{16},\ \frac{e}{4} + \frac{5}{16},\ \cdots\right) \right\|_\infty \\
&= \frac{4e+5}{16} > \frac{e+1}{4} = (e+1)\left\|\xi - \psi\right\|_\infty.
\end{aligned}$$

Therefore, G is not an e-ENM.

Remark 3.1 If $F(G) \neq \emptyset$ and $\xi^\star \in F(G)$, then (17) becomes

$$\|e(\xi - \xi^\star) + G^n\xi - G^n\xi^\star\| \leq (e+1)\varpi_n\|\xi - \xi^\star\|,\ \forall \xi \in C,\ \forall n \in N. \tag{22}$$

The class of mappings satisfying (22) is known as $(e,\ \{\varpi_n\}_{n=1}^\infty)$-enriched quasi-asymptotically nonexpansive $((e,\ \{\varpi_n\}_{n=1}^\infty))$-EQANM. Therefore, every $(e,\ \{\varpi_n\}_{n=1}^\infty)$-EANM whose fixed point set is nonempty is $(e,\ \{\varpi_n\}_{n=1}^\infty)$-EQANM.

Assumption Q

In this section, we consider the following assumptions:

(1) E_2 is a UCBS and 2-USBS endowed with the Opial property and the best smoothness constant $0 < k < \frac{1}{\sqrt{2}}$;

(2) E_1 is a real Banach space;

(3) $A : E_1 \longrightarrow E_2$ is a bounded linear operator and $A^\star$ is its adjoint;

(4) $\eth : E_1 \longrightarrow E_1$ is an $(\eta,\ \{\varpi_n\}_{n=1}^{\infty})$-EANM with $\{\varpi_n\}_{n=1}^{\infty} \subset [1,\ \infty)$ and $\varpi_n \to 1$ as $n \to \infty$. G is an $(e,\ \vartheta)$-ESPM with $F(G) \neq \emptyset$, G is demiclosed at zero, and $e \in [0,\ \infty)$.

Remark 3.2 In view of Assumption Q(1), E_1 is a real smooth, strictly convex, and reflexive (SCRBC) Banach space. Consequently, the normalised duality mapping $J_2 : E_1 \longrightarrow 2^{E_1{}^\star}$ is single-valued, injective, and surjective.

Now, we consider SCFPP for an $(e,\ \{\varpi_n\}_{n=1}^{\infty})$-EANM $\eth$ and an $(e,\ \vartheta)$-EQSPM G in two Banach space domains; that is, we seek a point

$$\xi^\star \in F(\eth) \quad \text{that assures} \quad A\xi^\star \in F(G). \tag{23}$$

In the sequel, the solution set of SCFPP for $\eth$ and G will be designated with Γ; to be precise,

$$\Gamma = \{\xi^\star \in F(\eth) : A\xi^\star \in F(G)\}. \tag{24}$$

Theorem 3.1 Let $E_1,\ E_2,\ A,\ \eth,\ G$, and $\{\varpi_n\}_{n=1}^{\infty}$ be as stated in Assumption Q. For each $\xi_1 \in E_1$, $\{\xi_n\}_{n=1}^{\infty}$ is given by

$$\begin{cases} {}_n = \xi_n + \gamma J_1^{-1} A^\star J_2 (G_\sigma - 1) A\xi_n, \\ \xi_{n+1} = (1-\alpha_n){}_n + \alpha_n \eth^n_{\sigma n}, \quad n \geq 1, \end{cases} \tag{25}$$

where $G_\sigma = (1-\sigma)I + \sigma G$, $\eth^n_\sigma = (1-\sigma)I + \sigma\eth^n$, $\{\alpha_n\}_{n=1}^{\infty} \subseteq (0,\ 1)$ with $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and γ being controlled by the inequality $0 < \gamma < \min\left\{\dfrac{1-2k^2}{(e+1)\|A\|^2},\ \dfrac{1-\pi}{(e+1)\|A\|^2}\right\}$, $\{\varpi_n\}_{n=1}^{\infty} \subset [1,\ \infty)$ with $L = \sup_{n\to\infty} \varpi_n$ and $\sum_{n=1}^{\infty} (\varpi_n - 1) < \infty$.

(I) If $\Gamma = \{\xi^\star \in F(\eth) : A\xi^\star \in F(G)\}$, then $\xi_n \rightharpoonup \xi^\star \in \Gamma$.

(II) In addition, if $\Gamma = \{\xi^\star \in F(\eth) : A\xi^\star \in F(G)\} \neq \emptyset$ and $\eth$ is semicompact, then $\xi_n \to \xi^\star \in \Gamma$.

Proof. We will break the proof into six different steps:

Step (1), we prove conclusion (I) by first showing that the limit $\lim_{n\to\infty} \|\xi_n - \xi^\star\|$ exists for each $\xi^\star \in \Gamma$. Since, for any $\xi^\star \in \Gamma$, we have $\xi^\star \in F(\eth)$ and $\xi^\star \in F(G)$, we obtain from (25) and Lemma 2.2 that

$$\begin{aligned} \|\hbar_n - \xi^\star\|^2 =& \|(\xi_n - \xi^\star) + \gamma J_1^{-1} A^\star J_2 (G_\sigma - 1) A\xi_n\|^2 \\ \leq& \|\gamma J_1^{-1} A^\star J_2 (G_\sigma - 1) A\xi_n\|^2 + 2\gamma\langle \xi_n - \xi^\star,\ A^\star J_2 (G_\sigma - 1) A\xi_n\rangle + 2k^2\|\xi_n - \xi^\star\|^2 \\ =& \frac{\gamma^2\|A\|^2}{(e+1)^2 n}\|(G-1)A\xi_n\|^2 + 2\gamma\left\langle A\xi_n - A\xi^\star,\ \frac{1}{e+1} J_2(GA\xi_n - A\xi_n)\right\rangle + 2k^2\|\xi_n - \xi^\star\|^2 \\ =& \frac{\gamma^2\|A\|^2}{(e+1)^2 n}\|(G-1)A\xi_n\|^2 + 2\gamma\Big\langle A\xi_n - [(e+G)A\xi_n - eA\xi_n] \end{aligned}$$

$$+[(e+G)A\xi_n - eA\xi_n] - A\xi^\star,\ \frac{1}{e+1}J_2((e+G)A\xi_n - (e+1)A\xi_n)\Big\rangle + 2k^2\|\xi_n - \xi^\star\|^2$$

$$= \frac{\gamma^2\|A\|^2}{(e+1)^2 n}\|(G-1)A\xi_n\|^2 + 2k^2\|\xi_n - \xi^\star\|^2 - \frac{2\gamma}{(e+1)}\|(G-1)A\xi_n\|^2$$

$$+\frac{2\gamma}{(e+1)}\langle[e(A\xi_n - A\xi^\star) + GA\xi_n - GA\xi^\star$$

$$-e(A\xi_n - A\xi^\star)],\ J_2((e+G)A\xi_n - (e+1)A\xi_n)\rangle$$

$$\leq \frac{\gamma^2\|A\|^2}{(e+1)^2}\|(G-1)A\xi_n\|^2 + 2k^2\|\xi_n - \xi^\star\|^2 - \frac{2\gamma}{(e+1)}\|(G-1)A\xi_n\|^2$$

$$+\frac{2\gamma}{(e+1)}\langle[e(A\xi_n - A\xi^\star) + GA\xi_n - GA\xi^\star,\ J_2((e+G)A\xi_n - (e+1)A\xi_n)\rangle$$

$$\leq \frac{\gamma^2\|A\|^2}{(e+1)^2}\|(G-1)A\xi_n\|^2 + 2k^2\|\xi_n - \xi^\star\|^2 - \frac{2\gamma}{(e+1)}\|(G-1)A\xi_n\|^2$$

$$+\frac{\gamma}{e+1}[\|e(A\xi_n - A\xi^\star r) + GA\xi_n - GA\xi^\star\|^2 + \|(G-I)A\xi_n\|^2]$$

$$\leq 2k^2\|\xi_n - \xi^\star\|^2 - \frac{\gamma}{e+1}(1-\gamma(e+1)\|A\|^2)\|(G-1)A\xi_n\|^2$$

$$+\frac{\gamma}{e+1}[(e+1)^2\|A\xi_n - A\xi^\star\|^2 + \vartheta\|(G-I)A\xi_n\|^2]$$

$$= [2k^2 + \gamma(e+1)\|A\|^2]\|\xi_n - \xi^\star\|^2 - \frac{\gamma}{e+1}(1-\mu-\gamma(e+1)\|A\|^2)\|(G-1)A\xi_n\|^2$$

$$\leq \|\xi_n - \xi^\star\|^2 - \frac{\gamma}{e+1}(1-\mu-\gamma(e+1)\|A\|^2)\|(G-1)A\xi_n\|^2. \tag{26}$$

Again, using (17), (26) and Lemma 2.1, we get

$$\|\xi_{n+1} - \xi^\star\|^2 = \|(1-\alpha_n)(\hbar_n - \xi^\star) + \alpha_n(\eth_\sigma\hbar_n - \xi^\star)\|^2$$

$$\leq (1-\alpha_n)\|\hbar_n - \xi^\star\|^2 + \alpha_n\|\eth_\sigma\hbar_n - \xi^\star\|^2 - \alpha_n(1\alpha_n)g(\|\hbar_n - \eth_\sigma^n\hbar_n\|)$$

$$= (1-\alpha_n)\|\hbar_n - \xi^\star\|^2 + \frac{\alpha_n}{(e+1)^2}\|e(\hbar_n - \xi^\star) + \eth\hbar_n - \eth^n\xi^\star\|^2 - \alpha_n(1-\alpha_n)g\left(\frac{1}{e+1}\|\hbar_n - \eth_\sigma^n\hbar_n\|\right)$$

$$\leq (1-\alpha_n)\|\hbar_n - \xi^\star\|^2 + \frac{\alpha_n \varpi_n}{e+1}\|\hbar_n - \xi^\star\|^2 - \alpha_n(1-\alpha_n) g\left(\frac{1}{e+1}\|\hbar_n - \eth_\sigma^n \hbar_n\|\right)$$

$$< \left(1 - \frac{\alpha_n}{e+1}\right)\|\hbar_n - \xi^\star\|^2 + \frac{\alpha_n \varpi_n}{e+1}\|_n - \xi^\star\|^2 - \alpha_n(1-\alpha_n) g\left(\frac{1}{e+1}\|\hbar_n - \eth_\sigma^n \hbar_n\|\right)$$

$$\leq \left(1 + \frac{\alpha_n(\varpi_n - 1)}{e+1}\right)\|\xi_n - \xi^\star\|^2 - \frac{\gamma}{e+1}(1 - \mu - \gamma(e+1)\|A\|^2)\|(G-1)A\xi_n\|^2$$

$$- \alpha_n(1-\alpha_n) g\left(\frac{1}{e+1}\|\hbar_n - \eth^n \hbar_n\|\right)$$

$$\leq \left(1 + \frac{\alpha_n(\varpi_n - 1)}{e+1}\right)\|\xi_n - \xi^\star\|^2, \tag{27}$$

which, by using the fact that $\sum_{n=1}^{\infty}(\varpi_n - 1) < \infty$, (26), and Lemma 2.3, leads to the conclusion that $\lim_{n\to\infty}\|\xi_n - \xi^\star\|$ exists. Consequently, $\{\xi_n\}_{n=1}^{\infty}$ is bounded.

Step (2), we show that $\lim_{n\to\infty}\|\xi_{n+1} - \xi_n\| = 0$ and $\lim_{n\to\infty}\|\hbar_{n+1} - \hbar_n\| = 0$. To do this, set

$$\Theta_n = \frac{\gamma}{e+1}(1 - \mu - \gamma(e+1)\|A\|^2)\|(G-1)A\xi_n\|^2 + \alpha_n(1-\alpha_n) g\left(\frac{1}{e+1}\|\hbar_n - \eth^n \hbar_n\|\right).$$

Then, we obtain from (27) that

$$\Theta_n \leq \left(1 + \frac{\alpha_n(\varpi_n - 1)}{e+1}\right)\|\xi_n - \xi^\star\|^2 - \|\xi_{n+1} - \xi^\star\|^2,$$

which subsequently yields

$$\lim_{n\to\infty}\|(G-I)A\xi_n\| = 0. \tag{28}$$

and

$$\lim_{n\to\infty} g\left(\frac{1}{e+1}\|\hbar_n - \eth^n \hbar_n\|\right) = 0. \tag{29}$$

By using Lemma 2.1 and the property of g, it follows from (29) that

$$\lim_{n\to\infty} \|\hbar_n - \eth^n \hbar_n\| = 0. \tag{30}$$

Since from (17)

$$\begin{aligned}\|\xi_{n+1} - \xi_n\| =& \|(\hbar_n - \xi_n) + \alpha_n(\eth^n_\sigma \hbar_n - \hbar_n)\| \\ \leq& \|\gamma J_1^{-1} A^\star J_2(\Im_\sigma - 1)A\xi_n\| + \alpha_n \|(\eth^n_\sigma \hbar_n - \hbar_n)\| \\ =& \frac{\gamma\|A\|}{e+1}\|(G-1)A\xi_n\| + \frac{\alpha_n}{e+1}\|(\eth^n \hbar_n - \hbar_n)\|,\end{aligned}$$

it follows from (28) and (30) that

$$\lim_{n\to\infty} \|\xi_{n+1} - \xi_n\| = 0. \tag{31}$$

Using a similar approach as in above, we get

$$\begin{aligned}\|\hbar_{n+1} - \hbar_n\| &\leq \|\xi_{n+1} - \xi_n\| + \|\gamma J_1^{-1} A^\star J_2(G_\sigma - 1)A\xi_{n+1}\| + \|\gamma J_1^{-1} A^\star J_2(G_\sigma - 1)A\xi_n\| \\ &\leq \|\xi_{n+1} - \xi_n\| + \frac{\gamma\|A\|}{e+1}\|(G-1)A\xi_{n+1}\| + \frac{\gamma\|A\|}{e+1}\|(G-1)A\xi_n\|,\end{aligned}$$

which on the application of (30) and (31), yields

$$\lim_{n\to\infty} \|\hbar_{n+1} - \hbar_n\| = 0. \tag{32}$$

Furthermore, since

$$\begin{aligned}\|\xi_n - \hbar_n\| = \|J_1(\xi_n - \hbar_n)\| &= \|\gamma A^\star J_2(G_\sigma - 1)A\xi_n\| \\ &\leq \frac{\gamma\|A\|}{e+1}\|(G-1)A\xi_n\|,\end{aligned}$$

it follows from (28) that

$$\lim_{n\to\infty} \|\xi_n - \hbar_n\| = 0. \tag{33}$$

Step (3), we show that $\lim_{n\to\infty}\|\hbar_n - ð\hbar_n\| = 0$.
Since ð is an enriched asymptotically nonexpansive mapping, by setting $L = \sup_{n\to\infty} \varpi_n$, it follows from (30) that

$$
\begin{aligned}
\|\hbar_n - \Im\hbar_n\| \leq & \|\hbar_n - \partial^n\hbar_n\| + \|\partial^n\hbar_n - \Im\hbar_n\| \\
= & \|\hbar_n - \partial^n\hbar_n\| + \left\|\partial\left(\partial^{n-1}\hbar_n\right) - \Im\hbar_n\right\| \\
= & \|\hbar n - \partial^n\hbar_n\| + \left\|e\left(\partial^{n-1}\hbar_n - \hbar_n\right) + \Im\left(\partial^{n-1}\hbar_n\right) - \Im\hbar_n - e\left(\partial^{n-1}\hbar_n - \hbar_n\right)\right\| \\
\leq & \|\hbar_n - \partial^n\hbar_n\| + \left\|e\left(\partial^{n-1}\hbar_n - \hbar_n\right) + \Im\left(\partial^{n-1}\hbar_n\right) - \Im\hbar_n\right\| + e\left\|\partial^{n-1}\hbar_n - \hbar_n\right\| \\
\leq & \|\hbar_n - \partial^n\hbar_n\| + (e+1)L\left\|\partial^{n-1}\hbar_n - \hbar_n\right\| + e\left\|\partial^{n-1}\hbar_n - \hbar_n\right\| \\
= & \|\hbar_n - \partial^n\hbar_n\| + [e(1+L)+L]\left\|\partial^{n-1}\hbar_n - \hbar_n\right\| \\
\leq & \|\hbar_n - \partial^n\hbar_n\| + [e(1+L)+L]\left\{\left\|\partial^{n-1}\hbar_n - \partial^{n-1}\hbar_{n-1}\right\| + \left\|\partial^{n-1}\hbar_{n-1} - \hbar_{n-1}\right\|\right. \\
& \left. + \|\hbar_{n-1} - \hbar_n\|\right\} \\
= & \|\hbar_n - \partial^n\hbar_n\| + [e(1+L)+L]\left\|\partial^{n-1}\hbar_n - \partial^{n-1}\hbar_{n-1}\right\| \\
& + [e(1+L)+L]\left\|\partial^{n-1}\hbar_{n-1} - \hbar_{n-1}\right\| + [e(1+L)+L]\|\hbar_{n-1} - \hbar_n\| \\
\leq & \|\hbar_n - \partial^n\hbar_n\| \\
& + [e(1+L)+L]\left\{\left\|e(\hbar_n - \hbar_{n-1}) + \partial^{n-1}\hbar_n - \partial^{n-1}\hbar_{n-1}\right\| + e\|\hbar_n - \hbar_{n-1}\|\right\} \\
& + [e(1+L)+L]\left\|\partial^{n-1}\hbar_{n-1} - \hbar_{n-1}\right\| + [e(1+L)+L]\|\hbar_{n-1} - \hbar_n\| \\
\leq & \|\hbar_n - \partial^n\hbar_n\| \\
& + [e(1+L)+L](e+1)L\|\hbar_n - \hbar_{n-1}\| + [e(1+L)+L]e\|\hbar_n - \hbar_{n-1}\|\} \\
& + [e(1+L)+L]\left\|\partial^{n-1}\hbar_{n-1} - \hbar_{n-1}\right\| + [e(1+L)+L]\|\hbar_{n-1} - \hbar_n\| \\
= & \|\hbar_n - \partial^n\hbar_n + [e(1+L)+L]\{(e+1)(1+L)\|\hbar_{n-1} - \hbar_n\| \\
& + \left\|\partial^{n-1}\hbar_{n-1} - \hbar_{n-1}\right\|\right\} \to 0 \quad (n\to\infty).
\end{aligned}
$$

That is,

$$\lim_{n\to\infty}\|_n - \eth\hbar_n\| = 0. \tag{34}$$

Step (4), we show that $\xi_n \rightharpoonup \xi^\star \in \Gamma$.

Since E_1 is a UCBS, it is reflexive. By considering the boundedness of $\{\xi_n\}_{n=1}^{\infty}$, there is a subsequence $\{\xi_{n_i}\}_{i=1}^{\infty}$ of $\{\xi_n\}_{n=1}^{\infty}$ for which $\xi_{n_i} \rightharpoonup \xi^\star$ as $i\to\infty$. By using (33), it follows that $\hbar_{n_i} \rightharpoonup \xi^\star \in \Gamma$ as well. Also, from (34),

$$\lim_{i\to\infty}\|\hbar_{n_i} - \eth\hbar_{n_i}\| = 0. \tag{35}$$

Following the demiclosedness property of the asymptotically nonexpansive mapping $\eth_\sigma$, we obtain

$$0 \leftarrow (I-\eth_\sigma)\xi_n = [1-((I-\sigma)I+\sigma)\eth]\xi_n = \sigma(I-\eth)\xi_n,$$

whenever $\{\xi_{n_i}\}_{i=1}^{\infty} \subset C$ with $\xi_n \rightharpoonup \xi^\star$. Consequently, $\xi^\star \in F(\eth)$.

On the other hand, in view of the fact that A is bounded, it follows that $A\xi_{n_i} \rightharpoonup A\xi^\star$ as $i\to\infty$. Thus, using (28), we obtain

$$\lim_{i\to\infty}\|(G-I)A\xi_{n_i}\| = 0. \tag{36}$$

From the demiclosedness property of G, we obtain that $A\xi^\star \in F(G)$. This, alongside the fact that $\xi^\star \in F(\eth)$, implies that $\xi^\star \in \Gamma$.

Now, we show that the convergence of the sequence $\{\xi_n\}_{n=1}^{\infty}$ to the point $\xi^\star \in \Gamma$ is unique. Here, we assume that $\{\xi_n\}_{n=1}^{\infty}$ admits another subsequence $\{\xi_{n_t}\}_{t=1}^{\infty}$ which converges to a point $\hbar^\star \in \Gamma$ different from $\xi^\star$ and then establish a contradiction. Indeed, considering Step 1 and the assumption that E_1 is endowed with the Opial property, we obtain

$$\begin{aligned}
\liminf_{n_i\to\infty}\|\xi_{n_i}-\xi^\star\| < \liminf_{n_i\to\infty}\|\xi_{n_i}-\hbar^\star\| &= \liminf_{n\to\infty}\|\xi_n-\hbar^\star\| \\
&= \liminf_{n_i\to\infty}\|\xi_{n_i}-\hbar^\star\| < \liminf_{n_i\to\infty}\|\xi_{n_i}-\xi^\star\| \\
&= \liminf_{n\to\infty}\|\xi_n-\xi^\star\| = \liminf_{n_i\to\infty}\|\xi_{n_i}-\xi^\star\|,
\end{aligned} \tag{37}$$

which is a contradiction. Hence, $\{\xi_n\}_{n=1}^{\infty}$ converges weakly to $\xi^\star$.

Step (5), we establish the validity of (II).

Indeed, since $\lim_{n\to\infty}\|(I-\eth)_n\| = 0$ and $\eth$ is semicompact, we can find a subsequeence $\{\hbar_{n_i}\}_{i=1}^{\infty}$ of $\{\hbar_n\}_{n=1}^{\infty}$ such that $\hbar_{n_i} \to \vartheta^\star \in E_1$. By (33), it is clear that the subsequence $\{\xi_{n_i}\}_{i=1}^{\infty}$ of $\xi_n \to \vartheta^\star$ too. Since $\{\hbar_n\}_{n=1}^{\infty}$ converges weakly to $\xi^\star$, it follows that $\xi^\star = \vartheta^\star$. Since $\liminf_{n\to\infty}\|\xi_n-\xi^\star\|$ exists and $\liminf_{i\to\infty}\|\xi_{n_i}-\xi^\star\| = 0$, it follows that $\wp_n \to \xi^\star \in \Gamma$' This establishes the validity of (II).

4. Application

Now, we apply our result to study the hierarchical variational inequality problem (HVIP, for short) in Banach space.

Let E be a Strictly Convex and Real reflexive Banach Space (SCRBS). Let C be a nonempty, closed, and convex subset of E. Then, for any $\xi \in E$, there exists a unique point $\hbar \in C$ that guarantees the inequality

$$\|\xi - \hbar\| \leq \|\xi - \omega\|, \quad \forall \omega \in C.$$

Putting $\hbar = P_C\xi$, we refer to P_C as the metric projection of E onto C.

Lemma 4.1 [31] Let E be a SCRBS and C be a nonempty, closed, and convex subset of E. Let $\xi \in E$ and $\hbar \in C$. Then, the subsequent conditions are equivalent:

(1) $\hbar \in P_C\xi$;

(2) $\langle \hbar - \omega, J(\xi - \hbar)\rangle \geq 0$, $\forall \omega \in C$.

Definition 4.1 Let E be a SCRBS and C, $Q \subset E$ be two nonempty, closed, and convex subsets of E. Let $\eth$, $\backepsilon : C \longrightarrow C$ be two nonlinear mappings which assure that $F(\eth)$ is a nonempty, closed, and convex subset of C. The HVIP for a mapping $\eth$ with respect to the mapping $\backepsilon$ in a real Banach space E is to find a point $\xi^\star \in F(\eth)$ satisfying the inequality

$$\langle \xi^\star - \omega, J(\backepsilon\xi^\star - \xi^\star)\rangle \geq 0, \ \forall \omega \in F(\eth). \tag{38}$$

By Lemma 4.1, the HVIP (38) in Banach space is equivalent to the fixed point equation below:

$$\xi^\star = P_{F(\eth)}\backepsilon(\xi^\star). \tag{39}$$

Let $C = F(\eth)$, $Q = F(P_{F(\eth)}\backepsilon)$ (the fixed point set of $P_{F(\eth)}\backepsilon$), and $A = I$ (the identity mapping on E). Then, the HVIP of problem (38) for a mapping $\eth$ in respect of the mapping $\backepsilon$ in a Banach space is equivalent to the SCFPP below:

$$\text{search for} \quad \xi^\star \in C \quad \text{which guarantees} \quad \xi^\star \in Q. \tag{40}$$

Consequently, the set of solution Γ_1 of the HVIP (38) is the same as the set of solutions to the SCFPP (40).

Hence, the following theorem is a direct consequence of Theorem 3.1.

Theorem 4.1 Let E be a UCBS and 2-USBS endowed with the Opial property and the best smoothness constant $0 < k < \frac{1}{\sqrt{2}}$. Let $\eth : E \longrightarrow E$ be an $(e,\ \{\varpi_n\}_{n=1}^{\infty})$-EANM with $\{\varpi_n\}_{n=1}^{\infty} \subset [1,\ \infty)$ such that $\varpi_n \to 1$ as $n \to \infty$ and $F(\eth) \neq \emptyset$. Let $\backepsilon : E \longrightarrow E$ be a mapping such that the mapping $G = P_{F(\eth)}\backepsilon$ is an $(e,\ \mu)$-EQSPM with $F(G) \neq \emptyset$ and G is demiclosed at zero. For each $\xi_1 \in E$, $\{\xi_n\}_{n=1}^{\infty}$ is a sequence generated from

$$\begin{cases} \hbar_n = \xi_n + \gamma(G_\sigma - 1)\xi_n, \\ \xi_{n+1} = (1-\alpha_n)\hbar_n + \alpha_n \eth_\sigma^n \hbar_n, \quad n \geq 1, \end{cases} \tag{41}$$

where $G_\sigma = (1-\sigma)I + \sigma G$, $\eth_\sigma^n = (1-\sigma)I + \sigma\eth^n$, $\{\alpha_n\}_{n=1}^{\infty} \subset (0,\ 1)$ with $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, and γ is being controlled by the inequality: $0 < \gamma < \min\left\{\frac{1-2k^2}{(e+1)},\ \frac{1-\pi}{(e+1)}\right\}$, $\{\varpi_n\}_{n=1}^{\infty} \subset [1,\ \infty)$ with $L = \sup_{n\to\infty} \varpi_n$ and $\sum_{n=1}^{\infty} (\varpi_n - 1) < \infty$.

(I) If $\Gamma_1 = \{\xi^\star \in F(\eth) : A\xi^\star \in F(G)\}$, then $\xi_n \rightharpoonup \xi^\star \in \Gamma$.

(II) In addition, if $\Gamma_1 = \{\xi^\star \in F(\eth) : A\xi^\star \in F(G)\} \neq \emptyset$ and $\eth$ is semicompact, then $\xi_n \to \xi^\star \in \Gamma_1$ as $n \to \infty$.

Proof. By setting $E_1 = E_2 = E,\ A = I,\ G = P_F(\eth) \circ \partial$ in Theorem 3.1, and noticing that $J_1 = J_2 = J$, the conclusion follows immediately from Theorem 3.1. □

5. Numerical examples

Example 5.1 [5] Let $E = \mathbb{R}^2$ be equipped with the Euclidean norm and

$$C = \{(\xi_1,\ \xi_2) \in \mathbb{R}^2,\ \xi_1,\ \xi_2 \geq 0,\ \xi_1^2 + \xi_2^2 \leq 1\}.$$

Let $G : C \longrightarrow C$ be given as

$$G(\hbar,\ \xi) = \left(\frac{\hbar}{2},\ \frac{\xi}{2}\right). \tag{42}$$

Notice that C a bounded, closed, and convex subset of the uniformly convex Banach space E. Let $e \in [0,\ +\infty)$ *and* $\vartheta \in [0,\ 1)$. Then, for all $\hbar,\ \xi \in C$, we have

$$\begin{aligned}
\| e(\hbar - \xi) + G\hbar - \xi \|^2 &= \| e[(\hbar_1,\ \xi_1) - (\hbar_2,\ \xi_2)] + G(_1,\ \xi_1) - G(\hbar_2,\ \xi_2) \|^2 \\
&= \left\| e(\hbar_1 - \hbar_2,\ \xi_1,\ \xi_2) + (\frac{\hbar_1 - \hbar_2}{2},\ \frac{\xi_1 - \xi_2}{2}) \right\|^2 \\
&= \left\| e(\hbar_1 - \hbar_2) + \frac{\hbar_1 - \hbar_2}{2},\ e(\xi_1 - \xi_2) + \frac{\xi_1 - \xi_2}{2} \right\|^2 \\
&= \left\| \frac{2e\hbar_1 - 2e\hbar_2 + \hbar_1 - \hbar_2}{2},\ \frac{2e\xi_1 - 2e\xi_2 + \xi_1 - \xi_2}{2} \right\|^2 \\
&= \left\| \frac{(2e+1)\hbar_1 - (2e+1)\hbar_2}{2},\ \frac{(2e+1)\xi_1 - (2e+1)\xi_2}{2} \right\|^2 \\
&= \left(\frac{2e+1}{2}\right)^2 \| \hbar_1 -_2,\ \xi_1 - \xi_2 \|^2 \\
&= \left(\frac{2e+1}{2}\right)^2 \| (\hbar_1,\ \xi_1) - (\hbar_2,\ \xi_2) \|^2 .
\end{aligned} \tag{43}$$

Also, by setting $Q = (e+1)^2 \| \hbar - \xi \|^2 + \vartheta \| \hbar - \xi - (G\hbar - G\xi) \|^2$, we obtain that

$$
\begin{aligned}
Q =&(e+1)^2 \parallel (\hbar_1,\ \xi_1) - (\xi_1,\ \xi_2) \parallel^2 + \vartheta \left\| (\hbar_1,\ \xi_1) - (\hbar_2,\ \xi_2) - \left(\frac{\hbar_1 - \hbar_2}{2},\ \frac{\xi_1 - \xi_2}{2} \right) \right\|^2 \\
=&(e+1)^2 \parallel (\hbar_1,\ \xi_1) - (\xi_1,\ \xi_2) \parallel^2 + \vartheta \left\| (\hbar_1 - \hbar_2,\ \xi_1 - \xi_2) - \left(\frac{\hbar_1 - \hbar_2}{2},\ \frac{\xi_1 - \xi_2}{2} \right) \right\|^2 \\
=&(e+1)^2 \parallel (\hbar_1,\ \xi_1) - (\xi_1,\ \xi_2) \parallel^2 + \vartheta \left\| \left(\hbar_1 - \hbar_2 - \left(\frac{\hbar_1 - \hbar_2}{2} \right) \right) - \left(\xi_1 - \xi_2 - \left(\frac{\xi_1 - \hbar_2}{2} \right) \right) \right\|^2 \\
=&(e+1)^2 \parallel (\hbar_1,\ \xi_1) - (\xi_1, xi_2) \parallel^2 + \vartheta \left\| \frac{2\hbar_1 - 2\hbar_2 - \hbar_1 +_2}{2} - \frac{2\xi_1 - 2\xi_2 - \xi_1 + \xi_2}{2} \right\|^2 \\
=&(e+1)^2 \parallel (\hbar_1,\ \xi_1) - (\xi_1,\ \xi_2) \parallel^2 + \vartheta \left\| \frac{\hbar_1 - \hbar_2}{2} - \frac{\xi_1 - \xi_2}{2} \right\|^2 \\
=&(e+1)^2 \parallel (\hbar_1,\ \xi_1) - (\xi_1,\ \xi_2) \parallel^2 + \frac{\vartheta}{4} \parallel \hbar_1 - \hbar_2,\ \xi_1 - \xi_2 \parallel^2 \\
=&(e+1)^2 \parallel (\hbar_1,\ \xi_1) - (\hbar_1,\ \hbar_2) \parallel^2 + \frac{\vartheta}{4} \parallel (\hbar_1,\ \xi_1) - (\hbar_2,\ \xi_2) \parallel^2 \\
=&\left[(e+1)^2 + \frac{\vartheta}{4} \right] \parallel (\hbar_1,\ \xi_1) - (\hbar_2,\ \xi_2) \parallel^2 .
\end{aligned}
\tag{44}
$$

Equations (43) and (44) imply

$$
\begin{aligned}
&\parallel e(\hbar - \xi) + G\hbar - G\xi \parallel^2 \\
&= \left(\frac{2e+1}{2} \right)^2 \parallel (\hbar_1,\ \xi_1) - (\hbar_2,\ \hbar_2) \parallel^2 \\
&\leq \left[(e+1)^2 + \left(\frac{\vartheta}{4} \right) \right] \parallel (\hbar_1,\ \xi_1) - (\hbar_2,\ \hbar_2) \parallel \\
&= (e+1)^2 \parallel \hbar - \xi \parallel^2 + \vartheta \parallel \hbar - \xi - (G\hbar - G\xi) \parallel^2 .
\end{aligned}
\tag{45}
$$

Therefore, G is an $(e,\ \vartheta)$-enriched strictly pseudocontractive mapping. Further, observe that $F(G) = (0,\ 0)$, so that G becomes an $(e,\ \vartheta)$-enriched strictly quasi-pseudocontractive mapping.

Example 5.2 Let G be a selfmap of $\mathbb{R}$. For each $\xi \in \mathbb{R}$, define

$$G\xi = -3\xi. \tag{46}$$

Here G is an $(e,\ \vartheta)$-enriched strictly pseudocontractive mapping with $e = 0$ and $F(G) = \{0\}$.

Let $E = R$ and $\eth : R^2 \longrightarrow R^2$ be a mapping defined by

$$G(\xi_1,\ \xi_2) = \frac{1}{\sqrt{2}}(\xi_1,\ \xi_2), \quad \forall \xi \in R^2, \tag{47}$$

where $\xi = (\xi_1,\ \xi_2)$. In Example 3.4, we have proved that G is an $(e,\ \{\varpi_n\}_{n=1}^{\infty})$-EANM and $F(G) = (0,\ 0)$. Therefore, Theorem 5.1 below follows from Theorem 3.1.

Theorem 5.1 Let $E_1 = E_2 = R^2$, C be as described above. Let $\eth : C \longrightarrow C$ $(e,\ \{\varpi_n\}_{n=1}^{\infty})$-EANM and Let $G : C \longrightarrow C$ be the $(e,\ \vartheta)$-ESPM given by (45) and (46), respectively. Let $A : C \longrightarrow C$ be a bounded linear operator and $A^{\star}$ is its adjoint. For each $\xi_1 \in C$, let $\{\xi_n\}_{n=1}^{\infty}$ be a sequence developed from

$$\begin{cases} \hbar_n = \xi_n + \gamma A^{\star}(G_\sigma - 1)A\xi_n, \\ \xi_{n+1} = (1-\alpha_n)\hbar_n + \alpha_n \eth_\sigma^n \hbar_n, \quad n \geq 1, \end{cases}$$

where $G_\sigma = (1-\sigma)I + \sigma G$, $\eth_\sigma^n = (1-\sigma)I + \sigma\eth^n$ and $\{\alpha_n\}_{n=1}^{\infty} \subset (0,\ 1)$ with $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and γ being controlled by the inequality $0 < \gamma < \min\left\{\dfrac{1-2k^2}{(e+1)\|A\|^2},\ \dfrac{1-\pi}{(e+1)\|A\|^2}\right\}$. If $\sum_{n=1}^{\infty}(\varpi_n - 1) < \infty$, then the results of Theorem 3.1 follows.

6. Conclusions

This paper initiated the concept of $(e,\ \{\varpi_n\}_{n=1}^{\infty})$-EANM in a real Banach space domain. Further, we used this concept to study the split common fixed point problem of enriched strictly pseuocontractive and asymptotically nonexpansive mappings, and subsequently showed that weak and strong convergence theorems of the SCFPP for $(e,\ \vartheta)$-ESPM and $(e,\ \{\varpi_n\}_{n=1}^{\infty})$-EANM could be established in such spaces using a modified Ishkawa iterative method. As the classes of enriched strictly pseudocontractive and asymptotically nonexpansive mappings cover the classes of strictly pseudocontractive and asymptotically nonexpansive mappings, our result contributes in the following ways:

1. The mappings we considered in this paper are larger than the mappings considered for the results in [4, 5, 8, 21].
2. The problems considered by Cui et al. [21] and Tang et al. [24] are consequences of our main result.

As an application, the iteration sequence studied in this paper was shown to be suitable for approximating the solution of the hierarchical variational inequality problem which is clearly more general than classical variational inequality problem and the fixed point problem. Our theorem therefore improves, complements, and unifies the results of [11, 24] and several other results announced recently.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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