Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Reckoning Common Fixed Point of Enriched Pseudocontractive Mappings: An Iterative Approach

Doaa Filali[1], Mohammad Dilshad[2], Imo Kalu Agwu[3], Mohammad Akram[4*]

[1] Department of Mathematical Science, College of Sciences, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh, 11671, Saudi Arabia
[2] Department of Mathematics, University of Tabuk, Tabuk, 71491, Saudi Arabia
[3] Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Umuahia Abia State Nigeria
[4] Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah, 42351, Saudi Arabia
E-mail: akramkhan_20@rediffmail.com

Received: 23 July 2025; **Revised:** 17 October 2025; **Accepted:** 17 October 2025

Abstract: We extend the notion of (a, k)-enriched strictly pseudocontractive mappings to the notion of the more general a-enriched pseudocontractive mappings. It is shown with examples that the class of a-enriched pseudocontractive mappings is more general than the classes of (a, k)-enriched strictly pseudocontractive and pseudocontractive mappings. Some fundamental properties of the class a-enriched pseudocontractive mappings are proved. In particular, it is shown that the fixed point set of certain class of a-enriched pseudocontractive self-mappings of a nonempty closed convex subset of a real Hilbert space is closed and convex. Demiclosedness property of such class of a-enriched pseudocontractive mappings is proved. Certain strong convergence theorems are then proved for the iterative approximation of fixed points of the class a-enriched pseudocontractive mappings.

Keywords: fixed point, enriched nonlinear mapping, pseudocontractive mapping, accretive operators, closed ball, Lipschitizian, iterative scheme, convergence

MSC: 47H09, 47H10, 47J05, 65J15

1. Introduction

The notations $\mathbb{N}$, $\mathbb{R}$ and C will denote the set of natural numbers, the set of real numbers and a nonempty closed and convex subset of a real Hilbert space H, respectively. We denote the set $\{r \in H\colon Tr = r\}$ of fixed point of T by $F(T)$. Solving a fixed point problem of a mapping T is to show that $F(T) \neq \emptyset$. Fixed Point Theory (FPT) is one of the most powerful and fruitful tools in modern mathematics and a core subject of nonlinear analysis. In FPT, Banach's fixed-point theorem played a vital role in solving equation of the form $Tr = r$, where T is a nonlinear mapping. It is well known that most of the nonlinear equations can be transformed into fixed-point problem.

Banach's fixed-point theorem has attracted and continued to attract the attention of several researches due to its applications in varieties of scientific problems, such as equilibrium problems, selection and matching problems, image processing (see, for example, [1]), the study of existence and uniqueness of solutions of integral and differential equations.

DOI: https://doi.org/10.37256/cm.6620257960

Recall that differential equation, whose essence is preeminent in mathematics, science and engineering, provides a robust model that can be used to study several systems, including celestial mechanics and biological processes (see, for example, [2]). Aside this, fixed point theory provides mathematical tools, like the Brouwer's and Kakutani's fixed-point theorems, to prove the existence of market and economic equilibria, where supply matches demand (see, for example, [1, 3]). Further, in the context of Banach algebra, the approximation of fixed point of the product of two operators has been involved in addressing complex equations (see, for example, [4]); fixed point theorem has also contributed consequential platforms to model self-similarity, convergence behavour, memory effects and nonlinear dynamics that occur in a variety of deciplines (see, for example, [1] for more detail). Admittedly, the recurring expansion in the field of nonlinear operators and nonlinear analysis is intimately connected to fixed point theory. The above illustrations underscore the richness and diversity of fixed point theorems in real-world applications.

Fixed point theorems for nonexpansive mappings and their invariant points have been known to be instrumental in convex feasibility problems, differential equations, game theory, economy, optimal control, dynamics and several other physical problems. Consequently, considerable research effort has been made toward generalization of nonexpansive mappings.

In [5], Saleem et al. studied the class of (a, Φ_T)-enriched Lipschtzian mappings in real Banach spaces. They referred to a mapping T as (a, Φ_T)-enriched Lipschtzian if, for all $r, s \in C$, there exist $a \in [0, \infty)$ and a continuous nondecreasing function $\Phi_T\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\Phi(0) = 0$ such that

$$\|a(r-s)+Tr-Ts\| \leq (a+1)\Phi_T(\|r-s\|). \tag{1}$$

Note that if $a = 0$ and $\Phi_T(\omega) = L\omega$, for $L > 0$, then T is called L-Lipschitzian, where L is the Lipschitz constant. Specifically, if $a = 0$, $\Phi_T(\omega) = L\omega$ and $L = 1$, then T is referred to as nonexpansive. However, if $a > 0$ and $b = (1+a)^{-1}$, then $b \in (0, 1)$. In this case, (1) is equivalently written as

$$\|(1-b)r+bTr-((1-b)s+bTs)\| \leq \Phi_T(\|r-s\|). \tag{2}$$

For each $r, s \in C$, (2) can also be written as

$$\|T_br-T_bs\| \leq \Phi_T(\|r-s\|), \tag{3}$$

where the average mapping $T_b = (1-b)I+bT$ is Φ_T-Lipschitizian in the terminology of Hicks and Kubecek [6].

Remark 1 Every Lipschitz mapping is invariably Φ_T-Lipschitzian but the converse may not be true; see, for example, Hicks and Kubecek [6]. Also, every Φ_T-Lipschitz mapping is $(0, \Phi_T)$-enriched Lipschitz mapping. Again, if Φ_T is not necessarily nondecreasing and fulfills $\Phi_T(\omega) < \omega$, for $\omega > 0$, then T is called a nonlinear contraction on C.

Example 1 Let T, a self-map of $\mathbb{R}$, be given as

$$Tr = \sqrt{|r|} \quad \forall r \in \mathbb{R}.$$

Consider $\Phi_T(\omega) = \sqrt{\omega}$, $\omega \geq 0$. It can easily be seen that Φ_T is continuous and nondecreasing. Observe that T is subadditive. Indeed, let $r, s \in \mathbb{R}$. Then.

$$\begin{aligned}(T(r+s))^2 &= |r+s| \\ &\leq (\sqrt{|r|}+\sqrt{|s|})^2 \\ &= (Tr+Ts)^2.\end{aligned}$$

By the subadditivity property of T, the last inequality yields

$$|Tr-Ts| \leq T(r-s) = \Phi_T(|r-s|).$$

Therefore, T is $(0, \Phi_T)$-enriched Lipschitzian mapping with Φ_T as the Φ_T-function. Consider T as being Lipschitizian with Lipschitiz constant $L > 0$. Then, $\forall r, s \in \mathbb{R}$, with $s = 0$ and $r \neq 0$, we obtain $Tr \leq L|r|$. Hence, $\forall r \neq 0$, $L \geq \dfrac{1}{\sqrt{|r|}}$. Lettting $r \to 0$, we get a contradiction. Consequently, T is not Lipschitizian mapping.

Now, if we approach the above selection of a in a more holistic way, say by giving a an arbitrary point in $[0, \infty)$ and considering $\Phi_T(\omega) = \omega$, then (1) reduces to

$$\|a(r-s)+Tr-Ts\| \leq (a+1)\|r-s\|, \tag{4}$$

and it is called an a-Enriched Nonexpansive Mapping (a-ENM). This class of mappings was initiated in [7] as a superior category of certain class of nonexpansive mappings.

In [8], Berinde introduced a class of nonexpansive-type mappings which is more general than the one considered in [7]. In the words of Browder and Petryshyn [9], he referred to a selfmap T of C as being (a, k)-Enriched Strictly Pseudocontractive Mapping ((a, k)-ESPM) if $\forall r, s \in C$ there exist $a \in [0, \infty)$ and $k \in [0, 1)$ such that,

$$\|a(r-s)+Tr-Ts\|^2 \leq (a+1)^2\|r-s\|^2 + k\|r-s-(Tr-Ts)\|^2. \tag{5}$$

They established some inclusive properties of this map in respect of a-ENM and k-Strictly Pseudocontractive Mappings (k-SPM) (Recall that a mapping T is called (k-SPM) if $\forall r, s \in C$, we can find a $k \in [0, 1)$ which assures $\|Tr-Ts\|^2 \leq \|r-s\|^2 + k\|r-s-(Tr-Ts)\|^2$. Notice that if $k = 1$, we obtain a pseudocontraction). In [8], it was shown that a Krasnoselskii's type sequence (see [10]) converges strongly to the fixed point of (a, k)-ESPM.

Remark 2 We observe from (5) that if $a = 0$ and $k = 1$, then we obtain a class of mappings defined by

$$\|Tr-Ts\|^2 \leq \|r-s\|^2 + \|r-s-(Tr-Ts)|^2 \tag{6}$$

and it is called a pseudocontraction. This class of mappings is intimately connected to the class of monotone mappings, where a selfmap A of C is considered to be monotone if it satisfies the inequality

$$\langle r-s, Ar-As\rangle \geq 0, \quad \forall r, s \in C. \tag{7}$$

It is on record (see, for instance, Browder and Petryshyn [9]) that T is a pseudocontraction provided $A = I - T$ is a monotone. As a consequence, $F(T) = \{r \in C: Tr = r\} = N(A) = \{r \in C: Ar = 0\}$, where $N(A)$ denotes the set of zeros of A. It is also on record (see, for example, [11]) that if A is monotone, then the points that solve $Ar = 0$ corresponds to the stability stance of certain evolution system.

Subsequently, several results have been announced in the domain of enriched nonlinear maps. Agwu and Igbokwe [12] proved demiclosedness principle and further obtained weakly and strongly convergent sequences of Mann and Ishikawa-type for (a, k)-ESPM. Razzaque et al. [13] studied multiple-set split feasibility problems for a finite family of (a, k)-ESPM in real Hilbert spaces, Agwu et al. [14] introduced a method that initiated approximation procedures for common fixed points of finite families (a, k)-ESPM and β-enriched strictly pseudononspreading mappings, and strong convergence result was later established in real Hilbert space. Agwu et al. [15] established the fixed point result for enriched pseudocontractive mappings, Berinde and Saleh [16] presented new average type iterative schemes for exploring split common fixed point of demicontractive mappings in the Hilbert spaces. Agwu et al. [17] initiated the notion of β-enriched strictly pseudo-nonspreading mappings and proved convergence theorems of the sequence initiated by Halpern type schemes. Their results solved the open problem posed by Kurokawa and Takahashi in [18]. They further established some basic properties of the map studied.

In many results involving (a, k)-enriched strictly pseudocontractive mappings, it is discovered that k is confined in the set $(0, 1)$. This restriction limits the applicability of the map in different settings. Motivated by the need to broaden the domain of k and the results of the authors mentioned above, it is our purpose in this paper to introduce a new nonlinear mapping called a-enriched pseudocontractive mapping. Clearly, the class of enriched nonlinear mappings, having been proven to be more general than the class of mappings from which they inherited their family name, forms richer bases in applications. Inspired by this and following the conclusions reached in [19], our contribution in this paper is as follows:

(1) We further generalize the concept of (a, k)-ESPM to that of a-Enriched Pseudocontractive Mappings (a-EPM) in order to extend and generalize fixed point theory to a broader class of functions, leading to more stable and efficient algorithms for solving nonlinear equations and optimization problems. It is believed that the new concept will provide a more flexible structure that can be applied to problems where neither pseudocontractive nor (a, k)-enriched strictly pseudocontractive mappings may not be suitable, thus improving convergent properties and widening applicability.

(2) We illustrate with examples that a-EPM is larger than (a, k)-ESPM.

(3) we established some basic properties of a-EPM. Specifically, for Lipschitz a-ESPM, we established that the fixed point set is closed and convex. Additionally, we proved demiclosedness principle for the class of mappings studied.

(4) We established strong convergence results involving Ishikawa-type sequence embedded with a-EPM. Finally, we conducted a numerical experiment to confirm our convergence results.

The rest of the paper is organized as follows: Section 2 contains the preliminary concepts and results regarding fixed point theory. Section 3 comprises strong convergence result of the proposed algorithm. Section 4 deals with the numerical example to support our convergence result.

2. Preludes

Next, we present the following lemmas in order to accomplish the goal.

Lemma 1 [19] For given elements $r, s \in H$, the inequality below remains valid:

$$\|r+s\|^2 \leq \|r\|^2 + 2\langle s, r+s\rangle.$$

Lemma 2 [20] Consider C as a convex subset of H. Let $r \in H$. Then, $r_0 = P_C r$ if and only if

$$\langle t - r_0, r_0 - r\rangle \geq 0, \quad \forall t \in C.$$

Lemma 3 [20] Let $\{\alpha_m\}$ be a sequence of nonnegative real numbers guaranteeing the inequality

$$\alpha_{m+1} \leq (1-\alpha_m)a_m + \alpha_m \delta_m, \quad m \geq m_0,$$

where $\{\alpha_m\} \subset (0, 1)$ and $\{\delta_m\} \subset \mathbb{R}$ fulfill the following condition: $\lim_{m\to\infty} \alpha_m = 0$, $\sum_{m=1}^{\infty} \alpha_m = \infty$, and $\limsup_{m\to\infty} \delta_m \leq 0$. Then, $\lim_{m\to\infty} a_m = 0$.

Lemma 4 [21] Let $\{a_m\}$ be a sequence of real numbers that assures the existence of a subsequence $\{m_i\}$ of $\{m\}$ for which $a_{m_i} < a_{m_i+1}$ $\forall i \in N$. Consider the integer $\{n_s\}$ given by

$$n_s = \max\{t \leq s \colon a_t < a_{t+1}\}.$$

Then, $\{n_s\}$ is a nondecreasing sequence fulfilling $\lim_{n\to\infty} n_n = \infty$, and for every $t \in \mathbb{N}$, the subsequent inequalities are validated:

$$a_{n_s} \leq a_{n_s+1} \quad \text{and} \quad a_s \leq a_{n_s+1}.$$

Lemma 5 [22] Consider a Hilbert space H. For $(\{r_t\}_{t=1}^{\infty}, \{\alpha_t\}_{t=1}^{\infty}) \in H \times [0, 1]$, the following equality is validated

$$\|\alpha_0 r_0 + \alpha_1 r_1 + \cdots + \alpha_t r_t\|^2 = \sum_{k=0}^{t} \alpha_k \|r_k\|^2 - \sum_{0\leq k,\, i\leq t} \alpha_k \alpha_i \|r_k - r_i\|^2.$$

Lemma 6 [17] Consider a Hilbert space H. Then
(i) $\|as + (1-a)r\|^2 = a\|s\|^2 + (1-a)\|r\|^2 - a(1-a)\|s-r\|^2$ for all $s, r \in H$ and $a \in [0, 1]$.
(ii) $\|s+r\|^2 \leq \|s\|^2 + 2\langle r, s+r\rangle$, for all $s, r \in H$.
(iii) If $\{s_n\}$ is a sequence in H such that $s_n \rightharpoonup q \in H$, then

$$\limsup_{n\to\infty} \|s_n - s\|^2 = \limsup_{n\to\infty} \|s_n - q\|^2 + \|q-s\|^2, \quad \forall s \in H.$$

3. Results and discussion

Now, we shall present our main results of this paper.

Definition 1 Let H be a Banach space and $C \subset H$. A selfmap T of C is known as a-EPM if for every $r, s \in C$, we can find an $a \in [0, \infty)$ that verifies the inequality:

$$\|a(r-s) + Tr - Ts\|^2 \leq (a+1)^2\|r-s\|^2 + \|(I-T)r - (I-T)s\|^2. \tag{8}$$

Note that 0-enriched pseudocontractive mappings is pseudocontractive with $a = 0$. Therefore, inequality (8) reduces to inequality (1) when $a = 0$.

Remark 3 Set $a = \frac{1}{b} - 1 \in [0, \infty)$ for $b \in (0, 1]$. Then, inequality (8) becomes

$$\|a(r-s)+Tr-Ts\|^2 \leq (a+1)^2\|r-s\|^2+\|(I-T)r-(I-T)s\|^2$$

$$\Leftrightarrow \quad \left\|\frac{(1-b)}{b}(r-s)+Tr-Ts\right\|^2 \leq \frac{1}{b^2}\|r-s\|^2+\|(I-T)r-(I-T)s\|^2$$

$$\Leftrightarrow \quad \|(1-b)r+bTr-[(1-b)s+Ts]\|^2 \leq \|r-s\|^2+\|(I-T)r-(I-T)s\|^2$$

$$\Leftrightarrow \quad \|T_br-T_bs]\|^2 \leq \|r-s\|^2+\|(I-T)r-(I-T)s\|^2,$$

where $T_b = (1-b)I+bT$ with I representing the identity operator on C.

It, therefore, becomes clear from Remark 3 that the average operator T_b is a pseudocontraction whenever T is an enriched pseudocontractive mappings.

The existence of the class of an $(a,\, L)$-EPM is illustrated below.

Example 2 Consider $H = \mathbb{R}$, with the usual norm. Given that the selfmap T of $\mathbb{R}$ as

$$T\hat{r} = \begin{cases} \hat{r}+\hat{r}^2, & \text{if} \quad \hat{r} \in [-2,\, 0] \\ \hat{r}, & \text{if} \quad \hat{r} \in (0,\, 1]. \end{cases}$$

Clearly, $F(T) = 0$. It is interesting to know that T is 0-enriched pseudocontractive mapping. To see this, observe that if $\hat{r},\, \hat{s} \in [-2,\, 0]$, then, we get

$$|a(\hat{r}-\hat{s})+T\hat{r}-T\hat{s}|^2 = |0(\hat{r}-\hat{s})+T\hat{r}-T\hat{s}|^2 = |1+\hat{r}+\hat{s}|^2|\hat{r}-\hat{s}|^2$$

and

$$|\hat{r}-T\hat{r}-(\hat{s}-T\hat{s})|^2 = |\hat{r}-(\hat{r}+\hat{r}^2)|^2 = |-(\hat{r}^2-\hat{s})|^2 = |\hat{r}^2-\hat{s}^2|^2 = (\hat{r}-\hat{s})^2(\hat{r}+\hat{s})^2.$$

Thus,

$$\begin{aligned} (0+1)^2|\hat{r}-\hat{s}|^2+|\hat{r}-T\hat{r}-(\hat{s}-T\hat{s})|^2 &= (\hat{r}-\hat{s})^2+(\hat{r}-\hat{s})^2(\hat{r}+\hat{s})^2 \\ &= (1+(\hat{r}+\hat{s})^2)|\hat{r}-\hat{s}|^2 \\ &\geq |1+\hat{r}+\hat{s}|^2|\hat{r}-\hat{s}|^2 \\ &= |0(\hat{r}-\hat{s})+T\hat{r}-T\hat{s}|^2. \end{aligned}$$

If $\hat{r},\ \hat{s} \in (0,\ 1]$, then

$$|0(\hat{r}-\hat{s})+T\hat{r}-T\hat{s}|^2 = |\hat{r}-\hat{s}|^2 = |\hat{r}-\hat{s}|^2+0 = (0+1)^2|\hat{r}-\hat{s}|^2+|\hat{r}-T\hat{r}-(\hat{s}-T\hat{s})|^2.$$

Lastly, if $\hat{r} \in [-2,\ 0]$ and $\hat{s} \in (0,\ 1]$, then

$$|0(\hat{r}-\hat{s})+T\hat{r}-T\hat{s}|^2 = |\hat{r}+\hat{r}^2-\hat{s}|^2 = |\hat{r}-\hat{s}+\hat{r}^2|^2$$

and

$$|\hat{r}-T\hat{r}-(\hat{s}-T\hat{s})|^2 = \hat{r}^2.$$

Thus,

$$\begin{aligned}(0+1)^2|\hat{r}-\hat{s}|^2+|\hat{r}-T\hat{r}-(\hat{s}-T\hat{s})|^2 &= |\hat{r}-\hat{s}|^2+\hat{r}^2 \geq |\hat{r}-\hat{s}+\hat{r}^2|^2 \\ &= |0(\hat{r}-\hat{s})+T\hat{r}-T\hat{s}|^2,\end{aligned}$$

which shows that the mapping 0-enriched pseudocontractive.

Next, we show T is 0-enriched Lipschitzian with $L = 5$. If $\hat{r},\ \hat{s} \in [-2,\ 0]$, then we have

$$|0(\hat{r}-\hat{s})+T\hat{r}-T\hat{s}| = |\hat{r}+\hat{r}^2-\hat{s}-\hat{r}^2| = |(\hat{r}+\hat{s})+1||\hat{r}-\hat{s}| \leq (0+1)3|\hat{r}-\hat{s}|.$$

If $\hat{r},\ \hat{s} \in (0,\ 1]$, then we have

$$|0(\hat{r}-\hat{s})+T\hat{r}-T\hat{s}| = (0+1)|\hat{r}-\hat{s}|. \tag{9}$$

If $\hat{r} \in [-2,\ 0]$ and $\hat{s} \in (0,\ 1]$, then we obtain

$$\begin{aligned}|0(\hat{r}-\hat{s})+T\hat{r}-T\hat{s}| &= |\hat{r}-\hat{s}+\hat{r}^2| = |\hat{r}-\hat{s}+\hat{r}^2-\hat{s}^2+\hat{s}^2| \\ &= |\hat{r}-\hat{s}+\hat{r}^2-\hat{s}^2|+\hat{s}^2 \\ &\leq |\hat{r}+\hat{s}+1||\hat{r}-\hat{s}|+|\hat{s}-\hat{r}| = |2+\hat{r}+\hat{s}||\hat{r}-\hat{s}| \\ &\leq (0+1)3|\hat{r}-\hat{s}|.\end{aligned}$$

If $\hat{s} \in [-2, 0]$ and $\hat{r} \in (0, 1]$, then we have

$$\begin{aligned} |0(\hat{r}-\hat{s})+T\hat{r}-T\hat{s}| &= |\hat{r}-(\hat{s}+\hat{s}^2)| = |\hat{r}-\hat{s}-\hat{s}^2+\hat{r}^2-\hat{r}^2| \\ &= |\hat{r}-\hat{s}+(\hat{r}-\hat{s})(\hat{r}+\hat{s})-\hat{r}^2| \\ &\leq |1+\hat{r}+\hat{s}||\hat{r}-\hat{s}|+\hat{r}^2 \\ &\leq |1+\hat{r}+\hat{s}||\hat{r}-\hat{s}|+(\hat{r}^2-\hat{s})^2 \\ &\leq [|1+\hat{r}+\hat{s}|+|\hat{r}-\hat{s}|]|\hat{r}-\hat{s}| \\ &\leq (0+1)5|\hat{r}-\hat{s}|. \end{aligned}$$

Thus, we have that T is a $(0, 5)$-enriched Lipschitzian pseudocontraction with $L = 5$.

Also, it is worthy fact illustrated below that the class of a-enriched pseudocontractions is a superclass of the class of pseudocontractions.

Example 3 Let T, a selfmap of $\mathbb{R}$, be given as

$$T\hat{r} = \begin{cases} 0, & if\ \hat{r} \in (-\infty, 2] \\ 1, & if\ \hat{r} \in (2, \infty). \end{cases}$$

Then, every $\hat{r}, \hat{s} \in (-\infty, 2]$ and $a = 1$, we have

$$\begin{aligned} (a+1)^2|\hat{r}-\hat{s}|^2+|\hat{r}-T\hat{r}-(\hat{s}-T\hat{s})|^2 &= 5|\hat{r}-\hat{s}|^2 \\ &> |\hat{r}-\hat{s}|^2 = |a(\hat{r}-\hat{s})+T\hat{r}-T\hat{s}|^2. \end{aligned}$$

Further, for all $\hat{r}, \hat{s} \in (2, \infty)$ and $a = 1$, we get

$$\begin{aligned} (a+1)^2|\hat{r}-\hat{s}|^2+|\hat{r}-T\hat{r}-(\hat{s}-T\hat{s})|^2 &= 5|\hat{r}-\hat{s}|^2 \\ &> |\hat{r}-\hat{s}|^2 = |a(\hat{r}-\hat{s})+T\hat{r}-T\hat{s}|^2; \end{aligned}$$

and for $\hat{r} \in (-\infty, 2]$ and $\hat{s} \in (1, \infty)$, we obtain

$$
\begin{aligned}
(a+1)^2|\hat{r}-\hat{s}|^2+|\hat{r}-T\hat{r}-(\hat{s}-T\hat{s})|^2 &= 4|\hat{r}-\hat{s}|^2+|\hat{r}-(\hat{s}-1)|^2 \\
&= 4|\hat{r}-\hat{s}|^2+|\hat{r}-\hat{s}+1|^2 \\
&> |\hat{r}-\hat{s}-1|^2 = |a(\hat{r}-\hat{s})+T\hat{r}-T\hat{s}|^2.
\end{aligned}
$$

Thus, for all $\hat{r}$, $\hat{s} \in \mathbb{R}$ and $a \geq 1$, we obtain

$$
\|a(\hat{r}-\hat{s})+T\hat{r}-T\hat{s}\|^2 \leq (a+1)^2\|\hat{r}-\hat{s}\|^2+|(I-T)\hat{r}-(I-T)\hat{s}\|^2
$$

and T is an a-enriched pseudocontractive mapping. However, T is not a pseudocontraction since for $\hat{r}=2$ and $\hat{s}=\frac{5}{2}$, we get

$$
|T\hat{r}-T\hat{s}|^2 > \frac{1}{2} = |\hat{r}-\hat{s}|^2+|\hat{r}-T\hat{r}-(\hat{s}-T\hat{s})|^2.
$$

The following example illustrate the fact that the class of class of a-EPMs contains $(a,\, k)$-ESPMs.

Example 4 Consider R^2 as a 2-dimensional Euclidean plane. Define $T\colon R^2 \longrightarrow R^2$ by

$$
\begin{aligned}
Tr &= (r_1,\, r_2)+(r_2,\, -r_1) \\
&= (r_1+r_2,\, r_2-r_1),\ \forall r=(r_1,\, r_2) \in R^2.
\end{aligned}
$$

Then, for $r=(r_1,\, r_2)$, $s=(s_1,\, s_2) \in R^2$ and $a=1$, we get

$$
\begin{aligned}
\|a(r-s)+Tr-Ts\|^2 &= \|a((r_1,\, r_2)-(s_1,\, s_2))+(r_1+r_2,\, r_2-r_1)-(s_1+s_2,\, s_2-s_1)\|^2 \\
&= \|a((r_1-s_1),\, (r_2-,\, s_2))+(r_1+r_2,\, r_2-r_1)-(s_1+s_2,\, s_2-s_1)\|^2 \\
&= (2(r_1-s_1)+(r_2-s_2))^2+(2(r_2-s_2)-(r_1-s_1))^2 \\
&= 5[(r_1-s_1)^2+(r_2-s_2)^2] \\
&= 5\|r-s\|^2 = (a+1)^2\Phi_T(\|r-s\|).
\end{aligned}
$$

Hence, T is 1-enriched Φ_T-Lipshitz mapping with $\Phi_T(p^2)=\dfrac{5p^2}{4}$.

Now, observe that $\|a(r-s)+Tr-Ts\|^2 = 5\|r-s\|^2$, $s-Ts=-(s_2,\, -s_1)$ and $r-Tr=-(r_2,\, -r_1)$ so that

$$
\begin{aligned}
\|a(r-s)+Tr-Ts\|^2 &= \|-(r_2-r_1)-(-(s_2-s_1))\|^2 \\
&= \|-[(r_2-r_1)-(s_2-s_1)]\|^2 \\
&= \|(r_2-s_2),\, -(r_1-s_1)]\|^2 \\
&= (r_1-s_1)^2+(r_2-s_2)^2 \\
&= \|r-s\|^2.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
(a+1)^2\|r-s\|^2+\|r-Tr-(s-Ts)\|^2 &= 4\|r-s\|^2+\|r-s\|^2 \\
&= 5\|r-s\|^2 \\
&= \|a(r-s)+Tr-Ts\|^2
\end{aligned}
$$

Therefore, T is a Φ_T-Lipshitz and an a-enriched pseudocontractive mapping. However, T is not an (a, k)-enriched strictly pseudocontractive since for $r=(1, 1)$, $s=(-1, -1)\in R^2$, we have $\|r-s\|^2=8$ and $\|a(r-s)+Tr-Ts\|^2=40$, so that

$$
(a+1)^2\|r-s\|^2+k\|r-Tr-(s-Ts)\|^2=32+8k<40=\|a(r-s)+Tr-Ts\|^2,\ \forall k\in(0, 1).
$$

Lemma 7 Let $T\colon C\longrightarrow C$ be an (a, L)-enriched Lipschitz pseudocontractive mapping. Then, $F(T)=\{r\in C\colon Tr=r\}$ is closed and convex.

Proof. If $F(T)=\emptyset$, then it is done. Suppose otherwise and let $\{u_m\}_{m=1}^{\infty}\subseteq F(T)$ so that $u_m\to q$. We show that $q\in F(T)$.

$$
\begin{aligned}
\|q-Tq\| &\leq \|q-Tu_m\|+\|Tq-Tu_m\| \\
&\leq \|q-u_m\|+(a+1)L\|q-u_m\| \\
&= [1+(a+1)L]\|q-u_m\|\to 0 \quad \text{as} \quad m\to\infty.
\end{aligned}
$$

Therefore, $q\in F(T)$, and $F(T)$ is closed. Next, we prove the convexity of T. If $F(T)$ is singleton or empty, then it is convex. Suppose the converse and let $q_1, q_2\in F(T)$ and $\alpha\in[0, 1]$ be arbitrary, and $q=\alpha q_1+(1-\alpha)q_2$. We demonstrate that $q\in F(T)=F(T_b)$. Notice that

$$\|q - q_1\| = (1-\alpha)\|q_1 - q_2\|; \; \|q - q_2\| = \alpha\|q_1 - q_2\|. \tag{10}$$

For $\vartheta \in \left(0, \dfrac{1}{1+\sqrt{1+L^2}}\right)$, set $S_\vartheta r = T_b[(1-\vartheta)r + \vartheta T_b r]$, where $T_b = (1-b)I + bT$ with I representing the identity operator on C. Then, $S_\vartheta q_1 = q_1$ and $S_\vartheta q_2 = q_2$. Now, since

$$\begin{aligned}\|q - S_\vartheta q\|^2 &= \|\alpha(q_1 - S_\vartheta q) + (1-\alpha)(q_2 - S_\vartheta q)\|^2 \\ &= \alpha\|q_1 - S_\vartheta q\|^2 + (1-\alpha)\|q_2 - S_\vartheta q\|^2 - \alpha(1-\alpha)\|q_1 - q_2\|^2 \end{aligned} \tag{11}$$

and

$$\begin{aligned}\|S_\vartheta q - q_1\|^2 &= \Big\| \frac{a}{a+1}\Big[(1-\vartheta)q + \vartheta T_b q\Big] + \frac{1}{a+1} T\Big[(1-\vartheta)q + \vartheta T_b q\Big] \\ &\quad - \Big[\frac{a}{a+1} q_1 + \frac{1}{a+1} T q_1\Big]\Big\|^2 \\ &= \frac{1}{(a+1)^2}\|a[(1-\vartheta)q + \vartheta T_b q - q_1] + T[(1-\vartheta)q + \vartheta T_b q] - Tq_1\|^2 \\ &\le \frac{1}{(a+1)^2}[(a+1)^2\|(1-\vartheta)q + \vartheta T_b q - q_1\|^2 \\ &\quad + \|(1-\vartheta)q + \vartheta T_b q - S_\vartheta q - (q_1 - Tq_1)\|^2] \\ &\le \|(1-\vartheta)(q - q_1) + \vartheta(T_b q - q_1)\|^2 \\ &\quad + \|(1-\vartheta)(q - S_\vartheta q) + \vartheta(T_b q - S_\vartheta q)\|^2 \\ &= (1-\vartheta)\|q - q_1\|^2 + \frac{\vartheta}{(a+1)^2}\|a(q - q_1) + Tq - Tq_1\|^2 - \frac{\vartheta(1-\vartheta)}{(a+1)^2}\|q - Tq\|^2 \\ &\quad + (1-\vartheta)\|q - S_\vartheta q\|^2 + \frac{\vartheta}{(a+1)^2}\|a(q - ((1-\vartheta)q + \vartheta T_b q)) \\ &\quad + Tq - T((1-\vartheta)q + \vartheta T_b q))\|^2 - \frac{\vartheta(1-\vartheta)}{(a+1)^2}\|q - Tq\|^2 \\ &\le (1-\vartheta)\|q - q_1\|^2 + \vartheta\|q - q_1\|^2 + \frac{\vartheta}{(a+1)^2}\|q - Tq\|^2 - \frac{\vartheta(1-\vartheta)}{(a+1)^2}\|q - Tq\|^2 \end{aligned}$$

$$
\begin{aligned}
&+(1-\vartheta)\|q-S_\vartheta q\|^2+\vartheta L^2\|q-((1-\vartheta)q+\vartheta T_b q)\|^2-\frac{\vartheta(1-\vartheta)}{(a+1)^2}\|q-Tq\|^2 \\
= \quad &\|q-q_1\|^2-\frac{\vartheta}{(a+1)^2}(1-2\vartheta-\vartheta^2L^2)\|q-Tq\|^2+(1-\vartheta)\|q-S_\vartheta q\|^2,
\end{aligned}
$$

it follows that

$$
\|S_\vartheta q-q_1\|^2\le\|q-q_1\|^2+(1-\vartheta)\|q-S_\vartheta q\|^2. \tag{12}
$$

Using the same approach as above, we obtain

$$
\|S_\vartheta q-q_2\|^2\le\|q-q_2\|^2+(1-\vartheta)\|q-S_\vartheta q\|^2. \tag{13}
$$

Putting (12) and (13) into (11), we get

$$
\begin{aligned}
\|q-S_\vartheta q\|^2 \quad \le \quad &\alpha[\|q-q_1\|^2+(1-\vartheta)\|q-S_\vartheta q\|^2] \\
&+(1-\alpha)[\|q-q_2\|^2+(1-\vartheta)\|q-S_\vartheta q\|^2]-\alpha(1-\alpha)\|q_1-q_2\|^2 \\
= \quad &(1-\vartheta)\|q-S_\vartheta q\|^2.
\end{aligned}
$$

It, therefore, follows from the last inequality that $0\le\|q-S_\vartheta q\|^2\le 0$, and as a consequence, we have $S_\vartheta q=q$. Observe that

$$
\begin{aligned}
\|T_b q-q\| \quad &\le \quad \|q-S_\vartheta q\|+\|S_\vartheta q-T_b q\| \\
&= \quad \|q-S_\vartheta q\|+\|T_b[(1-\vartheta)q+\vartheta T_b q]-T_b q\| \\
&= \quad \|q-S_\vartheta q\|+\frac{1}{(a+1)}\|a[(1-\vartheta)q+\vartheta T_b q-q]+T[(1-\vartheta)q+\vartheta T_b q]-Tq\| \\
&\le \quad \|q-S_\vartheta q\|+L\|(1-\vartheta)q+\vartheta T_b q-q\| \\
&= \quad \|q-S_\vartheta q\|+L\vartheta\|q-T_b q\|.
\end{aligned}
$$

Consequently, $0\le[1-L\vartheta]\|T_b q-q\|\le 0$, and hence $T_b q=Tq=q$. □

Lemma 8 If a mapping $T\colon C \to C$ is an $(a,\, L)$-enriched Lipschitz pseudocontraction, then (I-T) is demiclosed at zero.

Proof. Let $\{r_m\}_{m=1}^{\infty} \in C$ assures $r_n \rightharpoonup q$ and $r_m - Tr_m \to 0$. We show that $q - Tq = 0$. For each $r \in H$, define $f\colon H \longrightarrow \mathbb{R}^+$ by

$$f(r) = \limsup_{m\to\infty} \|r_m - r\|^2.$$

Recalling Lemma 6(iii), we acquire

$$f(r) = \limsup_{m\to\infty} \|r_m - q\|^2 + \|q - r\|^2.$$

Consequently,

$$f(r) = f(q) + \|q - r\|^2 \quad \forall r \in H.$$

Therefore,

$$f(S_\vartheta q) = f(q) + \|q - S_\vartheta q\|^2. \tag{14}$$

Notice that

$$\begin{aligned}
\|S_\vartheta r_m - r_m\| &\leq \|S_\vartheta r_m - T_b r_m\| + \|r_m - T_b r_m\| \\
&= \|(1-b)[(1-\vartheta) r_m + \vartheta T_b r_m] + bT[(1-\vartheta) r_m + \vartheta T_b r_m] \\
&\quad -[(1-b) r_m + bTr_m]\| + \|r_m - T_b r_m\| \\
&= \frac{1}{a+1}\|a[(1-\vartheta) r_m + \vartheta T_b r_m - r_m] + T[(1-\vartheta) r_m + \vartheta T_b r_m] - Tr_m\| \\
&\quad + \|r_m - T_b r_m\| \\
&= L\|(1-\vartheta) r_m + \vartheta T_b r_m - r_m\| + \|r_m - T_b r_m\| \\
&= \left(\frac{L\vartheta}{a+1} + 1\right)\|r_m - Tr_m\| \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Moreover,

$$\|Tr_m - S_\vartheta r_m\| \quad \leq \quad \|Tr_m - r_m\| + \|r_m - S_\vartheta r_m\| \to 0 \quad \text{as} \quad n \to \infty.$$

Thus

$$\begin{aligned}
f(S_\vartheta q) &= \limsup_{m\to\infty}\|r_m - S_\vartheta q\|^2 \\
&= \limsup_{m\to\infty}\|r_m - S_\vartheta r_m + S_\vartheta r_m - S_\vartheta q\|^2 \\
&= \limsup_{m\to\infty}\|S_\vartheta r_m - S_\vartheta q\|^2 \\
&= \limsup_{m\to\infty}\|T_b[(1-\vartheta)r_m + \vartheta T_b r_m] - T_b[(1-\vartheta)q + \vartheta T_b q]\|^2 \\
&= \frac{1}{(a+1)^2}\limsup_{m\to\infty}\|a[(1-\vartheta)r_m + \vartheta T_b r_m - ((1-\vartheta)q + \vartheta T_b q)] + T[(1-\vartheta)r_m + \vartheta T_b r_m] \\
&\quad -T[(1-\vartheta)q + \vartheta T_b q]\|^2 \\
&\leq \frac{1}{(a+1)^2}\limsup_{m\to\infty}[(a+1)^2\|(1-\vartheta)r_m + \vartheta T_b r_m - ((1-\vartheta)q + \vartheta T_b q)\|^2 \\
&\quad +\|(1-\vartheta)r_m + \vartheta T_b r_m - T[(1-\vartheta)r_m + \vartheta T_b r_m] \\
&\quad -\{((1-\vartheta)q + \vartheta T_b q) - T[(1-\vartheta)q + \vartheta T_b q]\}\|^2] \\
&= \frac{1}{(a+1)^2}\limsup_{m\to\infty}(a+1)^2\|(1-\vartheta)r_m + \vartheta T_b r_m - ((1-\vartheta)q + \vartheta T_b q)\|^2 \\
&\quad +\frac{1}{(a+1)^2}\|(1-\vartheta)r_m + \vartheta T_b r_m - T[(1-\vartheta)r_m + \vartheta T_b r_m] \\
&\quad -\{((1-\vartheta)q + \vartheta T_b q) - T[(1-\vartheta)q + \vartheta T_b q]\}\|^2 \\
&< \limsup_{m\to\infty}\|(1-\vartheta)r_m + \vartheta T_b r_m - ((1-\vartheta)q + \vartheta T_b q)\|^2 \\
&\quad +\|(1-\vartheta)r_m + \vartheta T_b r_m - T[(1-\vartheta)r_m + \vartheta T_b r_m] \\
&\quad -\{((1-\vartheta)q + \vartheta T_b q) - T[(1-\vartheta)q + \vartheta T_b q]\}\|^2
\end{aligned}$$

$$
\begin{aligned}
= \quad & \limsup_{m\to\infty}[\|(1-\vartheta)r_m+\vartheta T_b r_m-((1-\vartheta)q+\vartheta T_b q)\|^2 \\
& +\|(1-\vartheta)r_m+\vartheta T_b r_m-\{(1-b)(1-\vartheta)r_m+\vartheta T_b r_m+bT[(1-\vartheta)r_m+\vartheta T_b r_m]\} \\
& -\{(1-\vartheta)q+\vartheta T_b q-\{(1-b)(1-\vartheta)q+\vartheta T_b q+bT[(1-\vartheta)q+\vartheta T_b q]\}\}\|^2] \\
= \quad & \limsup_{m\to\infty}\{(1-\vartheta)\|r_m-q\|^2+\vartheta\|T_b r_m-T_b q\|^2-\vartheta(1-\vartheta)\|r_m-T_b r_m-(q-T_b q)\|^2 \\
& +(1-\vartheta)\|q-S_\vartheta q\|^2+\vartheta\|T_b q-S_\vartheta q\|^2-\vartheta(1-\vartheta)\|q-T_b q\|^2\} \\
= \quad & \limsup_{m\to\infty}\{(1-\vartheta)\|r_m-q\|^2+\frac{\vartheta}{(a+1)^2}\|a(r_m-q)+Tr_m-Tq\|^2-\frac{\vartheta(1-\vartheta)}{(a+1)^2}\|q-Tq\|^2 \\
& +(1-\vartheta)\|q-S_\vartheta q\|^2+\frac{\vartheta}{(a+1)^2}\|a(q-((1-\vartheta)q+\vartheta T_b q))+Tq-T((1-\vartheta)q+\vartheta T_b q)\|^2 \\
& -\frac{\vartheta(1-\vartheta)}{(a+1)^2}\|q-Tq\|^2\} \\
\leq \quad & \limsup_{m\to\infty}\{(1-\vartheta)\|r_m-q\|^2+\vartheta\|r_m-q\|^2+\frac{\vartheta}{(a+1)^2}\|r_m-Tr_m-(q-Tq)\|^2 \\
& -\frac{\vartheta(1-\vartheta)}{(a+1)^2}\|q-Tq\|^2+(1-\vartheta)\|q-S_\vartheta q\|^2+\vartheta L^2\|q-((1-\vartheta)q+\vartheta T_b q)\|^2 \\
& -\frac{\vartheta(1-\vartheta)}{(a+1)^2}\|q-Tq\|^2\} \\
= \quad & \limsup_{m\to\infty}\{\|r_m-q\|^2-\frac{\vartheta}{(a+1)^2}(1-2\vartheta-\vartheta^2L^2)\|q-Tq\|^2+(1-\vartheta)\|q-S_\vartheta q\|^2\} \\
\leq \quad & \limsup_{m\to\infty}\{\|r_m-q\|^2+(1-\vartheta)\|q-S_\vartheta q\|^2\}.
\end{aligned}
$$

Thus,

$$
f(S_\vartheta q)\leq f(q)+(1-\vartheta)\|q-S_\vartheta q\|^2. \tag{15}
$$

It follows from (14) and (15) that

$$f(S_\vartheta q) = f(q) + \|q - S_\vartheta q\|^2 \leq f(q) + (1-\vartheta)\|q - S_\vartheta q\|^2.$$

Thus, $0 \leq \vartheta\|q - S_\vartheta q\|^2 \leq 0$, and we get $\|q - S_\vartheta q\| = 0$.

Observe that

$$\begin{aligned}
0 \leq \|T_b q - q\| &\leq \|q - S_\vartheta q\| + \|S_\vartheta q - T_b q\| \\
&= \|q - S_\vartheta q\| + \|T_b[(1-\vartheta)q + \vartheta T_b q] - T_b q\| \\
&= \|q - S_\vartheta q\| + \frac{1}{(a+1)}\|a[(1-\vartheta)q + \vartheta T_b q - q] + T[(1-\vartheta)q + \vartheta T_b q] - Tq\| \\
&\leq \|q - S_\vartheta q\| + L\|(1-\vartheta)q + \vartheta T_b q - q\| \\
&= \|q - S_\vartheta q\| + L\vartheta\|q - T_b q\|.
\end{aligned}$$

Consequently, $0 \leq (1-\vartheta L)\|q - T_b q\| \leq \|q - S_\vartheta q\| = 0$, and we have $T_b q = Tq = q$. □

Subsequently, we establish convergence results for iterative computation of invariant points of a class of enriched Lipschitz pseudocontractions in the domain of real Hilbert spaces.

Theorem 1 Let T_1 and T_2 be selfmaps of C. Consider T_1 and T_2 as two $(a,\ \{L_i\}_{i=1}^2)$-enriched Lipschitz pseudocontractions where L_1 and L_2, are the Lipschitz constants. Suppose $\Gamma = F(T_2) \cap F(T_2) \neq \emptyset$. Let $r_0,\ u \in C$ be arbitrary and define $\{r_m\}$ by

$$\begin{cases} z_m = (1-c_m)r_m + c_m T_{b,\,2} r_m; \\ y_m = (1-\beta_m)r_m + \beta_m T_{b,\,1} r_m; \\ r_{m+1} = \alpha_m u + (1-\alpha_m)[\vartheta_m r_m + \delta_m T_{b,\,1} y_m + \gamma_m T_{b,\,2} z_m], \end{cases} \tag{16}$$

where $T_{b,\,i} = (1-b)I + bT_i\ (i = 1,\,2)$, $\{\delta_m\}$, $\{\vartheta_m\}$, $\{\gamma_m\} \subset [a,\,b] \subset (0,\,1)$, $\{\alpha_m\} \subset (0,\,c) \subset (0,\,1)$ fulfill the following requirements: (i) $\vartheta_m + \delta_m + \gamma_m = 1$; (ii) $\lim_{m\longrightarrow\infty} \alpha_m = 0$, $\sum_{m=1}^{\infty} \alpha_m = \infty$; $\delta_m + \gamma_m \leq c_m$, $\beta_m \leq \beta < \frac{1}{\sqrt{L^2+1}+1}$, $\forall m \geq 1$, for $L = \max\{L_1,\,l_2\}$. Then, $\{r_m\}$ admits strong convergence to a common point of T_1 and T_2,that is fixed nearest to u.

Proof. Let $q \in \Gamma$. Since from (8) and Lemma 5, we get

$$\begin{aligned}
\|r_{m+1} - q\|^2 &= \|\alpha_m u + (1-\alpha_m)[\vartheta_m r_m + \delta_m T_{b,\,1} y_m + \gamma_m T_{b,\,2} z_m] - q\|^2 \\
&\leq \alpha_m\|u - q\|^2 + (1-\alpha_m)\|\delta_m(T_{b,\,1} y_m - q) + \vartheta_m(r_m - q) + \gamma_m(T_{b,\,2} z_m - q)\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_m\|u-q\|^2+(1-\alpha_m)[\delta_m\|T_{b,\,1}y_m-q\|^2+\vartheta_m\|r_m-q\|^2+\gamma_m\|T_{b,\,2}z_m-q\|^2\\
&\quad -(1-\alpha_m)\delta_m\vartheta_m\|T_{b,\,1}y_m-r_m\|^2-(1-\alpha_m)\vartheta_m\gamma_m\|T_{b,\,2}z_m-r_m\|^2]\\
&= \alpha_m\|u-q\|^2+(1-\alpha_m)[\delta_m\|(1-b)y_m+bT_1y_m-q\|^2+\vartheta_m\|r_m-q\|^2\\
&\quad +\gamma_m\|(1-b)z_m+bT_2z_m-q\|^2-(1-\alpha_m)\delta_m\vartheta_m\|(1-b)y_m+bT_1y_m-r_m\|^2\\
&\quad -(1-\alpha_m)\vartheta_m\gamma_m\|(1-b)z_m+bT_2z_m-r_m\|^2]\\
&\leq \alpha_m\|u-q\|^2+\frac{(1-\alpha_m)\delta_m}{(a+1)^2}\|a(y_m-q)+T_1y_m-Tq\|^2+(1-\alpha_m)\vartheta_m\|r_m-q\|^2\\
&\quad +\frac{(1-\alpha_m)\gamma_m}{(a+1)^2}\|a(z_m-q)+T_2z_m-Tq\|^2-\frac{(1-\alpha_m)\delta_m\vartheta_m}{(a+1)^2}\|T_1y_m-r_m\|^2\\
&\quad -\frac{(1-\alpha_m)\vartheta_m\gamma_m}{(a+1)^2}\|T_2z_m-r_m\|^2,
\end{aligned}
$$

it follows from the a-enriched pseudocontractivity of T_1 and T_2 that

$$
\begin{aligned}
&\leq \alpha_m\|u-q\|^2+\frac{(1-\alpha_m)\delta_m}{(a+1)^2}[(a+1)^2\|y_m-q\|^2+\|y_m-T_1y_m\|^2]\\
&\quad +(1-\alpha_m)\vartheta_m\|r_m-q\|^2+\frac{(1-\alpha_m)\gamma_m}{(a+1)^2}[(a+1)^2\|z_m-q\|^2+\|z_m-T_2z_m\|^2]\\
&\quad -\frac{(1-\alpha_m)\delta_m\vartheta_m}{(a+1)^2}\|T_1y_m-r_m\|^2-\frac{(1-\alpha_m)\vartheta_m\gamma_m}{(a+1)^2}\|T_2z_m-r_m\|^2\\
&= \alpha_m\|u-q\|^2+(1-\alpha_m)\delta_m\|y_m-q\|^2+\frac{(1-\alpha_m)\delta_m}{(a+1)^2}\|y_m-T_1y_m\|^2\\
&\quad +(1-\alpha_m)\vartheta_m\|r_m-q\|^2+(1-\alpha_m)\gamma_m\|z_m-q\|^2+\frac{(1-\alpha_m)\gamma_m}{(a+1)^2}\|z_m-T_2z_m\|^2\\
&\quad -\frac{(1-\alpha_m)\delta_m\vartheta_m}{(a+1)^2}\|T_1y_m-r_m\|^2-\frac{(1-\alpha_m)\vartheta_m\gamma_m}{(a+1)^2}\|T_2z_m-r_m\|^2. \qquad (17)
\end{aligned}
$$

Furthermore, from (16), Lemma 5 and enriched pseudocontractivity of T_1, we obtain

$$
\begin{aligned}
\|y_m - q\|^2 &= \|(1-\beta_m)(r_m - q) + \beta_m(T_{b,1}r_m - q)\|^2 \\
&= (1-\beta_m)\|r_m - q\|^2 + \beta_m\|T_{b,1}r_m - q\|^2 - \beta_m(1-\beta_m)\|r_m - T_{b,1}r_m\|^2 \\
&= (1-\beta_m)\|r_m - q\|^2 + \frac{\beta_m}{(a+1)^2}\|a(r_m - q) + T_1 r_m - Tq\|^2 \\
&\quad -\frac{\beta_m(1-\beta_m)}{(a+1)^2}\|r_m - T_1 r_m\|^2 \\
&\leq (1-\beta_m)\|r_m - q\|^2 + \frac{\beta_m}{(a+1)^2}[(a+1)^2\|r_m - q\|^2 + \|r_m - T_1 r_m\|^2] \\
&\quad -\frac{\beta_m(1-\beta_m)}{(a+1)^2}\|r_m - T_1 r_m\|^2 \\
&= \|r_m - q\|^2 + \frac{\beta_m^2}{(a+1)^2}\|r_m - T_1 r_m\|^2. \qquad (18)
\end{aligned}
$$

Using tha same approach as in (18), we get that

$$
\|z_m - q\|^2 = \|r_m - q\|^2 + \frac{c_m^2}{(a+1)^2}\|r_m - T_1 r_m\|^2. \qquad (19)
$$

Now, since from (16) and Lemma 5

$$
\begin{aligned}
\|y_m - T_{b,1}y_m\|^2 &= \|(1-\beta_m)(r_m - T_{b,1}y_m) + \beta_m(T_{b,1}r_m - T_{b,1}y_m)\|^2 \\
&= (1-\beta_m)\|r_m - T_{b,1}y_m\|^2 + \beta_m\|T_{b,1}r_m - T_{b,1}y_m\|^2 \\
&\quad -\beta_m(1-\beta_m)\|r_m - T_{b,1}r_m\|^2 \\
&= (1-\beta_m)\|r_m - T_{b,1}y_m\|^2 + \frac{\beta_m}{(a+1)^2}\|a(r_m - y_m) + T_1 r_m - T_1 y_m\|^2 \\
&\quad \frac{-\beta_m(1-\beta_m)}{(a+1)^2}\|r_m - T_1 r_m\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\beta_m)\|r_m - T_{b,\,1}y_m\|^2 + \beta_m L^2\|r_m - y_m\|^2 \\
&\quad \frac{-\beta_m(1-\beta_m)}{(a+1)^2}\|r_m - T_1 r_m\|^2 \\
&= (1-\beta_m)\|r_m - [(1-b)y_m + bT_1 y_m]\|^2 \\
&\quad +\beta_m L^2\|r_m - [(1-\beta_m)r_m + \beta_m T_{b,\,1} r_m]\|^2 - \frac{\beta_m(1-\beta_m)}{(a+1)^2}\|r_m - T_1 r_m\|^2 \\
&\leq \frac{(1-\beta_m)a^2}{(a+1)^2}\|r_m - y_m\|^2 + \frac{(1-\beta_m)}{(a+1)^2}\|T_1 y_m - r_m)\|^2 + \frac{\beta_m^3 L^2}{(a+1)^2}\|r_m - T_1 r_m\|^2 \\
&\quad -\frac{\beta_m(1-\beta_m)}{(a+1)^2}\|r_m - T_1 r_m\|^2 \\
&= \frac{(1-\beta_m)a^2\beta_m^2}{(a+1)^4}\|r_m - T_1 r_m]\|^2 + \frac{(1-\beta_m)}{(a+1)^2}\|T_1 y_m - r_m)\|^2 \\
&\quad +\frac{\beta_m^3 L^2}{(a+1)^2}\|r_m - T_1 r_m\|^2 - \frac{\beta_m(1-\beta_m)}{(a+1)^2}\|r_m - T_1 r_m\|^2 \\
&\leq \frac{\beta_m^2}{(a+1)^2}\|r_m - T_1 r_m]\|^2 + \frac{(1-\beta_m)}{(a+1)^2}\|T_1 y_m - r_m)\|^2 + \frac{\beta_m^3 L^2}{(a+1)^2}\|r_m - T_1 r_m\|^2 \\
&\quad -\frac{\beta_m(1-\beta_m)}{(a+1)^2}\|r_m - T_1 r_m\|^2 \\
&= \frac{(1-\beta_m)}{(a+1)^2}\|T_1 y_m - r_m)\|^2 - \frac{\beta_m}{(a+1)^2}[1 - 2\beta_m - \beta_m^2 L^2]\|r_m - T_1 r_m\|^2,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\|y_m - T_1 y_m\|^2 &\leq (1-\beta_m)\|T_1 y_m - r_m)\|^2 \\
&\quad -\beta_m[1 - 2\beta_m - \beta_m^2 L^2]\|r_m - T_1 r_m\|^2. \qquad (20)
\end{aligned}
$$

By employing the same approach as above, we get

$$\|z_m - T_2 z_m\|^2 \leq (1-c_m)\|T_2 z_m - r_m)\|^2 - c_m[1 - 2c_m - c_m^2 L^2]\|r_m - T_2 r_m\|^2. \quad (21)$$

Putting (18), (19), (20) and (21) into (17), we get

$$\begin{aligned}
\|r_{m+1} - q\| &\leq \alpha_m\|u-q\|^2 + (1-\alpha_m)\delta_m[\|r_m - q\|^2 + \frac{\beta_m^2}{(a+1)^2}\|r_m - T_1 r_m\|^2] \\
&\quad + \frac{(1-\alpha_m)\delta_m}{(a+1)^2}[(1-\beta_m)\|T_1 y_m - r_m\|^2 \\
&\quad - \beta_m[1 - 2\beta_m - \beta_m^2 L^2]\|r_m - T_1 r_m\|^2] + (1-\alpha_m)\vartheta_m\|r_m - q\|^2 \\
&\quad + (1-\alpha_m)\gamma_m[\|r_m - q\|^2 + \frac{c_m^2}{(a+1)^2}\|r_m - T_2 r_m\|^2] \\
&\quad + \frac{(1-\alpha_m)\gamma_m}{(a+1)^2}[(1-c_m)\|T_2 z_m - r_m\|^2 \\
&\quad - c_m[1 - 2c_m - c_m^2 L^2]\|r_m - T_2 r_m\|^2] \\
&\quad - \frac{(1-\alpha_m)\delta_m\vartheta_m}{(a+1)^2}\|T_1 y_m - r_m\|^2 - \frac{(1-\alpha_m)\vartheta_m\gamma_m}{(a+1)^2}\|T_2 z_m - r_m\|^2 \\
&\leq \alpha_m\|u-q\|^2 + (1-\alpha_m)\|r_m - q\|^2 - \beta_m[1 - 3\beta_m - \beta_m^2 L^2]\|r_m - T_1 r_m\|^2 \\
&\quad - \frac{(1-\alpha_m)\gamma_m}{(a+1)^2}c_m[1 - 3c_m - c_m^2 L^2]\|r_m - T_2 r_m\|^2 \\
&\quad + \frac{(1-\alpha_m)\delta_m}{(a+1)^2}(1 - \beta_m - \vartheta_m)\|T_1 y_m - r_m\|^2 \\
&\quad + \frac{(1-\alpha_m)\gamma_m}{(a+1)^2}(1 - c_m - \vartheta_m)\|T_2 z_m - r_m\|^2 \\
&= \alpha_m\|u-q\|^2 + (1-\alpha_m)\|r_m - q\|^2 - \beta_m[1 - 3\beta_m - \beta_m^2 L^2]\|r_m - T_1 r_m\|^2 \\
&\quad - \frac{(1-\alpha_m)\gamma_m}{(a+1)^2}c_m[1 - 3c_m - c_m^2 L^2]\|r_m - T_2 r_m\|^2
\end{aligned}$$

$$+\frac{(1-\alpha_m)\delta_m}{(a+1)^2}(\delta_m+\gamma_m-\beta_m)\|T_1y_m-r_m\|^2$$

$$+\frac{(1-\alpha_m)\gamma_m}{(a+1)^2}(\delta_m+\gamma_m-c_m)\|T_2z_m-r_m\|^2. \tag{22}$$

Now, from (ii) of the hypothesis, we obtain that

$$1-3\beta_m-L_2\beta_m^2\geq 1-3\beta-L^2\beta^2>0,$$

$$1-3c_m-L_2c_m^2\geq 1-3\beta-L^2\beta^2>0 \tag{23}$$

and

$$\delta_m+\gamma_m-\beta_m\leq 0,\quad \delta_m+\gamma_m-c_m\leq 0,\quad \forall m\geq 1. \tag{24}$$

It, therefore, follows from (22), (23) and (24) that

$$\|r_{m+1}-q\|^2 \quad\leq\quad \alpha_m\|u-q\|^2+(1-\alpha_m)\|r_m-q\|^2. \tag{25}$$

Thus, using induction, we get

$$\|r_{m+1}-q\|^2 \quad\leq\quad \max\{\|u-q\|^2,\, \|r_0-q\|^2\},\quad \forall m\geq 0.$$

Therefore, the sequence $\{r_m\}$ is bounded, and so is the sequence $\{y_m\}$. □

Let $r^\star=P_\Gamma(u)$. Then, by (16), Lemma 1 and employing the same approach as in (18) and (19), we obtain

$$\begin{aligned}
\|r_{m+1}-r^\star\|^2 &= \|\alpha_m(u-r^\star)+(1-\alpha_m)[\vartheta_m r_m+\delta_m T_{b,1}y_m+\gamma_m T_{b,2}z_m-r^\star]\|^2\\
&\leq (1-\alpha_m)\|\delta_m T_{b,1}y_m+\vartheta_m r_m+\gamma_m T_{b,2}z_m-r^\star\|^2\\
&\quad +2\alpha_m\langle u-r^\star,\, r_{m+1}-r^\star\rangle\\
&\leq (1-\alpha_m)\delta_m\|T_{b,1}y_m-r^\star\|^2+(1-\alpha_m)\vartheta_m\|r_m-r^\star\|^2\\
&\quad +(1-\alpha_m)\gamma_m\|T_{b,2}z_m-r^\star\|^2-(1-\alpha_m)\delta_m\vartheta_m\|T_{b,1}y_m-r_m\|^2
\end{aligned}$$

$$
\begin{aligned}
& -(1-\alpha_m)\vartheta_m\gamma_m\|T_{b,\,2}z_m - r_m\|^2 + 2\alpha_m\langle u - r^\star,\, r_{m+1} - r^\star\rangle \\
= \;& \frac{(1-\alpha_m)\delta_m}{(a+1)^2}\|a(y_m - r^\star) + T_1 y_m - T r^\star\|^2 + (1-\alpha_m)\vartheta_m\|r_m - r^\star\|^2 \\
& + \frac{(1-\alpha_m)\gamma_m}{(a+1)^2}\|a(z_m - r^\star) + T_2 z_m - T r^\star\|^2 \\
& - \frac{(1-\alpha_m)\delta_m\vartheta_m}{(a+1)^2}\|a(y_m - r_m) + T_1 y_m - r_m\|^2 \\
& - \frac{(1-\alpha_m)\vartheta_m\gamma_m}{(a+1)^2}\|a(z_m - r_m) + T_2 z_m - r_m\|^2 + 2\alpha_m\langle u - r^\star,\, r_{m+1} - r^\star\rangle \\
\leq \;& \frac{(1-\alpha_m)\delta_m}{(a+1)^2}[(a+1)^2\|y_m - r^\star\|^2 + \|y_m - T_1 y_m\|^2] + (1-\alpha_m)\vartheta_m\|r_m - r^\star\|^2 \\
& + \frac{(1-\alpha_m)\gamma_m}{(a+1)^2}[(a+1)^2\|z_m - r^\star\|^2 + \|z_m - T_2 z_m\|^2] \\
& - \frac{(1-\alpha_m)\delta_m\vartheta_m}{(a+1)^2}\|T_1 y_m - r_m\|^2 - \frac{(1-\alpha_m)\vartheta_m\gamma_m}{(a+1)^2}\|T_2 z_m - r_m\|^2 \\
& + 2\alpha_m\langle u - r^\star,\, r_{m+1} - r^\star\rangle \\
\leq \;& (1-\alpha_m)\delta_m[\|r_m - r^\star\|^2 + \frac{\beta_m^2}{(a+1)^2}\|r_m - T_1 r_m\|^2] \\
& + \frac{(1-\alpha_m)\delta_m}{(a+1)^2}[(1-\beta_m)\|T_1 y_m - r_m\|^2 - \beta_m[1 - 2\beta_m - \beta_m^2 L^2]\|r_m - T_1 r_m\|^2] \\
& + (1-\alpha_m)\vartheta_m\|r_m - r^\star\|^2 + (1-\alpha_m)\gamma_m[\|r_m - q\|^2 + \frac{c_m^2}{(a+1)^2}\|r_m - T_1 r_m\|^2] \\
& + \frac{(1-\alpha_m)\gamma_m}{(a+1)^2}[(1-c_m)\|T_2 z_m - r_m\|^2 - c_m[1 - 2c_m - c_m^2 L^2]\|r_m - T_2 r_m\|^2] \\
& - \frac{(1-\alpha_m)\delta_m\vartheta_m}{(a+1)^2}\|T_1 y_m - r_m\|^2 - \frac{(1-\alpha_m)\vartheta_m\gamma_m}{(a+1)^2}\|T_2 z_m - r_m\|^2 \\
& + 2\alpha_m\langle u - r^\star,\, r_{m+1} - r^\star\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\le (1-\alpha_m)\|r_m - r^\star\|^2 + \frac{(1-\alpha_m)\delta_m}{(a+1)^2}(\delta_m + \gamma_m - \beta_m)\|T_1 y_m - r_m\|^2 \\
&-\frac{(1-\alpha_m)\delta_m\beta_m}{(a+1)^2}[1-3\beta_m - \beta_m^2 L^2]\|r_m - T_1 r_m\|^2 \\
&+\frac{(1-\alpha_m)\gamma_m}{(a+1)^2}(\delta_m + \gamma_m - c_m)\|T_2 z_m - r_m\|^2 \\
&-\frac{(1-\alpha_m)\gamma_m c_m}{(a+1)^2}[1-3c_m - c_m^2 L^2]\|r_m - T_2 r_m\|^2 \\
&+2\alpha_m\langle u - r^\star, r_{m+1} - r^\star\rangle \qquad (26) \\
&\le (1-\alpha_m)\|r_m - r^\star\|^2 + 2\alpha_m\langle u - r^\star, r_{m+1} - r^\star\rangle. \qquad (27)
\end{aligned}
$$

Now, we enlist the following cases:

Case A. Suppose we can find $n_0 \in \mathbb{N}$ which assures that $\{\|r_m - r^\star\|\}$ is decreasing for $n \ge n_0$. Then, we obtain that $\{\|r_m - r^\star\|\}$ is convergent. It, therefore, follows from (23), (24) and (26) that

$$
r_m - T_1 r_m \to 0, \quad r_m - T_2 r_m \to 0 \quad \text{as} \quad n \to \infty. \tag{28}
$$

Further, in view of the fact that $\{\|r_{m+1} - r^\star\|\}$ is a bounded subset of a reflexive space H, it is possible to choose a subsequence $\{\|r_{m_k+1} - r^\star\|\}$ of $\{\|r_{m+1} - r^\star\|\}$ that assures $r_{m+1} \rightharpoonup r$ and $\limsup_{m\to\infty}\langle u - r^\star, r_{m+1} - r^\star\rangle = \lim_{k\to\infty}\langle u - r^\star, r_{m_k+1} - r^\star\rangle$. Then, from (28), Lemma 7 and Lemma 8, we obtain that $r \in F(T_1)$ and $r \in F(T_2)$. Hence, by Lemma 2, we get

$$
\limsup_{m\to\infty}\langle u - r^\star, r_{m+1} - r^\star\rangle = \lim_{k\to\infty}\langle u - r^\star, r_{m_k+1} - r^\star\rangle = \langle u - r^\star, r - r^\star\rangle \le 0. \tag{29}
$$

Therefore, we conclude from (26), (29) and Lemma 3 that $\|r_m - r^\star\| \to 0$ as $m \to \infty$. Consequent upon this, $r_m \to r^\star = P_\Gamma(u)$.

Case B. Suppose we can find a subsequence $\{n_k\}$ of $\{n\}$ that guarantees the inequality

$$
\|r_{m_k} - r^\star\| < \|r_{m_k+1} - r^\star\| \quad \forall k \in \mathbb{N}.
$$

Then, from Lemma 4, we can find a non-decreasing sequence $\{n_k\} \subset \mathbb{N}$ which guarantees $n_k \to\to \infty$, and

$$
\|r_{n_k} - r^\star\| \le \|r_{n_k+1} - r^\star\| \quad \text{and} \quad \|r_k - r^\star\| \le \|r_{n_k+1} - r^\star\| \tag{30}
$$

for all $k \in \mathbb{N}$. Now, from (23), (24) and (26), we have $r_{n_k} - T_1 r_{n_k} \to 0$ and $r_{n_k} - T_2 r_{n_k} \to 0$ as $n \to \infty$. Thus, using the same approach as in Case A, we have

$$\limsup_{k\to\infty} \langle u - r^\star, r_{m_k+1} - r^\star \rangle \leq 0. \tag{31}$$

Now, from (27), we get

$$\|r_{n_k+1} - r^\star\|^2 \leq (1-\alpha_{n_k})\|r_{n_k} - r^\star\|^2 + 2\alpha_{n_k}\langle u - r^\star, r_{n_k+1} - r^\star \rangle. \tag{32}$$

and hence (30) and (32) imply that

$$\begin{aligned} \alpha_{n_k}\|r_{n_k} - r^\star\|^2 &\leq \|r_{n_k} - r^\star\|^2 - \|r_{n_k+1} - r^\star\|^2 + 2\alpha_{n_k}\langle u - r^\star, r_{n_k+1} - r^\star \rangle \\ &\leq \alpha_{n_k}\langle u - r^\star, r_{n_k+1} - r^\star \rangle. \end{aligned}$$

Using the fact that $\alpha_{n_k} > 0$ and (31), we have

$$\limsup_{k\to\infty} \|r_{n_k} - r^\star\| \leq 0,$$

and hence $\|r_{n_k} - r^\star\| \to 0$ as $k \to \infty$. But $\|r_k - r^\star\| \leq \|r_{n_k+1} - r^\star\|$ $\forall k \in \mathbb{N}$. Therefore, we get $r_k \to r^\star$. From the forgoing cases, we admit that $\{r_n\}$ yields to strong convergence of the common fixed point of T_1 and T_2 nearest to u. This ends the proof.

It is worthy to mention that the technique used to prove the above Theorem could be employed for a finite family of enriched Lipchitzian pseudocontractions. To be precise, the following theorem is presented.

Theorem 2 Let $\{T_i\}_{i=1}^N$ be selfmaps of C. Consider $\{T_i\}_{i=1}^N$ as $(a, \{\lambda_i\}_{i=1}^N)$-enriched Lipschitz pseudocontractions with Lipschitz constants $\{\lambda_i\}_{i=1}^N$, respectively. Suppose $\Gamma = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. For arbitrary $r_0, u \in C$, let $\{r_m\}$ be developed from

$$\begin{cases} y_m = (1-v_m)r_m + v_m T_{b,i} r_m, \; i = 1, 2, \cdots, N; \\ r_{m+1} = \alpha_m u + (1-\alpha_m)[u_{m_0} r_m + \sum_{i=1}^N u_{m_i} T_{b,i} y_m], \end{cases} \tag{33}$$

where $T_{b,i} = (1-b)I + bT_i$ $(i = 1, 2)$, $\{u_{m_i}: i = 0, 1, \cdots, N\} \subset (a, b) \subset (0, 1)$, $\{\alpha_m\} \subset (0, c) \subset (0, 1)$ ensures that: $(i) \sum_{i=0}^N v_{m_i} = 1$; $(ii) \lim_{m\longrightarrow\infty} \alpha_m = 0$, $\sum_{m=1}^\infty \alpha_m = \infty$; $\sum_{i=1}^N u_{m_i} \leq v_m \leq v < \frac{1}{\sqrt{\lambda^2+1}+1}$, $\forall m \geq 1$, for $\lambda = \max\{\lambda_i: i = 0, 1, \cdots, N\}$. Then, $\{r_m\}$ converges strongly to a common fixed points of $T_i(: i = 0, 1, \cdots, N)$ nearest to u.

If in Theorem 1, we take T as an a-ENM, then we have that T is an (a, L)-EPM with $L = 1$, and consequent upon this, the following corollary ensues.

Corollary 1 Let T_1 and T_2 be $(a, \{\lambda_i\}_{i=1}^2)$-enriched Lipschitz pseudocontractive selfmaps of C with Lipschitz constants λ_1 and λ_2, respectively. Suppose $\Gamma = F(T_2) \cap F(T_2) \neq \emptyset$. For arbitrary r_0, $u \in C$, let $\{r_m\}$ be developed from

$$\begin{cases} \hat{z}_m = (1-c_m)r_m + c_m T_{b,2} r_m; \\ y_m = (1-\hat{\beta}_m)r_m + \hat{\beta}_m T_{b,1} r_m; \\ r_{m+1} = \hat{\alpha}_m u + (1-\hat{\alpha}_m)[\vartheta_m r_m + \hat{\delta}_m T_{b,1} y_m + \hat{\gamma}_m T_{b,2} \hat{z}_m], \end{cases} \tag{34}$$

where $T_{b,i} = (1-b)I + bT_i$ $(i = 1, 2)$, $\{\hat{\delta}_m\}$, $\{\hat{\vartheta}_m\}$, $\{\hat{\gamma}_m\} \subset [a, b] \subset (0, 1)$, $\{\hat{\alpha}_m\} \subset (0, c) \subset (0, 1)$ ensures that: (i) $\hat{\vartheta}_m + \hat{\delta}_m + \hat{\gamma} m = 1$; (ii) $\lim_{m \longrightarrow \infty} \hat{\alpha}_m = 0$, $\sum_{m=1}^{\infty} \hat{\alpha}_m = \infty$; $\hat{\delta}_m + \hat{\gamma}_m \leq c_m$, $\beta_m \leq \beta < (\sqrt{2}-1)$, $\forall m \geq 1$, for $\lambda = \max\{\lambda_1, \lambda_2\}$. Then, $\{r_m\}$ admits strong convergence to a common fixed points of T_1 and T_2 nearest to u.

In what follows, we establish convergence results for common zeros of a family of enriched monotone operators.

Corollary 2 Let H be as stated above. Let $\{A_i\}_{i=1}^2$: $C \longrightarrow C$ be a sequence of $(a, \{\lambda_i\}_{i=1}^2)$-enriched Lipschitz monotone operators with Lipschitz constants λ_1 and λ_2, respectively. Suppose $\Gamma = \bigcap_{i=1}^{2} N(A_i) \neq \emptyset$. For arbitrary r_0, $u \in C$, let $\{r_m\}$ be developed from

$$\begin{cases} z_m = r_m - c_m A_{b,2} r_m; \\ y_m = r_m - \beta_m A_{b,1} r_m; \\ r_{m+1} = \alpha_m u + (1-\alpha_m)[\vartheta_m r_m + \delta(I - A_{b,1}) y_m + \gamma_m (I - A_{b,2}) z_m], \end{cases} \tag{35}$$

where $T_{b,i} = (1-b)I + bT_i$ $(i = 1, 2)$, $\{\delta_m\}$, $\{\vartheta_m\}$, $\{\gamma_m\} \subset [a, b] \subset (0, 1)$, $\{\alpha_m\} \subset (0, c) \subset (0, 1)$ fulfill the following requirements: (i) $\vartheta_m + \delta_m + \gamma_m = 1$; (ii) $\lim_{m \longrightarrow \infty} \alpha_m = 0$, $\sum_{m=1}^{\infty} \alpha_m = \infty$; $\delta_m + \gamma_m \leq c_m$, $\beta_m \leq \beta < \dfrac{1}{\sqrt{\lambda^2+1}+1}$, $\forall m \geq 1$, for $\lambda = \max\{\lambda_1, \lambda_2\}$. Then, $\{r_m\}$ admits strong convergence to a common zero points of A_1 and A_2 nearest to u.

Proof. Let $\{T_{b,i}\}_{i=1}^2 = (I - \{A_{b,i}\}_{i=1}^2)$. The, every $\{T_{b,i}\}_{i=1}^2$ is an a-enriched Lipshitz pseudocontraction with the Lipshitz constant $\lambda_i' = (1+\lambda_i)$ and $\bigcap_{i=1}^{2} F(T_i) = \bigcap_{i=1}^{2} (A_i) \neq \emptyset$. Furthermore, when $\{A_{b,i}\}_{i=1}^2$ is replaced with $(I - \{T_{b,i}\}_{i=1}^2)$, then the iterative method of (35) reduces to (16), and hence, the conclusion follows immediately from Theorem 1. □

We can also obtain the corollary below for a finite family of enriched monotone operators.

Corollary 3 Let Hilbert space. Let $\{A_i\}_{i=1}^N$: $C \longrightarrow C$ be a sequence of $(a, \{\lambda_i\}_{i=1}^N)$-enriched Lipschitzian monotone mappings with Lipschitz constants λ_i, $i = 1, 2, \cdots, N$, respectively. Assume that $\Gamma = \bigcap_{i=1}^{N} N(A_i)$ is nonempty. Let a sequence $\{r_m\}$ be developed from an arbitrary r_0, $u \in C$ by

$$\begin{cases} y_m = r_m - v_m A_{b,i} r_m, \ i = 1, 2, \cdots, N; \\ r_{m+1} = \alpha_m u + (1-\alpha_m)[u_{m_0} r_m + \sum_{i=1}^{N} u_{m_i} (I - A_{b,i}) y_m], \end{cases} \tag{36}$$

where $T_{b,\,i} = (1-b)I + bT_i$ $(i = 1,\, 2)$, $\{\vartheta_{m_i}\}_{i=1}^{N} \subset (a,\, b) \subset (0,\, 1)$, $\{\alpha_m\} \subset (0,\, c) \subset (0,\, 1)$ ensure that: (i) $\sum_{i=0}^{N} u_{m_i} = 1$; (ii) $\lim_{m\longrightarrow\infty} \alpha_m = 0$, $\sum_{m=1}^{\infty} \alpha_m = \infty$; $\sum_{i=1}^{N} u_{m_i} \leq v_m \leq v < \frac{1}{\sqrt{\lambda^2+1}+1}$, $\forall m \geq 1$, for $\lambda = \max\{\lambda_i\colon i = 0,\, 1,\, \cdots,\, N\}$. Then, $\{r_m\}$ admits strong convergence to a common zero points of $A_i(\colon i = 0,\, 1,\, \cdots,\, N)$ nearest to u.

In Corollary 3, by taking a single Lipschitz monotone operator, we obtain the corollary as:

Corollary 4 Let $A\colon C \longrightarrow C$ be a sequence of $(a,\, L)$-enriched Lipschitzian monotone operator with Lipschitz constants λ. Suppose $\Gamma = N(A) \neq \emptyset$. For arbitrary $r_0,\, u \in C$, let $\{r_m\}$ be developed from

$$\begin{cases} y_m = r_m - c_m A_b r_m; \\ r_{m+1} = d_m u + (1-d_m)[(1-\gamma_m)r_m + \gamma_m(I-A_b)y_m], \end{cases} \tag{37}$$

where $T_b = (1-b)I + bT$, $\{\gamma_m\} \subset (a,\, b) \subset (0,\, 1)$, $\{\alpha_m\} \subset (0,\, c) \subset (0,\, 1)$ ensure that: (i) $\lim_{m\longrightarrow\infty} d_m = 0$, $\sum_{m=1}^{\infty} d_m = \infty$; (ii) $\gamma_m \leq c_m \leq c < \frac{1}{\sqrt{\lambda^2+1}+1}$, $\forall m \geq 1$. hen, $\{r_m\}$ admits strong convergence to a common zero points of A nearest to u.

4. Numerical example

The purpose of this section is to give a numerical example to support our result. The following example is given to support Theorem 1.

Example 5 Let $\mathbb{R}$ be the set of real numbers and let $H = \mathbb{R}$. Let $C = [-2,\, 1]$ and define $T_1,\, T_2\colon C \longrightarrow C$ by

$$T_1 r = \begin{cases} r + r^2, & \text{if} \quad r \in [-2,\, 0]; \\ r, & \text{if} \quad r \in (0,\, 1], \end{cases}$$

and

$$T_2 r = \begin{cases} r, & \text{if} \quad r \in \left[-2,\, \frac{1}{2}\right]; \\ r - 2\left(r - \frac{1}{2}\right)^2 & \text{if} \quad r \in \left(\frac{1}{2},\, 1\right]. \end{cases}$$

Let $\{r_m\}$, $\{y_m\}$ and $\{z_m\}$ be generated by (16), where $\alpha_m = \frac{1}{m+20}$, $c_m = \beta_m = \frac{1}{100} + \frac{1}{m+200}$, $\gamma_m = \delta_m = \frac{1}{2}\left(\frac{1}{100} + \frac{1}{m+200}\right)$ and $\vartheta_m = 1 - \left(\frac{1}{100} + \frac{1}{m+200}\right)$ for every $m \in \mathbb{N}$. Then, the sequence $\{r_m\}$ converges to a common fixed point of T_1 and T_2.

Solution Observe that $\Gamma = F(T_1) \cap F(T_2) = [0,\, 1] \cap \left[-2,\, \frac{1}{2}\right]$, and that for all $r,\, s \in C$, we have

$$(0+1)^2|r-s|^2+|r-T_1r-(s-T_1s)|^2 = |r-s|^2+r^2 \geq |r-s+r^2|^2$$

$$= |0(r-s)+T_1r-T_1s|^2,$$

which shows that the mapping T_1 is a $(0, 5)$-enriched Lipshitzian pseudocontractive mapping (see Example 2). Similarly, we can show that T_2 is a $(0, 10)$-enriched Lipshitzian pseudocontractive mapping.

For every $m \in \mathbb{N}$, $\alpha_m = \frac{1}{m+20}$, $c_m = \beta_m = \frac{1}{100} + \frac{1}{m+200}$, $\gamma_m = \delta_m = \frac{1}{2}\left(\frac{1}{100} + \frac{1}{m+200}\right)$ and $\vartheta_m = 1 - \left(\frac{1}{100} + \frac{1}{m+200}\right)$. Then, the sequences $\{\alpha_m\}$, $\{c_m\}$, $\{\beta_m\}$, $\{\gamma_m\}$, $\{\delta_m\}$ and $\{\vartheta_m\}$ satisfy all the condition of Theorem 1. Using the algorithm (16) and choosing $u = 0.7$ and $r_0 = -1$, we have the numerical (and graphical) results in Table 1 and Figure 1 (scenario 1). Also, from algorithm (16) and choosing $u = -1$ and $r_0 = 0.6$, we obtain the numerical (and graphical) results in Table 2 and Figure 1 (scenario 2).

Remark 4 1. The sequence $\{r_m\}$ converges to 0.5 when $u = 0.7$ and $r_0 = -1$ as shown in in Table 1 and Figure 1 (scenario 1). When $u = -1$ and $r_0 = 0.6$, the sequence $\{r_m\}$ converges to 0 as shown in Table 2 and Figure 1 (scenario 2).

2. From Theorem 1, we can conclude that the sequence $\{r_m\}$ in Example 4.1, converges to $[0, 1] \cap \left[-2, \frac{1}{2}\right]$.

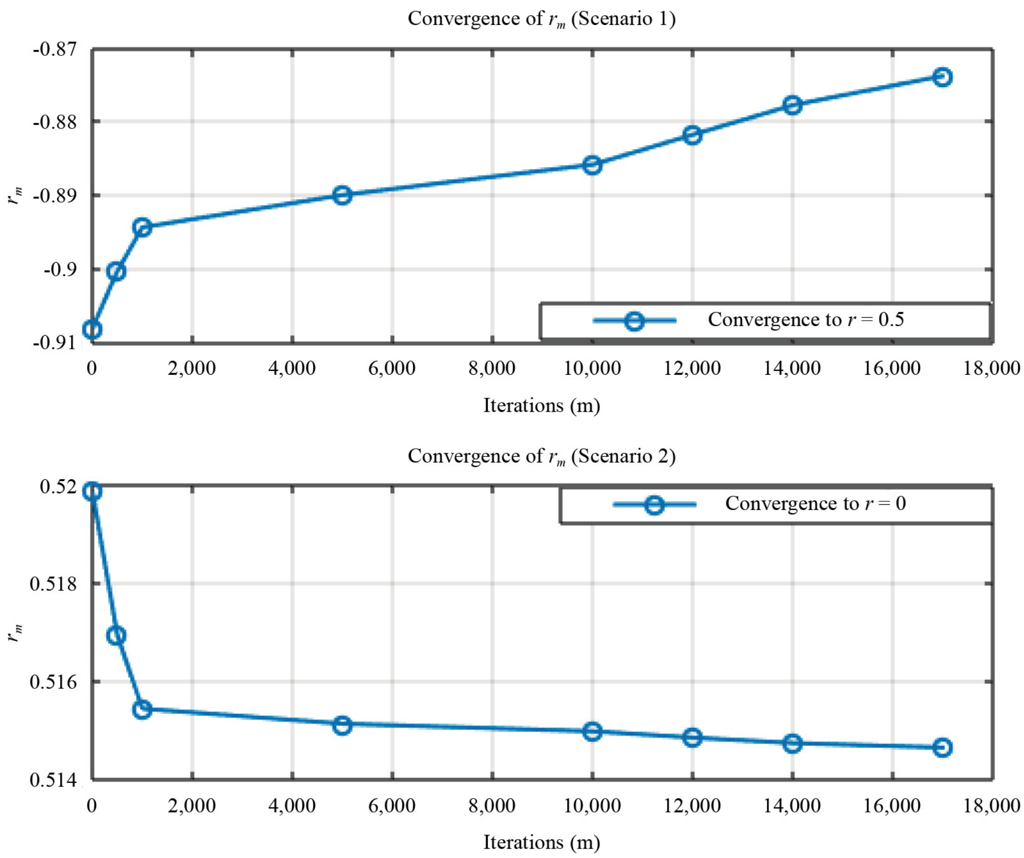

Figure 1. Depicts the above scenarios 1, 2

Table 1. Convergence of r_m for Scenario 1 ($u = 0.7$, $r_0 = -1$)

Iterations (m)	r_m
0	-0.907980272
500	-0.900229384
1,000	-0.894312711
5,000	-0.889952467
10,000	-0.885826455
12,000	-0.881769705
14,000	-0.877771988
17,000	-0.873833840

Table 2. Convergence of r_m for Scenario 2 ($u = -1$, $r_0 = 0.6$)

Iterations (m)	r_m
0	0.519856216
500	0.516928871
1,000	0.515438553
5,000	0.515134221
10,000	0.514980675
12,000	0.514852353
14,000	0.514742061
17,000	0.514650857

5. Conclusion

The class pseudocontractive mappings are very important class of nonlinear mappings due to their intimate connection with another class of nonlinear mappings called accretive mappings. Here, we introduced a more general class of a-enriched pseudocontractive mappings which contains, the class of pseudocontracive mappings. We proved convergence results in a real Hilbert space. Several examples are given to support our results. Unlike several works in the literature, we do not require strict assumptions on the space or the operator in order to obtain our results. Indeed, this results represent the vanguard for other results in this direction. It is hoped that the introduction of this new class of nonlinear mappings will provide a new direction for the evaluation of common solution of variational inequality and fixed point problems with several practical applications in operator theory.

Acknowledgements

This research project was funded by Deanship of Scientific Research and Libraries, Princess Nourah bint Abdulrahman University, through the program of Research Project Funding after publication, grant No. (RPFAP-81-1445).

Conflict of interest

The authors declare no competing financial interest.

References

[1] Urai K. Fixed point theorems and the existence of economic equilibria based on conditions for local directions of mappings. In: Kusuoka S, Maruyama T. (eds.) *Advances in Mathematical Economics*. Tokyo: Springer; 2000. p.87-118. Available from: https://doi.org/10.1007/978-4-431-67909-7-5.

[2] Turab A, Nescolarde-Selva JA, Ullah F, Montoyo A, Alfiniyah C, Sintunavarat W, et al. Deep neural networks and stochastic methods for cognitive modeling of rat behavioral dynamics in $\mathbb{T}$-mazes. *Cognitive Neurodynamics*. 2025; 19: 66. Available from: https://doi.org/10.1007/s11571-025-10247-9.

[3] Agwu IK. A novel iteration algorithm for a hybrid pair of total symptotically nonexpansive single-valued and total asymptotically quasi-nonexpansive non-self mappings in Banach spaces. *Results in Fixed Point Theory and Applications*. 2020; 2020: 1-28. Available from: https://doi.org/10.30697/rfpta-2020-007.

[4] Turab A, Montoyo A, Nescolarde-Selva JA. Stability and numerical solutions for second-order ordinary differential equations with application in mechanical systems. *Journal of Applied Mathematics and Computing*. 2024; 70(5): 5103-5128. Available from: https://doi.org/10.1007/s12190-024-02175-4.

[5] Saleem N, Agwu IK, Ishtiaq U, Radenovic S. Strong convergence theorems for a finite family of (b, k)-enriched strictly pseudocontractive mappings and Φ_T-enriched Lipschitizian mappings using a new modified mixed-type Ishikawa iteration scheme with error. *Symmetry*. 2022; 14(5): 1032. Available from: https://doi.org/10.3390/sym14051032.

[6] Hicks TL, Kubicek JD. On the Mann iteration process in Hilbert space. *Journal of Mathematical Analysis and Applications*. 1977; 59: 498-504. Available from: https://doi.org/10.1016/0022-247X(77)90076-2.

[7] Berinde V. Approximating fixed points of enriched nonexpansive mappings by Krasnoselskii iteration in Hilbert spaces. *Carpathian Journal of Mathematics*. 2019; 35(3): 293-304. Available from: https://doi.org/10.37193/CJM.2019.03.04.

[8] Berinde V. Weak and strong convergence theorems for the Krasnoselskii iteration algorithm in the class of enriched strictly pseudocontractive operators. *Annals of West University of Timisoara-Mathematics and Computer Science*. 2018; 56: 13-37. Available from: https://doi.org/10.2478/awutm-2018-0013.

[9] Browder FE, Petryshyn WV. Construction of fixed points of nonlinear mappings in Hilbert space. *Journal of Mathematical Analysis and Applications*. 1967; 20: 197-228. Available from: https://doi.org/10.1016/0022-247X(67)90085-6.

[10] Krasnoselskii MA. Two remarks about the method of successive approximations. *Russian Mathematical Surveys*. 1955; 10: 123-127.

[11] Zeidler E. *Nonlinear Functional Analysis and Its Applications, Part II: Monotone Operators*. Berlin: Springer; 1985. Available from: https://doi.org/10.1007/978-1-4612-0985-0.

[12] Agwu IK, Igbokwe DI. Convergence theorems and demiclosedness principle for enriched strictly pseudocontractive mappings in real Banach spaces. *International Journal of Nonlinear Analysis and Applications*. 2023; 14(6): 345-355.

[13] Razzaque A, Saleem N, Agwu IK, Ishtiaq U, Alphane M. Strong and weak convergence theorems for the split feasibility problem of (β, k)-enriched strict pseudocontractive mappings with an application in Hilbert spaces. *Symmetry*. 2024; 16: 546. Available from: https://doi.org/10.3390/sym16050546.

[14] Agwu IK, Isik H, Igbokwe DI. Fixed point results involving a finite family of enriched strictly pseudocontractive and pseudononspreading mappings. *Journal of Inequalities and Applications*. 2024; 2024: 58. Available from: https://doi.org/10.1186/s13660-024-03120-6.

[15] Agwu IK, Igbokwe DI, Okeke GA, Olaoluwa H. Existence of fixed point for a new class of enriched pseudocontractive mappings in Banach space. *Journal of the Nigerian Mathematical Society*. 2024; 43(3): 221-235. Available from: https://ojs.ictp.it/jnms/index.php/jnms/article/view/1079.

[16] Berinde V, Saleh K. New averaged type algorithms for solving split common fixed point problems for demicontractive mappings. *Arabian Journal of Mathematics*. 2024; 13: 679-688. Available from: https://doi.org/10.1007/s40065-024-00476-x.

[17] Agwu IK, Isik H, Igbokwe DI. Weak and strong convergence theorems for a new class of enriched strictly pseudononspreading mappings in Hilbert spaces. *Fixed Point Theory and Algorithms for Sciences and Engineering*. 2024; 2024: 14. Available from: https://doi.org/10.1186/s13663-024-00770-5.

[18] Kurokawa Y, Takahashi W. Weak and strong convergence theorems for nonspreading mappings in Hilbert spaces. *Nonlinear Analysis*. 2010; 73: 1562-1568. Available from: https://doi.org/10.1016/j.na.2010.04.060.

[19] Zegeye H, Shahzad N. An algorithm for a common fixed point of a family of pseudocontractive mappings. *Fixed Point Theory and Applications*. 2013; 2013(1): 234. Available from: https://doi.org/10.1186/1687-1812-2013-234.

[20] Xu HK. Another control condition in an iterative method for nonexpansive mappings. *Bulletin of the Australian Mathematical Society*. 2002; 65: 109-113. Available from: https://doi.org/10.1017/S0004972700020116.

[21] Mainge PE. Strong convergence of projected subgradient methods for nonsmooth and non-strictly convex minimization. *Set-Valued Analysis*. 2008; 16: 899-912. Available from: https://doi.org/10.1007/s11228-008-0102-z.

[22] Zegeye H, Shahzad N. Convergence of Mann's type iteration method for generalized asymptotically nonexpansive mappings. *Computers & Mathematics with Applications*. 2011; 62: 4007-4014. Available from: https://doi.org/10.1016/j.camwa.2011.09.018.