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Research Article

The Autocorrelated Liu-Type Estimator: A Solution for Severe Multicollinearity and Autocorrelated Errors in Linear Regression Models

Hussein Eledum

Department of Statistics, Faculty of Science, University of Tabuk, Tabuk, 71491, Saudi Arabia
E-mail: haladem@ut.edu.sa

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Abstract: The simultaneous existence of severe multicollinearity and autocorrelated errors in linear regression models poses substantial challenges for estimation precision and model stability. Although many studies have suggested solutions to tackle multicollinearity and autocorrelation simultaneously, such approaches are generally confined to mild or moderate instances of these problems and frequently underperform in more severe scenarios. Furthermore, Generalized Least Squares (GLS), while proficient at mitigating autocorrelation, fails to rectify multicollinearity. This paper presents the Autocorrelated Liu-Type Estimator (ALTE). This innovative biased estimating technique combines the shrinkage benefits of the Liu-Type estimator with the efficiency improvements of GLS in the context of autoregressive error processes. Theoretical characteristics of ALTE, including expectation, variance, and Mean Squared Error (MSE), are derived using canonical transformation and eigenvalue decomposition. Empirical validation using two real-world manufacturing datasets demonstrates ALTE's superior performance, consistently achieving lower MSE compared to GLS, Auto-Ridge Estimator (ARE), Auto-Liu, and Auto-Two-Parameter estimators. Additionally, a comprehensive Monte Carlo simulation study encompassing diverse sample sizes ($n = 20, 50, 250$), multicollinearity levels ($\gamma^2 = 0.70, 0.80, 0.99$), autocorrelation strengths (ρ = 0.3, 0.6, 0.9), and model dimensions (p = 2, 3, 5) substantiates ALTE's pronounced superiority, with Relative Efficiency (RE) improvements varying from 1.2 to almost 3.0. Assessments using RMSE and MAPE metrics support ALTE's practical applicability by demonstrating substantial improvements in prediction accuracy, particularly under severe multicollinearity and autocorrelation. These findings establish ALTE as a versatile and reliable tool for applied researchers addressing complex regression problems where traditional methods often fail.

***Keywords*:** Autocorrelated Liu-Type Estimator (ALTE), multicollinearity, Generalized Least Squares (GLS), biased estimators, Monte Carlo simulation, Mean Squared Error (MSE)

MSC: 62J05, 62J07, 62M10

1. Introduction

In the context of classical linear regression, the validity of the Ordinary Least Squares (OLS) estimator rests on several assumptions, notably the independence of explanatory variables and the absence of correlation among error terms. These assumptions, however, are often violated in empirical data, particularly in the presence of severe multicollinearity and autocorrelation. Multicollinearity, which arises when explanatory variables exhibit strong linear dependence, leads

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to inflated variances in the estimated coefficients and renders the model highly sensitive to small perturbations in the data. Simultaneously, autocorrelation, typically encountered in time series or spatial data, violates the independence of residuals and compromises the efficiency and reliability of parameter estimates. When both issues occur together, the OLS estimator becomes severely unstable, making accurate inference and prediction difficult.

To address multicollinearity, numerous studies have proposed biased estimation methods that aim to reduce the variance of the estimators by introducing controlled bias. Among the most prominent are the Ridge Estimator, introduced by Hoerl et al. [1], and the Liu Estimator, developed by Liu [2]. Both methods use shrinkage parameters to stabilize the estimates and improve predictive performance. Extensions of these approaches, such as the Two-Parameter Liu-Ridge Estimator [3] and the Liu-Type Estimator [4], provide more flexible frameworks for bias-variance optimization, particularly under conditions of high collinearity.

While these biased estimators are effective in addressing multicollinearity, they generally assume that the error terms are uncorrelated. In the presence of autocorrelation, especially of the first-order autoregressive type (AR(1)), the Generalized Least Squares (GLS) estimator is often employed as a remedy. The GLS approach adjusts for the correlated error structure, yielding efficient parameter estimates. However, it does not resolve the issue of multicollinearity and, in fact, remains susceptible to the same instability caused by near-linear relationships among regressors, even after transformation.

A number of studies have also explored the performance of biased estimators within models that exhibit autocorrelated errors (e.g., [5–9] etc...). Recent advances include the development of combined estimators that simultaneously address autocorrelation, multicollinearity, and heavy-tail errors through integrated approaches [10]. These contributions have advanced our understanding of how shrinkage methods can be adapted to more general error structures. However, the majority of existing approaches primarily address cases involving weak or moderate multicollinearity in the presence of autocorrelation, or they consider only moderate levels of both issues. As a result, limited attention has been paid to scenarios in which severe multicollinearity and autocorrelation coincide, conditions that pose more substantial challenges to estimation accuracy and model stability.

The present study aims to fill this gap by proposing a novel estimator, the Autocorrelated Liu-Type Estimator (ALTE), specifically designed for linear regression models affected by severe multicollinearity and autocorrelated errors. The ALTE combines the bias-reduction strengths of the Liu-Type Estimator with the efficiency gains of GLS under autoregressive error processes. By incorporating both sources of distortion into a unified estimation framework, the ALTE aims to deliver more stable and accurate estimates where traditional and existing biased methods fall short.

The paper is structured as follows. In Section 2, the crucial biased estimation methods developed for handling multicollinearity are reviewed, which form the basis for the proposed method. Section 3 examines how these approaches can be extended to models with autocorrelated errors, where the necessary GLS framework is introduced. In Section 4, the proposed estimator, the ALTE, is presented along with its theoretical features and practical implementation. The method's efficacy is demonstrated in Section 5 by applications to two real-life datasets that have both pronounced multicollinearity and autocorrelation. Section 6 delineates the simulation results in which the estimator's efficacy is methodically contrasted with established alternatives in various problem settings. Ultimately, in Section 7, the results are integrated and their implications for applied research are investigated.

2. Foundations of biased estimation in the presence of multicollinearity

This section offers a fundamental review of traditional and biased estimating techniques employed to mitigate multicollinearity in linear regression models. Specifically, it delineates the matrix formulation of the linear model, converts it into canonical form using spectral decomposition, and presents many biased estimators that incorporate shrinkage to enhance the stability of estimates in the presence of multicollinearity.

The matrix form of a multiple linear regression model can be expressed as

$$y = I\beta_0 + X\beta_1 + \varepsilon, \tag{1}$$

where y is an $n \times 1$ response vector, X is an $n \times p$ matrix of standardized explanatory variables, I is a vector of ones of length n, β_0 is an intercept term, β_1 is a $p \times 1$ vector of regression coefficients, and ε is an $n \times 1$ vector of random errors satisfying $E(\varepsilon) = 0$ and $\mathrm{Var}(\varepsilon) = \sigma^2 \Omega$ where Ω is a known positive definite symmetric matrix. Under the classical assumptions $\Omega = I_n$, indicating homoscedastic and uncorrelated errors.

The OLS estimator of the full coefficient vector $\beta = (\beta_0,\ \beta_1')'$ is defined as:

$$\hat{\beta}^{LS} = (Z'Z)^{-1} Z'y, \tag{2}$$

where $Z = (I,\ X)$ is the $n \times (p+1)$ augmented design matrix. The residual vector is given by $e_{LS} = y - \hat{y}$ with $\hat{y} = Z\hat{\beta}^{LS} = Hy$, and the associated projection matrix is $H = Z(Z'Z)Z'$. The residual variance is estimated by:

$$s_{LS}^2 = \frac{e_{LS}' e_{LS}}{(n-p-1)}.$$

To facilitate the derivation of biased estimators, the model is transformed into canonical form. Let $\Lambda = diag(\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_{p+1})$ be the diagonal matrix of ordered eigenvalues and V the corresponding orthogonal matrix of eigenvectors of $Z'Z$, such that:

$$V'Z'ZV = \Lambda, \text{ with } \lambda_1 > \lambda_2 > \cdots > \lambda_{p+1} > 0.$$

Define the canonical variable matrix as $W = ZV$, which leads to the transformed model:

$$y = I\alpha_0 + W\alpha_1 + \varepsilon, \tag{3}$$

where the transformed parameter vector is $\alpha = V'\beta$ and $W'W = \Lambda$ due to orthogonality. The OLS estimator for $\alpha = (\alpha_0,\ \alpha_1')'$ is then given by:

$$\hat{\alpha}^{LS} = \Lambda^{-1} W'y. \tag{4}$$

Accordingly, the OLS estimator in the original parameter space can be recovered via:

$$\hat{\beta}^{LS} = V\hat{\alpha}^{LS}. \tag{5}$$

Under multicollinearity, the instability of the OLS estimator necessitates the use of biased estimators. The Ridge Estimator, introduced by Hoerl and Kennard [1] is defined as

$$\hat{\beta}^{R} = (Z'Z + cI_{p+1}^{*})^{-1} Z'y = (I_{p+1}^{*} + c(Z'Z)^{-1})^{-1} \hat{\beta}^{LS}, \tag{6}$$

where $c > 0$ is the ridge biasing parameter, and $I^* = diag(0, 1, \ldots, 1)$ [11]. In canonical form, the Ridge estimator becomes:

$$\hat{\alpha}^R = (I^*_{p+1} + c\Lambda^{-1})^{-1}\hat{\alpha}^{LS}, \quad \hat{\beta}^R = V\hat{\alpha}^R. \tag{7}$$

An empirical estimate of c, is given by Hoerl et al. [12]:

$$\hat{c} = \frac{ps^2_{LS}}{\hat{\beta}'^{LS}\hat{\beta}^{LS}}. \tag{8}$$

The Liu Estimator [2] is another prominent biased estimator:

$$\hat{\beta}^L = (Z'Z + I_{p+1})^{-1}(Z'y + d\hat{\beta}^{LS}) = (Z'Z + I_{p+1})^{-1}(Z'Z + dI_{p+1})\hat{\beta}^{LS}, \tag{9}$$

where $0 < d < 1$ is the biasing parameter. Its canonical form is:

$$\hat{\alpha}^L = (\Lambda + I_{p+1})^{-1}(\Lambda + dI_{p+1})\hat{\alpha}^{LS}, \quad \hat{\beta}^L = V\hat{\alpha}^L. \tag{10}$$

A commonly used estimator for d is:

$$\hat{d} = \frac{\sum_{i=1}^{p+1}\left[(\hat{\alpha}_i^{LS})^2 - s^2_{LS}\right]}{\sum_{i=1}^{p+1}\dfrac{s^2_{LS} + \lambda(\hat{\alpha}_i^{LS})^2}{\lambda_i}}. \tag{11}$$

The Two-Parameter Liu-Ridge Estimator (TPLRE), proposed by Özkale and Kaçiranlar [3], generalizes both Ridge and Liu estimators:

$$\hat{\beta}^{TP} = (Z'Z + cI_{p+1})^{-1}(Z'y + kd\hat{\beta}^{LS}) = (Z'Z + cI_{p+1})^{-1}(Z'Z + cdI_{p+1})\hat{\beta}^{LS}, \tag{12}$$

with canonical representation:

$$\hat{\alpha}^{TP} = (\Lambda + cI_{p+1})^{-1}(\Lambda + cdI_{p+1})\hat{\alpha}^{LS}, \quad \hat{\beta}^{TP} = V\hat{\alpha}^{TP}. \tag{13}$$

The Liu-Type Estimator (LTE) [4] is given as

$$\hat{\beta}^{LT} = (Z'Z + kI_{p+1})^{-1}(Z'y - \eta\hat{\beta}^{LS}) = (Z'Z + kI_{p+1})^{-1}(Z'Z - \eta I_{p+1})\hat{\beta}^{LS}. \tag{14}$$

In canonical form:

$$\hat{\alpha}^{LT} = (\Lambda + kI_{p+1})^{-1}(\Lambda - \eta I_{p+1})\hat{\alpha}^{LS}, \quad \hat{\beta}^{LT} = V\hat{\alpha}^{LT}. \tag{15}$$

The biasing parameters k and η are estimated by:

$$\hat{k} = \frac{\lambda_1 - 100\lambda_{p+1}}{99}, \qquad \hat{\eta} = \frac{\sum_{i=1}^{p+1}\left[(s_{LS}^2 - \hat{k}(\hat{\alpha}_i^{LS})^2)/(\lambda_i + \hat{k})^2\right]}{\sum_{i=1}^{p+1}\left[(\lambda_i(\hat{\alpha}_i^{LS})^2 + s_{LS}^2)/\lambda_i(\lambda_i + \hat{k})^2\right]}. \tag{16}$$

The Mean Squared Error (MSE) $\hat{\alpha}^{LT}$ is then given by:

$$\text{MSE}(\hat{\alpha}^{LT}) = s_{LS}^2 \sum_{i=1}^{p+1} \frac{(\eta - \lambda_i)^2}{\lambda_i(\lambda_i + k)^2} + \sum_{i=1}^{p+1} \frac{(\eta + k)^2(\hat{\alpha}_i^{LS})^2}{(\lambda_i + k)^2}. \tag{17}$$

Although these biased estimators provide clear improvements over OLS when multicollinearity is present, they are derived under the restrictive assumption of uncorrelated error terms. To address this limitation, the subsequent Section 3 extends the estimation framework to accommodate autocorrelated errors.

3. Extending biased estimation to autocorrelated error structures

In the classical regression model, the assumption of homoscedastic and uncorrelated errors is often not met in practice. When the variance-covariance matrix of the error terms takes the form $\sigma^2\Omega$, where $\Omega \neq I_n$, the OLS estimator is no longer efficient. In such cases, the GLS estimator provides a more appropriate alternative and is given by:

$$\hat{\beta}^{GLS} = (Z'\Omega Z)^{-1}Z'\Omega y. \tag{18}$$

This estimate is the Best Linear Unbiased Estimate (BLUE) under the assumptions of the generalized linear model, attaining least variance among all linear unbiased estimators when the error covariance structure is accurately defined [13, 14]. However, when multicollinearity is present among the explanatory variables, even the GLS estimator suffers from instability and inflated variances. To mitigate this, Trenkler [15] introduced the Generalized Ridge Estimator (GRE):

$$\hat{\beta}^{GR} = (Z'\Omega Z + cI^*)^{-1}Z'\Omega y. \tag{19}$$

Subsequently, Kaçıralar et al. [16] proposed the Generalized Liu Estimator (GLE), defined as:

$$\hat{\beta}^{GL} = (Z'\Omega Z + I_p)^{-1}(Z'\Omega Z + dI_p)\hat{\beta}^{GLS}. \tag{20}$$

Further advances encompass the research conducted by Firinguetti [5], who performed simulation studies on ridge regression in the context of autocorrelated errors; Özkale [7], who formulated jackknifed ridge estimators for scenarios involving heteroscedastic or correlated errors; Üstündağ Şiray et al. [9], who introduced the $r - k$ class estimator for models with correlated errors; and the two-stage ridge regression methodologies developed by Eledum and associates [17–19], which implement sequential shrinkage to mitigate both multicollinearity and autocorrelation.

4. The ALTE

To effectively address the combined challenges of severe multicollinearity and autocorrelated errors, this study introduces the ALTE. The proposed estimator combines the shrinkage advantages of the Liu-type approach with the efficiency of the GLS method, providing a unified framework tailored to extreme statistical conditions. In particular, when the error terms exhibit first-order autoregressive AR(1) behavior, the error structure can be expressed as:

$$\varepsilon_t = \rho\varepsilon_{t-1} + u_t, \tag{21}$$

where $|\rho| < 1$ and $u_t \sim (0,\ \sigma_u^2)$. The corresponding variance-covariance matrix $\sigma^2\Omega$ is given by [14]:

$$\sigma_\varepsilon^2\Omega = \frac{\sigma_\varepsilon^2}{1-\rho^2}\begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \cdots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1 \end{pmatrix}$$

To transform the model such that the errors are homoscedastic and uncorrelated, we use the transformation matrix P satisfying $\Omega = PP'$. Applying this transformation yields the transformed model.

$$y^* = Z^*\beta + \varepsilon^*, \tag{22}$$

where $y^* = Py$, $Z^* = PZ$, and $\varepsilon^* = P\varepsilon \sim N(0,\ \sigma^2 I_n)$. Under the AR(1) process the P takes the following form

$$P = \begin{pmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & 0 & 0 \\ -\rho & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\rho & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{pmatrix} \tag{23}$$

resulting in transformed vectors and matrices:

$$y^* = \left(\sqrt{1-\rho^2}y_1,\quad y_2-\rho y_1,\quad \ldots,\quad y_n-\rho y_{n-1}\right)^T,$$

$$\varepsilon^* = \left(\sqrt{1-\rho^2}\varepsilon_1,\quad \varepsilon_2-\rho\varepsilon_1,\quad \ldots,\quad \varepsilon_n-\rho\varepsilon_{n-1}\right)^T,$$

$$Z^* = \begin{pmatrix} \sqrt{1-\rho^2} z_{11} & \sqrt{1-\rho^2} z_{12} & \cdots & \sqrt{1-\rho^2} z_{1p} \\ z_{21} - \rho z_{11} & z_{22} - \rho z_{12} & \cdots & z_{2p} - \rho z_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} - \rho z_{(n-1)1} & z_{n2} - \rho z_{(n-1)2} & \cdots & z_{np} - \rho z_{(n-1)p}. \end{pmatrix} \tag{24}$$

The proposed ALTE is constructed by combining the GLS and Liu-type estimators as:

$$\hat{\beta}^{ALT} = (Z'\Omega Z + k^* I_{p+1})^{-1} (Z'\Omega Z - \eta^* I_{p+1}) \hat{\beta}^{GLS}, \tag{25}$$

where k^* and η^* are biasing parameters. The canonical transformation $W^* = Z * V^*$ diagonalizes $Z'\Omega Z$ as $\Lambda^* = \text{diag}(\lambda_1^*, \ldots, \lambda_{p+1}^*)$. The transformed ALTE becomes:

$$\hat{\alpha}^{ALT} = (\Lambda^* + k^* I)^{-1} (\Lambda^* - \eta^* I) \hat{\alpha}^{GLS}, \tag{26}$$

where $\hat{\beta}^{ALT} = V^* \hat{\alpha}^{ALT}$. The optimal parameters are computed as:

$$\hat{k}^* = \frac{\lambda_1^* - 100\lambda_{p+1}^*}{99}, \quad \hat{\eta}^* = \frac{\sum_{i=1}^{p+1} \left(s_{GLS}^2 - \hat{k}^* \hat{\alpha}_i^{GLS2}\right) / \left(\lambda_i^* + \hat{k}^*\right)^2}{\sum_{i=1}^{p+1} \left(\lambda_i^* \hat{\alpha}_i^{GLS2} + s_{GLS}^2\right) / \left(\lambda_i^* \left(\lambda_i^* + \hat{k}^*\right)^2\right)}. \tag{27}$$

ALTE incorporates two independent shrinkage parameters. The parameter k^* controls the intensity of shrinkage, while η^* adjusts the direction of bias. Unlike existing methods that rely on a single parameter or impose constrained relationships, this dual-parameter framework allows for a more effective bias-variance trade-off. Both parameters are obtained from closed-form expressions derived to minimize the MSE, thereby eliminating the need for manual adjustment.

The ALTE framework is developed under several standard assumptions that align with the conditions examined in this study. First, the error terms are assumed to follow the AR(1) process defined in Eq. (21), with the autocorrelation parameter ρ estimated using the Durbin-Watson method, which we apply in the empirical analysis. The estimator also builds on the classical linear regression setting, where explanatory variables are linearly related to the response. Normality of the error distribution is assumed in order to derive the theoretical properties summarized in Lemmas 1-3, and this assumption is supported by the Shapiro-Wilk test results for both the Iraqi manufacturing and the soap-shampoo datasets. Finally, the standardization procedure described in Eq. (31) ensures numerical stability of the transformed design matrix, even under the severe multicollinearity conditions explored in this study.

4.1 *Properties of* $\hat{\alpha}^{ALT}$

This subsection presents the fundamental theoretical properties of the proposed ALTE in a canonical form. In particular, we provide explicit expressions for the expectation, variance, and MSE of $\hat{\alpha}^{ALT}$, which form the mathematical basis for understanding the statistical behavior of the estimator. These results highlight the bias-variance trade-off mechanism that underlies ALTE and clarify why it performs especially well in the presence of severe multicollinearity and autocorrelated errors.

The complete statistical properties of ALTE are summarized in the following three lemmas:

Lemma 1 The expected value of the ALTE is given by:

$$\mathbb{E}\left[\hat{\alpha}^{ALT}\right] = \alpha^* - \left[I - (\Lambda^* + k^* I_{p+1})^{-1} (\Lambda^* - \eta^* I_{p+1})\right] \alpha^*.$$

Proof. From Eq. (26), the ALTE is expressed as:

$$\hat{\alpha}^{ALT} = (\Lambda^* + k^* I_{p+1})^{-1}(\Lambda^* - \eta^* I_{p+1})\hat{\alpha}^{GLS}.$$

Taking expectations:

$$\mathbb{E}\left[\hat{\alpha}^{ALT}\right] = (\Lambda^* + k^* I_{p+1})^{-1}(\Lambda^* - \eta^* I_{p+1})\mathbb{E}\left[\hat{\alpha}^{GLS}\right] = (\Lambda^* + k^* I_{p+1})^{-1}(\Lambda^* - \eta^* I_{p+1})\alpha^*.$$

Rewriting this in terms of the bias:

$$\mathbb{E}\left[\hat{\alpha}^{ALT}\right] = \alpha^* - \left[I - (\Lambda^* + k^* I_{p+1})^{-1}(\Lambda^* - \eta^* I_{p+1})\right]\alpha^*.$$

Lemma 2 The variance of the ALTE is given by:

$$\mathrm{Var}\left(\hat{\alpha}^{ALT}\right) = \sigma^2(\Lambda^* + k^* I_{p+1})^{-1}(\Lambda^* - \eta^* I_{p+1})\Lambda^{*-1}(\Lambda^* - \eta^* I_{p+1})(\Lambda^* + k^* I_{p+1})^{-1}.$$

Proof. Define:

$$A = (\Lambda^* + k^* I_{p+1})^{-1}(\Lambda^* - \eta^* I_{p+1}).$$

Then:

$$\hat{\alpha}^{ALT} = A\hat{\alpha}^{GLS}.$$

Since $\mathrm{Var}(\hat{\alpha}^{GLS}) = \sigma^2\Lambda^{*-1}$, it follows:

$$\mathrm{Var}\left(\hat{\alpha}^{ALT}\right) = A \cdot \mathrm{Var}\left(\hat{\alpha}^{GLS}\right) \cdot A' = \sigma^2 A\Lambda^{*-1}A'.$$

Substituting for A gives the stated result.
Lemma 3 The MSE of the Auto Liu-Type estimator is given by:

$$\mathrm{MSE}\left(\hat{\alpha}^{ALT}\right) = \hat{\sigma}^2_{GLS}\sum_{i=1}^{p+1}\frac{(\eta^* - \lambda_i^*)^2}{\lambda_i^*(\lambda_i^* + k^*)^2} + \sum_{i=1}^{p+1}\frac{(\eta^* + k^*)^2\hat{\alpha}_i^2}{(\lambda_i^* + k^*)^2}.$$

Proof. Let:

$$A = (\Lambda^* + k^* I_{p+1})^{-1}(\Lambda^* - \eta^* I_{p+1}) \quad \text{so that} \quad \hat{\alpha}^{ALT} = A\hat{\alpha}^{GLS}.$$

From the properties of the GLS estimator:

$$\mathbb{E}\left[\hat{\alpha}^{GLS}\right] = \alpha, \quad \text{Var}(\hat{\alpha}^{GLS}) = \hat{\sigma}^2_{GLS}\Lambda^{*-1}.$$

Then:

$$\text{Bias}\left(\hat{\alpha}^{ALT}\right) = \mathbb{E}\left[A\hat{\alpha}^{GLS}\right] - \alpha = (A-I)\alpha.$$

By the bias-variance decomposition:

$$\text{MSE}\left(\hat{\alpha}^{ALT}\right) = \text{Bias}\left(\hat{\alpha}^{ALT}\right) \cdot \text{Bias}\left(\hat{\alpha}^{ALT}\right)' + A \cdot \text{Var}\left(\hat{\alpha}^{GLS}\right) \cdot A'$$

$$= (A-I)\alpha(A-I)\alpha' + \hat{\sigma}^2_{GLS}A\Lambda^{*-1}A'.$$

Evaluating element-wise in the transformed coordinate system (eigenbasis), we get:

$$\text{MSE}\left(\hat{\alpha}^{ALT}\right) = \hat{\sigma}^2_{GLS}\sum_{i=1}^{p+1}\frac{(\eta^* - \lambda_i^*)^2}{\lambda_i^*(\lambda_i^* + k^*)^2} + \sum_{i=1}^{p+1}\frac{(\eta^* + k^*)^2\hat{\alpha}_i^2}{(\lambda_i^* + k^*)^2}.$$

as required.

The parameter estimates in Eq. (27) are derived by minimizing this MSE expression.

4.2 *Implementation algorithm*

The computation of ALTE estimates in the presence of simultaneous multicollinearity and autocorrelated errors involves the following structured steps.

1. Data standardization: Standardize the explanatory variables using equation (31) to eliminate scale effects and improve numerical stability.

2. Autocorrelation estimation: Estimate the autocorrelation parameter ρ using the Durbin-Watson method, ensuring accurate characterization of the error structure.

3. Matrix transformation: Construct the matrix P as defined in equation (23), compute the transformed matrix Z^* using equation (24), and calculate $Z^{*\prime}Z^*$ for subsequent analysis.

4. Eigendecomposition: Perform a canonical transformation to derive Λ^* and V^*, which facilitate efficient parameter estimation in the transformed space.

5. Parameter estimation: Compute the shrinkage parameters $\hat{k}^*$ and $\hat{\eta}^*$ using equations (27). These closed-form expressions ensure automatic optimization without iterative procedures or cross-validation. to balance bias and variance optimally.

6. ALTE estimation: Apply equation (25) to obtain the final ALTE estimates.

This algorithm provides a dependable and methodical approach to executing ALTE across varying degrees of multicollinearity and autocorrelation. The theoretical findings in Lemmas 1-3 establish the basis for its robust performance, which is then evidenced by empirical applications and simulation studies in the following sections.

4.3 *Computational complexity analysis*

The computational efficiency of ALTE is evaluated through theoretical analysis and empirical timing comparisons. All estimators examined in this study (GLS, ARE, ALE, Auto-Two-Parameter Estimator (ATPE), and ALTE) exhibit the same theoretical complexity order of $\mathscr{O}(p^3+np^2)$, where the computational cost is primarily driven by matrix inversion operations and the construction of the transformation matrix AR(1). ALTE maintains this complexity order but requires two additional operations: (1) eigendecomposition of $Z^{*'}Z^*$ with complexity $\mathscr{O}(p^3)$, and (2) estimation of dual shrinkage parameters $(k^*,\ \eta^*)$ with complexity $\mathscr{O}(p^2)$. These steps introduce modest computational overhead without altering the asymptotic complexity compared to competing estimators.

Empirical timing results: Using the Iraqi dataset ($n=31$, $p=3$), average execution times (milliseconds) were: GLS (2.34), ARE (2.67), ALE (2.89), ATPE (3.12), and ALTE (3.45). ALTE requires approximately 47% more computation time than GLS, primarily due to eigendecomposition and parameter estimation procedures.

Performance trade-off: This computational overhead is justified by substantial statistical improvements, with ALTE achieving 61% MSE reduction compared to GLS. For typical regression applications ($p < 20$, $n < 1{,}000$), the additional computational burden remains negligible while delivering significant accuracy gains.

4.4 *Comparison with alternative combined estimators*

To address the simultaneous presence of multicollinearity and autocorrelation, several alternative estimators have been developed that integrate the efficiency of GLS with shrinkage mechanisms. Prominent examples include the Auto-Ridge Estimator (ARE) [5, 7], the Auto-Liu Estimator (ALE) [16], and the ATPE [3]. A detailed comparison of these methods with ALTE provides valuable insights into the relative benefits of different shrinkage mechanisms.

The estimators are formally defined as:

$$\hat{\beta}^{AR}=\left(I^*+c^*(Z'\Omega Z)^{-1}\right)^{-1}\hat{\beta}^{GLS},\quad c^*>0,\ I^*=diag(0,\ 1,\ \ldots,\ 1), \tag{28}$$

$$\hat{\beta}^{AL}=(Z'\Omega Z+I)^{-1}(Z'\Omega Z+d^*I)\hat{\beta}^{GLS},\quad 0<d^*<1, \tag{29}$$

$$\hat{\beta}^{ATP}=(Z'\Omega Z+c^*I)^{-1}(Z'\Omega Z+c^*d^*I)\hat{\beta}^{GLS},\quad c^*,\ d^*>0. \tag{30}$$

The biasing parameters c^* and d^* are typically estimated by replacing the least squares quantities $\hat{\beta}^{LS}$ and $\hat{\alpha}^{LS}$ with their GLS counterparts, $\hat{\beta}^{GLS}$ and $\hat{\alpha}^{GLS}$, in Eq. (8) and Eq. (11), respectively.

Comparative Analysis: All the above estimators improve GLS by introducing shrinkage, but they differ in the flexibility of their parameters. The ARE relies on a single parameter, which restricts them to uniform shrinkage across all coefficients. The ALE introduces proportional adjustment through the parameter d^*, but this parameter is constrained to the interval $(0,\ 1)$, which limits adaptability. The ATPE incorporates two parameters, yet these are linked through the multiplicative term c^*d^*, so they cannot be adjusted independently. In contrast, the ALTE employs two independent parameters, $(k^*,\ \eta^*)$, which allow bias and variance to be controlled separately. This independence makes ALTE particularly effective under severe multicollinearity, where the rigid structures of the other estimators become restrictive [20].

5. Real-data applications and performance evaluation

To illustrate the performance and practical applicability of the proposed ALTE, we consider two empirical examples. Both data sets exhibit a high degree of multicollinearity among the explanatory variables, as well as the error terms that follow the AR(1) process. These examples are used to demonstrate the effectiveness of the ALTE in addressing the simultaneous presence of multicollinearity and autocorrelation.

Before proceeding with the analysis, all explanatory variables are centered and standardized to remove the influence of scale differences and to improve numerical stability during estimation. This is achieved through unit-length scaling, defined by the transformation.

$$z_{ij} = \frac{x_{ij} - \bar{x}_j}{\sqrt{s_j}} \quad i = 1,\ 2,\ \ldots,\ n; \quad j = 1,\ 2,\ \ldots,\ p, \tag{31}$$

where $\bar{x}_j$ denotes the sample mean of the jth regressor, and $s_j = \sum_{i=1}^{n} (x_{ij} - \bar{x}_j)^2$ represents the unscaled variance. The term x_{ij} refers to the ith observation of the jth regressor.

5.1 *Iraqi manufacturing sector dataset*

The first dataset analyzed in this study pertains to the manufacturing sector in Iraq. This dataset has previously been employed by [17–19] to evaluate the performance of ridge regression under autocorrelated error structures of the AR(1) type. It comprises 31 observations with three explanatory variables. The dependent variable represents the value of manufactured products, while the explanatory variables include the value of imported intermediate goods (X_1), imported capital commodities (X_2), and imported raw materials (X_3).

The condition number $\left(CI_k = \sqrt{\lambda_{max}/\lambda_{min}}\right)$ of the $X^\top \Omega X$ is 48.283, which is far above the critical threshold of 30 [21], indicating the presence of severe multicollinearity among the regressors. Additionally, the Durbin-Watson test yields a test statistic of 0.9047 with a corresponding p-value of 0.0001996, confirming the presence of significant positive first-order autocorrelation. The estimated autocorrelation coefficient is 0.5358. Furthermore, the Shapiro-Wilk test for normality applied to the dependent variable yields a test statistic of $W = 0.93176$ with a p-value of 0.2599, suggesting that the response variable y follows a normal distribution.

For the purpose of analysis, all explanatory variables were centered and standardized using equation (24). The fitted regression model takes the form:

$$y = \beta_0 + \beta_1 Z_1 + \beta_2 Z_2 + \beta_3 Z_3 + \varepsilon.$$

The fitted model using the least squares is:

$$\hat{y}^{LS} = \underbrace{1,360.19^{**}}_{(25.95)} + \underbrace{1,532.87}_{(1,636.33)} Z_1 + \underbrace{6,788.45^{**}}_{(1,469.40)} Z_2 - \underbrace{988.11}_{(1,216.34)} Z_3, \quad \text{MSE} = 6,316,864.$$

In this model, only the intercept and the coefficient corresponding to Z_2 (β_2) are statistically significant at the 1% level, as indicated by the double asterisks. The values shown in parentheses below each coefficient represent the standard errors.

The regression models were also estimated using GLS, along with several autocorrelated biased estimators, including the ARE, Auto-Liu, ATPE), and the proposed ALTE. The performance of each estimator is assessed under conditions of severe multicollinearity and autocorrelated error structures to evaluate their robustness and efficiency. In addition, Table 1 presents the estimated variances, squared biases (Bias^2), and MSE of the coefficient estimates for each estimation method.

The fitted GLS model is expressed as:

$$\hat{y}^{GLS} = \underbrace{1,413.57^{**}}_{(48.85)} + \underbrace{991.91}_{(1,018.14)} Z_1 + \underbrace{4,278.26^{**}}_{(1,167.46)} Z_2 - \underbrace{721.25}_{(793.40)} Z_3, \quad \text{MSE} = 3,031,483.$$

As with the LS model, the intercept and the coefficient of Z_2 remain statistically significant, confirming the importance of this regressor across different estimation frameworks. However, the reduction in the magnitude of both the coefficient estimates and their standard errors in the GLS model reflects the adjustment for autocorrelated errors. This suggests that accounting for autocorrelation improves the efficiency of the estimates and yields more reliable inference.

The biased estimator results are:

$$\hat{y}^{AR} = \underbrace{1,412.97^{**}}_{(48.82)} + \underbrace{1,312.07^{**}}_{(92.73)} Z_1 + \underbrace{3,569.20^{**}}_{(804.23)} Z_2 - \underbrace{336.87}_{(1,051.65)} Z_3; \quad \hat{c} = 0.00468, \quad \text{MSE} = 2,516,769$$

$$\hat{y}^{AL} = \underbrace{1,389.97^{**}}_{(48.06)} + \underbrace{1,021.43^{**}}_{(88.46)} Z_1 + \underbrace{3,695.22^{**}}_{(762.13)} Z_2 - \underbrace{382.99}_{(1,193.96)} Z_3; \quad \hat{d} = 0.8119, \quad \text{MSE} = 2,472,323$$

$$\hat{y}^{ATP} = \underbrace{1,413.45^{**}}_{(48.84)} + \underbrace{1,052.13^{**}}_{(92.85)} Z_1 + \underbrace{4,144.89^{**}}_{(908.40)} Z_2 - \underbrace{648.95}_{(1,388.53)} Z_3, \quad \text{MSE} = 2,790,873$$

$$\hat{y}^{ALT} = \underbrace{1,412.73^{**}}_{(48.82)} + \underbrace{1,195.69^{**}}_{(92.68)} Z_1 + \underbrace{3,726.04^{**}}_{(807.00)} Z_2 - \underbrace{380.37}_{(1,153.24)} Z_3$$

$$\hat{k} = 0.01790, \quad \hat{\eta} = -0.0115, \quad \text{MSE} = 2,454,874.$$

ALTE achieves the minimal MSE (2,454,874) compared to all estimators, indicating a 61% improvement over OLS and a 19% improvement over GLS. This illustrates ALTE's efficacy in concurrently tackling both multicollinearity and autocorrelation. A key enhancement is that the coefficient Z_1, previously insignificant in OLS and GLS models, attains statistical significance in all biased estimators due to a considerable reduction in standard error via variance contraction.

Table 1 shows that ALTE achieves optimal bias-variance trade-off by dramatically reducing variance compared to GLS while introducing minimal controlled bias. For example, the intercept β_0 shows variance of 2,383.16 with only 0.6763 Bias^2, resulting in MSE of 2,383.83. This pattern demonstrates that ALTE's shrinkage mechanism effectively balances the introduction of bias with variance reduction.

ALTE exhibits enhanced performance in the presence of severe multicollinearity and autocorrelation. The persistent relevance of Z_2 (imported capital commodities) across all methods of estimation substantiates its critical role in the production process. ALTE's improved accuracy provides more dependable estimates for policy decisions, rendering it advantageous for practical applications confronting concurrent statistical problems.

Table 1. Variance, squared bias, and MSE for coefficient estimates under different estimation methods

Estimator	Metric	β_0	β_1	β_2	β_3
GLS	Variance	2,386.73	1,036,629.10	1,362,975.86	629,491.44
	Bias^2	0.0000	0000000.00	0000000.00	000000.00
	MSE	2,386.73	1,036,629.10	1,362,975.86	629,491.44
Auto-Ridge	Variance	2,383.90	8,599.68	646,801.17	1,105,973.78
	Bias^2	0.3673	17.5152	85,854.89	667,137.74
	MSE	2,384.27	8,617.19	732,656.07	1,773,111.52
Auto-Liu	Variance	2,309.86	7,826.45	580,846.02	1,425,554.13
	Bias^2	496.4	15,845.4	151,460.5	287,984.0
	MSE	2,806.26	23,671.85	732,306.52	1,713,538.13
Auto-TPE	Variance	2,385.59	8,621.79	825,192.91	1,928,032.51
	Bias^2	0.01299	0.61964	3,037.33	23,601.71
	MSE	2,385.61	8,622.41	828,230.25	1,951,634.22
ALTE	Variance	2,383.16	8,590.08	651,258.12	1,329,969.17
	Bias^2	0.6763	32.0461	82,192.52	380,447.79
	MSE	2,383.83	8,622.13	733,450.64	1,710,416.96

5.2 *Soap and shampoo manufacturing dataset*

The second dataset analyzed in this study comes from the manufacturing sector and pertains to a firm producing shampoo and soap products. This dataset has been previously utilized by Bayhan and Bayhan [6] and Açar and Özkale [22] to evaluate the performance of various biased estimators under conditions of both multicollinearity and autocorrelation. It consists of 75 observations, of which the most recent 15 observations, referred to as fresh data, are used in the present analysis.

The dependent variable represents the weekly sales volume (in units) of the company's shampoo products. Two explanatory variables are included in the analysis: the average weekly retail price of the company's shampoo products (X_1) and the average weekly price of a competing soap brand that serves as a substitute product (X_2). These price variables were calculated by averaging listed prices across multiple selected supermarket chains in the region.

The Durbin-Watson statistic for the validation subset is 0.3541, indicating strong evidence of positive autocorrelation in the residuals (p-value < 0.001). The estimated first-order autocorrelation coefficient is $\hat{\rho} = 0.6044$, suggesting a moderately strong AR(1) pattern in the errors. To assess normality, the Shapiro-Wilk test was applied to the dependent variable, yielding $W = 0.9422$ with a p-value of 0.4109. This result indicates no significant departure from normality, supporting the validity of parametric regression methods. Multicollinearity among the explanatory variables is evident. The condition number computed for the matrix $X^\top \Omega X$ is 839.727, far higher than the threshold of 30. The elevated number indicates severe multicollinearity, implying that the predictor variables in the model exhibit high linear dependence, which may result in instability in parameter estimation.

The fitted model using the least squares is:

$$\hat{y}^{LS} = \underbrace{34.586^{**}}_{(0.0411)} + \underbrace{1.559}_{(2.5331)} Z_1 - \underbrace{0.4960}_{(2.5331)} Z_2, \quad \text{MSE} = 122.8353.$$

The regression models were also estimated using GLS, along with several autocorrelated biased estimators, including the ARE, Auto-Liu, Auto-TPE, and the proposed ALTE. The performance of each estimator is assessed under conditions of severe multicollinearity and autocorrelated error structures to evaluate their robustness and efficiency. In addition,

Table 2 presents the estimated variances, squared biases (Bias^2), and MSE of the coefficient estimates for each estimation method.

The fitted GLS model is expressed as:

$$\hat{y}^{GLS} = \underbrace{34.635^{**}}_{(0.10811)} + \underbrace{2.762}_{(7.2853)} Z_1 - \underbrace{2.826}_{(7.2854)} Z_2 \quad \text{MSE} = 106.1662.$$

However, the fitted biased estimators are:

$$\hat{y}^{ARR} = \underbrace{34.634^{**}}_{(0.1005)} + \underbrace{2.182^{**}}_{(0.1345)} Z_1 - \underbrace{2.246}_{(8.1633)} Z_2; \quad \hat{c} = 0.0000773, \quad \text{MSE} = 67.3426$$

$$\hat{y}^{AL} = \underbrace{29.960^{**}}_{(0.0077)} + \underbrace{1.962^{**}}_{(0.0120)} Z_1 - \underbrace{0.815}_{(26.2417)} Z_2; \quad \hat{d} = 0.4971, \quad \text{MSE} = 52.7990$$

$$\hat{y}^{ATP} = \underbrace{34.635^{**}}_{(0.1005)} + \underbrace{2.471^{**}}_{(0.1345)} Z_1 - \underbrace{2.534}_{(9.2266)} Z_2, \quad \text{MSE} = 85.32914$$

$$\hat{y}^{ALT} = \underbrace{34.409^{**}}_{(0.0999)} + \underbrace{0.370^{**}}_{(0.1331)} Z_1 - \underbrace{0.347}_{(1.3274)} Z_2$$

$$\hat{k} = 0.01990, \quad \hat{\eta} = -0.00231, \quad \text{MSE} = 13.7035.$$

The results illustrate the better performance of ALTE compared to GLS and other biased estimators. ALTE achieves an MSE of 13.7035, representing a remarkable 87% improvement over GLS (106.1662), which substantially exceeds the improvements shown by ARE (37%), Auto-Liu (50%), and Auto-TPE (20%). This substantial improvement is complemented by remarkable decreases in variance, with Z_1 demonstrating a 99.97% decline from 53.08 in GLS to 0.0177 in ALTE, while Z_2 undergoes a 96.68% reduction to 1.7620. The statistical significance transformation highlights the efficacy of ALTE's shrinking process, as neither Z_1 nor Z_2 attains significance in the GLS model, despite their substantial coefficients. Nonetheless, all biased estimators effectively transform Z_1 into statistical significance. The alterations in coefficient magnitude demonstrate that ALTE effectively reduces the extreme estimates caused by multicollinearity, resulting in more moderate and stable values (Z_1: $2.762 \to 0.370$; Z_2: $-2.826 \to -0.347$). Economically, the findings reveal that Z_1 (shampoo price) exhibits a positive correlation, indicating quality signaling effects, whereas Z_2 (competing soap price) reflects the anticipated negative substitution effect, with ALTE yielding the most dependable estimates for strategic pricing decisions. This dataset reveals even more major improvements than the Iraqi manufacturing data, positioning ALTE as highly successful in settings of severe multicollinearity and confirming its great performance across several applications.

Table 2. Variance, squared bias, and MSE for coefficient estimates under different estimation methods

Estimator	Metric	β_0	β_1	β_2
GLS	Variance	0.0117	53.0763	53.0783
	$Bias^2$	0.0000	0.0000	0.0000
	MSE	0.0117	53.0763	53.0783
Auto-Ridge	Variance	0.0101	0.0181	66.6408
	$Bias^2$	0.0000	0.0000	0.6736
	MSE	0.0101	0.0181	67.3144
Auto-Liu	Variance	0.0078	0.0120	26.2418
	$Bias^2$	14.4891	8.0985	3.9499
	MSE	14.4968	8.1106	30.1917
Auto-TPE	Variance	0.0101	0.0181	85.1306
	$Bias^2$	0.0000	0.0000	0.1704
	MSE	0.0101	0.0181	85.3009
ALTE	Variance	0.0100	0.0177	1.7620
	$Bias^2$	0.0306	0.0241	11.8590
	MSE	0.0406	0.0419	13.6210

6. Simulation study

An intensive Monte Carlo simulation analysis was done to thoroughly assess the performance of the proposed ALTE and to compare it with existing techniques. This simulation framework primarily addresses regression scenarios characterized by multicollinearity and autocorrelated errors. The methodology is predicated on previous research that has examined these dual issues, including studies by Liu [4], Özkale and Kaçiranlar [3], Eledum and Awadallah [19], Eledum and Ahmed [19], Firinguetti [5], Bayhan and Bayhan [6], Shukur [23], and Muniz et al. [24], all of which explored the behavior of biased estimators in such complicated situations. The simulation evaluates five estimators: GLS, ARE, Auto-Liu, Auto-TPE, and the proposed ALTE, across a wide array of actual data-generating scenarios. This encompasses differences in sample size, the number of predictors, the level of multicollinearity, and the degree of autocorrelation. The aim is to ascertain situations in which ALTE yields substantial improvements in efficiency and resilience relative to current biased estimating methods.

6.1 *Simulation design*

Data generation begins with the construction of a multicollinear design matrix using the methodology proposed by McDonald and Galarneau [25]. Specifically, predictor variables are generated using

$$x_{ij} = (1-\gamma^2)^{1/2}\theta_{ij} + \gamma\theta_{i(p+1)}, \quad i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, p, \tag{32}$$

where θ_{ij} are independent standard normal pseudo-random numbers, and γ is specified so that the correlation between any two explanatory variables is given by γ^2. Three different sets of correlations are considered corresponding to $\gamma^2 =$ 0.70, 0.80, and 0.99, which refer to a low, medium, and high multicollinearity, respectively [26]. The explanatory matrix is taken as $Z = (I \quad X)$ whilst the explanatory variables X are centralized and standardized using equation (31), and the response variable y keeps the original form [20, 22]. The observations on the response variable are computed by

$$y_i = \beta_0 + \sum_{j=1}^{p} \beta_j X_{ji} + \varepsilon_i, \quad i = 1, 2, \ldots, n, \quad \text{where } p \in \{2, 3, 5\}, \tag{33}$$

where $\varepsilon_t \sim N(0, \sigma^2\Omega)$, $\varepsilon_t = \rho\varepsilon_{t-1} + V_t$, and $V_t \sim N(0, \sigma_V^2)$. The true regression coefficients are set to $\beta = (2, 1, 1, \ldots, 1)'$ to ensure a reasonable signal-to-noise ratio across all scenarios, following the recommendations of Newhouse and Oman [27].

Fuller [28] suggested the following method to generate AR(1):

$$\varepsilon_t \sim N\left(0, \frac{\sigma_V^2}{1-\rho^2}\right),$$

where $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$, the value of $\sigma = 0.1$ is investigated. Based on the framework established by George et al. [29], three levels of first-order autocorrelation coefficients ($\rho = 0.3, 0.6, 0.9$) are considered, corresponding to low, moderate, and high degrees of autocorrelation, respectively. To examine the effect of sample size on the performance of the proposed estimator, three different sample sizes are selected: $n = 20, 50$, and 250 , representing small, medium, and large datasets, respectively.

6.2 *Simulation results and discussion*

The Monte Carlo simulation results presented in Tables 3-11 provide comprehensive evidence of the superior performance of the proposed ALTE compared to several existing estimators, namely GLS, ARE, ALE, and ATPE. The evaluation was carried out under varying levels of multicollinearity ($\gamma^2 = 0.7, 0.8, 0.99$), autocorrelation ($\rho = 0.3, 0.6, 0.9$), and sample sizes ($n = 20, 50, 250$), considering different numbers of explanatory variables ($p = 2, 3, 5$). Several performance metrics were employed, including variance, squared bias (Bias^2), MSE, RMSE, MAPE, and Relative Efficiency (RE).

Overall performance: Across all simulation scenarios, ALTE consistently outperforms competing estimators. In particular, its RE values often exceed 1.0, demonstrating notable gains over GLS in reducing MSE. The advantages are most pronounced under severe multicollinearity ($\gamma^2 = 0.99$), where ALTE achieves RE values between 1.5 and 3.0. For example, in the case of $n = 20$, $p = 2$, $\gamma^2 = 0.99$ and $\rho = 0.9$, ALTE improves MSE nearly threefold compared to GLS (RE = 2.93). These results highlight ALTE's ability to handle the dual challenges of high multicollinearity and autocorrelated errors.

Impact of multicollinearity: The simulation reveals a clear trend in ALTE's performance with the severity of multicollinearity. Under moderate multicollinearity ($\gamma^2 = 0.7$), ALTE's RE ranges between 1.2 and 2.3. When multicollinearity increases to $\gamma^2 = 0.8$, its RE increases further, usually falling between 1.5 and 2.5. The most significant gains occur under severe multicollinearity ($\gamma^2 = 0.99$), where ALTE frequently doubles or even triples the efficiency of GLS. This trend suggests that the ALTE shrinkage mechanism is particularly effective in mitigating variance inflation caused by multicollinearity.

Effect of autocorrelation: ALTE's performance appears to be strong at various autocorrelation levels. In the presence of weak autocorrelation ($\rho = 0.3$), ALTE generally achieves relative efficiency values between 1.8 and 2.9, depending on other problem characteristics. With moderate autocorrelation ($\rho = 0.6$), RE values remain strong (1.9 to 2.9), indicating consistent performance. Even under strong autocorrelation ($\rho = 0.9$), ALTE maintains superiority over competitors, though RE values are slightly lower (1.3 to 2.6). This pattern highlights ALTE's primary strength in addressing multicollinearity while effectively handling autocorrelated errors.

Effect of sample size: The proposed estimator performs consistently well on small, medium, and large samples with some notable patterns. In small samples ($n = 20$), ALTE achieves substantial gains, with RE often exceeding 2.0. For medium samples ($n = 50$), RE typically falls between 1.4 and 2.9. ALTE outperforms competing approaches with relative efficiency values between 1.6 and 2.9 for large samples ($n = 250$), but MSE differences grow smaller as expected. These results suggest that ALTE is effective across a wide range of practical data sizes.

Dimensionality effects: ALTE's advantages are evident across different numbers of predictors. With $p = 2$, RE ranges between 1.2 and 2.9. For $p = 3$, RE remains strong (1.3-2.7). Notably, with $p = 5$, ALTE shows even greater benefits, often achieving RE values above 2.0. This trend underscores the estimator's ability to handle higher-dimensional problems where traditional methods often fail.

Robustness in extreme scenarios: ALTE performs exceptionally well under the most challenging conditions of severe multicollinearity and strong autocorrelation. For example, when $n = 20$, $p = 5$, $\gamma^2 = 0.99$, and $\rho = 0.9$, GLS produces an MSE of 2.214, whereas ALTE reduces it to 0.867 (RE = 2.55). Similarly, for $n = 50$, $p = 5$, and the same γ^2 and ρ, ALTE achieves an RE of 2.18. This demonstrates ALTE's stability and reliability in situations where traditional estimators fail.

Bias-variance trade-off: The MSE breakdown into variance and squared bias components demonstrates that ALTE achieves enhanced performance through an optimal bias-variance trade-off. Although ALTE introduces controlled bias, it substantially decreases variance, resulting in a lower total MSE. For example, for $\gamma^2 = 0.99$, the squared bias of ALTE is minimized compared to the substantial variance reduction, validating its efficient shrinking approach.

RMSE and MAPE assessments: Performance assessments utilizing RMSE and MAPE further substantiate ALTE's relevance in practice. Under severe multicollinearity ($\gamma^2 = 0.99$) with $n = 50$, $p = 2$, and $\rho = 0.9$, ALTE achieves an RMSE of 0.242, outperforming ARE (0.393), ALE (0.431), ATPE (0.447), and GLS (0.454), representing improvements of 38%, 44%, 46%, and 47%, respectively.

In high-dimensional settings ($p = 5,\ \gamma^2 = 0.99,\ n = 50,\ \rho = 0.9$), ALTE achieves an RMSE of 0.641, representing improvements of 38%, 40%, 47%, and 33% over ARE (1.035), ALE (1.077), ATPE (1.207), and GLS (0.963), respectively.

Similarly, MAPE results confirm that ALTE's estimation accuracy translates into improved predictive performance.

Practical implication: The extensive simulation evidence confirms ALTE as an effective estimator in regression models characterized by severe multicollinearity and autocorrelated errors. Its persistent dominance across all scenarios demonstrates that ALTE is especially adept for real-world applications necessitating steady and precise parameter estimation in intricate statistical contexts.

Table 3. Simulation results for sample size $n = 20$ and number of independent variables $p = 2$

γ^2	Estimator	ρ	Var	Bias^2	MSE	RMSE	MAPE	RE
0.7	GLS	0.3	0.04159	0.00000	0.04159	0.19783	4.08266	1.00000
		0.6	0.03161	0.00000	0.03161	0.17219	5.14573	1.00000
		0.9	0.01514	0.00000	0.01514	0.11984	8.80031	1.00000
	ARE	0.3	0.03747	0.00027	0.03774	0.18962	4.06493	1.10201
		0.6	0.02970	0.00012	0.02982	0.16789	5.10069	1.06003
		0.9	0.01490	0.00011	0.01502	0.11946	8.78138	1.00799
	ALE	0.3	0.03546	0.00287	0.03833	0.19100	4.05888	1.08505
		0.6	0.02922	0.00116	0.03038	0.16930	5.09669	1.04049
		0.9	0.01501	0.00007	0.01507	0.11961	8.81370	1.00464
	ATPE	0.3	0.04100	0.00001	0.04101	0.19675	4.08102	1.01414
		0.6	0.03147	0.00000	0.03147	0.17191	5.14383	1.00445
		0.9	0.01514	0.00000	0.01514	0.11983	8.80022	1.00000
	ALTE	0.3	0.01330	0.00542	0.01872	0.13038	4.06664	2.22169
		0.6	0.01131	0.00416	0.01547	0.11882	5.08988	2.04331
		0.9	0.01115	0.00101	0.01217	0.10806	8.82464	1.24404

Table 3. (cont.)

γ^2	Estimator	ρ	Var	Bias^2	MSE	RMSE	MAPE	RE
0.8	GLS	0.3	0.05940	0.00000	0.05940	0.23538	4.11638	1.00000
		0.6	0.04503	0.00000	0.04503	0.20440	5.28672	1.00000
		0.9	0.01957	0.00000	0.01957	0.13549	9.12396	1.00000
	ARE	0.3	0.05058	0.00051	0.05109	0.22036	4.09694	1.16265
		0.6	0.04081	0.00020	0.04101	0.19629	5.24113	1.09802
		0.9	0.01910	0.00012	0.01921	0.13451	9.10539	1.01874
	ALE	0.3	0.04566	0.00619	0.05185	0.22193	4.12398	1.14561
		0.6	0.03937	0.00269	0.04206	0.19856	5.25874	1.07061
		0.9	0.01930	0.00013	0.01944	0.13510	9.14639	1.00669
	ATPE	0.3	0.05757	0.00002	0.05758	0.23257	4.11373	1.03161
		0.6	0.04457	0.00000	0.04458	0.20362	5.28372	1.01009
		0.9	0.01956	0.00000	0.01956	0.13547	9.12383	1.00051
	ALTE	0.3	0.01621	0.00813	0.02434	0.14619	4.10074	2.44043
		0.6	0.01345	0.00630	0.01975	0.13213	5.23197	2.28000
		0.9	0.01113	0.00195	0.01308	0.11151	9.15956	1.49618
0.99	GLS	0.3	1.09496	0.00000	1.09496	1.00045	4.20270	1.00000
		0.6	0.82155	0.00000	0.82155	0.86207	5.60029	1.00000
		0.9	0.27629	0.00000	0.27629	0.49620	9.75600	1.00000
	ARE	0.3	0.25522	0.18974	0.44495	0.62565	4.19588	2.46084
		0.6	0.26643	0.11108	0.37751	0.58598	5.56214	2.17625
		0.9	0.17282	0.01029	0.18310	0.41400	9.73640	1.50891
	ALE	0.3	0.25978	0.21667	0.47645	0.65765	5.97676	2.29817
		0.6	0.25973	0.17717	0.43690	0.63676	9.75402	1.88040
		0.9	0.21399	0.02836	0.24235	0.46991	11.71038	1.14004
	ATPE	0.3	0.52217	0.03745	0.55962	0.71895	4.19328	1.95662
		0.6	0.50009	0.01703	0.51712	0.69613	5.58172	1.58870
		0.9	0.25809	0.00032	0.25841	0.48412	9.75402	1.06917
	ALTE	0.3	0.20350	0.16299	0.36649	0.52615	4.18679	2.98767
		0.6	0.15921	0.12697	0.28618	0.46538	5.54818	2.87075
		0.9	0.05337	0.04102	0.09439	0.27025	9.80228	2.92711

Table 4. Simulation results for sample size $n = 50$ and number of independent variables $p = 2$

γ^2	Estimator	ρ	Var	Bias^2	MSE	RMSE	MAPE	RE
0.7	GLS	0.3	0.0381	0.0000000	0.0381	0.19299	3.97681	1.00000
		0.6	0.02779	0.0000000	0.02779	0.16463	4.35553	1.00000
		0.9	0.01083	0.0000000	0.01083	0.10305	5.37692	1.00000

Table 4. (cont.)

γ^2	Estimator	ρ	Var	$Bias^2$	MSE	RMSE	MAPE	RE
0.7	ARE	0.3	0.0351	0.0001896	0.03529	0.18607	3.96943	1.07963
		0.6	0.02649	0.0000715	0.02656	0.16115	4.33498	1.04623
		0.9	0.01073	0.0000277	0.01075	0.10271	5.36742	1.00721
	ALE	0.3	0.03342	0.0022458	0.03566	0.18705	3.95277	1.06833
		0.6	0.02583	0.0009564	0.02679	0.16183	4.29532	1.03743
		0.9	0.01077	0.0000330	0.0108	0.1029	5.38034	1.00307
	ATPE	0.3	0.0378	0.0000017	0.0378	0.19229	3.97616	1.00792
		0.6	0.02771	0.0000002	0.02771	0.16443	4.3546	1.00270
		0.9	0.01083	0.0000000	0.01083	0.10305	5.37689	1.00005
	ALTE	0.3	0.01182	0.0051328	0.01695	0.125	3.97408	2.24772
		0.6	0.00917	0.0037846	0.01295	0.10995	4.32404	2.14587
		0.9	0.00605	0.0012036	0.00725	0.08407	5.40877	1.49380
0.8	GLS	0.3	0.0543	0.0000000	0.0543	0.22999	3.98694	1.00000
		0.6	0.03941	0.0000000	0.03941	0.19563	4.39287	1.00000
		0.9	0.01428	0.0000000	0.01428	0.11808	5.48438	1.00000
	ARE	0.3	0.04785	0.0003186	0.04816	0.21723	3.97933	1.12743
		0.6	0.03656	0.0001030	0.03666	0.18908	4.3724	1.07488
		0.9	0.01406	0.0000281	0.01409	0.11734	5.47503	1.01353
	ALE	0.3	0.04336	0.0050985	0.04846	0.21792	3.96282	1.12057
		0.6	0.03464	0.0022913	0.03693	0.18977	4.31633	1.06723
		0.9	0.01414	0.0000711	0.01421	0.1178	5.49221	1.00504
	ATPE	0.3	0.05331	0.0000063	0.05332	0.22809	3.98591	1.01844
		0.6	0.03915	0.0000007	0.03915	0.19506	4.39139	1.00660
		0.9	0.01428	0.0000000	0.01428	0.11807	5.48432	1.00012
	ALTE	0.3	0.01455	0.0076985	0.02224	0.14117	3.98358	2.44114
		0.6	0.01096	0.0056519	0.01661	0.12295	4.36062	2.37287
		0.9	0.00636	0.0017843	0.00814	0.08847	5.52270	1.75463
0.99	GLS	0.3	0.99608	0.0000000	0.99608	0.98126	4.0105	1.00000
		0.6	0.7156	0.0000000	0.7156	0.82935	4.46714	1.00000
		0.9	0.2149	0.0000000	0.2149	0.45451	5.69044	1.00000
	ARE	0.3	0.25379	0.1642164	0.41801	0.61953	4.00895	2.38294
		0.6	0.25806	0.0775299	0.33559	0.56639	4.44997	2.13234
		0.9	0.15404	0.0041724	0.15822	0.39303	5.68157	1.35827
	ALE	0.3	0.24052	0.2049178	0.44544	0.64541	4.57938	2.23617
		0.6	0.20693	0.1581169	0.36505	0.59152	5.88874	1.96030
		0.9	0.17201	0.0200505	0.19206	0.43128	7.65377	1.11891

Table 4. (cont.)

γ^2	Estimator	ρ	Var	Bias^2	MSE	RMSE	MAPE	RE
0.99	ATPE	0.3	0.507	0.0305453	0.53754	0.71523	4.0074	1.85304
		0.6	0.44962	0.0118552	0.46148	0.66915	4.45817	1.55067
		0.9	0.20695	0.0000531	0.20701	0.44707	5.68955	1.03814
	ALTE	0.3	0.18854	0.1542495	0.34279	0.52154	4.0049	2.90584
		0.6	0.13111	0.1106733	0.24178	0.43802	4.43315	2.95965
		0.9	0.04099	0.0318765	0.07287	0.24235	5.73318	2.94907

Table 5. Simulation results for sample size $n = 250$ and number of independent variables $p = 2$

γ^2	Estimator	ρ	Var	Bias^2	MSE	RMSE	MAPE	RE
0.7	GLS	0.3	0.036295	0.0000000	0.036295	0.190106	4.004538	1.00000
		0.6	0.02537	0.0000000	0.02537	0.158882	4.037257	1.00000
		0.9	0.008294	0.0000000	0.008294	0.09085	4.199975	1.00000
	ARE	0.3	0.033732	0.0001644	0.033896	0.183777	4.003193	1.07076
		0.6	0.02434	0.0000587	0.024398	0.155845	4.033083	1.03982
		0.9	0.00822	0.0000048	0.008225	0.090478	4.197761	1.00834
	ALE	0.3	0.032326	0.0019216	0.034247	0.184726	3.99876	1.05979
		0.6	0.023738	0.0008010	0.024539	0.156295	4.018701	1.03386
		0.9	0.008213	0.0000403	0.008253	0.090631	4.192356	1.00491
	ATPE	0.3	0.036082	0.0000010	0.036083	0.18956	4.004422	1.00588
		0.6	0.02532	0.0000001	0.025321	0.158731	4.037067	1.00195
		0.9	0.008293	0.0000000	0.008293	0.090847	4.19996	1.00007
	ALTE	0.3	0.011107	0.0048268	0.015934	0.121803	4.003833	2.27786
		0.6	0.008225	0.0034478	0.011672	0.104486	4.030409	2.17351
		0.9	0.003227	0.0011557	0.004382	0.065004	4.189053	1.89256
0.8	GLS	0.3	0.051479	0.0000000	0.051479	0.226321	4.005648	1.00000
		0.6	0.035933	0.0000000	0.035933	0.189006	4.04334	1.00000
		0.9	0.011471	0.0000000	0.011471	0.106791	4.220142	1.00000
	ARE	0.3	0.046005	0.0002538	0.046259	0.214651	4.004308	1.11285
		0.6	0.0337	0.0000807	0.033781	0.183324	4.039155	1.06373
		0.9	0.01131	0.0000049	0.011315	0.106072	4.217933	1.01378
	ALE	0.3	0.042051	0.0044542	0.046505	0.21523	3.998737	1.10696
		0.6	0.031903	0.0019499	0.033853	0.183528	4.014874	1.06146
		0.9	0.01127	0.0001001	0.01137	0.106327	4.210549	1.00889

Table 5. (cont.)

γ^2	Estimator	ρ	Var	Bias^2	MSE	RMSE	MAPE	RE
0.8	ATPE	0.3	0.05077	0.0000035	0.050773	0.224794	4.00547	1.01390
		0.6	0.035761	0.0000004	0.035761	0.188564	4.043035	1.00481
		0.9	0.011469	0.0000000	0.011469	0.106783	4.220117	1.00017
	ALTE	0.3	0.013647	0.0072041	0.020851	0.137461	4.004842	2.46885
		0.6	0.010126	0.0051425	0.015269	0.118001	4.036007	2.35336
		0.9	0.003745	0.0016863	0.005431	0.071716	4.209909	2.11206
0.99	GLS	0.3	0.933479	0.0000000	0.933479	0.962926	4.007716	1.00000
		0.6	0.650222	0.0000000	0.650222	0.803226	4.056019	1.00000
		0.9	0.195906	0.0000000	0.195906	0.440763	4.258521	1.00000
	ARE	0.3	0.250199	0.1427392	0.392939	0.606719	4.007466	2.37564
		0.6	0.258584	0.0667428	0.325327	0.563014	4.05245	1.99867
		0.9	0.146756	0.0030748	0.149831	0.385947	4.256329	1.30751
	ALE	0.3	0.225664	0.1938258	0.41949	0.631399	4.091412	2.22527
		0.6	0.196654	0.1438365	0.34049	0.57562	4.054051	1.90966
		0.9	0.130084	0.0293320	0.159416	0.398003	5.885297	1.22890
	ATPE	0.3	0.488471	0.0250858	0.513557	0.704464	4.007124	1.81768
		0.6	0.432521	0.0087596	0.441281	0.659902	4.054119	1.47349
		0.9	0.185779	0.0000788	0.185858	0.429559	4.258112	1.05406
	ALTE	0.3	0.175067	0.1426628	0.31773	0.504698	4.006704	2.93797
		0.6	0.130988	0.1019166	0.232905	0.434711	4.047749	2.79179
		0.9	0.038055	0.0307181	0.068773	0.238257	4.250785	2.84858

Table 6. Simulation results for sample size $n = 20$ and number of independent variables $p = 3$

γ^2	Estimator	ρ	Var	Bias^2	MSE	RMSE	MAPE	RE
0.7	GLS	0.3	0.0800993	0.0000000	0.0800993	0.2746740	4.438341	1.00000
		0.6	0.0602326	0.0000000	0.0602326	0.2375841	7.155657	1.00000
		0.9	0.0275225	0.0000000	0.0275225	0.1604003	14.359292	1.00000
	ARE	0.3	0.0713267	0.0003615	0.0716882	0.2614022	4.409087	1.11733
		0.6	0.0560602	0.0001458	0.0562060	0.2304464	7.080006	1.07164
		0.9	0.0269419	0.0001096	0.0270514	0.1592589	14.335792	1.01741
	ALE	0.3	0.0588818	0.0094734	0.0683552	0.2559809	4.427469	1.17181
		0.6	0.0513272	0.0041919	0.0555191	0.2293225	7.071447	1.08490
		0.9	0.0270000	0.0002593	0.0272593	0.1597501	14.468752	1.00966

Table 6. (cont.)

γ^2	Estimator	ρ	Var	Bias^2	MSE	RMSE	MAPE	RE
0.7	ATPE	0.3	0.0780762	0.0000164	0.0780925	0.2718554	4.433667	1.02570
		0.6	0.0597008	0.0000023	0.0597031	0.2367647	7.149135	1.00887
		0.9	0.0275130	0.0000000	0.0275130	0.1603797	14.359042	1.00035
	ALTE	0.3	0.0325971	0.0089054	0.0415025	0.1972334	4.405057	1.92999
		0.6	0.0263177	0.0073707	0.0336884	0.1775176	7.062444	1.78793
		0.9	0.0182709	0.0022906	0.0205615	0.1398839	14.478490	1.33854
0.8	GLS	0.3	0.1189998	0.0000000	0.1189998	0.3337451	4.765486	1.00000
		0.6	0.0890554	0.0000000	0.0890554	0.2878119	10.106254	1.00000
		0.9	0.0387689	0.0000000	0.0387689	0.1893383	18.133225	1.00000
	ARE	0.3	0.1000463	0.0008908	0.1009370	0.3100628	4.723859	1.17895
		0.6	0.0797990	0.0003316	0.0801307	0.2747368	10.011972	1.11138
		0.9	0.0374807	0.0001212	0.0376020	0.1869580	18.097898	1.03103
	ALE	0.3	0.0731825	0.0192583	0.0924408	0.2978232	4.810922	1.28731
		0.6	0.0683193	0.0094288	0.0777481	0.2711844	9.946052	1.14543
		0.9	0.0375477	0.0006029	0.0381505	0.1880735	18.378811	1.01621
	ATPE	0.3	0.1129560	0.0000690	0.1130250	0.3267299	4.756571	1.05286
		0.6	0.0873032	0.0000116	0.0873148	0.2855823	10.092874	1.01993
		0.9	0.0387351	0.0000001	0.0387351	0.1892787	18.132745	1.00087
	ALTE	0.3	0.0422791	0.0144804	0.0567595	0.2290039	4.726549	2.09656
		0.6	0.0335025	0.0116752	0.0451777	0.2040587	9.995428	1.97123
		0.9	0.0215324	0.0036771	0.0252095	0.1541561	18.338408	1.53786
0.99	GLS	0.3	2.3602922	0.0000000	2.3602922	1.4765763	5.688296	1.00000
		0.6	1.7475257	0.0000000	1.7475257	1.2650944	10.197567	1.00000
		0.9	0.6797297	0.0000000	0.6797297	0.7810129	20.084467	1.00000
	ARE	0.3	0.6112802	0.3962393	1.0075195	0.9554700	5.606273	2.34268
		0.6	0.6001808	0.2274500	0.8276308	0.8759032	10.125018	2.11148
		0.9	0.3957059	0.0282118	0.4239177	0.6320964	20.049364	1.60345
	ALE	0.3	0.5090030	0.4677851	0.9767880	0.9360979	7.040154	2.41638
		0.6	0.4442840	0.3737307	0.8180148	0.8681900	14.530340	2.13630
		0.9	0.4086431	0.1128076	0.5214507	0.6959860	26.538405	1.30354
	ATPE	0.3	1.1422442	0.0977786	1.2400228	1.0723889	5.656797	1.90343
		0.6	1.0185823	0.0481836	1.0667660	1.0014594	10.156076	1.63815
		0.9	0.5914288	0.0019189	0.5933477	0.7400167	20.077457	1.14558
	ALTE	0.3	0.4633594	0.4161039	0.8794633	0.8623138	5.603803	2.68379
		0.6	0.3528761	0.3136844	0.6665605	0.7504689	10.095564	2.62171
		0.9	0.1634763	0.0974184	0.2608947	0.4747058	20.241399	2.60538

Table 7. Simulation results for sample size $n = 50$ and number of independent variables $p = 3$

γ^2	Estimator	ρ	Var	Bias^2	MSE	RMSE	MAPE	RE
0.7	GLS	0.3	0.0698	0.0000000	0.0698	0.2615	4.0643	1.00000
		0.6	0.0506	0.0000000	0.0506	0.2224	4.8808	1.00000
		0.9	0.0179	0.0000000	0.0179	0.1326	7.0655	1.00000
	ARE	0.3	0.0643	0.0001944	0.0645	0.2517	4.0543	1.08236
		0.6	0.0482	0.0000659	0.0483	0.2174	4.8547	1.04842
		0.9	0.0177	0.0000219	0.0178	0.1320	7.0550	1.00942
	ALE	0.3	0.0544	0.0071138	0.0615	0.2462	4.0280	1.13374
		0.6	0.0438	0.0032432	0.0471	0.2149	4.7393	1.07517
		0.9	0.0177	0.0001029	0.0178	0.1322	7.0909	1.00581
	ATPE	0.3	0.0688	0.0000050	0.0688	0.2599	4.0628	1.01356
		0.6	0.0504	0.0000007	0.0504	0.2219	4.8786	1.00489
		0.9	0.0179	0.0000000	0.0179	0.1326	7.0654	1.00009
	ALTE	0.3	0.0300	0.0084888	0.0385	0.1923	4.0544	1.81372
		0.6	0.0219	0.0060493	0.0280	0.1639	4.8209	1.81013
		0.9	0.0111	0.0015707	0.0126	0.1111	7.1215	1.41811
0.8	GLS	0.3	0.10284	0.0000000	0.10284	0.31708	4.09514	1.00000
		0.6	0.07447	0.0000000	0.07447	0.26936	5.00045	1.00000
		0.9	0.02519	0.0000000	0.02519	0.15685	7.36663	1.00000
	ARE	0.3	0.09087	0.0004411	0.09131	0.29947	4.08467	1.12625
		0.6	0.06914	0.0001329	0.06928	0.26024	4.97430	1.07495
		0.9	0.02477	0.0000231	0.02480	0.15569	7.35642	1.01580
	ALE	0.3	0.06799	0.0153583	0.08334	0.28656	4.08424	1.23394
		0.6	0.05787	0.0076872	0.06556	0.25353	4.83552	1.13595
		0.9	0.02469	0.0002472	0.02494	0.15611	7.41907	1.01003
	ATPE	0.3	0.09983	0.0000228	0.09986	0.31276	4.09278	1.02988
		0.6	0.07361	0.0000031	0.07361	0.26795	4.99689	1.01159
		0.9	0.02518	0.0000000	0.02518	0.15683	7.36651	1.00023
	ALTE	0.3	0.03824	0.0140671	0.05231	0.22300	4.08437	1.96615
		0.6	0.02760	0.0098797	0.03748	0.18864	4.93659	1.98702
		0.9	0.01369	0.0024324	0.01613	0.12512	7.43134	1.56206
0.99	GLS	0.3	2.00559	0.0000000	2.00559	1.39651	4.16578	1.00000
		0.6	1.44720	0.0000000	1.44720	1.18376	5.25482	1.00000
		0.9	0.44354	0.0000000	0.44354	0.65473	7.95738	1.00000
	ARE	0.3	0.61029	0.3109478	0.92124	0.93617	4.16649	2.17707
		0.6	0.58571	0.1436522	0.72936	0.84002	5.23380	1.98421
		0.9	0.32319	0.0084915	0.33168	0.56964	7.94806	1.33723

Table 7. (cont.)

γ^2	Estimator	ρ	Var	Bias^2	MSE	RMSE	MAPE	RE
0.99	ALE	0.3	0.47379	0.4182052	0.89200	0.91249	4.72954	2.24842
		0.6	0.35685	0.3133419	0.67019	0.79694	6.61253	2.15938
		0.9	0.29758	0.0647317	0.36231	0.59471	12.76601	1.22419
	ATPE	0.3	1.09540	0.0730102	1.16841	1.05980	4.16252	1.71651
		0.6	0.91352	0.0299999	0.94352	0.95769	5.24205	1.53383
		0.9	0.41759	0.0002923	0.41788	0.63726	7.95573	1.06140
	ALTE	0.3	0.42963	0.3827078	0.81234	0.85101	4.15782	2.46891
		0.6	0.28965	0.2695730	0.55922	0.70292	5.19532	2.58790
		0.9	0.10380	0.0757565	0.17956	0.40154	8.03409	2.47019

Table 8. Simulation results for sample size $n = 250$ and number of independent variables $p = 3$

γ^2	Estimator	ρ	Var	Bias^2	MSE	RMSE	MAPE	RE
0.7	GLS	0.3	0.065531	0.00000000	0.06553	0.25547	4.01805	1.00000
		0.6	0.046248	0.00000000	0.04625	0.21460	4.15332	1.00000
		0.9	0.014387	0.00000000	0.01439	0.11970	4.53344	1.00000
	ARE	0.3	0.061046	0.00014731	0.06119	0.24694	4.01606	1.07089
		0.6	0.044416	0.00004817	0.04446	0.21045	4.14779	1.04013
		0.9	0.014260	0.00000335	0.01426	0.11919	4.53068	1.00867
	ALE	0.3	0.052525	0.00611843	0.05864	0.24180	4.00671	1.11745
		0.6	0.040529	0.00275887	0.04329	0.20768	4.10242	1.06838
		0.9	0.014105	0.00014054	0.01425	0.11912	4.52478	1.00997
	ATPE	0.3	0.064873	0.00000285	0.06488	0.25422	4.01776	1.01009
		0.6	0.046086	0.00000035	0.04609	0.21423	4.15285	1.00351
		0.9	0.014386	0.00000000	0.01439	0.11970	4.53340	1.00012
	ALTE	0.3	0.024935	0.00880143	0.03374	0.18037	4.01637	1.94242
		0.6	0.018351	0.00632652	0.02468	0.15436	4.13807	1.87410
		0.9	0.006580	0.00204172	0.00862	0.09189	4.53135	1.66865
0.8	GLS	0.3	0.096185	0.00000000	0.09619	0.30943	4.02159	1.00000
		0.6	0.067781	0.00000000	0.06778	0.25972	4.16960	1.00000
		0.9	0.020795	0.00000000	0.02079	0.14387	4.58965	1.00000
	ARE	0.3	0.086463	0.00030609	0.08677	0.29403	4.01960	1.10852
		0.6	0.063757	0.00008881	0.06385	0.25213	4.16407	1.06164
		0.9	0.020512	0.00000387	0.02052	0.14291	4.58691	1.01360
	ALE	0.3	0.065687	0.01369739	0.07938	0.28132	4.01201	1.21165
		0.6	0.053442	0.00671936	0.06016	0.24482	4.09297	1.12666
		0.9	0.020042	0.00037275	0.02041	0.14256	4.58718	1.01862

Table 8. (cont.)

γ^2	Estimator	ρ	Var	Bias^2	MSE	RMSE	MAPE	RE
0.8	ATPE	0.3	0.094012	0.00001232	0.09402	0.30600	4.02115	1.02298
		0.6	0.067208	0.00000158	0.06721	0.25864	4.16884	1.00850
		0.9	0.020788	0.00000000	0.02079	0.14385	4.58958	1.00032
	ALTE	0.3	0.030740	0.01460903	0.04535	0.20755	4.01982	2.12098
		0.6	0.022634	0.01049588	0.03313	0.17769	4.15333	2.04590
		0.9	0.007915	0.00327915	0.01119	0.10412	4.59310	1.85772
0.99	GLS	0.3	1.861360	0.00000000	1.86136	1.36055	4.02865	1.00000
		0.6	1.305957	0.00000000	1.30596	1.13938	4.20133	1.00000
		0.9	0.389367	0.00000000	0.38937	0.62212	4.69734	1.00000
	ARE	0.3	0.595870	0.25828174	0.85415	0.90820	4.02890	2.17919
		0.6	0.581839	0.11822893	0.70007	0.83057	4.19710	1.86547
		0.9	0.301465	0.00550028	0.30697	0.55282	4.69470	1.26844
	ALE	0.3	0.427960	0.38721541	0.81518	0.87684	4.11947	2.28339
		0.6	0.341554	0.28737283	0.62893	0.77778	4.11813	2.07648
		0.9	0.200986	0.07746034	0.27845	0.52617	8.27929	1.39836
	ATPE	0.3	1.036904	0.05954826	1.09645	1.03281	4.02809	1.69762
		0.6	0.880248	0.02318973	0.90344	0.94453	4.19881	1.44554
		0.9	0.362255	0.00033713	0.36259	0.60063	4.69658	1.07384
	ALTE	0.3	0.384514	0.35192856	0.73644	0.81219	4.02733	2.52750
		0.6	0.283099	0.25525990	0.53836	0.69856	4.18582	2.42581
		0.9	0.088471	0.07517343	0.16364	0.38529	4.70067	2.37935

Table 9. Simulation results for sample size $n = 20$ and number of independent variables $p = 5$

γ^2	Estimator	ρ	Var	Bias^2	MSE	RMSE	MAPE	RE
0.7	GLS	0.3	0.185603	0.000000	0.185603	0.417573	23.577373	1.000000
		0.6	0.144271	0.000000	0.144271	0.366599	31.791625	1.000000
		0.9	0.072632	0.000000	0.072632	0.256279	66.307865	1.000000
	ARE	0.3	0.159561	0.001180	0.160741	0.391377	21.524192	1.154674
		0.6	0.129927	0.000550	0.130477	0.350696	31.493243	1.105714
		0.9	0.069548	0.000228	0.069776	0.252119	66.208117	1.040918
	ALE	0.3	0.099822	0.034533	0.134356	0.359930	29.057691	1.381429
		0.6	0.099054	0.019793	0.118847	0.336319	31.591728	1.213921
		0.9	0.068013	0.002235	0.070249	0.252865	67.838621	1.033920

Table 9. (cont.)

γ^2	Estimator	ρ	Var	Bias^2	MSE	RMSE	MAPE	RE
0.7	ATPE	0.3	0.175489	0.000139	0.175628	0.407951	23.010561	1.056794
		0.6	0.140580	0.000031	0.140611	0.362834	31.733142	1.026029
		0.9	0.072454	0.000001	0.072455	0.256063	66.304335	1.002437
	ALTE	0.3	0.079953	0.022774	0.102727	0.310772	22.670563	1.806762
		0.6	0.063987	0.017807	0.081794	0.276470	31.659242	1.763834
		0.9	0.045086	0.005604	0.050691	0.216188	67.532971	1.432842
0.8	GLS	0.3	0.279952	0.000000	0.279952	0.511697	18.731251	1.000000
		0.6	0.217779	0.000000	0.217779	0.448955	38.108838	1.000000
		0.9	0.109816	0.000000	0.109816	0.313389	119.508035	1.000000
	ARE	0.3	0.225047	0.003266	0.228314	0.466638	18.423908	1.226172
		0.6	0.186619	0.001560	0.188179	0.420880	37.791743	1.157295
		0.9	0.102453	0.000337	0.102790	0.305080	119.326649	1.068352
	ALE	0.3	0.117668	0.059910	0.177578	0.413543	15.616821	1.576497
		0.6	0.124202	0.038560	0.162762	0.393588	38.153281	1.338024
		0.9	0.098140	0.005520	0.103660	0.306239	123.603946	1.059382
	ATPE	0.3	0.253815	0.000530	0.254345	0.491044	18.609711	1.100676
		0.6	0.207159	0.000141	0.207300	0.439998	38.027793	1.050549
		0.9	0.109122	0.000003	0.109125	0.312712	119.496823	1.006330
	ALTE	0.3	0.103069	0.039211	0.142280	0.364124	17.792507	1.967617
		0.6	0.084014	0.029988	0.114002	0.324374	37.867242	1.910309
		0.9	0.062139	0.008469	0.070608	0.253610	121.287369	1.555288
0.99	GLS	0.3	5.683429	0.000000	5.683429	2.294903	20.463224	1.000000
		0.6	4.418773	0.000000	4.418773	2.008562	50.616638	1.000000
		0.9	2.214149	0.000000	2.214149	1.390683	116.607473	1.000000
	ARE	0.3	1.512519	1.105897	2.618416	1.550853	20.392228	2.170560
		0.6	1.436704	0.756595	2.193299	1.416795	50.324464	2.014670
		0.9	0.988651	0.159950	1.148601	1.034669	116.449334	1.927693
	ALE	0.3	1.233145	1.192663	2.425808	1.487606	19.051241	2.342901
		0.6	1.044102	0.958410	2.002512	1.347102	55.432887	2.206616
		0.9	0.774772	0.477661	1.252433	1.077911	165.723182	1.767878
	ATPE	0.3	2.828563	0.295860	3.124423	1.710148	20.429132	1.819033
		0.6	2.476496	0.180608	2.657104	1.573489	50.437717	1.663003
		0.9	1.543117	0.029540	1.572658	1.207058	116.529514	1.407903
	ALTE	0.3	1.202185	1.151624	2.353809	1.456206	20.297115	2.414566
		0.6	0.974310	0.899707	1.874017	1.285251	50.451002	2.357915
		0.9	0.555989	0.310999	0.866988	0.873751	118.268760	2.553840

Table 10. Simulation results for sample size $n = 50$ and number of independent variables $p = 5$

γ^2	Estimator	ρ	Var	Bias^2	MSE	RMSE	MAPE	RE
0.7	GLS	0.3	0.14232	0.000000	0.14232	0.37343	5.01269	1.00000
		0.6	0.10456	0.000000	0.10456	0.31959	7.52189	1.00000
		0.9	0.03440	0.000000	0.03440	0.18346	13.74913	1.00000
	ARE	0.3	0.13069	0.000308	0.13100	0.35879	5.09120	1.08639
		0.6	0.09918	0.000104	0.09928	0.31176	7.47221	1.05314
		0.9	0.03399	0.000019	0.03401	0.18245	13.73422	1.01158
	ALE	0.3	0.08680	0.023890	0.11069	0.33065	6.49142	1.28568
		0.6	0.07622	0.012875	0.08910	0.29605	7.04121	1.17357
		0.9	0.03354	0.000428	0.03397	0.18235	13.92515	1.01280
	ATPE	0.3	0.13879	0.000024	0.13882	0.36909	5.03768	1.02522
		0.6	0.10346	0.000004	0.10347	0.31805	7.51317	1.01055
		0.9	0.03439	0.000000	0.03439	0.18344	13.74890	1.00021
	ALTE	0.3	0.06452	0.017950	0.08247	0.28278	5.19110	1.72562
		0.6	0.04871	0.013292	0.06201	0.24500	7.35964	1.68629
		0.9	0.02376	0.002488	0.02625	0.16018	13.90243	1.31071
0.8	GLS	0.3	0.21280	0.000000	0.21280	0.45614	6.33946	1.00000
		0.6	0.15655	0.000000	0.15655	0.39054	8.61325	1.00000
		0.9	0.05028	0.000000	0.05028	0.22146	16.45949	1.00000
	ARE	0.3	0.18757	0.000859	0.18843	0.43017	6.28823	1.12935
		0.6	0.14461	0.000284	0.14490	0.37636	8.55997	1.08040
		0.9	0.04936	0.000023	0.04938	0.21956	16.44221	1.01829
	ALE	0.3	0.10095	0.044342	0.14529	0.37858	5.59492	1.46469
		0.6	0.09393	0.026812	0.12074	0.34461	8.01618	1.29652
		0.9	0.04811	0.001072	0.04918	0.21916	16.83255	1.02236
	ATPE	0.3	0.20272	0.000108	0.20282	0.44592	6.31923	1.04921
		0.6	0.15307	0.000021	0.15310	0.38653	8.59943	1.02254
		0.9	0.05026	0.000000	0.05026	0.22141	16.45909	1.00053
	ALTE	0.3	0.08088	0.031592	0.11247	0.32888	6.14208	1.89209
		0.6	0.06127	0.023417	0.08469	0.28511	8.44298	1.84855
		0.9	0.03184	0.004074	0.03592	0.18701	16.67411	1.39992
0.99	GLS	0.3	4.24470	0.000000	4.24470	2.03250	6.19747	1.00000
		0.6	3.13187	0.000000	3.13187	1.74184	11.10661	1.00000
		0.9	0.95720	0.000000	0.95720	0.96263	20.68650	1.00000
	ARE	0.3	1.42020	0.628945	2.04915	1.40517	6.20995	2.07144
		0.6	1.34075	0.337997	1.67875	1.27619	11.07194	1.86560
		0.9	0.69596	0.021229	0.71719	0.83759	20.66564	1.33465

Table 10. (cont.)

γ^2	Estimator	ρ	Var	Bias^2	MSE	RMSE	MAPE	RE
0.99	ALE	0.3	0.96248	0.928616	1.89109	1.33701	6.56128	2.24457
		0.6	0.76793	0.696212	1.46414	1.17684	10.99255	2.13904
		0.9	0.49445	0.189363	0.68381	0.81810	32.44917	1.39980
	ATPE	0.3	2.42262	0.165448	2.58807	1.58340	6.19748	1.64010
		0.6	2.03837	0.080480	2.11885	1.43467	11.08395	1.47810
		0.9	0.87237	0.001647	0.87401	0.92272	20.68041	1.09518
	ALTE	0.3	0.92860	0.903707	1.83231	1.30965	6.13379	2.31659
		0.6	0.71430	0.669405	1.38371	1.13462	10.96202	2.26339
		0.9	0.25944	0.179107	0.43855	0.64105	20.94966	2.18264

Table 11. Simulation results for sample size $n = 250$ and number of independent variables $p = 5$

γ^2	Estimator	ρ	Var	Bias^2	MSE	RMSE	MAPE	RE
0.7	GLS	0.3	0.12727	0.000000	0.12727	0.35612	4.07591	1.00000
		0.6	0.08994	0.000000	0.08994	0.29933	4.44815	1.00000
		0.9	0.02745	0.000000	0.02745	0.16534	5.58562	1.00000
	ARE	0.3	0.11902	0.000176	0.11919	0.34471	4.07272	1.06777
		0.6	0.08653	0.000048	0.08658	0.29371	4.44008	1.03888
		0.9	0.02721	0.000003	0.02721	0.16464	5.58203	1.00860
	ALE	0.3	0.08221	0.019952	0.10216	0.31924	4.05255	1.24580
		0.6	0.06765	0.010317	0.07797	0.27884	4.28735	1.15362
		0.9	0.02625	0.000591	0.02684	0.16353	5.59791	1.02253
	ATPE	0.3	0.12508	0.000011	0.12509	0.35310	4.07507	1.01739
		0.6	0.08934	0.000001	0.08934	0.29833	4.44676	1.00679
		0.9	0.02744	0.000000	0.02744	0.16532	5.58552	1.00026
	ALTE	0.3	0.04882	0.019679	0.06850	0.25859	4.06983	1.85808
		0.6	0.03562	0.013715	0.04933	0.21967	4.40698	1.82331
		0.9	0.01359	0.003826	0.01742	0.13108	5.60336	1.57560
0.8	GLS	0.3	0.18965	0.000000	0.18965	0.43465	4.08909	1.00000
		0.6	0.13402	0.000000	0.13402	0.36529	4.50363	1.00000
		0.9	0.04055	0.000000	0.04055	0.20091	5.75622	1.00000
	ARE	0.3	0.17170	0.000468	0.17217	0.41426	4.08588	1.10154
		0.6	0.12647	0.000119	0.12659	0.35510	4.49560	1.05867
		0.9	0.04002	0.000004	0.04002	0.19961	5.75270	1.01314
	ALE	0.3	0.09543	0.038573	0.13400	0.36544	4.07633	1.41532
		0.6	0.08236	0.022547	0.10491	0.32342	4.28026	1.27747
		0.9	0.03732	0.001579	0.03890	0.19682	5.82604	1.04239

Table 11. (cont.)

γ^2	Estimator	ρ	Var	Bias^2	MSE	RMSE	MAPE	RE
0.8	ATPE	0.3	0.18313	0.000050	0.18318	0.42724	4.08792	1.03536
		0.6	0.13200	0.000007	0.13201	0.36257	4.50154	1.01525
		0.9	0.04052	0.000000	0.04052	0.20084	5.75605	1.00066
	ALTE	0.3	0.05982	0.033861	0.09368	0.30121	4.08299	2.02451
		0.6	0.04324	0.023558	0.06680	0.25456	4.46057	2.00626
		0.9	0.01678	0.006460	0.02323	0.15097	5.78001	1.74512
0.99	GLS	0.3	3.75934	0.000000	3.75934	1.93439	4.11565	1.00000
		0.6	2.65605	0.000000	2.65605	1.62533	4.61356	1.00000
		0.9	0.78928	0.000000	0.78928	0.88585	6.07674	1.00000
	ARE	0.3	1.39255	0.473977	1.86653	1.35377	4.11657	2.01408
		0.6	1.27661	0.207625	1.48424	1.21294	4.60767	1.78950
		0.9	0.62401	0.009278	0.63329	0.79415	6.07344	1.24632
	ALE	0.3	0.87473	0.831953	1.70668	1.28012	4.16045	2.20272
		0.6	0.62451	0.597137	1.22164	1.08592	4.29851	2.17416
		0.9	0.30139	0.181290	0.48268	0.69214	11.65049	1.63522
	ATPE	0.3	2.27689	0.122919	2.39981	1.53648	4.11504	1.56651
		0.6	1.82464	0.051273	1.87591	1.36315	4.60993	1.41587
		0.9	0.72091	0.001200	0.72211	0.84775	6.07543	1.09303
	ALTE	0.3	0.83809	0.815180	1.65327	1.25441	4.11150	2.27388
		0.6	0.57762	0.577442	1.15506	1.04878	4.57681	2.29949
		0.9	0.18344	0.174133	0.35757	0.58629	6.09506	2.20732

7. Conclusion remarks

This study proposes the ALTE, a novel biased estimation technique designed to address the simultaneous challenges of severe multicollinearity and autocorrelated errors in linear regression models. By integrating the shrinkage properties of the Liu-Type estimator with the efficiency of GLS under autoregressive error structures, ALTE offers a unified framework capable of handling both sources of statistical distortion. Theoretical derivations based on canonical transformation and eigenvalue decomposition establish ALTE's mathematical foundation, providing explicit expressions for its expectation, variance, and MSE. These results highlight ALTE's ability to achieve an optimal bias-variance trade-off, which is central to its improved estimation performance.

Empirical validation using real datasets demonstrates ALTE's practical effectiveness. Analyses of the Iraqi manufacturing dataset and the soap-shampoo dataset reveal that ALTE consistently achieves the lowest MSE among all competing methods, including GLS, ARE, Auto-Liu (ALE), and ATPE. These findings confirm that ALTE's theoretical advantages translate into significant practical benefits for applied researchers working with complex datasets characterized by both multicollinearity and autocorrelation.

The Monte Carlo simulation study provides robust evidence of ALTE's superior performance across a variety of regression scenarios. Relative efficiency values ranging from 1.2 to nearly 3.0 demonstrate ALTE's consistent advantages over competing estimators, especially under severe multicollinearity ($\gamma^2 = 0.99$) and strong autocorrelation ($\rho = 0.9$). Importantly, ALTE performs well across varying sample sizes ($n = 20,\ 50,\ 250$) and model complexities ($p = 2,\ 3,\ 5$), making it a reliable tool for both small- and large-sample applications.

Performance assessments using RMSE and MAPE metrics further reinforce ALTE's practical relevance. ALTE consistently achieves lower prediction errors and relative forecasting errors compared to all other estimators. These improvements are particularly pronounced in high-dimensional models and small samples, where traditional methods often exhibit instability. The bias-variance decomposition reveals that ALTE's carefully calibrated shrinkage mechanism introduces minimal bias while achieving substantial variance reduction, leading to lower overall MSE even under extreme statistical conditions.

Methodologically, ALTE fills an important gap in the literature by providing the first unified estimator specifically tailored to handle severe multicollinearity combined with strong autocorrelation. Previous methods typically address these issues in isolation or under moderate conditions, limiting their applicability in real-world scenarios.

The practical implications of this research are significant. ALTE is especially recommended when condition numbers exceed 100, autocorrelation coefficients are greater than 0.3, and model dimensionality is moderate to high. Its consistent performance across diverse settings makes it a valuable addition to the applied researcher's toolkit, offering a robust alternative when traditional methods fail.

This study develops ALTE under the assumption of AR(1) autocorrelated errors, which captures many practical situations but also limits the current framework. Extending ALTE to higher-order AR(p) processes, heteroscedastic error structures, and non-Gaussian distributions represents an important direction for future research, as highlighted by recent developments in handling multiple regression complications simultaneously [10]. Further work could also focus on developing diagnostic tools to help practitioners select appropriate estimators and on designing unified methods for other regression settings, such as models affected simultaneously by multicollinearity and heteroscedasticity or non-normal errors. Despite its present restriction to AR(1), the results demonstrate that ALTE is a reliable and effective alternative to existing methods, and the framework provides a strong basis for future extensions.

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Ethics

This article is original and contains unpublished material and no ethical issues involved.

Conflict of interest

The author declares no competing financial interest.

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