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Research Article

A Novel Kumaraswamy-Fréchet Poisson Distribution

Baqer Kareem Fahad1[1], Muhyiddin Izadi[1*], Abbas Lafta Kneehr[2]

[1] Department of Statistics, Razi University, Kermanshah, Iran
[2] Department of Statistics, Wasit University, Wasit, Iraq
E-mail: izadi_552@yahoo.com

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Abstract: This paper introduces a novel probability distribution by combining the Kumaraswamy-G-Poisson and Fréchet distributions, resulting in the six-parameter Kumaraswamy-Fréchet Poisson (KFP) distribution. The proposed model enhances flexibility for modeling diverse data types beyond traditional lifetime applications. We rigorously derive its key statistical properties and estimate parameters using Maximum Likelihood Estimation (MLE). To demonstrate its effectiveness, we apply the KFP distribution to real-world data, showing satisfactory goodness-of-fit when compared with competing distributions.

***Keywords*:** fréchet distribution, hazard rate function, maximum likelihood estimate, moments, quantile function

MSC: 60E05, 60Exx

1. Introduction

Statistical distributions have been widely employed by researchers as important tools in data analysis. In this regard, developing new distributions is an essential issue that has attracted the attention of many researchers. Although the classical distributions have a simpler form are more understandable via fewer parameters, they are not suitable for fitting the skewed data. To overcome this shortcoming, some generalized distributions have been introduced that are more flexible and usable than classical distributions, as follows: Generalized-Exponential (GE) [1], beta-generated distributions [2, 3], McDonald Generalized (Mc-G) distribution [4], gamma-generated type-I distribution [5], gamma-generated type-II distribution [6], exponentiated generalized distribution [7].

In this regard, the Kumaraswamy distribution is a statistical distribution used for data modeling and analysis in several fields such as insurance and finance. This distribution was first introduced by Kumaraswamy [8] for variables with lower and upper bounds. It is defined on the interval $(0,\ 1)$ and encompasses two non-negative parameters a and b, along with the cumulative distribution function (cdf), and the probability density function (pdf) of the Kumaraswamy distribution as follows

$$F(x;\ a,\ b) = 1 - [1 - x^a]^b, \quad 0 < x < 1,$$

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$$f(x;\ a,\ b) = abx^{a-1}\left(1-x^{a}\right)^{b-1},\quad 0<x<1.$$

Cordeiro and De Castro [9] introduced the Kumaraswamy-G family of distributions. Let G be an arbitrary baseline distribution with the corresponding the pdf g. Then the Kumaraswamy-G distribution has the cdf and pdf, respectively, given by

$$H(x;\ a,\ b) = 1-\ \left[1\ -[G(x)]^{a}\right]^{b},\quad x\in R \tag{1}$$

$$h(x;\ a,\ b) = abg(x)[G(x)]^{a-1}\left(1-[G(x)]^{a}\right)^{b-1},\quad x\in R.$$

Next, a new generalized family called the Kumaraswamy-G Poisson (Kw-GP) family, with three additional positive parameters to generalize the Kumaraswamy-G family, was proposed by Ramos et al. [10]. Later, Rocha et al. [11] proposed the family of sequential Kumaraswamy distributions to address the need for developing distributions with desirable properties and sufficient flexibility to model data with different characteristics. Nofal et al. [12] derived the three-parameter Kumaraswamy uniform distribution from the family of Kumaraswamy distributions with the uniform distribution. Then, some statistical properties of the proposed model were derived, such as the reliability/survival function, hazard rate function, quantile function, median, and mode. Silva et al. [13] developed a new family of univariate continuous distributions to represent the family of distributions called the Exponentiated Kumaraswamy-G family. Many of its properties were studied in detail, including moments, skewness, kurtosis, entropy, and some reliability characteristics. Arshad et al. [14] proposed a new class of distributions by merging the Kumaraswamy-G and odd gamma-G families introduced by Torabi and Montazari [15]. Tahir et al. [16] introduced a new family of generalized Kumaraswamy (G) distributions using a new generator, which serves as an alternative to the previously proposed Kumaraswamy-G family. They also derived some mathematical properties of the proposed distribution. More recently, Tharu et al. [17] studied the Kumaraswamy-G with the baseline distribution assumed to be a uniform distribution on the interval. Several statistical properties of the proposed distribution were derived. In this context, several researchers have utilized the Kumaraswamy-G family of distributions to extend various existing models. For example, Paranaíba et al. [18] proposed the Kumaraswamy Burr XII distribution, and Gomes et al. [19] introduced the Kumaraswamy generalized Rayleigh distribution. Rodrigues and Silva [20] presented the Kumaraswamy exponential distribution. Additionally, Jamal et al. [21] proposed the generalized inverted Kumaraswamy generated family of distributions.

In this paper, we derive a six-parameter distribution by combining the Kumaraswamy-G Poisson distribution (introduced by Ramos et al. [10]) with the Fréchet distribution. The Fréchet distribution is a special case of the generalized extreme value distribution and is widely used in extreme value theory to model the maxima of datasets. Due to its flexibility, the Fréchet distribution and its generalizations have been extensively employed by researchers to model data distributions. For instance, Alyami et al. [22] recently introduced the binomial Fréchet distribution. Similarly, Alotaibi et al. [23] and Alzeley et al. [24] conducted studies on the exponential Fréchet distribution, while Almetwally et al. [25] proposed the bivariate Fréchet distribution.

The remainder of this paper is organized as follows: In Section 2, first, we introduce the new distribution, which is a combination of the Kumaraswamy-G Poisson distribution with the Fréchet distribution, called the Kumaraswamy-Fréchet Poisson (KFP) distribution. In Section 3, some statistical properties of this new model are investigated. In Section 4, to demonstrate its effectiveness, we apply the KFP distribution to real-world data, showing satisfactory goodness-of-fit when compared with competing distributions.

2. The KFP distribution

For an arbitrary baseline distribution H with corresponding density function h, the Poisson-G distribution introduced by Abouelmagd et al. [26] has the cdf and pdf, respectively, as follows

$$F(x) = \frac{1}{1-e^{-\lambda}}\left[1-\ e^{-\lambda H(x)}\right], \qquad x \in R \tag{2}$$

$$f(x) = \frac{\lambda h(x) e^{-\lambda H(x)}}{1-e^{-\lambda}}, \qquad x \in R,$$

where $\lambda > 0$. By combining (2) and (1), the cdf of the Kumaraswamy-G Poisson distribution, as introduced by Ramos et al. [10], is derived as follows:

$$F(x) = \frac{1}{1-e^{-\lambda}}\ \left[1-e^{-\lambda\left(1-[1-G(x)^{a}]^{b}\right)}\right], \quad x \in R. \tag{3}$$

The Fréchet distribution is a continuous probability distribution widely used in extreme value theory to model the maxima of datasets. Its cdf and pdf are given by

$$G(x) = e^{-\left(\frac{x-\mu}{s}\right)^{-\alpha}}, \qquad x > \mu \tag{4}$$

$$g(x) = \frac{\alpha}{s}\left(\frac{x-\mu}{s}\right)^{-\alpha-1} e^{-\left(\frac{x-\mu}{s}\right)^{-\alpha}}, \quad x > \mu,$$

where μ is the location parameter, $s > 0$ is the scale parameter and $\alpha > 0$ is the shape parameter. Now, substituting (4) into (3), we obtain the cumulative distribution function

$$F(x) = \frac{1}{1-e^{-\lambda}}\left[1-e^{-\lambda\left(1-\left(1-e^{-a\left(\frac{x-\mu}{s}\right)^{-\alpha}}\right)^{b}\right)}\right], \ x > \mu, \tag{5}$$

where $a,\ b,\ \alpha,\ s,\ \lambda \in R^{+}$ and $\mu \in R$. This new family of distributions, defined by equation (5), is referred to as the Kumaraswamy-Fréchet Poisson (KFP) distribution and is denoted by KFP$(a,\ b,\ \alpha,\ \mu,\ s,\ \lambda)$. It is easy to see that the corresponding pdf is given by

$$f(x) = \frac{\lambda ab\frac{\alpha}{s}}{1-e^{-\lambda}}\left[\left(\frac{x-\mu}{s}\right)^{-1-\alpha} e^{-a\left(\frac{x-\mu}{s}\right)^{-\alpha}}\left(1-e^{-a\left(\frac{x-\mu}{s}\right)^{-\alpha}}\right)^{b-1} e^{-\lambda\left(1-\left(1-e^{-a\left(\frac{x-\mu}{s}\right)^{-\alpha}}\right)^{b}\right)}\right], \ x > \mu. \tag{6}$$

Figure 1 displays the pdf plots of the KFP distribution for various parameter values. These plots demonstrate that the KFP distribution can take on different shapes. It is characterized by an increasing-decreasing pattern and is always one-sided.

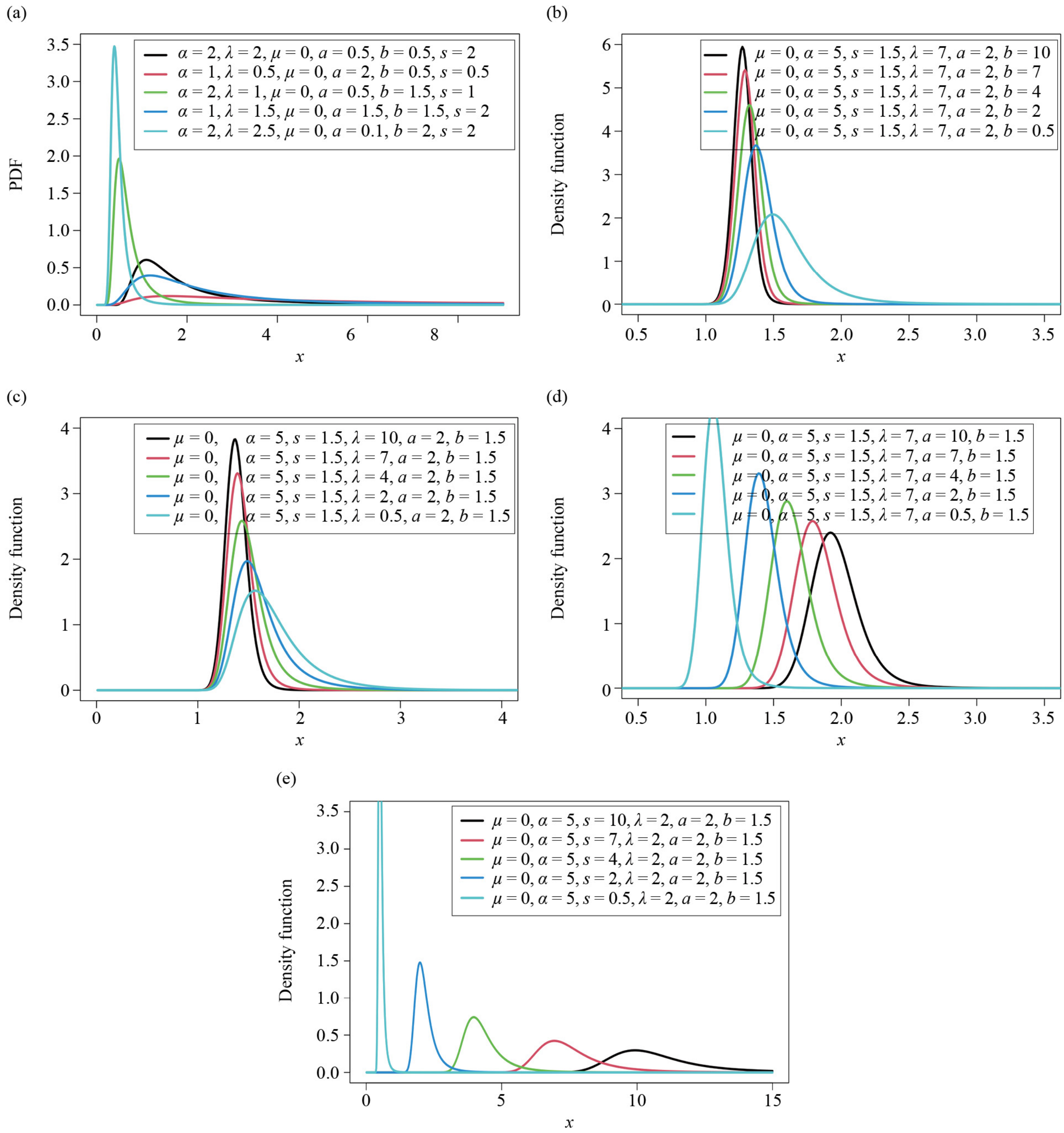

Figure 1. Comparison of density functions across parameter sets. Subfigures (a)-(e) illustrate the effects of varying $(a, b, \alpha, \mu, s, \lambda)$

3. Statistical properties

In this section, we obtain some important characteristics of the KFP distribution. First, we obtain the quantile function.

For every $0 < p < 1$, by solving $F(x_p) = p$, it is easy to see that the quantile p^{th} of the KFP distribution is as follows.

$$F^{-1}(p) = x_p = \mu + s\left(-\frac{1}{a}\ln\left(1-\left(1+\frac{1}{\lambda}\left(\ln(1-p(1-e^{-\lambda}))\right)^{\frac{1}{b}}\right)\right)\right)^{-\frac{1}{\alpha}}, \ \forall\ 0 < p < 1. \tag{7}$$

Using (7) and the probability integral transformation theorem, we can easily generate random samples from the KFP distribution using the following steps:

- Generate a uniform random number $u \sim \text{Unif}(0,\ 1)$.
- Transform u using $x = F^{-1}(u)$.

The hazard rate function (also known as the failure rate function) is a crucial concept in reliability theory and serves as an important characteristic of a distribution. It provides valuable insights into the behavior of a system or component over time, particularly in terms of its likelihood of failure. Let X be a random variable with distribution function F, survival function $\overline{F} = 1 - F$ and density function f. Then the hazard rate function of X, denoted by $h(x)$, is defined by $h(x) = \dfrac{f(x)}{\overline{F}(x)}$. Now, the hazard rate function of the KFP distribution is

$$h(x) = \frac{f(x)}{\overline{F}(x)} = \frac{\lambda ab\alpha s^{\alpha}\left[(x-\mu)^{-1-\alpha}\ e^{-a\left(\frac{x-\mu}{s}\right)^{-\alpha}}\left(1-e^{-a\left(\frac{x-\mu}{s}\right)^{-\alpha}}\right)^{b-1} e^{\lambda\left(1-e^{-a\left(\frac{x-\mu}{s}\right)^{-\alpha}}\right)^{b}}\right]}{e^{\lambda\left(1-e^{-a\left(\frac{x-\mu}{s}\right)^{-\alpha}}\right)^{b}}-1}, \ x > \mu.$$

Figure 2 illustrates the plots of the hazard rate function of the KFP distribution. From the figure, it is observed that the hazard rate function of the KFP distribution is unimodal for various values of parameters.

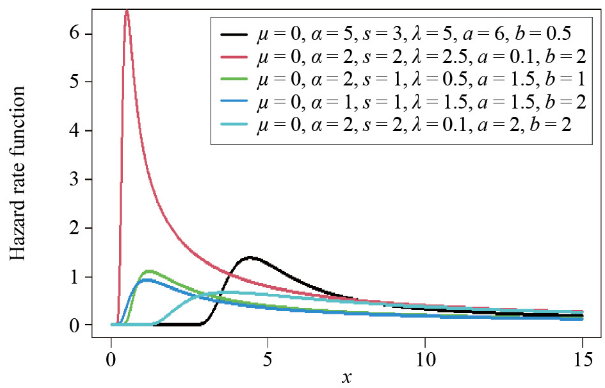

Figure 2. The hazard rate function of the KFP distribution for some parameter values

3.1 *Moments of the KFP distribution*

In this section, we obtain the moments of the KFP distribution. Let X be a random variable having KFP$(a,\ b,\ \alpha,\ \mu,\ s,\ \lambda)$ distribution. Then, for $r = 1,\ 2,\ \ldots,$

$$E(X^r) = \int_{-\infty}^{\infty} x^r\ f(x)dx$$

$$= \int_{\mu}^{\infty} \frac{ab\lambda\ \dfrac{\alpha}{s}}{1-e^{-\lambda}} \left[x^r \left(\frac{x-\mu}{s}\right)^{-1-\ \alpha} e^{-a\left(\frac{x-\mu}{s}\right)^{-\alpha}} \left(1-e^{-a\left(\frac{x-\mu}{s}\right)^{-\alpha}}\right)^{b-1} e^{-\lambda\left(1-\left(1-e^{-a\left(\frac{x-\mu}{s}\right)^{-\alpha}}\right)^{b}\right)} \right] dx.$$

Let $\dfrac{x-\mu}{s} = y$, then $x = sy+\mu$ and $dx = sdy$. So, we have,

$$E(X^r) = \frac{ab\lambda\alpha}{1-e^{-\lambda}} \int_0^{\infty} \left[(sy+\mu)^r\ y^{-1-\alpha}\ \ e^{-ay^{-\alpha}} \left(1-e^{-ay^{-\alpha}}\right)^{b-1} e^{-\lambda\left(1-\left(1-e^{-ay^{-\alpha}}\right)^{b}\right)} \right] dy.$$

By Taylor expansion $e^x = \sum_{k=0}^{\infty} \dfrac{x^k}{k!}$, we have,

$$E(X^r) = \frac{ab\lambda\alpha e^{-\lambda}}{1-e^{-\lambda}} \int_0^{\infty} \left[(sy+\mu)^r y^{-1-\alpha} e^{-ay^{-\alpha}} \left(1-e^{-ay^{-\alpha}}\right)^{b-1} \sum_{k=0}^{\infty} \frac{\lambda^k \left(1-e^{-ay^{-\alpha}}\right)^{kb}}{k!} \right] dy$$

$$= \frac{ab\lambda\alpha}{e^{\lambda}-1} \int_0^{\infty} \left[(sy+\mu)^r y^{-1-\alpha} e^{-ay^{-\alpha}} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left(1-e^{-ay^{-\alpha}}\right)^{kb+b-1} \right] dy$$

$$= \frac{ab\lambda\alpha}{e^{\lambda}-1} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_0^{\infty} \left[(sy+\mu)^r y^{-1-\alpha} e^{-ay^{-\alpha}} \left(1-e^{-ay^{-\alpha}}\right)^{b(k+1)-1} \right] dy.$$

By generalized binomial expansion $(1-x)^r = \sum_{w=0}^{\infty} (-1)^w \dbinom{r+w-1}{w} x^w$, we have,

$$E(X^r) = \frac{ab\lambda\alpha^{\lambda}}{e} - 1 \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_0^{\infty} \left[(sy+\mu)^r y^{-1-\alpha} e^{-ay^{-\alpha}} \sum_{w=0}^{\infty} (-1)^w \binom{b(k+1)+w-2}{w} e^{-ay^{-\alpha}w} \right] dy$$

$$= \frac{ab\lambda\alpha^{\lambda}}{e} - 1 \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \sum_{w=0}^{\infty} (-1)^w \binom{b(k+1)+w-2}{w} \int_0^{\infty} (sy+\mu)^r y^{-1-\alpha} e^{-ay^{-\alpha}(1+w)} dy.$$

By binomial expansion $(x+y)^n = \sum_{j=0}^{n} \dbinom{n}{j} x^{n-j} y^j$. So,

$$E(X^r) = \frac{ab\lambda\alpha}{e^{\lambda}-1}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\sum_{w=0}^{\infty}(-1)^w\binom{b(k+1)+w-2}{w}$$

$$\times\int_0^{\infty}\sum_{j=0}^{r}\binom{r}{j}(sy)^{r-j}\mu^j y^{-1-\alpha}e^{-ay^{-\alpha}(1+w)}dy$$

$$= \frac{ab\lambda\alpha}{e^{\lambda}-1}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\sum_{w=0}^{\infty}(-1)^w\binom{b(k+1)+w-2}{w}$$

$$\times\int_0^{\infty}\sum_{j=0}^{r}\binom{r}{j}\mu^j s^{r-j}y^{r-j-1-\alpha}e^{-ay^{-\alpha}(1+w)}dy.$$

By substitution $z = ay^{-\alpha}(1+w)$, we have

$$E(X^r) = \frac{ab\lambda\alpha}{e^{\lambda}-1}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\sum_{w=0}^{\infty}(-1)^w\binom{b(k+1)+w-2}{w}$$

$$\times\int_0^{\infty}\left(\sum_{j=0}^{r}\binom{r}{j}\mu^j s^{r-j}\frac{z^{\frac{-1}{\alpha}(r-j-\alpha-1)}}{(a(1+w))^{\frac{-1}{\alpha}(r-j-\alpha-1)}}e^{-z}\frac{z^{-\frac{1}{\alpha}}-1}{\alpha(a(1+w))^{-\frac{1}{\alpha}}}\right)dz$$

$$= \frac{ab\lambda}{e^{\lambda}-1}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\sum_{w=0}^{\infty}(-1)^w\binom{b(k+1)+w-2}{w}$$

$$\times\int_0^{\infty}\sum_{j=0}^{r}\binom{r}{j}\mu^j s^{r-j}\frac{z^{\frac{j-r}{\alpha}}e^{-z}}{(a(1+w))^{\frac{-1}{\alpha}(r-j-\alpha)}}dz.$$

If $\alpha > r$, then $\int_0^{\infty} z^{\frac{j-r}{\alpha}}e^{-z}dz$, $j = 0,\ 1,\ \ldots,\ r$, converges and equals $\Gamma\left(\frac{j-r}{\alpha}+1\right)$, where $\Gamma(.)$ is the Gamma function. Thus for $\alpha > r$, the expected value $E(X^r)$ is well-defined and equals

$$E(X^r) = \frac{ab\lambda}{e^{\lambda}-1}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\sum_{w=0}^{\infty}(-1)^w\binom{b(k+1)+w-2}{w}\sum_{j=0}^{r}\binom{r}{j}\frac{\mu^j s^{r-j}}{(a(1+w))^{\frac{-1}{\alpha}(r-j-\alpha)}}\Gamma\left(\frac{j-r}{\alpha}+1\right). \tag{8}$$

If $\alpha < r$, then $\int_0^{\infty} z^{\frac{j-r}{\alpha}}e^{-z}dz$, for $j = 0$, diverges, implying the r-th moment of the KFP is infinite. Using (8), for the case $r = 1$ and $r = 2$, we have, respectively,

$$E(X) = \frac{b\lambda}{e^{\lambda}-1}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\sum_{w=0}^{\infty}(-1)^w\binom{b(k+1)+w-2}{w}\left[\frac{sa^{\frac{1}{\alpha}}}{(1+w)^{-\frac{1}{\alpha}+1}}\Gamma\left(1-\frac{1}{\alpha}\right)+\frac{\mu}{1+w}\right],\quad \text{for } \alpha > 1 \tag{9}$$

and

$$E(X^2) = \frac{ab\lambda}{e^{\lambda}-1} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \sum_{w=0}^{\infty} (-1)^w \binom{b(k+1)+w-2}{w}$$

$$\times \left[\frac{s^2\Gamma\left(-\frac{2}{\alpha}+1\right)}{(a(1+w))^{-\frac{2}{\alpha}+1}} + \frac{2s\mu\Gamma\left(-\frac{1}{\alpha}+1\right)}{(a(1+w))^{-\frac{1}{\alpha}}} + \frac{\mu^2}{a(1+w)} \right], \quad \text{for } \alpha > 2. \tag{10}$$

Now, using (9) and (10), the variance of the KFP distribution for $\alpha > 2$ is as follows

$$Var(X) = \frac{ab\lambda}{e^{\lambda}-1} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \sum_{w=0}^{\infty} (-1)^w \binom{b(k+1)+w-2}{w} \left[\frac{s^2\Gamma\left(-\frac{2}{\alpha}+1\right)}{(a(1+w))^{-\frac{2}{\alpha}+1}} + \frac{2s\mu\Gamma\left(-\frac{1}{\alpha}+1\right)}{(a(1+w))^{-\frac{1}{\alpha}}} + \frac{\mu^2}{a(1+w)} \right]$$

$$-\left[\frac{b\lambda}{e^{\lambda}-1} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \sum_{w=0}^{\infty} (-1)^w \binom{b(k+1)+w-2}{w} \left(\frac{sa^{\frac{1}{\alpha}}}{(1+w)^{-\frac{1}{\alpha}+1}} \Gamma\left(1-\frac{1}{\alpha}\right) + \frac{\mu}{1+w} \right) \right]^2. \tag{11}$$

3.2 *Parameter estimation for the KFP distribution*

Let $x_1, \ldots, x_n$ be the observed values of the random samples $X_1, X_2, \ldots, X_n$ drawn from the KFP distribution with the parameters $\boldsymbol{\theta} = (a, b, \alpha, \mu, s, \lambda)$. The likelihood function of the observation $\mathbf{x} = (x_1, \ldots, x_n)$ is written as follows:

$$\mathscr{L}(\boldsymbol{\theta}|\mathbf{x}) = \prod_{i=1}^{n} f(x_i|\boldsymbol{\theta})$$

$$= \prod_{i=1}^{n} \left(\frac{\lambda ab\frac{\alpha}{s}}{1-e^{-\lambda}} \right) \left(\frac{x_i-\mu}{s} \right)^{-1-\alpha} e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}} \left(1-e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}}\right)^{b-1} e^{-\lambda\left(1-\left(1-e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}}\right)^{b}\right)}, \quad \mu < x_{(1)},$$

where $x_{(1)} = \min\{x_1, \ldots, x_n\}$. On taking logarithms of the likelihood function we have, for $\mu < x_{(1)}$,

$$l(\boldsymbol{\theta}) = \ln \mathscr{L}(\boldsymbol{\theta}|\mathbf{x})$$

$$= n\ln(ab\alpha\lambda/s) - n\ln\left(1-e^{-\lambda}\right) - (\alpha+1)\sum_{i=1}^{n} \ln\left(\frac{x_i-\mu}{s}\right)$$

$$-a\sum_{i=1}^{n}\left(\frac{x_i-\mu}{s}\right)^{-\alpha} + (b-1)\sum_{i=1}^{n}\ln\left(1-e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}}\right) - \lambda\sum_{i=1}^{n}\left(1-\left(1-e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}}\right)^{b}\right). \tag{12}$$

The partial derivatives of the log-likelihood function are supplied, respectively, with regard to the unknown parameters $a,\ b,\ \alpha,\ \mu,\ s$ and λ. These partial derivatives are given as:

$$\frac{\partial l(\boldsymbol{\theta})}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n} \left(\frac{x_i-\mu}{s}\right)^{-\alpha} + \lambda \sum_{i=1}^{n} b \left(1 - e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}}\right)^{b-1} e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}} \left(\frac{x_i-\mu}{s}\right)^{-\alpha}$$

$$\frac{\partial l(\boldsymbol{\theta})}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \ln\left(1 - e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}}\right) + \lambda \sum_{i=1}^{n} \left(1 - e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}}\right)^{b} \ln\left(1 - e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}}\right)$$

$$\frac{\partial l(\boldsymbol{\theta})}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \ln\left(\frac{x_i-\mu}{s}\right) + a \sum_{i=1}^{n} \left(\frac{x_i-\mu}{s}\right)^{-\alpha} \ln\left(\frac{x_i-\mu}{s}\right) - a(b-1) \sum_{i=1}^{n} \frac{\left(\frac{x_i-\mu}{s}\right)^{-\alpha} \ln\left(\frac{x_i-\mu}{s}\right) e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}}}{1 - e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}}}$$

$$- ab\lambda \sum_{i=1}^{n} \left(\frac{x_i-\mu}{s}\right)^{-\alpha} \ln\left(\frac{x_i-\mu}{s}\right) e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}} \left(1 - e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}}\right)^{b-1}$$

$$\frac{\partial l(\boldsymbol{\theta})}{\partial s} = \frac{n\alpha}{s} - a\alpha s^{\alpha-1} \sum_{i=1}^{n} (x_i-\mu)^{-\alpha} + (b-1) a\alpha s^{\alpha-1} \sum_{i=1}^{n} \frac{(x_i-\mu)^{-\alpha} e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}}}{1 - e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}}}$$

$$- a\alpha b\lambda s^{\alpha-1} \sum_{i=1}^{n} (x_i-\mu)^{-\alpha} e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}} \left(1 - e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}}\right)^{b-1}$$

$$\frac{\partial l(\boldsymbol{\theta})}{\partial \mu} = (\alpha+1) \sum_{i=1}^{n} \frac{1}{x_i-\mu} - a\alpha s^{-1} \sum_{i=1}^{n} \left(\frac{x_i-\mu}{s}\right)^{-\alpha-1} + (b-1) a\alpha s^{-1} \sum_{i=1}^{n} \frac{\left(\frac{x_i-\mu}{s}\right)^{-\alpha-1} e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}}}{1 - e^{-\alpha\left(\frac{x_i-\mu}{s}\right)^{-\alpha}}}$$

$$+ a\alpha b s^{-1} \lambda \sum_{i=1}^{n} \left(1 - e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}}\right)^{b-1} e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}} \left(\frac{x_i-\mu}{s}\right)^{-\alpha-1}$$

$$\frac{\partial l(\boldsymbol{\theta})}{\partial \lambda} = \frac{n}{\lambda} - \frac{ne^{-\lambda}}{1-e^{-\lambda}} - \sum_{i=1}^{n} \left(1 - \left(1 - e^{-a\left(\frac{x_i-\mu}{s}\right)^{-\alpha}}\right)^{b}\right).$$

The Maximum Likelihood Estimates (MLEs) $\hat{a},\ \hat{b},\ \hat{\alpha},\ \hat{s},\ \hat{\mu}$ and $\hat{\lambda}$ can be obtained by solving the estimating equations

$$\frac{\partial l(\boldsymbol{\theta})}{\partial a} = 0, \quad \frac{\partial l(\boldsymbol{\theta})}{\partial b} = 0, \quad \frac{\partial l(\boldsymbol{\theta})}{\partial \alpha} = 0, \quad \frac{\partial l(\boldsymbol{\theta})}{\partial s} = 0, \quad \frac{\partial l(\boldsymbol{\theta})}{\partial \mu} = 0, \quad \frac{\partial l(\boldsymbol{\theta})}{\partial \lambda} = 0.$$

or by direct maximization the likelihood function of (12). Clearly, the above estimating equations cannot be solved analytically. Therefore, numerical methods are used to calculate the MLE estimators.

4. Application

In this section, we consider data modeling and parameter estimation when a set of data becomes available. The data set consists of the waiting times between 65 consecutive eruptions of the Kiama Blowhole [27]. The raw data values are:

83, 51, 87, 60, 28, 95, 8, 27, 15, 10, 18, 16, 29, 54, 91, 8, 17, 55, 10, 35, 47, 77, 36, 17, 21, 36,

18, 40, 10, 7, 34, 27, 28, 56, 8, 25, 68, 146, 89, 18, 73, 69, 9, 37, 10, 82, 29, 8, 60, 61, 61, 18,

169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9, 12.

We conducted goodness-of-fit tests comparing the KFP distribution against several alternative distributions:

• Poisson-Fréchet (PF):

$$f(x;\,\mu,\,s,\,\alpha,\,\lambda)=\frac{\lambda\alpha}{s(1-e^{-\lambda})}\left(\frac{x-\mu}{s}\right)^{-\alpha-1}e^{-\left(\frac{x-\mu}{s}\right)^{-\alpha}}e^{-\lambda e^{-\left(\frac{x-\mu}{s}\right)^{-\alpha}}},\quad x>\mu.$$

• Kumaraswamy-G-Poisson with the baseline $U(0,\,c)$ distribution (KUP):

$$f(x;\,a,\,b,\,c,\,\lambda)=\frac{\lambda abx^{a-1}\left(1-\left(\frac{x}{c}\right)^{a}\right)^{b-1}\exp\left(-\lambda\left[1-\left(1-\left(\frac{x}{c}\right)^{a}\right)^{b}\right]\right)}{c^{a}\left(1-e^{-\lambda}\right)},\quad 0<x<c.$$

• Kumaraswamy-G with the baseline $U(0,\,c)$ distribution (KU):

$$f(x)=\frac{ab}{c^{a}}x^{a-1}\left(1-\left(\frac{x}{c}\right)^{\alpha}\right)^{a-1},\ 0<x<c.$$

• Fréchet (FR)

Table 1 presents the maximum likelihood estimates of the parameters for each distribution, along with the Kolmogorov-Smirnov Distance (KSD), its p-value, and the model performance metrics

• Akaike Information Criterion (AIC):

$$AIC=-2\ln(\mathscr{L})+2k$$

• Bayesian Information Criterion (BIC):

$$BIC=-2\ln(\mathscr{L})+k\ln(n)$$

where k is the number of parameters in the model and n is the sample size. From the table, it can be observed that the KFP distribution has the smallest Kolmogorov-Smirnov distance (0.098), indicating a strong fit to the data. Additionally, the KFP distribution yields the lowest AIC (593.02) value among all models and a competitively low BIC (605.98) value, second only to the FR distribution (603.6), suggesting it is the most efficient choice given the number of parameters. This demonstrates an optimal balance between goodness-of-fit and model complexity. The fitted Cumulative Distribution Function (CDF) of the models, along with the empirical distribution, is shown in Figure 3.

Table 1. Goodness-of-fit test results and parameter estimates

Distribution	Parameters	KS-D	KS p-value	AIC	BIC
KFP	$\hat{\mu} = 6.4570$ $\hat{s} = 3.8301$ $\hat{\alpha} = 0.2256$ $\hat{a} = 5.5683$ $\hat{b} = 34.3038$ $\hat{\lambda} = 0.0060$	0.098	0.565	593.02	605.98
PF	$\hat{\mu} = 3.9043$ $\hat{s} = 13.1878$ $\hat{\alpha} = 1.0072$ $\hat{\lambda} = 8.8661 \times 10^{-11}$	0.130	0.228	600.68	609.31
FR	$\hat{\mu} = 0.700$ $\hat{s} = 17.7853$ $\hat{\alpha} = 1.2735$	0.104	0.487	597.12	603.60
KUP	$\hat{c} = 337.9813$ $\hat{a} = 1.440626$ $\hat{b} = 8.312569$ $\hat{\lambda} = 2.607358$	0.110	0.417	600.14	608.78
KU	$\hat{c} = 338$ $\hat{a} = 1.210933$ $\hat{b} = 11.21113$	0.112	0.396	600.96	607.44

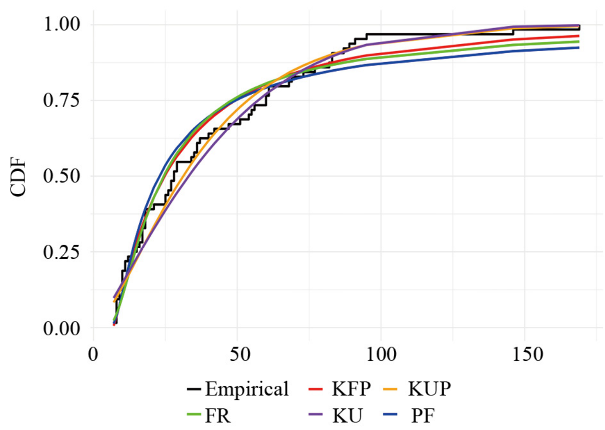

Figure 3. The cumulative distribution function of the fitted models and the empirical distribution

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Conflict of interest

The authors declare no competing financial interest.

References

[1] Gupta RD, Kundu D. Theory & methods: Generalized exponential distributions. *Australian & New Zealand Journal of Statistics*. 1999; 41(2): 173-188. Available from: https://doi.org/10.1111/1467-842X.00072.

[2] Eugene N, Lee C, Famoye F. Beta-normal distribution and its applications. *Communications in Statistics-Theory and Methods*. 2002; 31(4): 497-512. Available from: https://doi.org/10.1007/BF02602999.

[3] Jones M. Families of distributions arising from distributions of order statistics. *Test*. 2004; 13(1): 1-43. Available from: https://link.springer.com/article/10.1007/BF02602999.

[4] Alexander C, Cordeiro GM, Ortega EM, Sarabia JM. Generalized beta-generated distributions. *Computational Statistics & Data Analysis*. 2012; 56(6): 1880-1897. Available from: https://doi.org/10.1016/j.csda.2011.11.015.

[5] Amini M, MirMostafaee SMTK, Ahmadi J. Log-gamma-generated families of distributions. *Statistics*. 2014; 48(4): 913-932. Available from: https://doi.org/10.1080/02331888.2012.748775.

[6] Ristic MM, Balakrishnan N. The gamma-exponentiated exponential distribution. *Journal of Statistical Computation and Simulation*. 2012; 82(8): 1191-1206. Available from: https://doi.org/10.1080/00949655.2011.574633.

[7] Cordeiro GM, Ortega EM, Da Cunha DC. The exponentiated generalized class of distributions. *Journal of Data Science*. 2013; 11(1): 1-27. Available from: https://doi.org/10.6339/JDS.2013.11(1).1086.

[8] Kumaraswamy P. A generalized probability density function for double-bounded random processes. *Journal of Hydrology*. 1980; 46: 79-88. Available from: https://doi.org/10.1016/0022-1694(80)90036-0.

[9] Cordeiro GM, De Castro M. A new family of generalized distributions. *Journal of Statistical Computation and Simulation*. 2011; 81: 883-893. Available from: https://doi.org/10.1080/00949650903530745.

[10] Ramos MWA, Marinho PRD, Cordeiro GM, da Silva RV, Hamedani GG. The Kumaraswamy-G poisson family of distributions. *Journal of Statistical Theory and Applications*. 2015; 14(3): 222-239. Available from: https://doi.org/10.2991/jsta.2015.14.3.1.

[11] Rocha R, Nadarajah S, Tomazella V, Louzada F, Eudes A. New defective models based on the Kumaraswamy family of distributions with application to cancer data sets. *Statistical Methods in Medical Research*. 2017; 26(4): 1737-1755. Available from: https://doi.org/10.1177/096228021558797.

[12] Nofal ZM, Altun E, Afy AZ, Ahsanullah M. The generalized Kumaraswamy-G family of distributions. *Journal of Statistical Theory and Applications*. 2019; 18(4): 329-342. Available from: https://doi.org/%7B10.2991/jsta.d.191030.001%7D.

[13] Silva R, Gomes-Silva F, Ramos M, Cordeiro G, Marinho P, De Andrade TAN. The exponentiated Kumaraswamy-G class: General properties and application. *Colombian Journal of Statistics*. 2019; 42(1): 1-33. Available from: http://dx.doi.org/10.15446/rce.v42n1.66205.

[14] Arshad RMI, Tahir MH, Chesneau C, Jamal F. The gamma Kumaraswamy-G family of distributions: Theory, inference and applications. *Statistics in Transition New Series*. 2020; 21(5): 17-40. Available from: https://doi.org/10.21307stattrans-2020-053.

[15] Torabi H, Hedesh NM. The gamma-uniform distribution and its applications. *Kybernetika*. 2012; 48(1): 16-30. Available from: http://eudml.org/doc/246973.

[16] Tahir MH, Hussain MA, Cordeiro GM, El-Morshedy M, Eliwa MS. A new Kumaraswamy generalized family of distributions with properties, applications, and bivariate extension. *Mathematics*. 2020; 8(11): 1989. Available from: https://doi.org/10.3390/math8111989.

[17] Tharu RP, Dhungana GP, Joshi RK. Kumaraswamy uniform distribution: Model, properties, and applications. *Journal of Institute of Science and Technology*. 2023; 28(2): 117-124. Available from: https://doi.org/10.3126/jist.v28i2.51611.

[18] Paranaíba PF, Ortega EM, Cordeiro GM, Pascoa MAd. The Kumaraswamy Burr XII distribution: Theory and practice. *Journal of Statistical Computation and Simulation*. 2013; 83(11): 2117-2143. Available from: https://doi.org/10.1080/00949655.2012.683003.

[19] Gomes AE, da-Silva CQ, Cordeiro GM, Ortega EM. A new lifetime model: The Kumaraswamy generalized Rayleigh distribution. *Journal of Statistical Computation and Simulation*. 2014; 84(2): 290-309. Available from: https://doi.org/10.1080/00949655.2012.706813.

[20] Rodrigues JDA, Silva APCM. The exponentiated Kumaraswamy-exponential distribution. *British Journal of Applied Science & Technology*. 2015; 10(5): 1-12. Available from: https://doi.org/10.9734/BJAST/2015/16935.

[21] Jamal F, Arslan Nasir M, Ozel G, Elgarhy M, Mamode Khan N. Generalized inverted Kumaraswamy generated family of distributions: Theory and applications. *Journal of Applied Statistics*. 2019; 46(16): 2927-2944. Available from: https://doi.org/10.1080/02664763.2019.1623867.

[22] Alyami SA, Elgarhy M, Elbatal I, Almetwally EM, Alotaibi N, El-Saeed AR. Fréchet binomial distribution: Statistical properties, acceptance sampling plan, statistical inference and applications to lifetime data. *Axioms*. 2022; 11(8): 389. Available from: https://doi.org/10.3390/axioms11080389.

[23] Alotaibi R, Almetwally EM, Ghosh I, Rezk H. Classical and Bayesian inference on finite mixture of exponentiated Kumaraswamy Gompertz and exponentiated Kumaraswamy Fréchet distributions under progressive type II censoring with applications. *Mathematics*. 2022; 10(9): 1496. Available from: https://doi.org/10.3390/math10091496.

[24] Alzeley O, Almetwally EM, Gemeay AM, Alshanbari HM, Hafez EH, Abu-Moussa MH. Statistical inference under censored data for the new exponential-Fréchet distribution: Simulation and application to Leukemia data. *Computational Intelligence and Neuroscience*. 2021; 2021(1): 2167670. Available from: https://doi.org/10.1155/2021/2167670.

[25] Almetwally EM, Muhammed HZ. On a bivariate Fréchet distribution. *Journal of Statistics Applications and Probability*. 2020; 9(1): 71-91. Available from: http://dx.doi.org/10.18576/jsap/090108.

[26] Abouelmagd THM, Hamed MS, Abd El Hadi NE. The Poisson-G family of distributions with applications. *Pakistan Journal of Statistics and Operation Research*. 2017: 313-326. Available from: https://doi.org/10.18187/pjsor.v13i2.1740.

[27] Arshad MZ, Iqbal MZ, Al Mutairi A. A comprehensive review of datasets for statistical research in probability and quality control. *Journal of Mathematics and Computer Science*. 2021; 11(3): 3663-3728. Available from: https://doi.org/10.28919/jmcs/5692.