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Research Article

# A MABAC Model Based on Linguistic Bipolar Complex Fuzzy Soft Information for Production Management in Taiwan

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**Abstract:** Production management is dominant and essential in Taiwan because it confirms and certifies the effective use of timely delivery, resources, and high-quality output in an expert-driven and highly competitive manufacturing sector. With Taiwan being a world-famous player in industries, especially for precision machinery, semiconductors, and electronics. A modern difficulty in production management in Taiwan is the severe deficiency among significant manufacturing organizations, particularly in semiconductors, where over 30,000 posts remain empty, containing roles in maintenance, production, and quality control. For the valuation of the above problems, we consider the following production management systems in Taiwanese enterprises, such as the lean manufacturing system, smart production system, the automated inventory control system, the internet of things integrated manufacturing system, and the sustainable production management system. Therefore, we construct the procedure of the linguistic bipolar complex fuzzy soft multi-attribute border approximation area comparison model and linguistic bipolar complex fuzzy soft multi-attribute decision-making model based on the proposed operators. For the assessment of the above problem, we resolve some numerical examples based on the above two models and also derive the activity of the comparative analysis between proposed and existing ranking values to enhance the efficiency and rationality of the derived models.

***Keywords*:** bipolar complex fuzzy logic, decision-making analysis, linguistic sets, production management systems, soft sets

**MSC:** 65L05, 34K06, 34K28

## 1. Introduction

Taiwanese companies are known for their effective and creative production management systems, improving their worldwide marketability. Lean manufacturing principles are highlighted, enabling ongoing process optimization to eliminate inefficiencies. Robust supplier networks and regional production clusters, such as those in Hsinchu Science Park, enable smooth coordination and rapid market responses. Quality control is crucial. Taiwan's dedication to research and development guarantees that the production system adapts to new developments in technology, especially in the


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fields of semiconductors and electronics. The adaptability of Taiwanese producers enables rapid adjustment to worldwide supply chain interruptions, as demonstrated during the coronavirus outbreak. Employment training and interdisciplinary collaboration are emphasized, cultivating a skilled workforce. Furthermore, the government supports thorough procedures and incentives that encourage sustainable manufacturing practices. Taiwan's production management system is a blend of technological prowess, operational excellence, and strategic adaptability, setting a standard for global manufacturing efficiency. Liu et al. [1] conducted a systematic review of the use of Artificial Intelligence (AI) in production management. Elyasi et al. [2] researched production planning utilizing a flexible manufacturing system in the context of demand uncertainty. Their work investigates plans to optimize operations and adapt to varying market conditions. Garrido et al. [3] conducted a critical analysis and proposed a future agenda on operations management, sustainability, and Industry 5.0. Akhtar et al. [4] worked on AI-and generative AI-driven smart product platforms for production management systems.

Fuzzy sets are pivotal for addressing unpredictability and vagueness in real-world situations where conventional true/false reasoning is insufficient. For example, in temperature classification, instead of simply classifying a temperature as cold or hot, fuzzy sets enable us to describe 25 °C as partly cold (0.5) and partly hot (0.5). In terms of height, a person whose height is 175 cm can be considered 0.8 times tall and 0.2 times normal. In a medical context, a blood pressure reading of 150 mmHg might be represented as 0.85 high and 0.15 normal, reflecting varying risk levels. Additionally, when we say the road is slippery, in classical sets we say yes or no, but in fuzzy sets, we say 0.6 slippery and 0.4 not slippery, depending upon the moisture rather than simply being slippery or not. These examples show that fuzzy sets provide adjustable and reasonable ways to represent unpredictability or imprecise information.

Fuzzy sets have definite disadvantages. One problem is that defining a membership function can be personalized, which might result in an unreliable outcome. They can also be complicated to compute when there are many variables, reducing their effectiveness. Fuzzy outcomes may not be clear, leading to confusion about how different individuals define hot temperature in different ways. In medical evaluations, fuzzy results could be hard for patients to understand. Furthermore, developing accurate fuzzy results requires individual knowledge, which can be more time-consuming and expensive. The fuzzy set was proposed by Zadeh [5] in 1962. A fuzzy set is a generalization of a classical set that allows elements to have distinct degrees of membership, rather than simply stating that an element belongs to the set or not. Demir [6] developed the synergy fuzzy sets. Rahim et al. [7] introduced the concept of a bibliometric analysis of linguistic variables based on an intuitionistic fuzzy set.

Traditional fuzzy sets only measure one membership value; they are unable to solve challenges involving the simultaneous expression of positive and negative assessments. For example, treatment in a medical environment may have both beneficial (such as a recovery likelihood of 0.7) and detrimental (such as a risk factor of 0.3) side effects. This restriction is addressed by a bipolar fuzzy set, which captures both viewpoints by allocating both positive and negative membership. This dual representation ensures a more thorough review and is essential for balanced decision-making in situations such as risk-benefit analysis. In 1994, the concept of a bipolar fuzzy set was introduced by Zhang [8]. A Bipolar Fuzzy set is an interesting extension of a conventional fuzzy set that attributes two different membership functions, the first one is positive membership, and the second one is negative membership. Positive membership measures the degree to which an element belongs to a set based on satisfaction or agreement. A higher value specifies a stronger positive connection to the given set. On the other hand, negative membership measures the degree to which an element belongs to a set based on dissatisfaction or disagreement. A higher value specifies a stronger negative connection to the given set. Bipolar fuzzy sets are pivotal as they extend conventional fuzzy sets by independently capturing both positive (agreement) and negative (rejection) features of an element, allowing for more exact decision-making. Unlike intuitionistic fuzzy sets, which limit membership and non-membership to a sum $\leq 1$, Bipolar Fuzzy Sets (BFS) uses a separate scale ($\mu^+ \in [0, 1]$ for support and $\mu^- \in [-1, 0]$ for drawbacks). This flexibility makes them suitable for applied applications such as voting, medical diagnoses, and handling uncertainty. For example, a product may have satisfaction ($\mu^+ = 0.8$), but minor dissatisfaction ($\mu^- = -0.2$). BFS is especially valuable in multi-criteria decision-making, social networks, and dispute resolution, where conventional fuzzy methods may be insufficient.

Traditional fuzzy sets are limited in addressing problems that require both magnitude and angle information, as they only represent membership values as real numbers between 0 and 1. For example, in signal processing, the fuzzy set can capture a signal amplitude but fails to account for its phase angle, which is crucial for understanding interference.

Complex fuzzy sets address this limitation by using complex numbers, such as $\mu(x) = 0.9(x)e^{\top 60^\circ}$ where the magnitude (0.9) represents the membership and 60° provides additional directional or temporal information. This capability makes Complex Fuzzy Sets (CFSs) important for analyzing quantum systems, cyclic phenomena, and more accurate modeling. Ramot et al. [9] developed the complex fuzzy sets. A complex fuzzy set is the extension of the conventional fuzzy set by including a complex-valued membership function, where the membership grade is represented as $\mu_{\mathscr{A}}(x) = r(x)e^{\top\theta}$ where $r$ is the radius, which lies between 0 and 1, and the angle lies between 0° and 360°. This allows CFS to capture both the degree of truth and phase information, making them essential for practical applications involving quantum systems, signal processing, periodicity, and decision-making under phase-related unpredictability. For example, when examining a signal with both magnitude and angle, a CFS maintains important phase angle information that conventional fuzzy sets may overlook. CFS is highly used in signal processing to examine phase-sensitive data, such as radar and transmission signals. They are also applied in financial prediction to model periodic patterns and in multi-criteria decision-making within stock markets. Additionally, CFS plays the most important role in the medical field and AI by interpreting time-dependent biological signals like Electroencephalogram (EEG) and Electrocardiogram (ECG), as well as improving quantum-inspired neural networks for advanced pattern identification, addresses limitations of conventional fuzzy logic.

Symptoms' severity fluctuates over time(phase component), and it can have both positive and negative effects on medical diagnosis (supporting disease A and opposing disease B). Such bipolarity and dynamic behavior are not captured by traditional fuzzy sets; nevertheless, Bipolar Complex Fuzzy Soft (BCFS) captures both using complex-valued membership (amplitude + phase). Because of this, BCFS is essential for fields like AI, healthcare, and engineering, where opinions are subject to change and disagreement. Mahmood and Rehman [10] proposed a novel way to construct the bipolar complex fuzzy set and their application in generalized similarity measurements. Their work improved the method for comparing and analyzing dark data in complex circumstances. A bipolar complex fuzzy set is the extension of complex fuzzy sets to handle the uncertainties involving both positive and negative judgment, along with magnitude and phase information, where $\mu^{+}_{\mathscr{B}}(x)$ and $\mu^{-}_{\mathscr{B}}(x)$ represents the value of a complex-valued membership function that shows the degree of positive and negative membership, respectively. Representation of each membership function as $\mu(x) = r(x)e^{\top\theta}$ where $r$ is the radius lies between 0 and 1, angle lies between 0° and 360°. The necessity for BCFS emerges in situations where both positive and negative features need to be evaluated together.

In 1975, Zadeh [11] proposed the concept of linguistic variables and explored their application in approximation. It highlights how language can effectively represent uncertain or imprecise information in various fields. Tong [12] derived the linguistic fuzzy model. Dai [13] initiated the linguistic complex fuzzy models. Naz et al. [14] presented the 2-tuple linguistic bipolar fuzzy Heronian mean models. Molodtsov [15] gave the idea of soft set theory, which is a mathematical framework that generalizes traditional set theory, representing uncertainty through pairs of sets: a set of parameters and a corresponding mapping. Cagman et al. [16] developed the fuzzy soft system by integrating the fuzzy and soft models. Majumdar and Samanta [17] derived the modified fuzzy soft models. Thirunavukarasu et al. [18] designed the complex fuzzy soft system. Abdullah et al. [19] initiated the bipolar fuzzy soft system. Mahmood et al. [20] exposed the Bipolar Complex Fuzzy Soft Sets (BCFSSs) by integrating the model of bipolar fuzzy systems and soft systems with complex values. Ali and Yang [21] exposed the robust aggregation models for BCFSSs. Mahmood et al. [22] contributed to the study of pattern recognition and medical diagnosis using similarity measures in the context of the bipolar complex fuzzy soft set. Alqaraleh et al. [23] worked on a bipolar complex fuzzy soft set, gave new ideas on it, and discussed many applications using a bipolar complex fuzzy soft set. The Multi-Attribute Border Approximation Area (MABAC) method is a decision-making technique that assesses alternatives across multiple criteria. It emphasizes finding the ideal solution by evaluating how closely options align with a specified border area. Pamučar and Ćirović [24] proposed the idea of selecting transportation and handling resources in the logistic Centre using the MABAC method. Torkayesh et al. [25] worked on a systematic review of the MABAC method and its application. Mahmood et al. [26] derived the aggregation operators for BCFSSs. Jaleel et al. [27] worked on the analysis and application of a BCFSS using the Dombi aggregation operator. Kumar and Pamucar [28] derived the systematic review of the decision-making problems. Jamil and Riaz [29] developed the bipolar disorder diagnosis. Ali et al. [30] derived the possibility fuzzy bipolar soft information. Harl et al. [31] invented the interval-valued bipolar fuzzy hypersoft topological structure. Zararsiz [32] designed the bipolar fuzzy credibility number. Alolaiyan et al. [33] initiated the vehicle software selection based on bipolar fuzzy aggregation information. Sharma [34] invented the trapezoidal bipolar fuzzy VIKOR techniques. Kuppusamy et al. [35] introduced the bipolar Pythagorean fuzzy models. Yaqoob et al. [36] invented the Dombi operators for complex BFSs. Mohanta [37] derived the enhancement of the bipolar fuzzy leveraging models. These models are very effective and valuable, but not at all. During the decision-making activity, numerous scholars have faced the following problems, for instance

1. Why do we require a model of MABAC?
2. Why do we need to discuss the solution of the production management in Taiwan?
3. Why do we want to develop new operators and operations for proposed models?

These problems create a lot of difficulties for individuals because of uncertainty and problems. The construction of the "MABAC model" for "aggregation operators" based on "Linguistic Bipolar Complex Fuzzy Soft (LBCFS) information" is very efficient and reliable, and easily deals with the above problems. Anyhow, the main problem is that the idea of LBCFS information has not been proposed yet, because of the complexity and complication. The model of LBCFS is very efficient because of their characteristics, where the idea of fuzzy, bipolar fuzzy, complex fuzzy, bipolar complex fuzzy, soft, and their modified versions are the special cases of the proposed theory. Moreover, no one can derive the idea of "aggregation operators" for LBCFS information, because these operators can help us in the aggregation of information into a singleton set.

Further, these operators also help us in the construction of the MABAC models, which are used for the assessment of the best alternative in the collection of information. These models are not developed yet because of ambiguity and problems, where the construction of the MABAC model for aggregation operators based on LBCFS information is very important because of their characteristics and features. The construction of these models fully copes with the above queries and problems. The special cases of the invented model are as follows: for instance, the averaging operator, geometric operator, analytical hierarchical process, and MABAC models for fuzzy to Linguistic Bipolar Complex Fuzzy Soft Sets (LBCFSSs). The invented theory is very novel and no one can derive it yet, because the construction of the LBCFSS is very complex but efficient in coping with vague data. These models are part of the proposed information. Based on these advantages, the major contribution of the proposed theory is explained:

1. To design the model of LBCFSSs with algebraic operational laws.

2. To develop the "weighted averaging operator" and "weighted geometric operator" for linguistic bipolar complex fuzzy soft sets with their basic properties, called:

(a) LBCFS Weighted Averaging (LBCFSWA) operator.

(b) LBCFS Weighted Geometric (LBCFSWG) operator.

3. To construct the procedure of the LBCFS-MABAC model and the LBCFS-Multi-Attribute Decision Making (MADM) model based on the anticipated operators.

4. To resolve some numerical examples based on the above two models and also derive the activity of the comparative analysis between proposed and existing ranking values to enhance the efficiency and rationality of the derived models. The graphical abstract of the proposed theory is explained in Figure 1.

This manuscript is arranged in the following shape: In section 2, we revised the prevailing system of BCFSS with its basic operational laws. Further, we also described the information of Linguistic Term Sets (LTS) with a few important and valuable laws. In section 3, we developed the system of LBCFSSs and their basic laws. Then, we used these laws for the construction of the aggregation operators. Finally, using the aggregation operators, we developed the MABAC models. In section 4, we resolved some numerical examples based on the above two models and also derived the activity of the comparative analysis between the proposed and existing ranking values in section 5. Some concluding remarks are part of section 6.


**Figure 1.** Graphical abstract of the proposed theory

## 2. Preliminaries

In this section, we revised the prevailing system of BCFSS with its basic operational laws. Further, we also described the information of LTS with a few important and valuable laws.

**Definition 1** [11] Assume an LTS with odd cardinality, such as

$$\lfloor = \{s_i | i = 0,\ 1,\ \ldots,\ \llcorner\}. \tag{1}$$

For instance, we have seven linguistic scales, such as

$$S = \{s_0,\ s_1,\ s_2,\ s_3,\ s_4,\ s_5,\ s_6\} = \{\text{none, very low, low, medium, high, very high, perfect}\}.$$

But the linguistic set must satisfy the following conditions: if $s_i > s_j$, then $i > j$; if $Neg(s_i) = s_j$ with $j = \llcorner - i$; if $\max(s_i,\ s_j) = s_i \Leftrightarrow i \geq j$; if $\min(s_i,\ s_j) = s_i \Leftrightarrow i \leq j$. Assume any two LTSs $s_\alpha,\ s_\beta \in \bar{S}$, then

$$S_\alpha \oplus S_\beta = S_\beta \oplus S_\alpha = S_{\alpha+\beta} \tag{2}$$

$$S_\alpha \otimes S_\beta = S_\beta \otimes S_\alpha = S_{\alpha\beta}. \tag{3}$$

**Definition 2** [20] Consider any universe of discourse **U**, with a set of parameters $\eta$, and $\sigma \subseteq \eta$. The duplet of terms $(\Yup,\ \sigma)$, representing the BCFSSs, where $\Yup : \sigma \rightarrow \text{BCF}(U)$. The system $\text{BCF}(U)$ is called the group of BCFSs. The mathematical interpretation of the BCFSSs is initiated and deliberated by:

$$
\begin{aligned}
(\Yup,\ \sigma) = \Yup(\sigma_{\rtimes}) &= \{(\mathfrak{U}_{\ltimes},\ (\varphi_{\Yup}^{+},\ \varphi_{\Yup}^{-})) \mid \forall \mathfrak{U}_{\ltimes} \in U,\ \sigma_{\rtimes} \in \eta\} \\
&= \{(\mathfrak{U}_{\ltimes},\ (\Yup_{\Yup}^{+} + \top \Upsilon_{\Yup}^{+},\ \Yup_{\Yup}^{-} + \top \Upsilon_{\Yup}^{-})) \mid \forall \mathfrak{U}_{\ltimes} \in U,\ \sigma_{\rtimes} \in \eta\}.
\end{aligned}
\tag{4}
$$

The information $\Yup_{\Yup}^{+} + \top \Upsilon_{\Yup}^{+}$ and $\Yup_{\Yup}^{-} + \top \Upsilon_{\Yup}^{-}$ are representing the value of positive and negative truth functions. Throughout this manuscript, the interpretation of the Bipolar Complex Fuzzy Soft Numbers (BCFSN) is denoted and deliberated by: $(\Yup,\ \sigma) = \Yup_{\sigma_{\rtimes\ltimes}} = (\varphi_{\rtimes\ltimes}^{+},\ \varphi_{\rtimes\ltimes}^{-}) = (\Yup_{\rtimes\ltimes}^{+} + \top \Upsilon_{\rtimes\ltimes}^{+},\ \Yup_{\rtimes\ltimes}^{-} + \top \Upsilon_{\rtimes\ltimes}^{-})$.

**Definition 3** [21] Let $\Yup_{\sigma} = (\Yup^{+} + \top \Upsilon^{+},\ \Yup^{-} + \top \Upsilon^{-})$, $\Yup_{\sigma 11} = (\Yup_{11}^{+} + \top \Upsilon_{11}^{+},\ \Yup_{11}^{-} + \top \Upsilon_{11}^{-})$ and $\Yup_{\sigma 12} = (\Yup_{12}^{+} + \top \Upsilon_{12}^{+},\ \Yup_{12}^{-} + \top \Upsilon_{12}^{-})$ be a BCFSN. Thus

$$
\Yup_{\sigma 11} \oplus \Yup_{\sigma 12} = (\Yup_{11}^{+} + \Yup_{12}^{+} - \Yup_{11}^{+} \Yup_{12}^{+} + \top(\Upsilon_{11}^{+} + \Upsilon_{12}^{+} - \Upsilon_{11}^{+} \Upsilon_{12}^{+}),\ -(\Yup_{11}^{-} \Yup_{12}^{-}) + \top(-(\Upsilon_{11}^{-} \Upsilon_{12}^{-}))) \tag{5}
$$

$$
\Yup_{\sigma 11} \otimes \Yup_{\sigma 12} = (\Yup_{11}^{+} \Yup_{12}^{+} + \top(\Upsilon_{11}^{+} \Upsilon_{12}^{+}),\ \Yup_{11}^{-} + \Yup_{12}^{-} + \Yup_{11}^{-} \Yup_{12}^{-} + \top(\Upsilon_{11}^{-} + \Upsilon_{12}^{-} + \Upsilon_{11}^{-} \Upsilon_{12}^{-})) \tag{6}
$$

$$
\rho \Yup_{\sigma} = (1 - (1 - \Yup^{+})^{\rho}) + \top(1 - (1 - \Upsilon^{+})^{\rho}),\ -|\Yup^{-}|^{\rho} + \top(-|\Upsilon^{-}|^{\rho}) \tag{7}
$$

$$
\Yup_{\sigma}^{\rho} = (\Yup^{+\rho} + \top(\Upsilon^{+})^{\rho},\ -1 + (1 + \Yup^{-})^{\rho} + \top(-1 + (1 + \Upsilon^{-})^{\rho})) \tag{8}
$$

$$
(\Yup,\ \sigma)^{c} = (\Yup_{\sigma_{\rtimes\ltimes}})^{c} = \{(1 - \Yup_{\rtimes\ltimes}^{+} + \top(1 - \Upsilon_{\rtimes\ltimes}^{+}),\ -1 - \Yup_{\rtimes\ltimes}^{-} + \top(-1 - \Upsilon_{\rtimes\ltimes}^{-}))\}. \tag{9}
$$

**Definition 4** [20] Let $\Yup_{\sigma_{\rtimes\ltimes}} = (\Yup_{\rtimes\ltimes}^{+} + \top \Upsilon_{\rtimes\ltimes}^{+},\ \Yup_{\rtimes\ltimes}^{-} + \top \Upsilon_{\rtimes\ltimes}^{-})$ be any BCFSN. Thus

$$
\wp_{\Yup}(\Yup_{\sigma_{\rtimes\ltimes}}) = \frac{1}{4}(2 + \Yup_{\rtimes\ltimes}^{+} + \top \Upsilon_{\rtimes\ltimes}^{+} + \Yup_{\rtimes\ltimes}^{-} + \top \Upsilon_{\rtimes\ltimes}^{-}) \in [-1,\ 1] \tag{10}
$$

$$
H_{\Yup}(\Yup_{\sigma_{\rtimes\ltimes}}) = \frac{\Yup_{\rtimes\ltimes}^{+} + \top \Upsilon_{\rtimes\ltimes}^{+} - \Yup_{\rtimes\ltimes}^{-} - \top \Upsilon_{\rtimes\ltimes}^{-}}{4} \in [0,\ 1]. \tag{11}
$$

These ideas are representing the concept of score and accuracy values, which are useful for the interpretation of the difference between any two BCFSNs.

## 3. A MABAC model based on LBCFS information

This section is divided into three main sub-section, which are deliberated the innovation of the LBCFSSs, aggregation operators, and the MABAC model.

## 3.1 *LBCFS information*

In this sub-section, we deliberated the model of LBCFSS with basic laws for the construction of the aggregation operators.

**Definition 5** Consider any universe of discourse $U$, with a set of parameters $\eta$, and $\sigma \subseteq \eta$. The duplet of terms $(⅄, \sigma)$, representing the LBCFSSs, where $⅄ : \sigma \to \mathrm{LBCF}(U)$. The system $\mathrm{LBCF}(U)$ is called the group of LBCFSs. The mathematical interpretation of the LBCFSSs is initiated and deliberated by:

$$\begin{aligned}(⅄, \sigma) = ⅄(\sigma_{\ltimes}) &= \{(\mathfrak{U}_{\ltimes}, (S_{\varphi^+_{⅄}}, S_{\psi^-_{⅄}}))|\forall \mathfrak{U}_{\ltimes} \in U, \&\sigma_{\rtimes} \in \eta\} \\ &= \{(\mathfrak{U}_{\ltimes}, (\lfloor_{⅄^+_{⅄}+\top\Upsilon^+_{⅄}}, \lfloor_{⅄^-_{⅄}+\top\Upsilon^-_{⅄}}))|\forall \mathfrak{U}_{\ltimes} \in U, \&\sigma_{\rtimes} \in \eta\}.\end{aligned} \tag{12}$$

The information $\lfloor_{⅄^+_{⅄}+\top\Upsilon^+_{⅄}}$ and $\lfloor_{⅄^-_{⅄}+\top\Upsilon^-_{⅄}}$ are representing the value of positive and negative truth functions. Throughout this manuscript, the interpretation of the LBCFSN is denoted and deliberated by: $(⅄, \sigma) = ⅄_{\sigma_{\rtimes\ltimes}} = (\lfloor_{\varphi^+_{\rtimes\ltimes}}, \lfloor_{\psi^-_{\rtimes\ltimes}}) = (\lfloor_{⅄^+_{\rtimes\ltimes}+\top\Upsilon^+_{\rtimes\ltimes}}, \lfloor_{⅄^-_{\rtimes\ltimes}+\top\Upsilon^-_{\rtimes\ltimes}})$.

**Definition 6** Let $⅄_{\sigma} = (\lfloor_{⅄^+ + \top\Upsilon^+}, \lfloor_{⅄^- + \top\Upsilon^-})$, $⅄_{\sigma 11} = (\lfloor_{⅄^+_{11}+\top\Upsilon^+_{11}}, \lfloor_{⅄^-_{11}+\top\Upsilon^-_{11}})$ and $⅄_{\sigma 12} = (\lfloor_{⅄^+_{12}+\top\Upsilon^+_{12}}, \lfloor_{⅄^-_{12}+\top\Upsilon^-_{12}})$ be any LBCFSNs. Thus

$$⅄_{\sigma 11} \oplus ⅄_{\sigma 12} = \left(\lfloor_{\llcorner(\frac{⅄^+_{11}}{\llcorner}+\frac{⅄^+_{12}}{\llcorner}-\frac{⅄^+_{11}}{\llcorner}\frac{⅄^+_{11}}{\llcorner})+\top\llcorner(\frac{\Upsilon^+_{11}}{\llcorner}+\frac{\Upsilon^+_{12}}{\llcorner}-\frac{\Upsilon^+_{11}}{\llcorner}\frac{\Upsilon^+_{11}}{\llcorner})}, \lfloor_{\llcorner(\frac{-⅄^-_{11}}{\llcorner}\frac{⅄^-_{12}}{\llcorner})+\top\llcorner(\frac{-(\Upsilon^-_{11})}{\llcorner}\frac{(\Upsilon^-_{12})}{\llcorner})}\right) \tag{13}$$

$$⅄_{\sigma 11} \otimes ⅄_{\sigma 12} = \left(\lfloor_{\llcorner(\frac{⅄^+_{11}}{\llcorner}\frac{⅄^+_{12}}{\llcorner})+\top\llcorner(\frac{\Upsilon^+_{11}}{\llcorner}\frac{\Upsilon^+_{12}}{\llcorner})}, \lfloor_{\llcorner(\frac{⅄^-_{11}}{\llcorner}+\frac{⅄^-_{12}}{\llcorner}+\frac{⅄^-_{11}}{\llcorner}\frac{⅄^-_{12}}{\llcorner})+\top\llcorner(\frac{\Upsilon^-_{11}}{\llcorner}+\frac{\Upsilon^-_{12}}{\llcorner}+\frac{\Upsilon^-_{11}}{\llcorner}\frac{\Upsilon^-_{12}}{\llcorner})}\right) \tag{14}$$

$$\rho ⅄_{\sigma} = \left(\lfloor_{\llcorner\left(1-\left(1-\frac{⅄^+}{\llcorner}\right)^{\rho}\right)+\top\llcorner\left(1-\left(1-\frac{\Upsilon^+}{\llcorner}\right)^{\rho}\right)}, \lfloor_{-\llcorner\left|\frac{⅄^-}{\llcorner}\right|^{\rho}+\top\llcorner\left(-\left|\frac{\Upsilon^-}{\llcorner}\right|^{\rho}\right)}\right) \tag{15}$$

$$⅄^{\rho}_{\sigma} = \left(\lfloor_{⅄^{+\rho}+\top(\Upsilon^+)^{p}}, \lfloor_{\llcorner\left(-1+\left(1+\frac{⅄^-}{\llcorner}\right)^{\rho}\right)+\top\llcorner\left(-1+\left(1+\frac{\Upsilon^-}{\llcorner}\right)^{\rho}\right)}\right) \tag{16}$$

$$(⅄, \sigma)^c = (⅄_{\sigma\rtimes\ltimes})^c = \left\{\left(\lfloor_{(\llcorner-⅄^+_{\rtimes\ltimes})+\top(\llcorner-\Upsilon^+_{\rtimes\ltimes})}, \lfloor_{(\llcorner-⅄^-_{\rtimes\ltimes})+\top(\llcorner-\Upsilon^-_{\rtimes\ltimes})}\right)\right\}. \tag{17}$$

**Definition 7** Let $⅄_{\sigma\rtimes\ltimes} = \left(\lfloor_{⅄^+_{\rtimes\ltimes}+\top\Upsilon^+_{\rtimes\ltimes}}, \lfloor_{⅄^-_{\rtimes\ltimes}+\top\Upsilon^-_{\rtimes\ltimes}}\right)$ be a LBCFSN. Then

$$\wp_{⅄}(⅄_{\sigma\rtimes\ltimes}) = \frac{1}{4}(⅄^+_{\rtimes\ltimes} + \Upsilon^+_{\rtimes\ltimes} + ⅄^-_{\rtimes\ltimes} + \Upsilon^-_{\rtimes\ltimes}) \in [-1, 1] \tag{18}$$

$$H_{⅄}(⅄_{\sigma\rtimes\ltimes}) = \frac{⅄^+_{\rtimes\ltimes} + \Upsilon^+_{\rtimes\ltimes} - ⅄^-_{\rtimes\ltimes} - \Upsilon^-_{\rtimes\ltimes}}{4} H_{⅄}(⅄_{\sigma}) \in [0, 1]. \tag{19}$$

These ideas represent the concept of score and accuracy values, which are useful for interpretation of the difference between any two LBCFSNs.

### 3.2 *LBCFS-robust aggregation operators*

In this section, we used the group of LBCFSNs $⅄_{\sigma \rtimes \ltimes} = \left(\lfloor_{\varphi^{+}_{\rtimes\ltimes}},\ \lfloor_{\psi^{+}_{\rtimes\ltimes}}\right) = \left(\lfloor_{⅄^{+}_{\rtimes\ltimes}+\top\Upsilon^{+}_{\rtimes\ltimes}},\ \lfloor_{⅄^{-}_{\rtimes\ltimes}+\top\Upsilon^{-}_{\rtimes\ltimes}}\right)$ $(\rtimes = 1,\ 2,\ 3,\ \ldots,\ m : \ltimes = 1,\ 2,\ 3,\ \ldots,\ n)$ with weight vectors $\eta_{\rtimes}0$, $\Sigma^{m}_{\rtimes=1}\eta_{\rtimes} = 1$ and $\rho_{\rtimes}0$, $\Sigma^{n}_{\rtimes=1}\rho_{\rtimes} = 1$ for the construction of the LBCFSWA operator and LBCFSWG operator. Some basic properties are also discussed in detail.

**Definition 8** The defined and deliberated structure of the LBCFSWA operator is assessed by:

$$\text{LBCF}\lfloor\text{WA}: ⅄^{n} \rightarrow ⅄. \tag{20}$$

Where,

$$\begin{aligned}
&\text{LBCFSWA}(⅄_{\sigma 11},\ ⅄_{\sigma 12},\ \ldots,\ ⅄_{\sigma mn}) \\
&= \underset{\ltimes=1}{\overset{n}{\oplus}} \rho_{\ltimes} \left( \underset{\rtimes=1}{\overset{m}{\oplus}} \eta_{\rtimes} ⅄_{\sigma\rtimes\ltimes} \right) \\
&= \left( \begin{array}{l}
\lfloor_{\llcorner\left(1-\Pi^{m}_{\rtimes=1}\left(\Pi^{n}_{\ltimes=1}\left(1-\frac{⅄^{+}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right)+\top\llcorner\left(1-\Pi^{m}_{\rtimes=1}\left(\Pi^{n}_{\ltimes=1}\left(1-\frac{\Upsilon^{+}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right)}, \\
\lfloor_{-\llcorner\Pi^{m}_{\rtimes=1}\left(\Pi^{n}_{\ltimes=1}\left(1-\frac{⅄^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}+\top\llcorner\left(-\Pi^{m}_{\rtimes=1}\left(\Pi^{n}_{\ltimes=1}\left(1-\frac{\Upsilon^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right)}
\end{array} \right).
\end{aligned} \tag{21}$$

The Proof of Eq. (21) is given in Appendix A.

**Property 1** If $⅄_{\sigma\rtimes\ltimes} = ⅄_{\sigma}$ for all possible values of $\rtimes,\ \ltimes$ then $\text{LBCFSWA}(⅄_{\sigma 11},\ ⅄_{\sigma 12},\ \ldots,\ ⅄_{\sigma mn})$ (idempotency). The Proof of Property 1 is given in Appendix B.

**Property 2** If

$$⅄^{-}_{\rtimes\ltimes} = \left(\lfloor_{\underset{\rtimes}{\min}\ \underset{\ltimes}{\min}\{\varphi^{+}_{\rtimes\ltimes}\}},\ \lfloor_{\underset{\rtimes}{\max}\ \underset{\ltimes}{\max}\{\psi^{-}_{\rtimes\ltimes}\}}\right) = \left(\lfloor_{\underset{\rtimes}{\min}\ \underset{\ltimes}{\min}\{⅄^{+}_{\rtimes\ltimes}\}+\top\underset{\rtimes}{\min}\ \underset{\ltimes}{\min}\{\Upsilon^{+}_{\rtimes\ltimes}\}},\ \lfloor_{\underset{\rtimes}{\max}\ \underset{\ltimes}{\max}\{⅄^{-}_{\rtimes\ltimes}\}+\top\underset{\rtimes}{\max}\ \underset{\ltimes}{\max}\{\Upsilon^{-}_{\rtimes\ltimes}\}}\right)$$

and

$$⅄^{+}_{\rtimes\ltimes} = \left(\lfloor_{\underset{\rtimes}{\max}\ \underset{\ltimes}{\max}\{\varphi^{+}_{\rtimes\ltimes}\}},\ \lfloor_{\underset{\rtimes}{\min}\ \underset{\ltimes}{\min}\{\psi^{-}_{\rtimes\ltimes}\}}\right) = \left(\lfloor_{\underset{\rtimes}{\max}\ \underset{\ltimes}{\max}\{⅄^{+}_{\rtimes\ltimes}\}+\top\underset{\rtimes}{\max}\ \underset{\ltimes}{\max}\{\Upsilon^{+}_{\rtimes\ltimes}\}},\ \lfloor_{\underset{\rtimes}{\min}\ \underset{\ltimes}{\min}\{⅄^{-}_{\rtimes\ltimes}\}+\top\underset{\rtimes}{\min}\ \underset{\ltimes}{\min}\{\Upsilon^{-}_{\rtimes\ltimes}\}}\right).$$

Then

$$\lfloor_{⅄^{-}_{\rtimes\ltimes}} \leq \text{LBCFSWA}(⅄_{\sigma_{11}},\ ⅄_{\sigma_{12}},\ \ldots,\ ⅄_{\sigma_{mn}}) \leq \lfloor^{+}_{⅄_{\rtimes\ltimes}}\ \text{(Boundedness)}.$$

**Property 3** If $⅄^{\#}_{\rtimes\ltimes} \leq ⅄^{*}_{\rtimes\ltimes}$, then $\text{LBCFSWA}\left(⅄^{\#}_{\sigma_{11}},\ ⅄^{\#}_{\sigma_{12}},\ \ldots,\ ⅄^{\#}_{\sigma_{mn}}\right) \leq \text{LBCFSWA}\left(⅄^{*}_{\sigma_{11}},\ ⅄^{*}_{\sigma_{12}},\ \ldots,\ ⅄^{*}_{\sigma_{mn}}\right)$ (Monotonicity).

**Definition 9** The defined and deliberated structure of the LBCFSWG operator is assessed by:

$$\mathrm{LBCFSWG}: \curlywedge^{n} \rightarrow \curlywedge.$$

Where,

$$\mathrm{LBCFSWG}\left(\curlywedge_{\sigma_{11}}, \curlywedge_{\sigma_{12}}, \ldots, \curlywedge_{\sigma_{mn}}\right) = \bigotimes_{\rtimes=1}^{m}\left(\bigotimes_{\ltimes=1}^{n} \curlywedge_{\sigma_{\rtimes\ltimes}}^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}$$

$$= \left(\lfloor_{\llcorner\left(\Pi_{\rtimes=1}^{m}\left(\Pi_{\ltimes=1}^{n}\frac{\curlywedge_{\rtimes\ltimes}^{+}}{\llcorner}\right)^{\eta_{\rtimes}}\right)+\top\llcorner\left(\Pi_{\rtimes=1}^{m}\left(\Pi_{\ltimes=1}^{+}\frac{\curlyvee_{\rtimes\ltimes}^{+}}{\llcorner}\right)^{\eta_{\rtimes}}\right)}, \right. \quad (22)$$

$$\left. \lfloor_{\llcorner\left(-1+\Pi_{\rtimes=1}^{m}\left(\Pi_{\ltimes=1}^{n}\left(1+\frac{\curlywedge_{\rtimes\ltimes}^{-}}{\llcorner}\right)\right)^{\eta_{\rtimes}}\right)+\top\llcorner\left(-1+\Pi_{\rtimes=1}^{m}\left(\Pi_{\ltimes=1}^{n}\left(1+\frac{\curlyvee_{\rtimes\ltimes}^{-}}{\llcorner}\right)\right)\right)}\right).$$

The Proof of Eq. (22) is given in Appendix C.

**Property 4** If $\curlywedge_{\sigma\rtimes\ltimes} = \curlywedge_{\sigma}$ all possible values of $\rtimes, \ltimes$ then $\mathrm{LBCFSWG}(\curlywedge_{\sigma_{11}, \sigma_{12}, \ldots, \sigma_{mn}}) = \curlywedge_{\sigma}$ (idempotency).

**Property 5** If

$$\curlywedge_{\rtimes\ltimes}^{-} = \left(\lfloor_{\min_{\rtimes} \min_{\ltimes}\{\varphi_{\rtimes\ltimes}^{+}\}}, \lfloor_{\max_{\rtimes} \max_{\ltimes}\{\psi_{\rtimes\ltimes}^{-}\}}\right) = \left(\lfloor_{\min_{\rtimes} \min_{\ltimes}\{\curlywedge_{\rtimes\ltimes}^{+}\}+\top \min_{\rtimes} \min_{\ltimes}\{\curlyvee_{\rtimes\ltimes}^{+}\}}, \lfloor_{\max_{\rtimes} \max_{\ltimes}\{\curlywedge_{\rtimes\ltimes}^{-}\}+\top \max_{\rtimes} \max_{\ltimes}\{\curlyvee_{\rtimes\ltimes}^{-}\}}\right)$$

and

$$\curlywedge_{\rtimes\ltimes}^{+} = \left(\lfloor_{\max_{\rtimes} \max_{\ltimes}\{\varphi_{\rtimes\ltimes}^{+}\}}, \lfloor_{\min_{\rtimes} \min_{\ltimes}\{\psi_{\rtimes\ltimes}^{-}\}}\right) = \left(\lfloor_{\max_{\rtimes} \max_{\ltimes}\{\curlywedge_{\rtimes\ltimes}^{+}\}+\top \max_{\rtimes} \max_{\ltimes}\{\curlyvee_{\rtimes\ltimes}^{+}\}}, \lfloor_{\min_{\rtimes} \min_{\ltimes}\{\curlywedge_{\rtimes\ltimes}^{-}\}+\top \min_{\rtimes} \min_{\ltimes}\{\curlyvee_{\rtimes\ltimes}^{-}\}}\right).$$

Then

$$\lfloor_{\curlywedge_{\rtimes\ltimes}^{-}} \leq \mathrm{LBCFSWA}(\curlywedge_{\sigma_{11}}, \curlywedge_{\sigma_{12}}, \ldots, \curlywedge_{\sigma_{mn}}) \leq \lfloor_{\curlywedge_{\rtimes\ltimes}}^{+} \text{ (Boundedness).}$$

**Property 6** If $\curlywedge_{\rtimes\ltimes}^{\#} \leq \curlywedge_{\rtimes\ltimes}^{*}$, then $\mathrm{LBCFSWA}\left(\curlywedge_{\sigma_{11}}^{\#}, \curlywedge_{\sigma_{12}}^{\#}, \ldots, \curlywedge_{\sigma_{mn}}^{\#}\right) \leq \mathrm{LBCFSWA}\left(\curlywedge_{\sigma_{11}}^{*}, \curlywedge_{\sigma_{12}}^{*}, \ldots, \curlywedge_{\sigma_{mn}}^{*}\right)$ (Monotonicity).

**Definition 10** In this section, we deliberated and demonstrated the system of the MABAC technique for LBCFSS for the assessment of the production management in Taiwan. The MABAC model is very operative and efficient, because this technique can help us to find the best decision with the help of operators and measures collectively. For this purpose, we required a group of alternatives. $p_1, p_1, \ldots, ps$, and for each of the alternatives, we analyze the groups of attributes $p_1^{AT}, p_2^{AT}, \ldots, p_t^{AT}$, With a weight vector $\alpha_i \in [0, 1]$ such that $\sum_{i=1}^{t} \alpha_i = 1$. Given the above theory, the main steps of the MABAC model is briefly described:

**Step 1:** Construct the decision matrix by incorporating the terms of LBCFSNs. If the matrix includes cost-related information, we will normalize it accordingly, such as

$$N = \begin{cases} p = \langle S_{\varphi^+_{\text{⅄}}},\ S_{\psi^-_{\text{⅄}}} \rangle \quad \text{benefits} \\ \\ p^c = \langle S_{\psi^-_{\text{⅄}}},\ S_{\varphi^+_{\text{⅄}}} \rangle \quad \text{costs.} \end{cases} \tag{23}$$

If the matrix includes benefit-related information, then we do not need to normalize.

**Step 2:** Calculate the weighted normalized decision information data using a scaler and geometric multiplication technique.

$$\rho \text{⅄}_\sigma = \left( \llcorner_{\llcorner\left(1-\left(1-\frac{\text{⅄}^+}{\llcorner}\right)^\rho\right)+\top\llcorner\left(1-\left(1-\frac{\Upsilon^+}{\llcorner}\right)^\rho\right)},\ \llcorner_{-\llcorner\left|\frac{\text{⅄}^-}{\llcorner}\right|^\rho+\top\llcorner\left(-\left|\frac{\Upsilon^-}{\llcorner}\right|^\rho\right)} \right) \tag{24}$$

$$\text{⅄}_\sigma^\rho = \left( \llcorner_{\text{⅄}^{+\rho}+\top(\Upsilon^+)^\rho},\ \llcorner_{\llcorner\left(-1+\left(1+\frac{\text{⅄}^-}{\llcorner}\right)^\rho\right)+\top\llcorner\left(-1+\left(1+\frac{\Upsilon^-}{\llcorner}\right)^\rho\right)} \right). \tag{25}$$

**Step 3:** Compute the (Bipolar Aggregation Algorithm (BAA)) according to the theory of the LBCFSWA operator and the LBCFSWG operator, such as

$$\text{LBCFSWA}(\text{⅄}_{\sigma 11},\ \text{⅄}_{\sigma 12},\ \ldots,\ \text{⅄}_{\sigma mn})$$

$$= \begin{pmatrix} \llcorner_{\llcorner\left(1-\Pi^m_{\rtimes=1}\left(\Pi^n_{\ltimes=1}\left(1-\frac{\text{⅄}^+_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_\ltimes}\right)^{\eta_\rtimes}\right)+\top\llcorner\left(1-\Pi^m_{\rtimes=1}\left(\Pi^n_{\ltimes=1}\left(1-\frac{\Upsilon^+_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_\ltimes}\right)^{\eta_\rtimes}\right)}, \\ \llcorner_{\llcorner\left(-\Pi^m_{\rtimes=1}\left(\Pi^n_{\ltimes=1}\left|\frac{\text{⅄}^-_{\rtimes\ltimes}}{\llcorner}\right|^{\rho_\ltimes}\right)^{\eta_\rtimes}\right)+\top\llcorner\left(-\Pi^m_{\rtimes=1}\left(\Pi^n_{\ltimes=1}\left|\frac{\Upsilon^-_{\rtimes\ltimes}}{\llcorner}\right|^{\rho_\ltimes}\right)^{\eta_\rtimes}\right)} \end{pmatrix} \tag{26}$$

$$\text{LBCFSWG}(\text{⅄}_{\sigma_{11}},\ \text{⅄}_{\sigma_{12}},\ \ldots,\ \text{⅄}_{\sigma_{mn}})$$

$$= \begin{pmatrix} \llcorner_{\llcorner\left(\Pi^m_{\rtimes=1}\left(\Pi^n_{\ltimes=1}\frac{\text{⅄}^+_{\rtimes\ltimes}}{\llcorner}\right)^{\eta_\rtimes}\right)+\top\llcorner\left(\Pi^m_{\rtimes=1}\left(\Pi^+_{\ltimes=1}\frac{\Upsilon^+_{\rtimes\ltimes}}{\llcorner}\right)^{\eta_\rtimes}\right)}, \\ \llcorner_{\llcorner\left(-1+\Pi^m_{\rtimes=1}\left(\Pi^n_{\ltimes=1}\left(1+\frac{\text{⅄}^-_{\rtimes\ltimes}}{\llcorner}\right)\right)^{\eta_\rtimes}\right)+\top\llcorner\left(-1+\Pi^m_{\rtimes=1}\left(\Pi^n_{\ltimes=1}\left(1+\frac{\Upsilon^-_{\rtimes\ltimes}}{\llcorner}\right)\right)\right)} \end{pmatrix}. \tag{27}$$

**Step 4:** Calculate the distance function based on the values obtained in Step 2 and Step 3.

$$d_{\sigma_\rtimes \sigma_\ltimes} = \begin{cases} d(\text{⅄}_{\sigma_\rtimes},\ \text{⅄}_{\sigma_\ltimes}), & \text{if } \text{⅄}_{\sigma_\rtimes} > \text{⅄}_{\sigma_\ltimes} \\ \\ 0, & \text{if } \text{⅄}_{\sigma_\rtimes} = \text{⅄}_{\sigma_\ltimes} \\ \\ -(\text{⅄}_{\sigma_\rtimes},\ \text{⅄}_{\sigma_\ltimes}), & \text{if } \text{⅄}_{\sigma_\rtimes} < \text{⅄}_{\sigma_\ltimes}. \end{cases} \tag{28}$$

The mathematical construction of the distance function is deliberated by:

$$\wp_{⅄}\left(⅄_{\sigma \rtimes \ltimes}\right)=\frac{1}{4\llcorner}\left(\left|⅄_{\rtimes}^{+}-⅄_{\ltimes}^{+}\right|+\left|\Upsilon_{\rtimes}^{+}-\Upsilon_{\ltimes}^{+}\right|+\left|⅄_{\rtimes}^{-}-⅄_{\ltimes}^{-}\right|+\left|\Upsilon_{\rtimes}^{-}-\Upsilon_{\ltimes}^{-}\right|\right). \tag{29}$$

**Step 5:** Calculate the score value by using the technique of averaging information, such as

$$S_{\ltimes}=\frac{1}{n}\sum_{\rtimes=1}^{n} d\left(⅄_{\sigma \rtimes},\ ⅄_{\sigma \ltimes}\right). \tag{30}$$

Finally, in the presence of the appraisal values, we deliberated the ranking values for the valuation and investigation of the best decision among the group of information. The geometrical interpretation of the MABAC model is described in Figure 2.

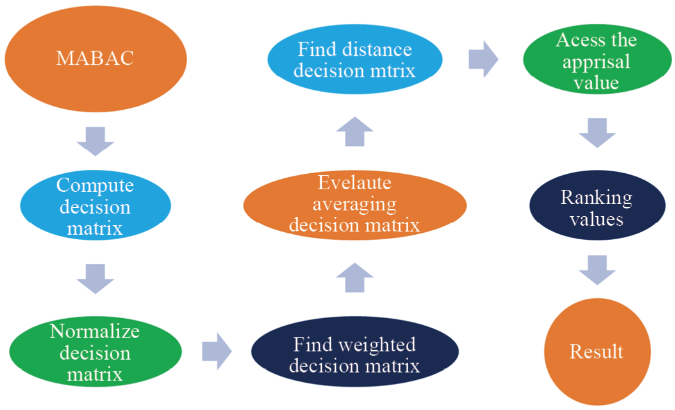


**Figure 2.** Geometrical abstract of the MABAC model

## 4. Analysis of production management system for Taiwan

To maintain their position as the world's leading manufacturers, Taiwanese businesses use a high-yield production management system that integrates with a Blockchain-enabled system, including just-in-time inventory control, lean productivity, and Industry 4.0 technologies such as AI, robotics, and Internet of Things (IoT). Manufacturing Execution System (MES) and Enterprise Resource Planning (ERP) software are used by companies like Foxconn and Taiwan Semiconductor Manufacturing Company (TSMC) to optimize operations, and adaptive processes and flexible production lines allow for a quick response to market requirements. Predictive analytics and digital twins for process optimization enhance the prominence of total quality management, data-driven decision-making, and kaizen (continuous improvement). Taiwanese companies' production management systems provide optimal resource implementation, cost control, and optimized workflow to increase productivity while reducing delays. They use advanced scheduling, real-time monitoring, and learning production rules to cut waste and achieve high productivity. Inventory management is streamlined to prevent shortages, and quality control methods ensure that products meet high standards. Digital and automation tools improve accuracy, while data analytics assists with performance tracking and demand predictions. Supplier management guarantees a consistent flow of raw materials, and workforce training enhances staff productivity. Compliance with rules and regulations, risk management strategies, and customer feedback integration refine processes, while continuous improvement plans for fostering innovation, energy-efficient practices, Just-in-Time (JIT) production, and prevention further reduce downtime and costs. Ultimately, in this way, the system improves the profitability, competitiveness, and long-term growth of Taiwanese businesses. Here are five alternative production management systems that Taiwanese companies could consider, such as:

1. Lean manufacturing system

The lean manufacturing system is a highly streamlined production method that aims to remove waste while increasing value and productivity. It streamlines operations by focusing on non-value-added activities such as unnecessary delays, excess inventory, and overproduction, using ideas such as continuous improvement, production, and resource optimization. Employee involvement is critical, as employees at all levels work together to uncover incompetence and implement solutions for workplace organizations using tools such as (sustain, shine, set in order). This approach not only lowers costs and decreases them, but also improves product quality, speeds up delivery, and increases customer happiness. Lean manufacturing is a critical approach for high-achieving and cost-effective production because it promotes a culture of efficiency and continuous improvement, which produces increased profitability and operational excellence.

2. Smart production system

The smart production system transforms production by integrating leading-edge technology, such as IoT, AI, and big data, to robotize repetitive operations, minimize human error, and improve operational efficiency. IoT sensors integrate exact and real-time data from machine learning for smooth production and line monitoring, while AI-powered analytics forecast maintenance needs to minimize downtime through predictive maintenance. This technology modified real-time visibility, enabling managers to make swift, data-informed decisions. It also uses machine learning algorithms to optimize production schedules and improve resource allocation. It is also used to optimize production schedules and machine learning algorithms and improve resource allocation. The digital process simulates the process for virtual testing and optimization before physical implementation. The system's flexibility allows for customized mass production to meet the development of client needs. Moreover, effective resource management reduces energy consumption and waste and fosters sustainability. By using these technologies, the smart production system significantly enhances productivity, competitiveness, and operational flexibility in the modern manufacturing landscape.

3. The automated inventory control system

The automated inventory control system transforms inventory management by exploiting digital tools to maintain the data for ideal inventory levels, avoid shortages, and lower excess inventory expenses. It employs barcode and Radio Frequency Identification (RFID) scanning for precise real-time tracking through the warehouse and supply network. Advanced analytics predict demand patterns, ensuring sufficient stock availability. Automated reorder triggers restock inventory when levels drop below-set thresholds, while cloud-based solutions offer comprehensive visibility across multiple locations for improved collaboration. It integrates with ERP and Point of Sale (POS) systems to align inventory with sales data, while a machine learning algorithm analyzes to adjust safety stock levels and identify slow-moving items. The system produces real-time reporting on key inventory, allowing for data-driven decisions to improve cash flow and increase customer satisfaction through consistent product availability.

4. IoT-integrated manufacturing system

The IoT-integrated manufacturing system modified the production set by linking the sensor, the machine, and the device in real-time to better operations. By continuously observing equipment performance with an embedded sensor, the system identifies the issues before they occur, allowing for forecasting maintenance that schedules repair during planned downtime to minimize disturbance. Machine-to-Machine (M2M) connectivity enables self-governance in production settings. Real-time data is routed to concentrated dashboards for rapid performance monitoring and decision-making. Technology enhances operational flexibility by dynamically redistributing resources during the outage and optimizing energy consumption across all linked devices. Additionally, benefits include digital work instructions provided directly to shop floor interfaces and an automated supply chain merger that prompts material restocking based on original use. This large-scale connection decreases downtime by 30-40%. Improve Overall Equipment Efficiency (OEE) and establish a continuous, data-driven production flow that will maximize efficiency and productivity across the whole production ecosystem.

5. The sustainable production management system

The sustainable production management system converts manufacturing by implementing ecologically friendly practices based on circular economic ideas that reduce waste through material reuse and recycling programs. It uses energy-efficient technology, including Solar power, and a smart Heating, Ventilation, and Air Conditioning (HVAC) system to reduce carbon emissions. Advanced water recycling technology and rainwater harvesting can lower freshwater consumption by as much as 45 percent. The technology also substitutes conventional material with non-toxic packaging and biodegradable packaging and optimizes greenhouse gas emissions in real-time. Combining lean production techniques to reduce resource misuse with extensive supplier Eco-friendly audits, ensures ethical material sourcing across the supply chain. Maintaining 14,001 certification ensures conformity with international environmental standards, while green programs for employees foster eco-awareness in day-to-day operations. Comprehensive lifecycle judgment examines the product from creation to discarding, driving continuous improvement in best performance-an approach that typically gives the result of around 20-25 percent operational cost savings over four years while future-proofing the business, and changing environmental rules and regulations.

Further, for the assessment of the above information, we also have some attributes. The lean manufacturing system focuses on cost efficiency and waste reduction to achieve high excellence. An IoT-focused on factory transmission and smart prediction systems prioritize AI. An automated control system ensures accuracy through digital tracking and stock optimization. IoT-integrated manufacturing system enhances decision-making and flexibility. A sustainable production management system balances productivity with environmental responsibility. Given the above theory, the main steps of the MABAC model are briefly described:

**Step 1:** Construct the decision matrix by incorporating the terms of Complex Pythagorean Fuzzy Numbers (CPFNs), see the data in Tables 1-5.

**Table 1.** LBCFS decision matrix for $A_1$

| | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ |
|---|---|---|---|---|
| $\mathbb{Y}_1$ | $\left(\lfloor 1+2\top, \lfloor -1-3\top\right)$ | $\left(\lfloor 2+2\top, \lfloor -3-3\top\right)$ | $\left(\lfloor 1+2\top, \lfloor -4-3\top\right)$ | $\left(\lfloor 1+5\top, \lfloor -4-4\top\right)$ |
| $\mathbb{Y}_2$ | $\left(\lfloor 2+2\top, \lfloor -3-3\top\right)$ | $\left(\lfloor 3+2\top, \lfloor -1-3\top\right)$ | $\left(\lfloor 4+2\top, \lfloor -2-3\top\right)$ | $\left(\lfloor 5+1\top, \lfloor -1-1\top\right)$ |
| $\mathbb{Y}_3$ | $\left(\lfloor 1+2\top, \lfloor -1-3\top\right)$ | $\left(\lfloor 1+2\top, \lfloor -1-2\top\right)$ | $\left(\lfloor 1+2\top, \lfloor -2-3\top\right)$ | $\left(\lfloor 4+3\top, \lfloor -3-2\top\right)$ |
| $\mathbb{Y}_4$ | $\left(\lfloor 1+3\top, \lfloor -1-3\top\right)$ | $\left(\lfloor 1+2\top, \lfloor -1-4\top\right)$ | $\left(\lfloor 2+2\top, \lfloor -2-3\top\right)$ | $\left(\lfloor 2+2\top, \lfloor -1-4\top\right)$ |

**Table 2.** LBCFS decision matrix for $A_2$

| | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ |
|---|---|---|---|---|
| $\mathbb{Y}_1$ | $\left(\lfloor 3+2\top, \lfloor -1-2\top\right)$ | $\left(\lfloor 1+3\top, \lfloor -2-3\top\right)$ | $\left(\lfloor 4+1\top, \lfloor -4-2\top\right)$ | $\left(\lfloor 2.1+3.2\top, \lfloor -2.1-3.5\top\right)$ |
| $\mathbb{Y}_2$ | $\left(\lfloor 3.1+2.1\top, \lfloor -0.9-1.8\top\right)$ | $\left(\lfloor 1.1+3.1\top, \lfloor -1.9-2.9\top\right)$ | $\left(\lfloor 4.1+1.1\top, \lfloor -3.9-1.9\top\right)$ | $\left(\lfloor 2.2+3.3\top, \lfloor -2-3.4\top\right)$ |
| $\mathbb{Y}_3$ | $\left(\lfloor 3.2+2.2\top, \lfloor -0.8-1.8\top\right)$ | $\left(\lfloor 1.2+3.2\top, \lfloor -1.8-2.8\top\right)$ | $\left(\lfloor 4.2+1.2\top, \lfloor -3.8-1.8\top\right)$ | $\left(\lfloor 2.3+3.4\top, \lfloor -1.9-3.3\top\right)$ |
| $\mathbb{Y}_4$ | $\left(\lfloor 3.3+2.3\top, \lfloor -0.7-1.7\top\right)$ | $\left(\lfloor 1.3+3.3\top, \lfloor -1.7-2.7\top\right)$ | $\left(\lfloor 4.3+1.3\top, \lfloor -3.7-1.7\top\right)$ | $\left(\lfloor 2.4+3.5\top, \lfloor -1.8-3.2\top\right)$ |

**Table 3.** LBCFS decision matrix for $A_3$

| | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ |
|---|---|---|---|---|
| $\mathbb{Y}_1$ | $\left(\lfloor 4+1\top, \lfloor -2-3\top\right)$ | $\left(\lfloor 2.1+3.3\top, \lfloor -2.2-3.7\top\right)$ | $\left(\lfloor 1.1+2.1\top, \lfloor -2.3-3.5\top\right)$ | $\left(\lfloor 2+3\top, \lfloor -1-4\top\right)$ |
| $\mathbb{Y}_2$ | $\left(\lfloor 4.2+1.1\top, \lfloor -1.9-2.9\top\right)$ | $\left(\lfloor 2.3+3.3\top, \lfloor -2.1-3.6\top\right)$ | $\left(\lfloor 1.3+2.2\top, \lfloor -2.2-3.4\top\right)$ | $\left(\lfloor 2.3+3.2\top, \lfloor -0.9-3.9\top\right)$ |
| $\mathbb{Y}_3$ | $\left(\lfloor 4.3+1.2\top, \lfloor -1.8-2.8\top\right)$ | $\left(\lfloor 2.4+3.3\top, \lfloor -2-3.5\top\right)$ | $\left(\lfloor 1.4+2.3\top, \lfloor -2.1-3.3\top\right)$ | $\left(\lfloor 2.24+3.3\top, \lfloor -0.8-3.8\top\right)$ |
| $\mathbb{Y}_4$ | $\left(\lfloor 4.4+1.3\top, \lfloor -1.7-2.7\top\right)$ | $\left(\lfloor 2.5+3.3\top, \lfloor -1.9-3.4\top\right)$ | $\left(\lfloor 1.5+2.4\top, \lfloor -2.0-3.2\top\right)$ | $\left(\lfloor 2.5+3.4\top, \lfloor -0.7-3.7\top\right)$ |

**Table 4.** LBCFS decision matrix for $A_4$

| | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ |
|---|---|---|---|---|
| $\mathbb{Y}_1$ | $(\lfloor 3.1+3.3\top, \lfloor -2.2-3.4\top)$ | $(\lfloor 3.2+3.5\top, \lfloor -3.3-3.4\top)$ | $(\lfloor 4.1+2.1\top, \lfloor -2.4-1.5\top)$ | $(\lfloor 1.5+2.4\top, \lfloor -2-1.4\top)$ |
| $\mathbb{Y}_2$ | $(\lfloor 3.3+3.3\top, \lfloor -2.1-3.3\top)$ | $(\lfloor 3.4+3.6\top, \lfloor -3.2-3.3\top)$ | $(\lfloor 4.3+2.2\top, \lfloor -2.3-1.4\top)$ | $(\lfloor 1.7+2.5\top, \lfloor -1.9-1.3\top)$ |
| $\mathbb{Y}_3$ | $(\lfloor 3.4+3.3\top, \lfloor -2-3.2\top)$ | $(\lfloor 3.5+3.7\top, \lfloor -3.1-3.2\top)$ | $(\lfloor 4.4+2.3\top, \lfloor -2.2-1.3\top)$ | $(\lfloor 1.8+2.6\top, \lfloor -1.8-1.2\top)$ |
| $\mathbb{Y}_4$ | $(\lfloor 3.5+3.3\top, \lfloor -1.9-3.1\top)$ | $(\lfloor 3.6+3.8\top, \lfloor -3-3.1\top)$ | $(\lfloor 4.5+2.4\top, \lfloor -2.1-1.2\top)$ | $(\lfloor 1.9+2.7\top, \lfloor -1.7-1.1\top)$ |

**Table 5.** LBCFS decision matrix for $A_5$

| | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ |
|---|---|---|---|---|
| $\mathbb{Y}_1$ | $(\lfloor 2.4+3.4\top, \lfloor -2.3-3.4\top)$ | $(\lfloor 1.5+2.9\top, \lfloor -1.5-3.3\top)$ | $(\lfloor 1.6+1.8\top, \lfloor -2.4-3.6\top)$ | $(\lfloor 1.5+2.4\top, \lfloor -2-1.4\top)$ |
| $\mathbb{Y}_2$ | $(\lfloor 2.8+3.5\top, \lfloor -2.2-3.3\top)$ | $(\lfloor 1.9+\top, \lfloor -1.4-3.3\top)$ | $(\lfloor 2+1.9\top, \lfloor -2.3-3.5\top)$ | $(\lfloor 1.7+2.5\top, \lfloor -1.9-1.3\top)$ |
| $\mathbb{Y}_3$ | $(\lfloor 2.9+3.6\top, \lfloor -2.1-3.2\top)$ | $(\lfloor 2+3.1\top, \lfloor -1.3-3.3\top)$ | $(\lfloor 2.1+2\top, \lfloor -2.2-3.4\top)$ | $(\lfloor 1.8+2.6\top, \lfloor -1.8-1.2\top)$ |
| $\mathbb{Y}_4$ | $(\lfloor 3+3.7\top, \lfloor -2-3.1\top)$ | $(\lfloor 2.1+3.2\top, \lfloor -1.2-3\top)$ | $(\lfloor 2.2+2.1\top, \lfloor -2.1-3.3\top)$ | $(\lfloor 1.9+2.7\top, \lfloor -1.7-1.1\top)$ |

After the assessment of the data in Table 1 to Table 5, we noticed that the data is benefit types, so, we do not need to be normalized.

**Step 2:** Calculate the weighted normalized decision information data using a scaler and geometric multiplication technique, see Tables 6-10.

**Table 6.** LBCFS weighted decision matrix for $A_1$

| | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ |
|---|---|---|---|---|
| $\mathbb{Y}_1$ | $(\lfloor 1.092851+2.17015\top, \lfloor -0.81225-2.71972\top)$ | $(\lfloor 2.17015+2.17015\top, \lfloor -2.71972-2.71972\top)$ | $(\lfloor 1.092851+2.17015\top, \lfloor -3.73213-2.71972\top)$ | $(\lfloor 1.09285+5.280279\top, \lfloor -3.773213-3.73213\top)$ |
| $\mathbb{Y}_2$ | $(\lfloor 2.17015+2.17015\top, \lfloor -2.71972-2.71972\top)$ | $(\lfloor 3.229565+2.17015\top, \lfloor -0.81225-2.71972\top)$ | $(\lfloor 4.267868+2.17015\top, \lfloor -1.7411-2.71972\top)$ | $(\lfloor 5.280279+1.09285\top, \lfloor -0.81225-0.81225\top)$ |
| $\mathbb{Y}_3$ | $(\lfloor 1.092851+2.17015\top, \lfloor -0.81225-2.71972\top)$ | $(\lfloor 1.092851+2.17015\top, \lfloor -0.81225-1.7411\top)$ | $(\lfloor 1.092851+2.17015\top, \lfloor -1.7411-2.71972\top)$ | $(\lfloor 4.267868+3.229565\top, \lfloor -2.71972-1.7411\top)$ |
| $\mathbb{Y}_4$ | $(\lfloor 1.092851+3.229565\top, \lfloor -0.81225-2.71972\top)$ | $(\lfloor 1.092851+2.17015\top, \lfloor -0.81225-3.73213\top)$ | $(\lfloor 2.17015+2.17015\top, \lfloor -1.7411-2.71972\top)$ | $(\lfloor 2.17015+2.17015\top, \lfloor -0.81225-3.73213\top)$ |

**Table 7.** LBCFS weighted decision matrix for $A_2$

| | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ |
|---|---|---|---|---|
| $\mathbb{Y}_1$ | $(\lfloor 3.229565+2.17015\top, \lfloor -0.81225-1.7411\top)$ | $(\lfloor 1.092851+3.229565\top, \lfloor -1.7411-2.71972\top)$ | $(\lfloor 4.267868+1.092851\top, \lfloor -3.73213-1.7411\top)$ | $(\lfloor 2.276941+3.439039\top, \lfloor -1.8371-3..2223\top)$ |
| $\mathbb{Y}_2$ | $(\lfloor 3.334409+0.276941\top, \lfloor -0.72337-1.6458\top)$ | $(\lfloor 1.201314+3.334409\top, \lfloor -1.64558-0.62017\top)$ | $(\lfloor 4.370372+1.201314\top, \lfloor -3.62963-1.64558\top)$ | $(\lfloor 2.383551+3.543451\top, \lfloor -1.7411-3.12117\top)$ |
| $\mathbb{Y}_3$ | $(\lfloor 3.439039+2.383551\top, \lfloor -0.63546-1.55057\top)$ | $(\lfloor 1.30962+3.439039\top, \lfloor -1.55057-2.52095\top)$ | $(\lfloor 4.472614+1.30962\top, \lfloor -3.52739-1.55057\top)$ | $(\lfloor 2.489978+3.647642\top, \lfloor -1.64558-3.02034\top)$ |
| $\mathbb{Y}_4$ | $(\lfloor 3.543454+2.489978\top, \lfloor -0.54865-1.45608\top)$ | $(\lfloor 1.417767+3.543451\top, \lfloor -1.45608-2.42209\top)$ | $(\lfloor 4.574587+1.417767\top, \lfloor -3.42541-1.45608\top)$ | $(\lfloor 2.596218+3.751606\top, \lfloor -1.55057-2.91982\top)$ |

**Table 8.** LBCFS weighted decision matrix for $A_3$

| | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ |
|---|---|---|---|---|
| $\mathbb{Y}_1$ | $\begin{pmatrix} \lfloor 4.267868+1.092851\top, \\ \lfloor -1.7411-2.71972\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.276941+3.543451\top, \\ \lfloor -1.93355-3.42541\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 1.201314+2.276941\top, \\ \lfloor -2.03045-3.2223\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.17015+3.229565\top, \\ \lfloor -0.81225-3.73213\top \end{pmatrix}$ |
| $\mathbb{Y}_2$ | $\begin{pmatrix} \lfloor 4.472614+1.201314\top, \\ \lfloor -1.64558-2.62017\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.489978+3.647642\top, \\ \lfloor -1.8371-3.32371\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 1.417767+2.383551\top, \\ \lfloor -1.93355-3.12117\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.383551+3.34409\top, \\ \lfloor -0.72337-3.62963\top \end{pmatrix}$ |
| $\mathbb{Y}_3$ | $\begin{pmatrix} \lfloor 4.574587+1.30962\top, \\ \lfloor -1.55057-2.52095\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.596218+3.751606\top, \\ \lfloor -1.7411-3.2223\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 1.525752+2.489978\top, \\ \lfloor -1.8371302034-1.633574\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.489978+3.439039\top, \\ \lfloor -0.63546-3.52739\top \end{pmatrix}$ |
| $\mathbb{Y}_4$ | $\begin{pmatrix} \lfloor 4.676285+1.417767\top, \\ \lfloor -1.45608-2.42209\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.702268+3.85534\top, \\ \lfloor -1.64558-3.12117\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.596218+1.7411\top, \\ \lfloor -2.91982-2.17015\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.596218+3.543451\top, \\ \lfloor -0.54865-3.42541\top \end{pmatrix}$ |

**Table 9.** LBCFS weighted decision matrix for $A_4$

| | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ |
|---|---|---|---|---|
| $\mathbb{Y}_1$ | $\begin{pmatrix} \lfloor 3.334409+3.543451\top, \\ \lfloor -1.93355-3.12117\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 3.439039+3.751606\top, \\ \lfloor -3.02034-3.12117\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 4.370372+2.276941\top, \\ \lfloor -2.12776-1.26879\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 1.633574+2.596218\top, \\ \lfloor -1.7411,-1.17607\top \end{pmatrix}$ |
| $\mathbb{Y}_2$ | $\begin{pmatrix} \lfloor 3.543451+3.647642\top, \\ \lfloor -1.8371-3.02034\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 3.647642+3.85534\top, \\ \lfloor -2.91982-3.02034\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 4.574587+2.383551\top, \\ \lfloor -2.03045-1.17607\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 1.848718+2.702268\top, \\ \lfloor -1.64558-1.084\top \end{pmatrix}$ |
| $\mathbb{Y}_3$ | $\begin{pmatrix} \lfloor 3.647642+3.751606\top, \\ \lfloor -1.7411-2.91982\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 3.751606+3.958838\top, \\ \lfloor -2.81961-2.91982\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 4.1676285+2.489978\top, \\ \lfloor -1.93355-1.084\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 1.956036+2.808126\top, \\ \lfloor -1.55057-0.99264\top \end{pmatrix}$ |
| $\mathbb{Y}_4$ | $\begin{pmatrix} \lfloor 3.751606+3.85534\top, \\ \lfloor -1.64558-2.81961\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 3.85534+3.958838\top, \\ \lfloor -2.71972-2.81961\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 4.777701+2.596218\top, \\ \lfloor -1.8371-0.99264\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.063181+2.913788\top, \\ \lfloor -1.45608-0.90203\top \end{pmatrix}$ |

**Table 10.** LBCFS weighted decision matrix for $A_5$

| | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ |
|---|---|---|---|---|
| $\mathbb{Y}_1$ | $\begin{pmatrix} \lfloor 2.596218+3.647642\top, \\ \lfloor -2.03045-3.12117\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 1.633574+3.124511\top, \\ \lfloor -1.26879-3.02034\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 1.74123+1.956036\top, \\ \lfloor -2.12776-3.32371\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 3.334409+3.85534\top, \\ \lfloor -2.12776,-3.2223\top \end{pmatrix}$ |
| $\mathbb{Y}_2$ | $\begin{pmatrix} \lfloor 3.019251+3.751606\top, \\ \lfloor -1.93355-3.02034\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.063182+3.229565\top, \\ \lfloor -1.17607-2.91982\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.17015+2.063181\top, \\ \lfloor -2.03045-3.2223\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 3.751606+3.958838\top, \\ \lfloor -2.03045-3.12117\top \end{pmatrix}$ |
| $\mathbb{Y}_3$ | $\begin{pmatrix} \lfloor 3.124511+3.85534\top, \\ \lfloor -1.8371-2.91982\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.17015+3.334409\top, \\ \lfloor -1.084-2.81961\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.276941+2.17015\top, \\ \lfloor -1.93355-3.12117\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 3.85534+4.062095\top, \\ \lfloor -1.93355-3.02034\top \end{pmatrix}$ |
| $\mathbb{Y}_4$ | $\begin{pmatrix} \lfloor 3.229565+3.958838\top, \\ \lfloor -1.7411-2.81961\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.276941+3.439039\top, \\ \lfloor -0.99264-2.71972\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.383551+2.276941\top, \\ \lfloor -1.8371-3.02034\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 3.958838+4.165107\top, \\ \lfloor -1.8371-2.91982\top \end{pmatrix}$ |

**Step 3:** Compute the (BAA) according to the theory of the LBCFSWA operator and LBCFSWG operator, see Table 11.

**Table 11.** LBCFS aggregated decision matrix

| | LBCFSWA | LBCFSWG |
|---|---|---|
| $A_1$ | $\begin{pmatrix} \lfloor 2.202778+2.498099\top, \\ \lfloor -1.3268-2.58017\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 1.740456+2.36411\top, \\ \lfloor -1.61215-2.74974\top \end{pmatrix}$ |
| $A_2$ | $\begin{pmatrix} \lfloor 3.186916+2.751988\top, \\ \lfloor -1.6749-2.07508\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.79325+2.443333\top, \\ \lfloor -2.08339-2.21466\top \end{pmatrix}$ |
| $A_3$ | $\begin{pmatrix} \lfloor 2.763228+2.873543\top, \\ \lfloor -1.26354-3.08022\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.443434+2.627155\top, \\ \lfloor -1.41066-3.11666\top \end{pmatrix}$ |
| $A_4$ | $\begin{pmatrix} \lfloor 3.593338+3.182161\top, \\ \lfloor -1.3268-2.58017\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 1.740456+2.36411\top, \\ \lfloor -1.96687-1.85401\top \end{pmatrix}$ |
| $A_5$ | $\begin{pmatrix} \lfloor 2.931886+3.381831\top, \\ \lfloor -1.69257-2.97686\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.784082+3.21072\top, \\ \lfloor -1.74405-1.85401\top \end{pmatrix}$ |

**Step 4:** Calculate the distance function based on the values obtained in Step 2 and Step 3, see Tables 12-16.

**Table 12.** Representation of distance measure for $A_1$

| | $\sigma_1$ | | $\sigma_2$ | | $\sigma_3$ | | $\sigma_4$ | |
|---|---|---|---|---|---|---|---|---|
| $\mathbb{Y}_1$ | 0.065374 | 0.052234 | 0.059158 | 0.055039 | 0.124461 | 0.093486 | 0.232794 | 0.208317 |
| $\mathbb{Y}_2$ | 0.059158 | 0.055039 | 0.062776 | 0.078531 | 0.09209 | 0.090011 | 0.211413 | 0.23589 |
| $\mathbb{Y}_3$ | 0.065374 | 0.052234 | 0.087234 | 0.082816 | 0.062241 | 0.031267 | 0.157142 | 0.172159 |
| $\mathbb{Y}_4$ | 0.077984 | 0.073218 | 0.097012 | 0.081995 | 0.028576 | 0.024457 | 0.063346 | 0.075186 |

**Table 13.** Representation of distance measure for $A_2$

| | $\sigma_1$ | | $\sigma_2$ | | $\sigma_3$ | | $\sigma_4$ | |
|---|---|---|---|---|---|---|---|---|
| $\mathbb{Y}_1$ | 0.05691 | 0.076699 | 0.102578 | 0.104187 | 0.160353 | 0.154606 | 0.090826 | 0.086436 |
| $\mathbb{Y}_2$ | 0.0626211 | 0.082396 | 0.098201 | 0.103948 | 0.159948 | 0.154202 | 0.084598 | 0.086207 |
| $\mathbb{Y}_3$ | 0.068266 | 0.08805 | 0.097955 | 0.103702 | 0.159533 | 0.153786 | 0.080224 | 0.085971 |
| $\mathbb{Y}_4$ | 0.073868 | 0.096568 | 0.097702 | 0.103448 | 0.159106 | 0.15336 | 0.079981 | 0.085728 |

**Table 14.** Representation of distance measure for $A_3$

| | $\sigma_1$ | | $\sigma_2$ | | $\sigma_3$ | | $\sigma_4$ | |
|---|---|---|---|---|---|---|---|---|
| $\mathbb{Y}_1$ | 0.128856 | 0.127691 | 0.067856 | 0.059826 | 0.095859 | 0.07243 | 0.064134 | 0.065299 |
| $\mathbb{Y}_2$ | 0.131991 | 0.130826 | 0.058263 | 0.053142 | 0.079576 | 0.056146 | 0.060317 | 0.061481 |
| $\mathbb{Y}_3$ | 0.131924 | 0.130759 | 0.052022 | 0.053541 | 0.0704452 | 0.0493 | 0.059281 | 0.063886 |
| $\mathbb{Y}_4$ | 0.131859 | 0.130964 | 0.045805 | 0.053541 | 0.063904 | 0.042752 | 0.059281 | 0.069995 |

**Table 15.** Representation of distance measure for $A_4$

| | $\sigma_1$ | | $\sigma_2$ | | $\sigma_3$ | | $\sigma_4$ | |
|---|---|---|---|---|---|---|---|---|
| $\mathbb{Y}_1$ | 0.069383 | 0.056892 | 0.10658 | 0.098546 | 0.068679 | 0.082749 | 0.096351 | 0.094742 |
| $\mathbb{Y}_2$ | 0.063816 | 0.066544 | 0.100404 | 0.102014 | 0.071586 | 0.085656 | 0.092176 | 0.090566 |
| $\mathbb{Y}_3$ | 0.067061 | 0.072907 | 0.100615 | 0.102224 | 0.071288 | 0.08744 | 0.091339 | 0.089729 |
| $\mathbb{Y}_4$ | 0.073405 | 0.079251 | 0.10083 | 0.10244 | 0.074852 | 0.093159 | 0.090472 | 0.088863 |

**Table 16.** Representation of distance measure for $A_5$

| | $\sigma_1$ | | $\sigma_2$ | | $\sigma_3$ | | $\sigma_4$ | |
|---|---|---|---|---|---|---|---|---|
| $\mathbb{Y}_1$ | 0.033865 | 0.032806 | 0.063215 | 0.054681 | 0.063215 | 0.054681 | 0.106203 | 0.094451 |
| $\mathbb{Y}_2$ | 0.023175 | 0.031355 | 0.049829 | 0.042827 | 0.049829 | 0.042827 | 0.083241 | 0.071488 |
| $\mathbb{Y}_3$ | 0.027116 | 0.035651 | 0.049218 | 0.048769 | 0.049218 | 0.048769 | 0.070373 | 0.05862 |
| $\mathbb{Y}_4$ | 0.033765 | 0.042484 | 0.052163 | 0.054678 | 0.052163 | 0.054678 | 0.057539 | 0.045786 |

**Step 5:** Calculate the score value by using the technique of averaging information, see data in Table 17.

**Table 17.** Representation of the appraisal values

| | $\sigma_1$ | | $\sigma_2$ | | $\sigma_3$ | | $\sigma_4$ | |
|---|---|---|---|---|---|---|---|---|
| $A_1$ | 0.066973 | 0.058181 | 0.076545 | 0.074595 | 0.076842 | 0.059805 | 0.166174 | 0.172888 |
| $A_2$ | 0.065414 | 0.085927 | 0.099109 | 0.103821 | 0.159735 | 0.153989 | 0.083907 | 0.086085 |
| $A_3$ | 0.131158 | 0.129993 | 0.055987 | 0.055115 | 0.077448 | 0.055157 | 0.060886 | 0.065165 |
| $A_4$ | 0.068416 | 0.068898 | 0.102107 | 0.101306 | 0.071601 | 0.087251 | 0.092585 | 0.090975 |
| $A_5$ | 0.02948 | 0.035574 | 0.053606 | 0.050239 | 0.079339 | 0.067586 | 0.057309 | 0.065577 |

The data in Table 18 contains the maximum values of the appraisal values for each alternative according to their attribute, and their graphical abstract is given in Figure 3.

**Table 18.** Final values for ranking information

| | Score values | |
|---|---|---|
| $A_1$ | 0.16617 | 0.17289 |
| $A_2$ | 0.15974 | 0.15399 |
| $A_3$ | 0.13116 | 0.12999 |
| $A_4$ | 0.10211 | 0.10131 |
| $A_5$ | 0.07934 | 0.06759 |

Finally, we rank all alternatives and assess the best decision among the collection of information to finalize the best one for showing the supremacy and validity of the designed theory, see the data in Table 19.

**Table 19.** Ranking values information

| Methods | Ranking values |
|---|---|
| MABAC-LBCFSWA | $A_1 > A_2 > A_3 > A_4 > A_5$ |
| MABAC-LBCFSWG | $A_1 > A_2 > A_3 > A_4 > A_5$ |

The best decision is $A_1$ for the proposed operators in the procedure of MABAC models.

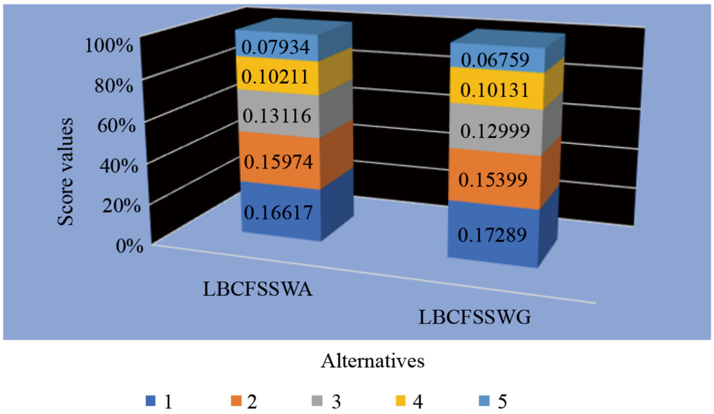


**Figure 3.** Graphical from the data in Table 18

Further, we check the supremacy of the proposed operators without the MABAC model, for this, we consider the data in Tables 1 to 5, and the aggregating values are described in Table 20.

**Table 20.** Aggregation information without the MABAC model

| | LBCFSWA | LBCFSWG |
|---|---|---|
| $A_1$ | $\begin{pmatrix} \lfloor 1.56056+2.086879\top, \\ \lfloor -1.61518-2.97186\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 1.348821+2.065938\top, \\ \lfloor -1.78287-3.0163\top \end{pmatrix}$ |
| $A_2$ | $\begin{pmatrix} \lfloor 2.959379+2.546946\top, \\ \lfloor -1.93075-2.34591\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.59277+2.260748\top, \\ \lfloor -2.33453-2.48236\top \end{pmatrix}$ |
| $A_3$ | $\begin{pmatrix} \lfloor 2.557564+2.661889\top, \\ \lfloor -1.49435-3.35942\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.260755+2.433175\top, \\ \lfloor -1.64375-3.39425\top \end{pmatrix}$ |
| $A_4$ | $\begin{pmatrix} \lfloor 3.347854+2.954852\top, \\ \lfloor -2.16433-1.80532\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 3.049792+2.862547\top, \\ \lfloor -2.23085-2.10126\top \end{pmatrix}$ |
| $A_5$ | $\begin{pmatrix} \lfloor 2.717147+3.145298\top, \\ \lfloor -1.94925-3.25678\top \end{pmatrix}$ | $\begin{pmatrix} \lfloor 2.579768+2.98559\top, \\ \lfloor -2.00053-3.26222\top \end{pmatrix}$ |

Thus, using the data in Table 20, the score values of that data are explained in Table 21, and the graphical abstract of the data in Table 21 is given in Figure 4.

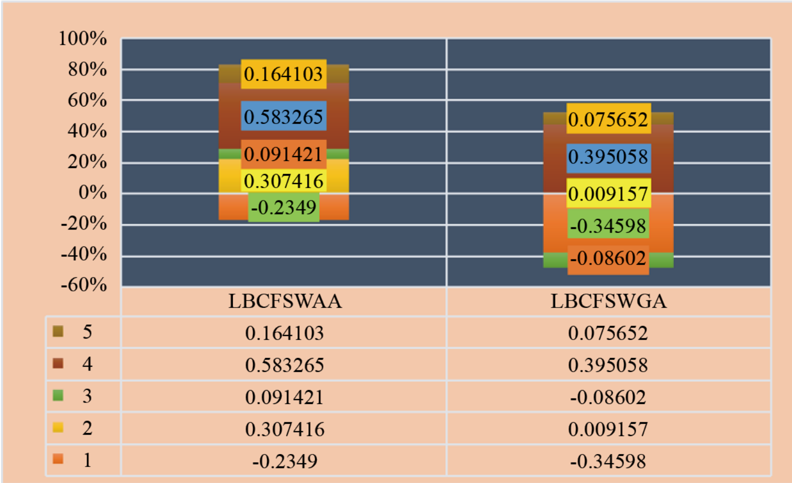


**Figure 4.** Graphical abstract of the data in Table 21

**Table 21.** Score values for the data in Table 20

| | LBCFSWAA | LBCFSWGA |
|---|---|---|
| $A_1$ | -0.2349 | -0.34598 |
| $A_2$ | 0.307416 | 0.009157 |
| $A_3$ | 0.091421 | -0.08602 |
| $A_4$ | 0.583265 | 0.395058 |
| $A_5$ | 0.164103 | 0.075652 |

Finally, we rank all alternatives and assess the best decision among the collection of information to finalize the best one for showing the supremacy and validity of the designed theory, see the data in Table 22.

**Table 22.** Ranking values information for the data in Table 21

| Methods | Ranking values |
|---|---|
| LBCFSWA | $A_4 > A_2 > A_5 > A_3 > A_1$ |
| LBCFSWG | $A_4 > A_5 > A_2 > A_3 > A_5$ |

The best decision is $A_4$ for the proposed operator in the procedure of MADM models. Further, we described the supremacy and validity of the designed theory by comparing our ranking values with the ranking values of the existing models.

## 5. Comparative analysis

In this section, we conduct the comparative analysis for the data in Tables 1 to 5 by using the information from the proposed models and the existing models. Comparative analysis is a unique way to interpret the proposed theory, and many researchers have analyzed this activity in numerous papers to assess the superiority and validity of the proposed theory. For comparative analysis, we consider the following existing models: Ali and Yang [21] exposed the robust aggregation models for BCFSSs. Mahmood et al. [22] contributed to the study of pattern recognition and medical diagnosis using similarity measures in the context of the bipolar complex fuzzy soft set. Alqaraleh et al. [23] worked on a bipolar complex fuzzy soft set, introduced new ideas, and discussed many applications using a bipolar complex fuzzy soft set. The MABAC method is a decision-making technique that assesses alternatives across multiple criteria. It emphasizes finding the ideal solution by evaluating how closely options align with a specified border area. Pamučar and Ćirović [24] proposed the idea of selecting transportation and handling resources in the logistic centre using the MABAC method. Torkayesh et al. [25] conducted a systematic review of the MABAC method and its applications. Mahmood et al. [26] derived the aggregation operators for BCFSSs. Jaleel et al. [27] worked on the analysis and application of a BCFSS using the Dombi aggregation operator. Finally, the comparative analysis is presented in Table 23 for the data in Tables 1-5.

**Table 23.** Interpretation of the comparative analysis

| Methods | Score values | Ranking values |
|---|---|---|
| Ali and Yang [21] | Do not evaluated | Do not evaluated |
| Mehmood et al. [22] | Do not evaluated | Do not evaluated |
| Alqaraleh et al. [23] | Do not evaluated | Do not evaluated |
| Pamučar and Ćirović [24] | Do not evaluated | Do not evaluated |
| Torkayesh et al. [25] | Do not evaluated | Do not evaluated |
| Mahmood et al. [26] | Do not evaluated | Do not evaluated |
| Jaleel et al. [27] | Do not evaluated | Do not evaluated |
| MABAC-LBCFSWA | 0.1661, 0.1597, 0.1311, 0.1021, 0.0793 | $A_1 > A_2 > A_3 > A_4 > A_5$ |
| MABAC-LBCFSWG | 0.1728, 0.1539, 0.1299, 0.1013, 0.0675 | $A_1 > A_2 > A_3 > A_4 > A_5$ |
| LBCFSWA | -0.2349, 0.307416, 0.091421, 0.583265, 0.164103 | $A_4 > A_2 > A_5 > A_3 > A_1$ |
| LBCFSWG | -0.34598, 0.009157, -0.08602, 0.395058, 0.075652 | $A_4 > A_5 > A_2 > A_3 > A_1$ |

The best decision is $A_1$ for the proposed operators in the procedure of MABAC models, but the best decision is $A_4$ for the proposed operators in the procedure of MADM models. From the above information, we observe that the existing models failed to cope with the data in Tables 1 to 5 because of ambiguity and complications. Since the proposed theory is novel and up-to-date, no one has proposed it yet. Therefore, the proposed models are more reliable and more efficient in handling vague data.

## 6. Conclusion

Production management is dominant and essential in Taiwan because it ensures the effective use of timely delivery, resources, and high-quality output in an expert-driven and highly competitive manufacturing sector. Taiwan is world-famous in industries, especially for precision machinery, semiconductors, and electronics. A modern difficulty in production management in Taiwan is the severe deficiency of personnel among significant manufacturing organizations, particularly in semiconductors, where over 30,000 positions remain vacant, including roles in maintenance, production, and quality control. To evaluate the above problems, we consider the following production management systems in Taiwanese enterprises: the lean manufacturing system, the smart production system, automated inventory control system, Internet of Things-an integrated manufacturing system and a sustainable production management system. The major contribution of the proposed theory is explained:

1. We designed the model of LBCFSSs with algebraic operational laws.

2. We developed the "weighted averaging operator" and "weighted geometric operator" for LBCFSSs with their basic properties, called:

(a) LBCFSWA operator.

(b) LBCFSWG operator.

3. We constructed the procedure of the LBCFS-MABAC model and the LBCFS-MADM model based on the anticipated operators.

4. We resolved some numerical examples based on the above two models and also derived the activity of the comparative analysis between proposed and existing ranking values.

The technique of LBCFSS is very effective and dominant, but in numerous cases, it does not work effectively. For instance, when an expert provides information in the form of positive and negative truth and falsity functions with linguistic scales, the existing techniques fail to cope with these types of data. For this reason, we need to develop the model of linguistic bipolar complex intuitionistic fuzzy soft sets and their extensions. In the future, we will work on the structure of LBCFSS and its extensions by proposing techniques of operators, measures, and methods based on Frank norms, Einstein norms, Dombi norms, Hamacher norms, and algebraic norms. We will also discuss their application in neuroscience, economics, statistics, data science, and green hydrogen to enhance the value of the invented models.

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## Conflict of interest

The authors declare no competing financial interest.

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## Appendix A

Assume that $m=1$ with the value of $\rho_1=1$, we have

$$\mathrm{LBCFSWA}(\curlywedge_{\sigma_{11}},\ \curlywedge_{\sigma_{12}},\ \ldots,\ \curlywedge_{\sigma_{mn}})=\bigoplus_{\rtimes=1}^{m}\eta_{\rtimes}\curlywedge_{\sigma_{\rtimes\ltimes}}$$

$$=\begin{pmatrix} \lfloor_{\llcorner}\left(1-\prod_{\rtimes=1}^{m}\left(\prod_{\ltimes=1}^{n}\left(1-\frac{\curlywedge^{+}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right)+\top_{\llcorner}\left(1-\prod_{\rtimes=1}^{m}\left(\prod_{\ltimes=1}^{n}\left(1-\frac{\Upsilon^{+}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right), \\ \lfloor_{\llcorner}\left(-\prod_{\rtimes=1}^{m}\left(\prod_{\ltimes=1}^{n}\left(\frac{\curlywedge^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right)+\top_{\llcorner}\left(-\prod_{\rtimes=1}^{m}\left(\prod_{\ltimes=1}^{n}\left(\frac{\Upsilon^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right) \end{pmatrix}$$

$$=\begin{pmatrix} \lfloor_{\llcorner}\left(1-\prod_{\rtimes=1}^{m}\left(\prod_{\ltimes=1}^{1}\left(1-\frac{\curlywedge^{+}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right)+\top_{\llcorner}\left(1-\prod_{\rtimes=1}^{m}\left(\prod_{\ltimes=1}^{1}\left(1-\frac{\Upsilon^{+}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right), \\ \lfloor_{\llcorner}\left(-\prod_{\rtimes=1}^{m}\left(\prod_{\ltimes=1}^{1}\left(\frac{\curlywedge^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right)+\top_{\llcorner}\left(-\prod_{\rtimes=1}^{m}\left(\prod_{\ltimes=1}^{1}\left(\frac{\Upsilon^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right) \end{pmatrix}.$$

Again, for $n=1$ and $\eta_1=1$ and hence,

$$\mathrm{LBCFSWA}(\curlywedge_{\sigma_{11}},\ \curlywedge_{\sigma_{12}},\ \ldots,\ \curlywedge_{\sigma_{mn}})=\bigoplus_{\ltimes=1}^{n}\rho_{\ltimes 1}(\curlywedge_{\sigma_{\ltimes 1}})$$

$$=\begin{pmatrix} \lfloor_{\llcorner}\left(1-\prod_{\ltimes=1}^{n}\left(1-\frac{\curlywedge^{+}_{\ltimes 1}}{\llcorner}\right)^{\rho_{\ltimes}}\right)+\top_{\llcorner}\left(1-\prod_{\ltimes=1}^{n}\left(1-\frac{\Upsilon^{+}_{\ltimes 1}}{\llcorner}\right)^{\rho_{\ltimes}}\right), \\ \lfloor_{\llcorner}\left(-\prod_{\ltimes=1}^{n}\left(\frac{\curlywedge^{-}_{\ltimes 1}}{\llcorner}\right)^{\rho_{\ltimes}}\right)+\top_{\llcorner}\left(-\prod_{\ltimes=1}^{n}\left(\frac{\Upsilon^{-}_{\ltimes 1}}{\llcorner}\right)^{\rho_{\ltimes}}\right) \end{pmatrix}$$

$$=\begin{pmatrix} \lfloor_{\llcorner}\left(1-\prod_{\rtimes=1}^{1}\left(\prod_{\ltimes=1}^{n}\left(1-\frac{\curlywedge^{+}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right)+\top_{\llcorner}\left(1-\prod_{\rtimes=1}^{1}\left(\prod_{\ltimes=1}^{n}\left(1-\frac{\Upsilon^{+}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right), \\ \lfloor_{\llcorner}\left(-\prod_{\rtimes=1}^{1}\left(\prod_{\ltimes=1}^{n}\left(\frac{\curlywedge^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right)+\top_{\llcorner}\left(-\prod_{\rtimes=1}^{1}\left(\prod_{\ltimes=1}^{n}\left(\frac{\Upsilon^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right) \end{pmatrix}.$$

This result is true for $m=1$ and $n=1$ let the given equation be held for $m=\ddot{I}\mathbb{Y}_1+1,\ n=\ddot{I}\mathbb{Y}_2+1$ and $m=\ddot{I}\mathbb{Y}_1,\ n=\ddot{I}\mathbb{Y}_2+1$. Then it follows that.

$$\bigoplus_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1}\eta_{\rtimes}\left(\bigoplus_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2}\rho_{\ltimes}\curlywedge_{\sigma_{\rtimes\ltimes}}\right)$$

$$=\begin{pmatrix} \lfloor_{\llcorner}\left(1-\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1}\left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2}\left(1-\frac{\curlywedge^{+}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right)+\top_{\llcorner}\left(1-\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1}\left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2}\left(1-\frac{\Upsilon^{+}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right), \\ \lfloor_{\llcorner}\left(-\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1}\left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2}\left(\frac{\curlywedge^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right)+\top_{\llcorner}\left(-\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1}\left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2}\left(\frac{\Upsilon^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right) \end{pmatrix}$$

$$\overset{\ddot{I}\mathbb{Y}_1}{\underset{\rtimes=1}{\oplus}} \eta_{\rtimes} \left( \overset{\ddot{I}\mathbb{Y}_2+1}{\underset{\ltimes=1}{\oplus}} \rho_{\ltimes} \curlywedge_{\sigma_{\rtimes\ltimes}} \right)$$

$$= \left( \begin{array}{l} \lfloor_{\llcorner} \left(1-\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1} \left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2+1} \left(1-\frac{\curlywedge^{+}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right) + \top_{\llcorner} \left(1-\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1} \left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2+1} \left(1-\frac{\curlyvee^{+}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right), \\ \lfloor_{\llcorner} \left(-\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1} \left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2+1} \left(\frac{\curlywedge^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right) + \top_{\llcorner} \left(-\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1} \left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2+1} \left(\frac{\curlyvee^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right) \end{array} \right).$$

Now for $m = \ddot{I}\mathbb{Y}_1 + 1,\ n = \ddot{I}\mathbb{Y}_2 + 1$, we obtained

$$\left( \begin{array}{l} \overset{\ddot{I}\mathbb{Y}_1+1}{\underset{\rtimes=1}{\oplus}} \eta_{\rtimes} \left( \overset{\ddot{I}\mathbb{Y}_2}{\underset{\ltimes=1}{\oplus}} \rho_{\ltimes} \curlywedge_{\sigma_{\rtimes\ltimes}} \right) \\ = \overset{\ddot{I}\mathbb{Y}_1+1}{\underset{\rtimes=1}{\oplus}} \eta_{\rtimes} \left( \overset{\ddot{I}\mathbb{Y}_2}{\underset{\ltimes=1}{\oplus}} \rho_{\ltimes} \curlywedge_{\sigma_{\rtimes\ltimes}} \oplus \rho_{\ddot{I}\mathbb{Y}_2+1} \curlywedge_{\sigma_{\rtimes(\ddot{I}\mathbb{Y}_2+1)}} \right) \\ = \overset{\ddot{I}\mathbb{Y}_1+1}{\underset{\rtimes=1}{\oplus}} \overset{\ddot{I}\mathbb{Y}_2}{\underset{\ltimes=1}{\oplus}} \eta_{\rtimes} \rho_{\ltimes} \curlywedge_{\sigma_{\rtimes\ltimes}} \overset{\ddot{I}\mathbb{Y}_1+1}{\underset{\rtimes=1}{\oplus}} \eta_{\rtimes} \rho_{\ddot{I}\mathbb{Y}_2+1} \curlywedge_{\sigma_{(\ddot{I}\mathbb{Y}_2+1)\rtimes}} \end{array} \right)$$

$$= \left( \begin{array}{l} \lfloor_{\llcorner} \left(1-\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1} \left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2} \left(1-\frac{\curlywedge^{+}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}} \oplus 1-\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1} \left(\left(1-\frac{\curlywedge^{+}_{(\ddot{I}\mathbb{Y}_2+1)\rtimes}}{\llcorner}\right)^{\rho_{\ddot{I}\mathbb{Y}_2+1}}\right)^{\eta_{\rtimes}}\right), \\ +\top_{\llcorner} \left(-\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1} \left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2} \left(1-\frac{\curlyvee^{+}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}} \oplus 1-\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1} \left(\left(1-\frac{\curlyvee^{+}_{(\ddot{I}\mathbb{Y}_2+1)\rtimes}}{\llcorner}\right)^{\rho_{\ddot{I}\mathbb{Y}_2+1}}\right)^{\eta_{\rtimes}}\right) \\ \lfloor_{\llcorner} \left(-\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1} \left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2} \left|\frac{\curlywedge^{-}_{\rtimes\ltimes}}{\llcorner}\right|^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}} \oplus -\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1} \left(\left|\frac{\curlywedge^{-}_{(\ddot{I}\mathbb{Y}_2+1)\rtimes}}{\llcorner}\right|^{\rho_{\ddot{I}\mathbb{Y}_2+1}}\right)^{\eta_{\rtimes}}\right) \\ +\top_{\llcorner} \left(-\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1} \left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2} \left|\frac{\curlyvee^{-}_{\rtimes\ltimes}}{\llcorner}\right|^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}} \oplus -\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1} \left(\left|\frac{\curlyvee^{-}_{(\ddot{I}\mathbb{Y}_2+1)\rtimes}}{\llcorner}\right|^{\rho_{\ddot{I}\mathbb{Y}_2+1}}\right)^{\eta_{\rtimes}}\right) \end{array} \right)$$

$$= \left( \begin{array}{l} \lfloor_{\llcorner} \left(1-\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1} \left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2+1} \left(1-\frac{\curlywedge^{+}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right) + \top_{\llcorner} \left(1-\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1} \left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2+1} \left(1-\frac{\curlyvee^{+}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right), \\ \lfloor_{\llcorner} \left(-\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1} \left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2+1} \left(\frac{\curlywedge^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right) + \top_{\llcorner} \left(-\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1} \left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2+1} \left(\frac{\curlyvee^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right) \end{array} \right).$$

The proposed model is hold for $m = \ddot{I}\mathbb{Y}_1 + 1,\ n = \ddot{I}\mathbb{Y}_2 + 1$.

## Appendix B

Since $⅄_{\sigma_{\rtimes\ltimes}} = ⅄_{\sigma} = \left(\lfloor_{\varphi^+},\ \lfloor_{\psi^+}\right) = \left(\lfloor_{⅄^+ + \top \Upsilon^+},\ \lfloor_{⅄^- + \top \Upsilon^-}\right)$ for all possible value of $\rtimes\ltimes$, then

$$\text{LBCFSWA}\left(⅄_{\sigma_{11}},\ ⅄_{\sigma_{12}},\ \ldots,\ ⅄_{\sigma_{n1}}\right)$$

$$= \begin{pmatrix} \lfloor_{\llcorner\left(1-\prod_{\rtimes=1}^{m}\left(\prod_{\ltimes=1}^{n}\left(1-\frac{⅄^+}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right)+\top\llcorner\left(1-\prod_{\rtimes=1}^{m}\left(\prod_{\ltimes=1}^{n}\left(1-\frac{\Upsilon^+}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right)}, \\ \lfloor_{\llcorner\left(-\prod_{\rtimes=1}^{m}\left(\prod_{\ltimes=1}^{n}\left(\frac{⅄^-}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right)+\top\llcorner\left(-\prod_{\rtimes=1}^{m}\left(\prod_{\ltimes=1}^{n}\left(\frac{\Upsilon^-}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \lfloor_{\llcorner\left(1-\left(\left(1-\frac{⅄^+}{\llcorner}\right)^{\sum_{\ltimes=1}^{n}\rho_{\ltimes}}\right)^{\sum_{\rtimes=1}^{m}\eta_{\rtimes}}\right)+\top\llcorner\left(1-\left(\left(1-\frac{\Upsilon^+}{\llcorner}\right)^{\sum_{\ltimes=1}^{n}\rho_{\ltimes}}\right)^{\sum_{\rtimes=1}^{m}\eta_{\rtimes}}\right)}, \\ \lfloor_{\llcorner\left(-\left(\left|\frac{⅄^-}{\llcorner}\right|^{\sum_{\ltimes=1}^{n}\rho_{\ltimes}}\right)^{\sum_{\rtimes=1}^{m}\eta_{\rtimes}}\right)+\top\llcorner\left(-\left|\frac{\Upsilon^-}{\llcorner}\right|^{\sum_{\ltimes=1}^{n}\rho_{\ltimes}}\right)^{\sum_{\rtimes=1}^{m}\eta_{\rtimes}}} \end{pmatrix}$$

$$= \begin{pmatrix} \lfloor_{\llcorner\left(1-\left(1-\frac{⅄^+}{\llcorner}\right)\right)+\top\llcorner\left(1-\left(1-\frac{\Upsilon^+}{\llcorner}\right)\right)}, \\ \lfloor_{(⅄^-)+\top(\Upsilon^-)} \end{pmatrix} = \left(\lfloor_{⅄^+ + \top\Upsilon^+}, \lfloor_{⅄^- + \top(\Upsilon^-)}\right) = \left(\lfloor_{\varphi^+},\ \lfloor_{\psi^-}\right).$$

## Appendix C

Assume that $n = 1$ and $\rho_1 = 1$, thus, we have

$$\text{LBCFSWA}(⅄_{\sigma_{11}}, ⅄_{\sigma_{12}}, \ldots, ⅄_{\sigma_{mn}}) = \bigoplus_{\rtimes=1}^{m} ⅄_{\sigma_{\rtimes\ltimes}}^{\rho_\ltimes}$$

$$= \left( \begin{array}{l} \llcorner \prod_{\rtimes=1}^{m} (⅄^{+}_{\rtimes 1})^{\eta_\rtimes} + \top \prod_{\rtimes=1}^{m} (\Upsilon^{+}_{\rtimes 1})^{\eta_\rtimes}, \\ \llcorner \left(-1+\prod_{\rtimes=1}^{m}\left(1+\frac{⅄^{-}_{\rtimes 1}}{\llcorner}\right)^{\eta_\rtimes}\right) + \top\llcorner \left(-1+\prod_{\rtimes=1}^{m}\left(1+\frac{\Upsilon^{-}_{\rtimes 1}}{\llcorner}\right)^{\eta_\rtimes}\right) \end{array}\right)$$

$$= \left( \begin{array}{l} \llcorner \prod_{\rtimes=1}^{m}\left(\prod_{\ltimes=1}^{1}\left|\frac{⅄^{+}_{\rtimes\ltimes}}{\llcorner}\right|^{\rho_\ltimes}\right)^{\eta_\rtimes} + \top\llcorner \prod_{\rtimes=1}^{m}\left(\prod_{\ltimes=1}^{1}\left|\frac{\Upsilon^{+}_{\rtimes\ltimes}}{\llcorner}\right|^{\rho_\ltimes}\right)^{\eta_\rtimes}, \\ \llcorner \left(-1+\prod_{\rtimes=1}^{m}\left(\prod_{\ltimes=1}^{1}\left(1+\frac{⅄^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_\ltimes}\right)^{\eta_\rtimes}\right) + \top\llcorner \left(-1+\prod_{\rtimes=1}^{m}\left(\prod_{\ltimes=1}^{1}\left(1+\frac{\Upsilon^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_\ltimes}\right)^{\eta_\rtimes}\right) \end{array}\right)$$

$$= \left( \begin{array}{l} \llcorner \prod_{\rtimes=1}^{1}\left(\prod_{\ltimes=1}^{n}\left|\frac{⅄^{+}_{\rtimes\ltimes}}{\llcorner}\right|^{\rho_\ltimes}\right)^{\eta_\rtimes} + \top\llcorner \left(\prod_{\rtimes=1}^{1}\left(\prod_{\ltimes=1}^{n}\left|\frac{\Upsilon^{+}_{\rtimes\ltimes}}{\llcorner}\right|^{\rho_\ltimes}\right)^{\eta_\rtimes}\right), \\ \llcorner \left(-1+\prod_{\rtimes=1}^{1}\left(\prod_{\ltimes=1}^{n}\left(1+\frac{⅄^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_\ltimes}\right)^{\eta_\rtimes}\right) + \top\llcorner \left(-1+\prod_{\rtimes=1}^{1}\left(\prod_{\ltimes=1}^{n}\left(1+\frac{\Upsilon^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_\ltimes}\right)^{\eta_\rtimes}\right) \end{array}\right).$$

For $n = 1$ and $\eta_{\rtimes=1}$ our theory is ok. Now for $m = \ddot{I}\mathbb{Y}_1 + 1$ and $n = \ddot{I}\mathbb{Y}_2 + 1$, we gained

$$\bigotimes_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1}\left(\bigotimes_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2+1} ⅄_{\sigma_{\rtimes\ltimes}}^{\rho_\ltimes}\right)^{\eta_\rtimes} = \bigotimes_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1}\left(\bigotimes_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2} ⅄_{\sigma_{\rtimes\ltimes}}^{\rho_\ltimes} \otimes ⅄_{\sigma_{(\ddot{I}\mathbb{Y}_2+1)\rtimes}}^{\rho_{\ddot{I}\mathbb{Y}_2+1}}\right)^{\eta_\rtimes}$$

$$= \bigotimes_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1}\left(\bigotimes_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2} ⅄_{\sigma_{\rtimes\ltimes}}^{\rho_\ltimes}\right) \bigotimes_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1}\left(⅄_{\sigma_{\ddot{I}\mathbb{Y}_2+1\rtimes}}^{\rho_{\ddot{I}\mathbb{Y}_2+1}}\right)$$

$$= \left( \begin{array}{l} \llcorner \left(\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1}\left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2}\left|\frac{⅄^{+}_{\rtimes\ltimes}}{\llcorner}\right|^{\rho_\ltimes}\right)^{\eta_\rtimes} + \top\llcorner \prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1}\left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2}\left|\frac{\Upsilon^{+}_{\rtimes\ltimes}}{\llcorner}\right|^{\rho_\ltimes}\right)^{\eta_\rtimes}\right) \\ \otimes \\ \llcorner \left(\left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_1+1}\left(\frac{⅄^{+}_{(\ddot{I}\mathbb{Y}_2+1)\rtimes}}{\llcorner}\right)^{\rho(\ddot{I}\mathbb{Y}_2+1)}\right)^{\eta_\rtimes} + \top\llcorner \left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_1+1}\left(\frac{⅄^{+}_{(\ddot{I}\mathbb{Y}_2+1)\rtimes}}{\llcorner}\right)^{\rho(\ddot{I}\mathbb{Y}_2+1)}\right)^{\eta_\rtimes}\right), \\ \llcorner \left(-1+\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1}\left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2}\left(1+\frac{⅄^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_\ltimes}\right)^{\eta_\rtimes}\right) + \top\llcorner \left(-1+\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1}\left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2}\left(1+\frac{\Upsilon^{-}_{\rtimes\ltimes}}{\llcorner}\right)^{\rho_\ltimes}\right)^{\eta_\rtimes}\right) \\ \otimes \\ \llcorner \left(\left(-1+\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_1+1}\frac{\left(⅄^{-}_{(\ddot{I}\mathbb{Y}_2+1)\rtimes}\right)^{\rho(\ddot{I}\mathbb{Y}_2+1)}}{\llcorner}\right)^{\eta_\rtimes} + \top\llcorner \left(-1+\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_1+1}\frac{\left(\Upsilon^{-}_{(\ddot{I}\mathbb{Y}_2+1)\rtimes}\right)^{\rho(\ddot{I}\mathbb{Y}_2+1)}}{\llcorner}\right)^{\eta_\rtimes}\right) \end{array}\right)$$

$$\text{LBCFSWG}\left(\curlywedge_{\sigma_{11}},\ \curlywedge_{\sigma_{12}},\ \ldots,\ \curlywedge_{\sigma_{mn}}\right) = \bigotimes_{\ltimes=1}^{n} \curlywedge_{\sigma_{\rtimes\ltimes}}^{\rho_{\ltimes}}$$

$$= \left( \begin{array}{l} \Big\lfloor_{\llcorner} \prod_{\ltimes=1}^{n}\left(\frac{\curlywedge_{\rtimes\ltimes}^{+}}{\llcorner}\right)^{\rho_{\ltimes}} + \top_{\llcorner}\left(\prod_{\ltimes=1}^{n}\left(\frac{\curlyvee_{\rtimes\ltimes}^{+}}{\llcorner}\right)^{\rho_{\ltimes}}\right), \\ \Big\lfloor_{\llcorner}\left(-1+\prod_{\ltimes=1}^{n}\left(\left(1+\frac{\curlywedge_{\rtimes\ltimes}^{-}}{\llcorner}\right)^{\rho_{\ltimes}}\right)\right) + \top_{\llcorner}\left(-1+\prod_{\ltimes=1}^{n}\left(\left(1+\frac{\curlyvee_{\rtimes\ltimes}^{-}}{\llcorner}\right)^{\rho_{\ltimes}}\right)\right) \end{array} \right)$$

$$= \left( \begin{array}{l} \Big\lfloor_{\llcorner} \prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1}\left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2}\left(\frac{\curlywedge_{\rtimes\ltimes}^{+}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}} + \top_{\llcorner}\prod_{\rtimes=1}^{\ddot{I}\mathbb{Y}_1+1}\left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2}\left(\frac{\curlyvee_{\rtimes\ltimes}^{+}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}, \\ \Big\lfloor_{\llcorner}\left(-1+\prod_{\rtimes=1}^{\ddot{i}\mathbb{Y}_1+1}\left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2}\left(\frac{\curlywedge_{\rtimes\ltimes}^{-}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}} + \top_{\llcorner}\left(-1+\prod_{\rtimes=1}^{\ddot{i}\mathbb{Y}_1+1}\left(\prod_{\ltimes=1}^{\ddot{I}\mathbb{Y}_2}\left(\frac{\curlyvee_{\rtimes\ltimes}^{-}}{\llcorner}\right)^{\rho_{\ltimes}}\right)^{\eta_{\rtimes}}\right)\right) \end{array} \right).$$

The invented model is hold for $m = \ddot{I}\mathbb{Y}_1 + 1$ and $n = \ddot{I}\mathbb{Y}_2 + 1$.