Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Mond-Weir Duality and Optimality Conditions for Nonsmooth Bilevel Optimization Under Uncertainty

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Received: 28 July 202; **Revised:** 22 September 2025; **Accepted:** 10 October 2025

Abstract: In this paper, we study a class of nondifferentiable bilevel optimization problems in which uncertainty is incorporated through both the upper- and lower-level constraints. By utilizing an optimal value reformulation, we reduce the original hierarchical model to an equivalent single-level nonsmooth optimization problem. Under the assumptions that the objective function is ∂_c-pseudoconvex and the constraints are ∂_c-quasiconvex, both characterized using Clarke subdifferentials, we derive sufficient optimality conditions for the reformulated problem. Moreover, we develop a Mond-Weir-type dual corresponding to the original bilevel model and derive several duality results under the same generalized convexity framework. To demonstrate the practical relevance of our theoretical contributions, we provide numerical examples of nonsmooth bilevel optimization problems in which uncertainty affects both the upper-level and lower-level constraints.

Keywords: robust optimization, bilevel programming, sufficient optimality conditions, duality, generalized convexity

MSC: 90C17, 90C26, 90C30, 90C46

1. Introduction

Many real-world optimization problems incorporate uncertainties; they might arise from measurement or manufacturing errors, incomplete information, and various fluctuations or disturbances. Due to that, there has been a high interest in taking up optimization problems under which uncertain data are presented. Robust optimization [1, 2] is one of the most effective ways to approach such a kind of difficulty; it seeks for solutions that can survive the worst-case scenarios caused by data uncertainty. Over the years, robust optimization has attracted much interest from researchers exploring theoretical problems and practical applications as seen in studies such as [3–12]. Recent contributions have significantly advanced the study of optimization under uncertainty and robustness from various perspectives. Thuy and Su [8] investigated robust optimality conditions and duality in nonsmooth multiobjective fractional semi-infinite programming problems under data uncertainty. Hung et al. [9] analyzed robust optimization problems involving intersections of closed sets, deriving corresponding optimality and duality results. Beck et al. [10] provided a comprehensive survey on bilevel

DOI: https://doi.org/10.37256/cm.7120268038

optimization under uncertainty, highlighting key challenges and developments in the field. Saini et al. [11] examined robust bilevel programming problems, establishing both optimality conditions and duality results. Similarly, Gadhi and Ohda [12] proposed sufficient optimality conditions for robust multiobjective problems, further enriching the theoretical foundation of robust optimization.

Bilevel programming is a prominent topic in contemporary optimization, driven by extensive applications across finance, economics, chemistry, and logistics. It models hierarchical decision making with two tiers: an upper-level leader problem and a lower-level follower problem, where the upper-level constraints depend on the solution set of the lower-level task. Over time, researchers have shown growing enthusiasm for bilevel programming [13–21]. Several recent studies have contributed significantly to the development of bilevel optimization theory and its applications. Chuong [16] analyzed nonsmooth multiobjective bilevel problems and derived optimality conditions, while Dempe [17] provided a comprehensive account of bilevel optimization, covering theoretical foundations, algorithms, and applications. Gadhi and Ohda [18, 19] addressed bilevel optimization by establishing necessary optimality conditions through approximation and applying tangential subdifferentials. Idrissi et al. [20] further advanced this area by employing directional upper semi-regular convexificators to derive optimality results. Dardour et al. [21] investigated primal and dual second-order necessary conditions for bilevel programming, offering a deeper understanding of higher-order analysis.

For each index $\mathfrak{r} \in \mathscr{R} = \{1, 2, \ldots, p\}$ and $\mathfrak{s} \in \mathscr{S} = \{1, 2, \ldots, q\}$, we consider nonempty, convex, and compact sets $\Omega_{\mathfrak{r}} \subseteq \mathbb{R}^{m_{\mathfrak{r}}}$ and $\Lambda_{\mathfrak{s}} \subseteq \mathbb{R}^{m_{\mathfrak{s}}}$, where $m_{\mathfrak{r}}$ and $m_{\mathfrak{s}}$ are positive integers representing the dimensions of the respective Euclidean spaces.

Our investigation focuses on a bilevel optimization problem $(\mathscr{BP})$ of the following form:

$$(\mathscr{BP}): \quad \begin{cases} \min\limits_{\mathfrak{z},\,\mathfrak{k}} & \Gamma(\mathfrak{z}, \mathfrak{k}) \\ \text{subject to} & \mathscr{T}_{\mathfrak{r}}(\mathfrak{z}, \xi_{\mathfrak{r}}) \leq 0, \quad \forall\, \mathfrak{r} \in \mathscr{R}, \ \mathfrak{k} \in F_0(\mathfrak{z}) \end{cases}$$

where $\xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}, \ \mathfrak{r} \in \mathscr{R} = \{1, 2, \ldots, p\}$ are uncertain parameters. For each $\mathfrak{z} \in \mathbb{R}^{n_1}$, the parametric optimization problem $(\mathscr{BP}_{\mathfrak{z}})$ admits a set of solutions, denoted by $F_0(\mathfrak{z})$,

$$(\mathscr{BP}_{\mathfrak{z}}): \quad \begin{cases} \min\limits_{\mathfrak{k}} & \Upsilon(\mathfrak{z}, \mathfrak{k}) \\ \text{subject to} & \zeta_{\mathfrak{s}}\big((\mathfrak{z}, \mathfrak{k}), \rho_{\mathfrak{s}}\big) \leq 0, \ \ \forall\, \mathfrak{s} \in \mathscr{S}, \end{cases}$$

where $\rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}, \ \mathfrak{s} \in \mathscr{S} = \{1, 2, \ldots, q\}$ are uncertain parameters, $\Gamma, \ \Upsilon \colon \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \longrightarrow \mathbb{R}, \ \ \mathscr{T}_{\mathfrak{r}} \colon \mathbb{R}^{n_1} \times \Omega_{\mathfrak{r}} \longrightarrow \mathbb{R}, \ \mathfrak{r} \in \mathscr{R}$ and $\zeta_{\mathfrak{s}} \colon \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \Lambda_{\mathfrak{s}} \longrightarrow \mathbb{R}, \ \mathfrak{s} \in \mathscr{S}$ are given functions, where p, q, n_1 and n_2 are integers. Note that uncertainty affects both upper- and lower-level constraints.

In this case, a robust strategy will be employed, which involves solving the bilevel optimization problem $(\mathscr{BP})$ through its robust counterpart $(\mathscr{RCBP})$, defined as follows:

$$(\mathscr{RCBP}): \quad \begin{cases} \min\limits_{\mathfrak{z},\,\mathfrak{k}} & \Gamma(\mathfrak{z}, \mathfrak{k}) \\ \text{subject to} & \mathscr{T}_{\mathfrak{r}}(\mathfrak{z}, \xi_{\mathfrak{r}}) \leq 0, \quad \forall\, \xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}} \ \forall\, \mathfrak{r} \in \mathscr{R}, \ \mathfrak{k} \in F(\mathfrak{z}), \end{cases}$$

where, for each $\mathfrak{z} \in \mathbb{R}^{n_1}$, the parametric optimization problem $(\mathscr{RCBP}_{\mathfrak{z}})$ admits a set of solutions, denoted by $F(\mathfrak{z})$,

$$(\mathscr{RCBP}_{\mathfrak{z}}): \quad \begin{cases} \min\limits_{\mathfrak{k}} \quad \Upsilon(\mathfrak{z}, \mathfrak{k}) \\ \\ \text{subject to} \qquad \zeta_{\mathfrak{s}}\big((\mathfrak{z}, \mathfrak{k}), \rho_{\mathfrak{s}}\big) \leq 0, \ \ \forall\, \rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}, \ \ \forall\, \mathfrak{s} \in \mathscr{S}. \end{cases}$$

It is important to highlight that the robust counterpart is designed to handle the worst-case scenario caused by uncertainty, without explicitly depending on the uncertain parameters. Let

$$\mathscr{G} := \Big\{ (\mathfrak{z}, \mathfrak{k}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \colon \ \mathscr{T}_{\mathfrak{r}}(\mathfrak{z}, \xi_{\mathfrak{r}}) \leq 0, \ \ \forall\, \xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}} \ \ \forall\, \mathfrak{r} \in \mathscr{R}, \ \ \mathfrak{k} \in F(\mathfrak{z}) \Big\} \tag{1}$$

is the feasible set of $(\mathscr{RCBP})$.

A pair $(\tilde{\mathfrak{z}}, \tilde{\mathfrak{k}})$ is called a robust feasible solution of $(\mathscr{BP})$, if it satisfies the feasibility conditions of its robust counterpart $(\mathscr{RCBP})$. Furthermore, a vector $(\tilde{\mathfrak{z}}, \tilde{\mathfrak{k}}) \in \mathscr{G}$ is called a robust optimal solution of $(\mathscr{BP})$ if

$$\Gamma(\mathfrak{z}, \mathfrak{k}) - \Gamma(\tilde{\mathfrak{z}}, \tilde{\mathfrak{k}}) \geq 0 \quad \forall\, (\mathfrak{z}, \mathfrak{k}) \in \mathscr{G}. \tag{2}$$

The study of robust bilevel optimization has gained significant attention in recent years due to its ability to address hierarchical decision-making under uncertainty. Gadhi and Ohda [22] examined this challenge and derived necessary optimality conditions by reformulating the bilevel model into a nonsmooth single-level program. They utilized Clarke subdifferentials and introduced appropriate constraint qualifications, providing a solid foundation for analyzing robust bilevel problems. Following this, Pandey et al. [23] established stabilized sufficient optimality conditions and Wolfe-type duality results under the assumptions of convexity. Their work extended the theoretical framework beyond just necessary conditions. However, the reliance on convexity significantly limits the applicability of these results, as many practical bilevel problems involving uncertainty do not meet such stringent structural assumptions.

To address this gap, the present work develops sufficient conditions for optimality together with Mond-Weir-type duality results for robust bilevel optimization problems based on the weaker assumptions of ∂_c-pseudoconvexity and ∂_c-quasiconvexity. By relaxing the requirement for convexity, our results generalize existing research and broaden the applicability of robust bilevel optimization theory to a wider range of uncertain and nonsmooth models.

This paper aims to establish sufficient optimality and duality results for the bilevel optimization problem $(\mathscr{BP})$ by employing Clarke subdifferentials within the framework of robust optimization. To achieve this, we first reformulate the bilevel problem into an equivalent single-level problem using the optimal value function of the lower-level problem, thereby preserving the structure and solution characteristics of the original model. Building upon the necessary optimality conditions provided in [22], and utilizing the notions of ∂_c-pseudoconvexity and ∂_c-quasiconvexity, we derive sufficient optimality conditions for $(\mathscr{BP})$. Additionally, we develop a Mond-Weir-type dual problem based on the primal formulation, incorporating Clarke subdifferentials of the associated nonsmooth functions. Several duality theorems are then established under generalized convexity assumptions defined via Clarke subdifferentiability. To make the effectiveness of the results concrete, a representative example is presented. As far as we are aware, no prior work has provided sufficient optimality results alongside Mond-Weir-type duality for bilevel optimization problems without requiring concavity assumptions, particularly in the presence of uncertainty at both hierarchical levels using ∂_c-pseudoconvex and ∂_c-quasiconvex assumptions.

The structure of this paper is as follows. Section 2 establishes the foundational definitions and preliminary results required for our analysis. Section 3 is dedicated to deriving sufficient optimality conditions for robust solutions. Following this, Section 4 proposes a Mond-Weir-type dual model and establishes duality theorems under assumptions of generalized convexity. We conclude with a summary of findings in the final section.

2. Preliminaries

The following section establishes the fundamental mathematical framework for this analysis. We define $\mathbb{R}^n$ as n-dimensional Euclidean space. The inner product in this space is denoted by $\langle\cdot, \cdot\rangle$, the closed line segment connecting two points $a, b \in \mathbb{R}^n$ is written as $[a, b] = \{\mu a + (1-\mu)b: 0 \leq \mu \leq 1\}$, while the open line segment from $a \in \mathbb{R}^n$ to $b \in \mathbb{R}^n$ is denoted by $(a, b) = \{\mu a + (1-\mu)b: 0 < \mu < 1\}$.

For a nonempty subset $\mathfrak{C} \subseteq \mathbb{R}^n$, we denote the topological boundary, topological interior, convex hull, closure, convex cone (including the origin), and cone of $\mathfrak{C}$ by $bd\,\mathfrak{C}$, $int\,\mathfrak{C}$, $co\,\mathfrak{C}$, $cl\,\mathfrak{C}$, $pos\,\mathfrak{C}$, and $cone\,\mathfrak{C}$, respectively.

The negative polar cone and the strictly negative polar cone of $\mathfrak{C}$ are defined, as follows:

(i) $\mathfrak{C}^\circ := \left\{\rho \in \mathbb{R}^n: \langle\rho, \kappa\rangle \leq 0, \ \forall\, \kappa \in \mathfrak{C}\right\}$.

(ii) $\mathfrak{C}^s := \left\{\rho \in \mathbb{R}^n: \langle\rho, \kappa\rangle < 0, \ \forall\, \kappa \in \mathfrak{C}\backslash\{0\}\right\}$.

It is straightforward to see that $\mathfrak{C}^s \subset \mathfrak{C}^\circ$ and when $\mathfrak{C}^s \neq \emptyset$, we have $cl\,\mathfrak{C}^s = \mathfrak{C}^\circ$. Moreover, by the bipolar theorem (see, for example, [24]), it follows that $\mathfrak{C}^{\circ\circ} = cl\,cone\,\mathfrak{C}$.

A function $\Gamma: \mathfrak{C} \subset \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called locally Lipschitz around $\bar{z} \in \operatorname{dom}\Gamma := \{z \in \mathbb{R}^n \mid \Gamma(z) \in \mathbb{R}\}$ if there is an open neighborhood $\mathscr{N}$ of $\bar{z}$ and $C > 0$ (called the Lipschitz constant) such that,

$$|\Gamma(z) - \Gamma(k)| \leq C\|z-k\|, \quad \forall z, k \in \mathscr{N},$$

where $\|\cdot\|$ denotes the standard Euclidean norm in $\mathbb{R}^n$.

Definition 1 [25] Let $\bar{\mathfrak{z}} \in \mathbb{R}^n$. For a locally Lipschitz $\Gamma: \mathbb{R}^n \to \mathbb{R}$, the Clarke directional derivative at $\bar{\mathfrak{z}}$ in direction $\delta \in \mathbb{R}^n$ is defined by

$$\Gamma^\circ(\bar{\mathfrak{z}}; \delta) := \limsup_{\substack{\mathfrak{z} \to \bar{\mathfrak{z}} \\ \mu \downarrow 0}} \frac{\Gamma(\mathfrak{z} + \mu\delta) - \Gamma(\mathfrak{z})}{\mu}. \tag{3}$$

The Clarke subdifferential of Γ at $\bar{\mathfrak{z}}$, denoted by $\partial_c\Gamma(\bar{\mathfrak{z}})$, is given by:

$$\partial_c\Gamma(\bar{\mathfrak{z}}) := \left\{\kappa^* \in \mathbb{R}^n: \left\langle\kappa^*, \delta\right\rangle \leq \Gamma^\circ(\bar{\mathfrak{z}}, \delta) \quad \forall\, \delta \in \mathbb{R}^n\right\}. \tag{4}$$

We recall the following properties from [25], which will be used in the subsequent analysis.

Lemma 1 [25] Let $\Gamma: \mathbb{R}^n \to \mathbb{R}$ be a function that is locally Lipschitz near a point $\bar{\mathfrak{z}} \in \mathbb{R}^n$. Then the following properties hold:

(i) The Clarke directional derivative $\Gamma^\circ(\bar{\mathfrak{z}}; \delta)$ is finite for all $\delta \in \mathbb{R}^n$, and satisfies:

- $\Gamma^\circ(\bar{\mathfrak{z}}; 0) = 0$,
- $\Gamma^\circ(\bar{\mathfrak{z}}; \delta)$ is positively homogeneous and subadditive in δ,
- $\Gamma^\circ(\bar{\mathfrak{z}}; \delta) = \max_{\kappa^* \in \partial_c\Gamma(\bar{\mathfrak{z}})}\langle\kappa^*, \delta\rangle$,
- $\partial(\Gamma^\circ(\bar{\mathfrak{z}}; \cdot))(0) = \partial_c\Gamma(\bar{\mathfrak{z}})$, where ∂ represents the subdifferential as defined in convex analysis.

(ii) The Clarke subdifferential $\partial_c\Gamma(\bar{\mathfrak{z}})$ is a nonempty, compact, and convex subset of $\mathbb{R}^n$.

(iii) If φ is also a locally Lipschitz function near $\bar{\mathfrak{z}}$, then:

$$\partial_c(\Gamma + \varphi)(\bar{\mathfrak{z}}) \subseteq \partial_c\Gamma(\bar{\mathfrak{z}}) + \partial_c\varphi(\bar{\mathfrak{z}}).$$

(iv) $\partial_c(\alpha\Gamma)(\bar{\mathfrak{z}}) = \alpha\,\partial_c\Gamma(\bar{\mathfrak{z}}),\ \forall\,\alpha \in \mathbb{R}$.

(v) If Γ is convex, then $\partial_c\Gamma(\bar{\mathfrak{z}}) = \partial\Gamma(\bar{\mathfrak{z}})$ in the usual sense of convex analysis. If Γ is continuously differentiable at $\bar{\mathfrak{z}}$, then: $\partial_c\Gamma(\bar{\mathfrak{z}}) = \{\nabla\Gamma(\bar{\mathfrak{z}})\}$.

(vi) If Γ is locally Lipschitz on $\mathbb{R}^n$, then the mapping $\mathfrak{z} \mapsto \partial_c\Gamma(\mathfrak{z})$ is upper semicontinuous set-valued function.

(vii) If Γ is locally Lipschitz on an open set containing $[x, y]$, then

$$\Gamma(x) - \Gamma(y) = \langle x^*, y - x\rangle$$

for some $a \in (x, y)$ and $x^* \in \partial_c\Gamma(a)$.

3. Sufficient optimality conditions

Let $(\mathfrak{z}, \mathfrak{k}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Define the functions:

$$\Phi_{\mathfrak{r}}(\mathfrak{z}) := \max_{\xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}} \mathscr{T}_{\mathfrak{r}}(\mathfrak{z}, \xi_{\mathfrak{r}}), \quad \text{for all } \mathfrak{r} \in \mathscr{R}, \tag{5}$$

$$\Psi_{\mathfrak{s}}(\mathfrak{z}, \mathfrak{k}) := \max_{\rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}} \zeta_{\mathfrak{s}}\big((\mathfrak{z}, \mathfrak{k}), \rho_{\mathfrak{s}}\big), \quad \text{for all } \mathfrak{s} \in \mathscr{S}. \tag{6}$$

These functions act as technical tools to handle the uncertainty present in both the upper and lower-level constraints of the bilevel optimization problem $(\mathscr{BP})$. Using these formulations, the bilevel problem $(\mathscr{RCBP})$ may be equivalently rewritten as:

$$\begin{cases} \min\limits_{\mathfrak{z},\,\mathfrak{k}} & \Gamma(\mathfrak{z}, \mathfrak{k}) \\ \text{subject to} & \Phi_{\mathfrak{r}}(\mathfrak{z}) \leq 0, \quad \forall\,\mathfrak{r} \in \mathscr{R},\ \ \mathfrak{k} \in F(\mathfrak{z}), \end{cases}$$

where, for each $\mathfrak{z} \in \mathbb{R}^{n_1}$, $F(\mathfrak{z})$ denotes the solution set of the following parametric optimization problem

$$\begin{cases} \min\limits_{\mathfrak{k}} & \Upsilon(\mathfrak{z}, \mathfrak{k}) \\ \text{subject to} & \Psi_{\mathfrak{s}}(\mathfrak{z}, \mathfrak{k}) \leq 0, \quad \forall\,\mathfrak{s} \in \mathscr{S}. \end{cases}$$

To proceed, we impose the following assumptions.

Let

$$\Delta := \{\mathfrak{z} \in \mathbb{R}^{n_1} \mid \mathscr{T}_\mathfrak{r}(\mathfrak{z}, \xi_\mathfrak{r}) \leq 0, \ \forall \xi_\mathfrak{r} \in \Omega_\mathfrak{r} \ \forall \mathfrak{r} \in \mathscr{R}\}.$$

• Assumption $(\mathscr{K})$ is said to hold for $\bar{\mathfrak{z}} \in \Delta$ if there exists an open neighborhood $\aleph_{\bar{\mathfrak{z}}}$ of $\bar{\mathfrak{z}}$ such that:

$\diamond$ $(\mathscr{K}_1)$: For each fixed $\mathfrak{z} \in \aleph_{\bar{\mathfrak{z}}}$, the function $\xi_\mathfrak{r} \in \Omega_\mathfrak{r} \longmapsto \mathscr{T}_\mathfrak{r}(\mathfrak{z}, \xi_\mathfrak{r}) \in \mathbb{R}$ is upper semicontinuous. Moreover, $\mathscr{T}_\mathfrak{r}$ is Lipschitz continuous in the first argument on $\aleph_{\bar{\mathfrak{z}}}$, that is there exists a constant $\mathscr{C}_\mathfrak{r} > 0$ such that

$$\left|\mathscr{T}_\mathfrak{r}(\mathfrak{z}_0, \xi_\mathfrak{r}) - \mathscr{T}_\mathfrak{r}(\mathfrak{z}_1, \xi_\mathfrak{r})\right| \leq \mathscr{C}_\mathfrak{r}\|\mathfrak{z}_0 - \mathfrak{z}_1\|, \quad \forall \mathfrak{z}_0, \mathfrak{z}_1 \in \aleph_{\bar{\mathfrak{z}}}, \ \forall \xi_\mathfrak{r} \in \Omega_\mathfrak{r}. \tag{7}$$

$\diamond$ $(\mathscr{K}_2)$: The multifunction

$$(\mathfrak{z}, \xi_\mathfrak{r}) \in \aleph_{\bar{\mathfrak{z}}} \times \Omega_\mathfrak{r} \rightrightarrows \partial_c \mathscr{T}_\mathfrak{r}(\cdot, \xi_\mathfrak{r})(\mathfrak{z}) \subset \mathbb{R}^{n_1}$$

is closed at every point $(\bar{\mathfrak{z}}, \bar{\xi}_\mathfrak{r})$, with $\bar{\xi}_\mathfrak{r} \in \Omega_\mathfrak{r}(\bar{\mathfrak{z}})$, where

$$\Omega_\mathfrak{r}(\bar{\mathfrak{z}}) := \left\{\xi_\mathfrak{r} \in \Omega_\mathfrak{r} \mid \mathscr{T}_\mathfrak{r}(\bar{\mathfrak{z}}, \xi_\mathfrak{r}) = \Phi_\mathfrak{r}(\bar{\mathfrak{z}})\right\}. \tag{8}$$

• Assumption $(\mathscr{V})$ is said to hold at a point $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \in \mathscr{G}$, if there exist open neighborhoods $\aleph_{\bar{\mathfrak{z}}}$ and $\aleph_{\bar{\mathfrak{k}}}$ of $\bar{\mathfrak{z}}$ and $\bar{\mathfrak{k}}$, respectively, such that:

$\diamond$ $(\mathscr{V}_1)$: For each $(\mathfrak{z}, \mathfrak{k}) \in \aleph_{\bar{\mathfrak{z}}} \times \aleph_{\bar{\mathfrak{k}}}$, the function $v \in \Lambda_\mathfrak{s} \longmapsto \zeta_\mathfrak{s}\big((\mathfrak{z}, \mathfrak{k}), v\big) \in \mathbb{R}$ is upper semicontinuous. In addition, $\zeta_\mathfrak{s}$ is Lipschitz continuous in the first argument on $\aleph_{\bar{\mathfrak{z}}} \times \aleph_{\bar{\mathfrak{k}}}$, that is there exists a constant $\mathscr{D}_\mathfrak{s} > 0$ such that

$$\left|\zeta_\mathfrak{s}\big((\mathfrak{z}_0, \mathfrak{k}_0), \rho_\mathfrak{s}\big) - \zeta_\mathfrak{s}\big((\mathfrak{z}_1, \mathfrak{k}_1), \rho_\mathfrak{s}\big)\right| \leq \mathscr{D}_\mathfrak{s}\|(\mathfrak{z}_0, \mathfrak{k}_0) - (\mathfrak{z}_1, \mathfrak{k}_1)\|$$

$$\forall\, (\mathfrak{z}_0, \mathfrak{k}_0), (\mathfrak{z}_1, \mathfrak{k}_1) \in \aleph_{\bar{\mathfrak{z}}} \times \aleph_{\bar{\mathfrak{k}}}, \ \forall\, \rho_\mathfrak{s} \in \Lambda_\mathfrak{s}. \tag{9}$$

$\diamond$ $(\mathscr{V}_2)$: The multifunction

$$\big((\mathfrak{z}, \mathfrak{k}), \rho_\mathfrak{s}\big) \in \big(\aleph_{\bar{\mathfrak{z}}} \times \aleph_{\bar{\mathfrak{k}}}\big) \times \Lambda_\mathfrak{s} \rightrightarrows \partial_c \zeta_\mathfrak{s}(\cdot, \rho_\mathfrak{s})(\mathfrak{z}, \mathfrak{k}) \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$$

is closed at every point $\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \bar{\rho_\mathfrak{s}}\big)$, where $\bar{\rho_\mathfrak{s}} \in \Lambda_\mathfrak{s}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$, and

$$\Lambda_\mathfrak{s}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) := \left\{\rho_\mathfrak{s} \in \Lambda_\mathfrak{s} \mid \zeta_\mathfrak{s}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_\mathfrak{s}\big) = \Psi_\mathfrak{s}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})\right\}. \tag{10}$$

It is important to note that under assumptions $(\mathscr{K}_1)$ and $(\mathscr{V}_1)$, together with the compactness of the sets $\Omega_\mathfrak{r}$ for all $\mathfrak{r} \in \mathscr{R}$ and $\Lambda_\mathfrak{s}$ for all $\mathfrak{s} \in \mathscr{S}$, the functions defined in (5) and (6) are well-defined and locally Lipschitz. Specifically, they are Lipschitz continuous with constants $\mathscr{C}_\mathfrak{r}$ (for each $\mathfrak{r} \in \mathscr{R}$) and $\mathscr{D}_\mathfrak{s}$ (for each $\mathfrak{s} \in \mathscr{S}$), respectively (see, for instance, [25, p.86]. Assumption $(\mathscr{K}_1)$ has been widely used in the literature for studying the subdifferential properties of supremum (or maximum) functions over compact sets. This can be seen in various references such as [26–29] as well as in the citations they include.

Outrata [30] demonstrated that the problem $(\mathscr{RCBP})$ can be equivalently reformulated as the following single-level optimization problem:

$$
(\mathscr{SP})^*: \begin{cases} \min\limits_{\mathfrak{z},\, \mathfrak{k}} & \Gamma(\mathfrak{z},\, \mathfrak{k}) \\ \text{subject to} & \Phi_{\mathfrak{r}}(\mathfrak{z}) \leq 0, \quad \forall\, \mathfrak{r} \in \mathscr{R}, \\ & \Psi_{\mathfrak{s}}(\mathfrak{z},\, \mathfrak{k}) \leq 0, \quad \forall\, \mathfrak{s} \in \mathscr{S}, \\ & \varphi(\mathfrak{z},\, \mathfrak{k}) \leq 0, \\ & (\mathfrak{z},\, \mathfrak{k}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \end{cases}
$$

where

$$
\varphi(\mathfrak{z},\, \mathfrak{k}) := \Upsilon(\mathfrak{z},\, \mathfrak{k}) - v(\mathfrak{z}),
$$

and for each $\mathfrak{z} \in \mathbb{R}^{n_1}$,

$$
v(\mathfrak{z}) = \min_{\mathfrak{k}} \left\{ \Upsilon(\mathfrak{z},\, \mathfrak{k}) \colon \Psi_{\mathfrak{s}}(\mathfrak{z},\, \mathfrak{k}) \leq 0, \quad \forall\, \mathfrak{s} \in \mathscr{S} \right\}
$$

is the optimal value function of $(\mathscr{RCBP}_{\mathfrak{z}})$.

However, due to the nondifferentiability of the value function, the optimization problem $(\mathscr{SP})^*$ is generally nonconvex and nonsmooth, even when all the functions involved are convex and continuously differentiable (see, for example, [31]). Moreover, the Mangasarian-Fromovitz Constraint Qualification typically fails to hold for $(\mathscr{SP})^*$, owing to its inherent bilevel structure (see, for instance, [32]). To address this challenge, we employ the partial calmness technique introduced by Ye and Zhu [33].

Definition 2 [34] Let $(\bar{\mathfrak{z}},\, \bar{\mathfrak{k}})$ be a local optimum of $(\mathscr{RCBP})$. The problem $(\mathscr{RCBP})$ is partially calm at $(\bar{\mathfrak{z}},\, \bar{\mathfrak{k}})$ if there exist $d > 0$ and $a^* > 0$ such that, for all $(\mathfrak{z},\, \mathfrak{k},\, z) \in \mathscr{B}_d(\bar{\mathfrak{z}},\, \bar{\mathfrak{k}},\, 0)$ satisfying

$$
\begin{cases} \Phi_{\mathfrak{r}}(\mathfrak{z}) \leq 0, & \mathfrak{r} \in \mathscr{R}, \\ \Psi_{\mathfrak{s}}(\mathfrak{z},\, \mathfrak{k}) \leq 0, & \mathfrak{s} \in \mathscr{S}, \\ \varphi(\mathfrak{z},\, \mathfrak{k}) \leq z, \end{cases}
$$

the following inequality holds: $\Gamma(\mathfrak{z},\, \mathfrak{k}) - \Gamma(\bar{\mathfrak{z}},\, \bar{\mathfrak{k}}) + a^*|z| \geq 0$.

Remark 1 Partial calmness of $(\mathscr{RCBP})$ at a local minimizer $(\bar{\mathfrak{z}},\, \bar{\mathfrak{k}})$ implies that the mapping $(\mathfrak{z},\, \mathfrak{k}) \longmapsto \varphi(\mathfrak{z},\, \mathfrak{k})$ acts as a locally exact penalty function for problem $(\mathscr{SP})^*$ at the point $(\bar{\mathfrak{z}},\, \bar{\mathfrak{k}})$; see [34, Lemma 3.1] for further details.

The idea of partial calmness is strongly connected to the notion of partial exact penalization, as highlighted by the result established in [33].

Theorem 1 [33] Suppose $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ is a local optimal solution of $(\mathscr{RCBP})$. Then, $(\mathscr{RCBP})$ is said to be partially calm at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ iff there exists $a^* > 0$ such that $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ is a local optimal solution of the following partially penalized problem:

$$(\mathscr{SP})_1^* \begin{cases} \min\limits_{\mathfrak{z}, \mathfrak{k}} & \Gamma(\mathfrak{z}, \mathfrak{k}) + a^* \varphi(\mathfrak{z}, \mathfrak{k}) \\ \text{subject to} & \Phi_{\mathfrak{r}}(\mathfrak{z}) \leq 0, \quad \mathfrak{r} \in \mathscr{R}, \\ & \Psi_{\mathfrak{s}}(\mathfrak{z}, \mathfrak{k}) \leq 0, \quad \mathfrak{s} \in \mathscr{S}. \end{cases}$$

Let $\mathbf{R}: = \{1, \ldots, p+q\}$. Consider the functions $\phi: \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and $\Theta: \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{p+q}$ defined by

$$\phi(\mathfrak{z}, \mathfrak{k}): = \Gamma(\mathfrak{z}, \mathfrak{k}) + a^* \varphi(\mathfrak{z}, \mathfrak{k})$$

and

$$\Theta_{\mathfrak{r}}(\mathfrak{z}, \mathfrak{k}): = \begin{cases} \Phi_{\mathfrak{r}}(\mathfrak{z}), & \text{if } \mathfrak{r} = 1, \ldots, p, \\ \Psi_{\mathfrak{r}-p}(\mathfrak{z}, \mathfrak{k}), & \text{if } \mathfrak{r} = p+1, \ldots, p+q. \end{cases}$$

Definition 3 [22] We say that the Extended Nonsmooth Mangasarian-Fromovitz Constraint Qualification (ENMFCQ) is satisfied at a point $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \in \mathscr{G}$ if there exists a nonzero direction $\delta \in \mathbb{R}^{n_1+n_2} \setminus \{0\}$ such that

$$\Theta_{\mathfrak{r}}^{\circ}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \delta\big) < 0, \quad \forall\, \mathfrak{r} \in \mathbf{R}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}),$$

where the active index set $\mathbf{R}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ is defined as:

$$\mathbf{R}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}): = \big\{\mathfrak{r} \in \mathbf{R}: \Theta_{\mathfrak{r}}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) = 0\big\}.$$

Gadhi and Ohda [22] have demonstrated the following result, which provides the necessary optimality conditions for the bilevel optimization problem $(\mathscr{BP})$.

Theorem 2 [22] Let $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \in \mathscr{G}$. Assume that (ENMFCQ) holds at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$, and that Γ and Υ are locally Lipschitz continuous. Also, suppose that $(\mathscr{RCBP})$ is partially calm at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$. If $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ is a local robust optimal solution of $(\mathscr{BP})$, then there exist $\mathfrak{x} > 0$, $a^* > 0$, $\mathfrak{y}_{\mathfrak{r}} \geq 0$ for all $\mathfrak{r} \in \mathscr{R}$, and $\mathfrak{f}_{\mathfrak{s}} \geq 0$ for all $\mathfrak{s} \in \mathscr{S}$ such that

$$(0, 0) \in \left\{ \begin{array}{c} \mathfrak{x}\partial_c \Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) + \mathfrak{x} a^* \partial_c \varphi(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) + \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}}\, co\Bigg(\bigcup\limits_{\xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}(\bar{\mathfrak{z}})} \partial_c \mathscr{T}_{\mathfrak{r}}(\cdot, \xi_{\mathfrak{r}})(\bar{\mathfrak{z}}) \times \{0\}\Bigg) \\ + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}}\, co\Bigg(\bigcup\limits_{\rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})} \partial_c \zeta_{\mathfrak{s}}\big(\cdot, \rho_{\mathfrak{s}}\big)(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})\Bigg) \end{array} \right\} \tag{11}$$

and

$$\mathfrak{y}_{\mathfrak{r}} \max_{\xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}} \mathscr{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}}) = 0, \quad \mathfrak{r} \in \mathscr{R}, \quad \text{and } \mathfrak{f}_{\mathfrak{s}} \max_{\rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}} \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}}\big) = 0, \quad \mathfrak{s} \in \mathscr{S}. \tag{12}$$

Before presenting the sufficient conditions for $(\mathscr{BP})$, we first introduce certain convexity concepts inspired by Pham [35].

Definition 4 We say that Γ is ∂_c-pseudoconvex on $\mathscr{G}$ at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \in \mathscr{G}$ if for all $(\mathfrak{z}, \mathfrak{k}) \in \mathscr{G}$, there exist $\vartheta \in \partial_c \Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ such that

$$\Big\langle \vartheta, \ (\mathfrak{z}, \mathfrak{k}) - (\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \Big\rangle \geq 0 \implies \Gamma(\mathfrak{z}, \mathfrak{k}) \geq \Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}).$$

Definition 5 We say that Γ is strict ∂_c-pseudoconvex on $\mathscr{G}$ at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \in \mathscr{G}$ if for all $(\mathfrak{z}, \mathfrak{k}) \in \mathscr{G}$, there exist $\vartheta \in \partial_c \Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ such that

$$\Big\langle \vartheta, \ (\mathfrak{z}, \mathfrak{k}) - (\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \Big\rangle > 0 \implies \Gamma(\mathfrak{z}, \mathfrak{k}) > \Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}).$$

Definition 6 We say that φ: $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \longrightarrow \mathbb{R}$, $\mathscr{T}_{\mathfrak{r}}$: $\mathbb{R}^{n_1} \times \Omega_{\mathfrak{r}} \longrightarrow \mathbb{R}$, $\mathfrak{r} \in \mathscr{R}$ and $\zeta_{\mathfrak{s}}$: $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \Lambda_{\mathfrak{s}} \longrightarrow \mathbb{R}$ are ∂_c-quasiconvex on $\mathscr{G}$ at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \in \mathscr{G}$ if for all $(\mathfrak{z}, \mathfrak{k}) \in \mathscr{G}$,

$$\varphi(\mathfrak{z}, \mathfrak{k}) - \varphi(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \leq 0 \implies \Big\langle \eta, \ (\mathfrak{z}, \mathfrak{k}) - (\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \Big\rangle \leq 0, \ \forall \eta \in \partial_c \varphi(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}),$$

$$\zeta_{\mathfrak{s}}\big((\mathfrak{z}, \mathfrak{k}), \rho_{\mathfrak{s}}\big) - \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}}\big) \leq 0 \implies \Big\langle \lambda_{\mathfrak{s}}, \ (\mathfrak{z}, \mathfrak{k}) - (\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \Big\rangle \leq 0, \ \forall \lambda_{\mathfrak{s}} \in \partial_c \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \ \rho_{\mathfrak{s}}\big), \forall \rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \ \forall \ \mathfrak{s} \in \mathscr{S},$$

$$\mathscr{T}_{\mathfrak{r}}(\mathfrak{z}, \xi_{\mathfrak{r}}) - \mathscr{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}}) \leq 0 \implies \Big\langle \beta_{\mathfrak{r}}, \ (\mathfrak{z}, 0) - (\bar{\mathfrak{z}}, 0) \Big\rangle \leq 0, \ \forall \beta_{\mathfrak{r}} \in \partial_c \mathscr{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}}), \forall \xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}(\bar{\mathfrak{z}}), \ \forall \ \mathfrak{r} \in \mathscr{R}.$$

We now establish sufficient optimality conditions for a feasible solution to be a robust optimal solution of the bilevel optimization problem $(\mathscr{BP})$.

Theorem 3 Let $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \in \mathscr{G}$. Assume that Γ is ∂_c-pseudoconvex at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ on $\mathscr{G}$, and that φ, $\zeta_{\mathfrak{s}}$, $\mathfrak{s} \in \mathscr{S}$ and $\mathscr{T}_{\mathfrak{r}}$, $\mathfrak{r} \in \mathscr{R}$ are ∂_c-quasiconvex at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ on $\mathscr{G}$, and that there exist $\mathfrak{x} > 0$, $a^* > 0$, $\mathfrak{y}_{\mathfrak{r}} \geq 0$, $\mathfrak{r} \in \mathscr{R}$ and $\mathfrak{f}_{\mathfrak{s}} \geq 0$, $\mathfrak{s} \in \mathscr{S}$, satisfying (11) and (12). Then, $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ is a robust optimal solution of $(\mathscr{BP})$.

Proof. Since $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \in \mathscr{G}$ satisfies (11) and (12), there exist $\mathfrak{x} > 0$, $\vartheta \in \partial_c \Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$, $a^* > 0$, $\eta \in \partial_c \varphi(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$, $\mathfrak{y}_{\mathfrak{r}} \geq 0$, $\mathfrak{r} \in \mathscr{R}$, $\bar{u}_{\mathfrak{r}i} \geq 0$, $\beta_{\mathfrak{r}i} \in \partial_c \mathscr{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i})$, $\xi_{\mathfrak{r}i} \in \Omega_{\mathfrak{r}}(\bar{\mathfrak{z}})$, $i \in I_{\mathfrak{r}} := \{1, \ldots, n_{\mathfrak{r}}\}$, $n_{\mathfrak{r}} \in \mathbb{N}$, and $\mathfrak{f}_{\mathfrak{s}} \geq 0$, $\mathfrak{s} \in \mathscr{S}$, $\bar{w}_{\mathfrak{s}j} \geq 0$, $\lambda_{\mathfrak{s}j} \in \partial_c \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}j}\big)$, $\rho_{\mathfrak{s}j} \in \Lambda_{\mathfrak{s}}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$, $j \in J_{\mathfrak{s}} := \{1, \ldots, n_{\mathfrak{s}}\}$, $n_{\mathfrak{s}} \in \mathbb{N}$, such that

$$\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i} = 1, \quad \sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j} = 1,$$

$$0 = \mathfrak{x}\,\vartheta + \mathfrak{x}a^*\eta + \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i}\, \beta_{\mathfrak{r}i} \right) + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j}\, \lambda_{\mathfrak{s}j} \right) \tag{13}$$

and

$$\mathfrak{y}_{\mathfrak{r}} \max_{\xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}} \mathscr{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}}) = 0, \quad \mathfrak{r} \in \mathscr{R}, \tag{14}$$

$$\mathfrak{f}_{\mathfrak{s}} \max_{\rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}} \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}}\big) = 0, \quad \mathfrak{s} \in \mathscr{S}. \tag{15}$$

- Since $\xi_{\mathfrak{r}i} \in \Omega_{\mathfrak{r}}(\bar{\mathfrak{z}})$,

$$\mathscr{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i}) = \max_{\xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}} \mathscr{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}}), \quad \forall\, i \in I_{\mathfrak{r}}, \ \forall\, \mathfrak{r} \in \mathscr{R}.$$

Thus, it follows by (14) that

$$\mathfrak{y}_{\mathfrak{r}} \mathscr{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i}) = 0, \forall\, i \in I_{\mathfrak{r}}, \ \forall\, \mathfrak{r} \in \mathscr{R}. \tag{16}$$

And since $\rho_{\mathfrak{s}j} \in \Lambda_{\mathfrak{s}}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$,

$$\zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}j}\big) = \max_{\rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}} \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}}\big), \quad \forall\, j \in J_{\mathfrak{s}}, \ \forall\, \mathfrak{s} \in \mathscr{S}.$$

Thus, it follows by (15) that

$$\mathfrak{f}_{\mathfrak{s}}\, \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}j}\big) = 0, \quad \forall\, j \in J_{\mathfrak{s}}, \ \forall\, \mathfrak{s} \in \mathscr{S}. \tag{17}$$

Using $(\mathfrak{z}_0, \mathfrak{k}_0) \in \mathscr{G},$ and $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \in \mathscr{G},$ we get the following relations

$$\begin{aligned}
&\mathfrak{y}_{\mathfrak{r}} \mathscr{T}_{\mathfrak{r}}(\mathfrak{z}_0, \xi_{\mathfrak{r}i}) \le \mathfrak{y}_{\mathfrak{r}} \mathscr{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i}) = 0 \ \ \forall\, i \in I_{\mathfrak{r}}, \ \forall\, \mathfrak{r} \in \mathscr{R}, \\
&\mathfrak{f}_{\mathfrak{s}}\, \zeta_{\mathfrak{s}}\big((\mathfrak{z}_0, \mathfrak{k}_0), \rho_{\mathfrak{s}j}\big) \le \mathfrak{f}_{\mathfrak{s}}\, \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}j}\big) = 0, \ \forall\, j \in J_{\mathfrak{s}}, \ \forall\, \mathfrak{s} \in \mathscr{S}, \\
&\varphi(\mathfrak{z}_0, \mathfrak{k}_0) \le \varphi(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) = 0.
\end{aligned} \tag{18}$$

Since $\mathscr{T}_{\mathfrak{r}},\ \mathfrak{r} \in \mathscr{R},\ \zeta_{\mathfrak{s}},\ \mathfrak{s} \in \mathscr{S}$ and φ are ∂_c-quasiconvex at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ on $\mathscr{G},$ by Definition 6, the inequalities above yield, respectively, for any $\beta_{\mathfrak{r}i} \in \partial_c \mathscr{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i}),\ \mathfrak{y}_{\mathfrak{r}} \ge 0,\ \mathfrak{r} \in \mathscr{R},\ \lambda_{\mathfrak{s}j} \in \partial_c \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}j}\big),\ \mathfrak{f}_{\mathfrak{s}} \ge 0,\ \mathfrak{s} \in \mathscr{S},$ and $\eta \in \partial_c \varphi(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}),$

$$\left\langle \mathfrak{y}_{\mathfrak{r}}\beta_{\mathfrak{r}i},\ (\mathfrak{z}_0, 0) - (\bar{\mathfrak{z}}, 0) \right\rangle \leq 0 \quad \forall\, i \in I_{\mathfrak{r}},\ \ \forall\, \mathfrak{r} \in \mathscr{R},$$

$$\left\langle \mathfrak{f}_{\mathfrak{s}}\lambda_{\mathfrak{s}j},\ \ (\mathfrak{z}_0, \mathfrak{k}_0) - (\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \right\rangle \leq 0 \quad \forall\, j \in J_{\mathfrak{s}},\ \ \forall\, \mathfrak{s} \in \mathscr{S}, \tag{19}$$

$$\left\langle \eta,\ (\mathfrak{z}_0, \mathfrak{k}_0) - (\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \right\rangle \leq 0.$$

As $\bar{u}_{\mathfrak{r}i} \geq 0,\ \forall\, i \in I_{\mathfrak{r}},\ \forall\, \mathfrak{r} \in \mathscr{R},\ \ \bar{w}_{\mathfrak{s}j} \geq 0,\ \forall\, j \in J_{\mathfrak{s}},\ \forall\, \mathfrak{s} \in \mathscr{S},\ \mathfrak{x} > 0$ and $a^* > 0$, therefore, the following inequality

$$\left\langle \sum_{\mathfrak{r}=1}^{p} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i}\mathfrak{y}_{\mathfrak{r}}\beta_{\mathfrak{r}i} \right) + \sum_{\mathfrak{s}=1}^{q} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j}\mathfrak{f}_{\mathfrak{s}}\lambda_{\mathfrak{s}j} \right) + \mathfrak{x}a^*\eta,\ (\mathfrak{z}_0, \mathfrak{k}_0) - (\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \right\rangle \leq 0 \tag{20}$$

holds for any $\beta_{\mathfrak{r}i} \in \partial_c \mathscr{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i}),\ \mathfrak{r} \in \mathscr{R},\ \ \xi_{\mathfrak{r}i} \in \Omega_{\mathfrak{r}}(\bar{\mathfrak{z}}),\ \lambda_{\mathfrak{s}j} \in \partial_c \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}j}\big),\ \mathfrak{s} \in \mathscr{S},\ \ \rho_{\mathfrak{s}j} \in \Lambda_{\mathfrak{s}}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ and $\eta \in \partial_c \varphi(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$. From (13), we can write

$$-\left\langle \mathfrak{x}\vartheta,\ (\mathfrak{z}_0, \mathfrak{k}_0) - (\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \right\rangle = \left\langle \sum_{\mathfrak{r}=1}^{p} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i}\mathfrak{y}_{\mathfrak{r}}\beta_{\mathfrak{r}i} \right) + \sum_{\mathfrak{s}=1}^{q} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j}\mathfrak{f}_{\mathfrak{s}}\lambda_{\mathfrak{s}j} \right) + \mathfrak{x}a^*\eta,\ (\mathfrak{z}_0, \mathfrak{k}_0) - (\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \right\rangle \leq 0. \tag{21}$$

That is

$$\left\langle \mathfrak{x}\vartheta,\ (\mathfrak{z}_0, \mathfrak{k}_0) - (\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \right\rangle \geq 0.$$

Also, since $\mathfrak{x} > 0$, we obtain

$$\left\langle \vartheta,\ (\mathfrak{z}_0, \mathfrak{k}_0) - (\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \right\rangle \geq 0, \qquad \forall\, \vartheta \in \partial_c \Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}). \tag{22}$$

Since Γ is a ∂_c-robust pseudoconvex at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ on $\mathscr{G}$, from (22),

$$\Longrightarrow \Gamma(\mathfrak{z}_0, \mathfrak{k}_0) - \Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \geq 0, \quad \forall\, (\mathfrak{z}_0, \mathfrak{k}_0) \in \mathscr{G}. \tag{23}$$

Which shows that $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ is a robust optimal solution of $(\mathscr{BP})$. □

Remark 2 A similar result can be derived for robust optimal solutions by assuming the strict ∂_c-pseudoconvexity of Γ, in place of mere ∂_c-pseudoconvexity.

To illustrate the optimality conditions derived in this section, we present a concrete example of a bilevel optimization problem.

Example 1 Let $\Omega_1 = [0, 1]$, $\Lambda_1 = [0, 1]$, $\Gamma(\mathfrak{z}, \mathfrak{k}) = \frac{1}{2}|\mathfrak{z}| + \frac{1}{2}|\mathfrak{k}| + \mathfrak{z} + \frac{1}{2}\mathfrak{k}$, $\mathscr{T}_1(\mathfrak{z}, \xi_1) = 1 - \exp \mathfrak{z} - \xi_1$, $\Upsilon(\mathfrak{z}, \mathfrak{k}) = \frac{5}{4}|\mathfrak{z}| + \mathfrak{k}$ and $\zeta_1\big((\mathfrak{z}, \mathfrak{k}), \rho_1\big) = \mathfrak{k}^2 - \mathfrak{k} - {\rho_1}^2$ (Figure 1). Consider the following nondifferentiable bilevel optimization problem defined by

$$(\mathscr{E}\mathscr{H}): \quad \begin{cases} \min\limits_{\mathfrak{z}, \mathfrak{k}} \quad \Gamma(\mathfrak{z}, \mathfrak{k}) \\ \\ \text{subject to} \quad \mathscr{T}_1(\mathfrak{z}, \xi_1) \leq 0, \\ \qquad\qquad\quad (\mathfrak{z}, \mathfrak{k}) \in \mathbb{R} \times \mathbb{R}, \quad \mathfrak{k} \in F_0(\mathfrak{z}), \end{cases}$$

where for each $\mathfrak{z} \in \mathbb{R}^{n_1}$, the parametric optimization problem $(\mathscr{E}\mathscr{H}_{\mathfrak{z}})$ has a set of solutions denoted by $F_0(\mathfrak{z})$

$$(\mathscr{E}\mathscr{H}_{\mathfrak{z}}): \quad \begin{cases} \min\limits_{\mathfrak{k}} \quad \Upsilon(\mathfrak{z}, \mathfrak{k}) \\ \\ \text{subject to} \quad \zeta_1\big((\mathfrak{z}, \mathfrak{k}), \rho_1\big) \leq 0, \end{cases}$$

with $\xi_1 \in \Omega_1$ and $\rho_1 \in \Lambda_1$.

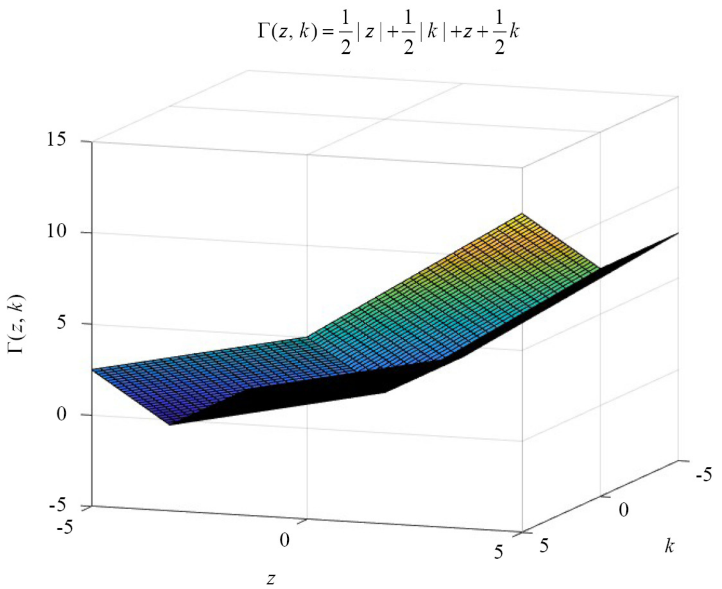

Figure 1. The graph of the objective function $\Gamma(\mathfrak{z}, \mathfrak{k}) = \frac{1}{2}|\mathfrak{z}| + \frac{1}{2}|\mathfrak{k}| + \mathfrak{z} + \frac{1}{2}\mathfrak{k}$ considered in Example 1

$\star$ The robust counterpart of $(\mathscr{E}\mathscr{H})$ is the bilevel optimization problem

$$(\mathscr{REH}): \quad \begin{cases} \min\limits_{\mathfrak{z},\, \mathfrak{k}} & \Gamma(\mathfrak{z}, \mathfrak{k}) \\ \\ \text{subject to} & \mathscr{T}_1(\mathfrak{z}, \xi_1) \leq 0, \quad \forall\, \xi_1 \in \Omega_1, \\ & \mathfrak{k} \in F(\mathfrak{z}), \end{cases}$$

where for each $\mathfrak{z} \in \mathbb{R}^{n_1}$, the parametric optimization problem $(\mathscr{REH}_{\mathfrak{z}})$ has a set of solutions denoted by $F(\mathfrak{z})$

$$(\mathscr{REH}_{\mathfrak{z}}): \quad \begin{cases} \min\limits_{\mathfrak{k}} & \Upsilon(\mathfrak{z}, \mathfrak{k}) \\ \\ \text{subject to} & \zeta_1\big((\mathfrak{z}, \mathfrak{k}), \rho_1\big) \leq 0, \quad \forall\, \rho_1 \in \Lambda_1. \end{cases}$$

- In this case, we have $\mathscr{R} = \{1\}$, $\mathscr{S} = \{1\}$, $F(\mathfrak{z}) = \{0\}$, $v(\mathfrak{z}) = \frac{5}{4}|\mathfrak{z}|$ and

$$\Phi_1(\mathfrak{z}) = \max_{\xi_1 \in \Omega_1} \mathscr{T}_1(\mathfrak{z}, \xi_1) = 1 - \exp \mathfrak{z},$$

$$\Psi_1(\mathfrak{z}, \mathfrak{k}) = \max_{\rho_1 \in \Lambda_1} \zeta_1\big((\mathfrak{z}, \mathfrak{k}), \rho_1\big) = \mathfrak{k}^2 - \mathfrak{k},$$

$$\varphi(\mathfrak{z}, \mathfrak{k}) = \mathfrak{k}.$$

As a consequence,

$$\mathscr{G} = \Big\{(x, 0) \in \mathbb{R} \times \mathbb{R} \mid x \in \mathbb{R}^+\Big\},$$

where $\mathscr{G}$ is shown in Figure 2, and

$$\Theta_1(\mathfrak{z}, \mathfrak{k}) = 1 - \exp \mathfrak{z}, \quad \Theta_2(\mathfrak{z}, \mathfrak{k}) = \mathfrak{k}^2 - \mathfrak{k}.$$

Observe that $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) = (0, 0) \in \mathscr{G}$ and that Assumption $(\mathscr{K})$ and $(\mathscr{V})$ are satisfied for $\bar{\mathfrak{z}}$ and $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$, respectively. Furthermore, we have

$$\Omega_1(\bar{\mathfrak{z}}) = \{0\}, \ \Lambda_1(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) = \{0\}, \text{ and } \mathbf{R}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) = \{1, 2\}.$$

Remark that

$$\partial_c\Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) = \left\{\left(\frac{3}{2}, 0\right), \left(\frac{3}{2}, 1\right), \left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, 1\right)\right\}, \ \partial_c\varphi(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) = \{(0, 1)\},$$

$$\partial_c\mathscr{T}_1(\cdot, 0)(\bar{\mathfrak{z}}) = \{-1\} \text{ and } \partial_c\zeta_2\big((\cdot, \cdot), 0\big)(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) = \{(0, -1)\}.$$

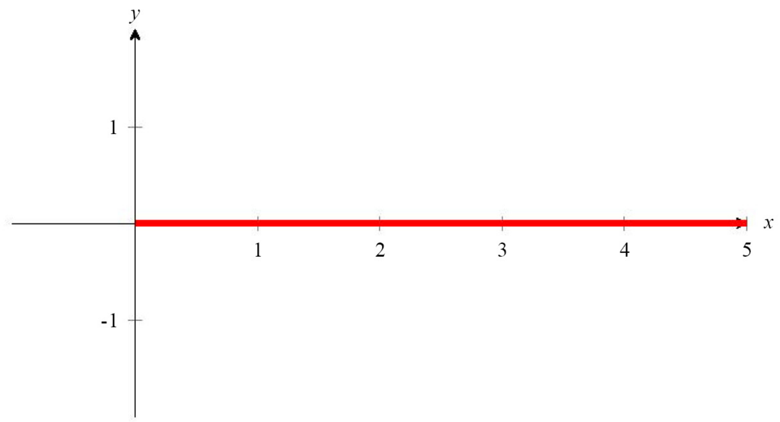

Figure 2. The feasible region $\mathscr{G} = \left\{(x, 0) \in \mathbb{R} \times \mathbb{R} \mid x \in \mathbb{R}^+\right\}$, considered in Example 1

$\star$ Note that $\Gamma(\cdot, \cdot)$ is ∂_c-pseudoconvex at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$, that $\varphi(\cdot, \cdot)$, $\mathscr{T}_1(\cdot, \xi_1)$ and $\zeta_1\big((\cdot, \cdot), \rho_1\big)$ are ∂_c-quasiconvex at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$.

- The constraint qualification (ENMFCQ) is satisfied at the point $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$. Specifically, by selecting $\delta = (\delta_1, \delta_2) = (2, 1) \neq (0, 0)$, we obtain

$$\Theta_1^\circ\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \delta\big) = -\delta_1 = -2 < 0$$

$$\Theta_2^\circ\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \delta\big) = -\delta_2 = -1 < 0.$$

- $(\mathscr{R}\mathscr{E}\mathscr{H})$ is partially calm at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$. Indeed, for $d = 1 > 0$ and $a^* = 2 > 0$ and $(\mathfrak{z}, \mathfrak{k}, z) \in \mathscr{B}_d(0, 0, 0)$ satisfying

$$\Phi_1(\mathfrak{z}) = 1 - \exp\mathfrak{z} \leq 0,$$

$$\Psi_1(\mathfrak{z}, \mathfrak{k}) = \mathfrak{k}^2 - \mathfrak{k} \leq 0,$$

$$\varphi(\mathfrak{z}, \mathfrak{k}) = \mathfrak{k} \leq z,$$

we have

$$\Gamma(\mathfrak{z},\mathfrak{k})-\Gamma(\bar{\mathfrak{z}},\bar{\mathfrak{k}})+a^*|z|=\frac{1}{2}|\mathfrak{z}|+\frac{1}{2}|\mathfrak{k}|+\mathfrak{z}+\frac{1}{2}\mathfrak{k}+2|z|$$

$$=\frac{1}{2}|\mathfrak{z}|+\frac{1}{2}|\mathfrak{k}|+\mathfrak{z}+\frac{1}{2}\mathfrak{k}+2|\mathfrak{k}|\geq 0.$$

⋆ For $\mathfrak{x}=\frac{1}{3}$, $a^*=2$, $\mathfrak{y}_1=\frac{1}{2}$ and $\mathfrak{f}_1=\frac{2}{3}$. As a result, inclusion (11) and equality (12) are valid. Hence, Theorem 3 implies that $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$ is a robust optimal solution of the problem $(\mathscr{E}\mathscr{H})$.

Algorithm 1. A method for determining robust optimal solutions to the problem $(\mathscr{B}\mathscr{P})$

Step 1. Provide Problem Data

Start by supplying the input data for the given $(\mathscr{B}\mathscr{P})$ problem:

• Input $\Gamma(\cdot,\cdot)$, $\Upsilon(\cdot,\cdot)$, $\mathscr{T}_{\mathfrak{r}}(\cdot,\xi_{\mathfrak{r}})$, $\xi_{\mathfrak{r}}\in\Omega_{\mathfrak{r}}$, $\mathfrak{r}\in\mathscr{R}$ and $\zeta_{\mathfrak{s}}\big((\cdot,\cdot),\rho_{\mathfrak{s}}\big)$, $\rho_{\mathfrak{s}}\in\Lambda_{\mathfrak{s}}$, $\mathfrak{s}\in\mathscr{S}$.

Step 2. Identify the Feasible Set

• Construct the feasible region as follows:

$$\mathscr{G}:=\Big\{(\mathfrak{z},\mathfrak{k})\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}:\ \mathscr{T}_{\mathfrak{r}}(\mathfrak{z},\xi_{\mathfrak{r}})\leq 0,\ \ \forall\,\xi_{\mathfrak{r}}\in\Omega_{\mathfrak{r}}\ \forall\,\mathfrak{r}\in\mathscr{R},\ \ \mathfrak{k}\in F(\mathfrak{z})\Big\}.$$

Step 3. Select a Feasible Point

• If the feasible set $\mathscr{G}$ is empty, terminate the algorithm.

• Otherwise, choose any point $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})\in\mathscr{G}$, and update the feasible set by removing this point: $\mathscr{G}=\mathscr{G}\setminus\{(\bar{\mathfrak{z}},\bar{\mathfrak{k}})\}$.

Step 4. Check the functions are locally Lipschitz continuous

• Verify whether each functions $\Gamma(\cdot,\cdot)$, $\Upsilon(\cdot,\cdot)$, $\mathscr{T}_{\mathfrak{r}}(\cdot,\xi_{\mathfrak{r}})$, $\xi_{\mathfrak{r}}\in\Omega_{\mathfrak{r}}$, $\mathfrak{r}\in\mathscr{R}$ and $\zeta_{\mathfrak{s}}\big((\cdot,\cdot),\rho_{\mathfrak{s}}\big)$, $\rho_{\mathfrak{s}}\in\Lambda_{\mathfrak{s}}$, $\mathfrak{s}\in\mathscr{S}$ are locally Lipschitz continuous at $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$.

• If all functions are locally Lipschitz continuous at $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$, proceed to Step 5.

• If any function fails this condition, return to Step 3.

Step 5. Verify Clarke subdifferentiability

• Compute the Clarke subdifferential of each function at $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$.

Step 6. Choose arbitrary subdifferentials

Choose arbitrary elements from the subdifferentials:

• $\vartheta^*\in\partial_c\Gamma(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$, $\eta^*\in\partial_c\varphi(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$, $\beta^*_{\mathfrak{r}i}\in\partial_c\mathscr{T}_{\mathfrak{r}}(\bar{\mathfrak{z}},\xi_{\mathfrak{r}i})$, $\mathfrak{r}\in\mathscr{R}$ and $\lambda^*_{\mathfrak{s}j}\in\partial_c\zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\rho_{\mathfrak{s}j}\big)$, $\mathfrak{s}\in\mathscr{S}$.

Step 7. Check the Extended Nonsmooth Mangasarian-Fromovitz Constraint Qualification (ENMFCQ)

• If the Extended Nonsmooth Mangasarian-Fromovitz constraint qualification holds at $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$, proceed to the next step.

• If not, return to Step 3.

Step 8. Verification of KKT Conditions

Attempt to find multipliers $\mathfrak{x}>0$, $a^*>0$, $\mathfrak{y}_{\mathfrak{r}}\geq 0$ for all $\mathfrak{r}\in\mathscr{R}$, and $\mathfrak{f}_{\mathfrak{s}}\geq 0$ satisfy (11) and (12):

• If such multipliers can be found, then $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$ is a local robust optimal solution of $(\mathscr{B}\mathscr{P})$.

• If not, return to Step 3.

Step 9. Verify ∂_c-pseudoconvexity and ∂_c-quasiconvexity Conditions
At the point $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$, confirm the following:
• The objective function Γ is ∂_c-pseudoconvex.
• The functions $\varphi,\ \zeta_{\mathfrak{s}},\ \mathfrak{s} \in \mathscr{S}$ and $\mathscr{T}_{\mathfrak{r}},\ \mathfrak{r} \in \mathscr{R}$ are ∂_c-quasiconvex.
If these conditions are not met, the problem cannot be solved using the current framework-return to Step 3.

Step 10. Output the solution
The point $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ obtained through this process is a robust optimal solution of $(\mathscr{BP})$.

4. Duality in robust bilevel optimization

Establishing duality results for robust bilevel optimization can be challenging, often requiring strict assumptions to ensure the validity of such results. In this section, we explore the connection between the primal and dual problems by examining their respective solutions.

We begin by defining

$$\mathbb{R}_+^{\mathbf{N}_1} := \left\{ \mathfrak{y} := (\mathfrak{y}_{\mathfrak{r}}, \bar{u}_{\mathfrak{r}i}),\ \mathfrak{r} = 1, \ldots, p,\ \ i = 1, \ldots, n_{\mathfrak{r}} : n_{\mathfrak{r}} \in \mathbb{N},\ \mathfrak{y}_{\mathfrak{r}} \geq 0,\ \bar{u}_{\mathfrak{r}i} \geq 0,\ \sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i} = 1 \right\}$$

and

$$\mathbb{R}_+^{\mathbf{N}_2} := \left\{ \mathfrak{f} := (\mathfrak{f}_{\mathfrak{s}}, \bar{w}_{\mathfrak{s}j}),\ \mathfrak{s} = 1, \ldots, q,\ \ j = 1, \ldots, n_{\mathfrak{s}} : n_{\mathfrak{s}} \in \mathbb{N},\ \mathfrak{f}_{\mathfrak{s}} \geq 0,\ \bar{w}_{\mathfrak{s}j} \geq 0,\ \sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j} = 1 \right\}.$$

Let $(u, v) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. For each $\xi_{\mathfrak{r}i} \in \Omega_{\mathfrak{r}}$ with $\mathfrak{r} \in \mathscr{R}$ and $\rho_{\mathfrak{s}j} \in \Lambda_{\mathfrak{s}}$ with $\mathfrak{s} \in \mathscr{S}$ the Mond-Weir-type uncertain dual problem $(\mathscr{RCBD})$ corresponding to $(\mathscr{RCBP})$ can be stated as follows:

$$(\mathscr{RCBD}) \begin{cases} \max\limits_{u,\, v} \Gamma(u, v) \\ \text{subject to} \\ \quad (0, 0) \in \mathfrak{x}\partial_c\Gamma(u, v) + \mathfrak{x}a^*\partial_c\varphi(u, v) + \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i}\, \beta_{\mathfrak{r}i} \right) \\ \quad + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j}\, \lambda_{\mathfrak{s}j} \right), \\ \quad \mathfrak{x}\varphi(u, v) \geq 0, \\ \quad \mathfrak{y}_{\mathfrak{r}} \mathscr{T}_{\mathfrak{r}}(u, \xi_{\mathfrak{r}i}) \geq 0, \quad \mathfrak{r} \in \mathscr{R}, \\ \quad \mathfrak{f}_{\mathfrak{s}} \zeta_{\mathfrak{s}}\big((u, v), \rho_{\mathfrak{s}j}\big) \geq 0, \quad \mathfrak{s} \in \mathscr{S}, \\ \quad \beta_{\mathfrak{r}i} \in \left\{ \cup\, \partial_c \mathscr{T}_{\mathfrak{r}}(u, \xi_{\mathfrak{r}i}),\ \ \xi_{\mathfrak{r}i} \in \Omega_{\mathfrak{r}}(u) \right\}, \\ \quad \lambda_{\mathfrak{s}j} \in \left\{ \cup\, \partial_c \zeta_{\mathfrak{s}}\big((u, v), \rho_{\mathfrak{s}j}\big),\ \ \rho_{\mathfrak{s}j} \in \Lambda_{\mathfrak{s}}(u, v) \right\}, \\ \quad \mathfrak{T}^* = \left\{ (\mathfrak{x},\ \mathfrak{y},\ \mathfrak{f}):\ \mathfrak{x} > 0,\ \mathfrak{y} \geq 0,\ \mathfrak{f} \geq 0 \right\}. \end{cases}$$

Here, $\Omega_{\mathfrak{r}}(u)$ is defined as in (8), with $\bar{\mathfrak{z}}$ replaced by u and $\Lambda_{\mathfrak{s}}(u, v)$ is defined as in (10), with $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ replaced by (u, v). The feasible set of $(\mathscr{RCBD})$ is defined as

$$\mathscr{F}_{\mathscr{RCBD}} = \Bigg\{ \left(u, v, \mathfrak{T}^*\right) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R} \times \mathbb{R}_+^{\mathbf{N}_1} \times \mathbb{R}_+^{\mathbf{N}_2} \colon (0, 0) \in \mathfrak{x}\partial_c \Gamma(u, v) + \mathfrak{x} a^* \partial_c \varphi(u, v)$$

$$+ \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i} \, \beta_{\mathfrak{r}i} \right) + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j} \, \lambda_{\mathfrak{s}j} \right), \tag{24}$$

$$\mathfrak{x}\varphi(u, v) \geq 0, \;\; \mathfrak{y}_{\mathfrak{r}} \mathscr{T}_{\mathfrak{r}}(u, \xi_{\mathfrak{r}i}) \geq 0, \;\; \mathfrak{r} \in \mathscr{R}, \;\; \mathfrak{f}_{\mathfrak{s}} \zeta_{\mathfrak{s}}\big((u, v), \rho_{\mathfrak{s}j}\big) \geq 0, \;\; \mathfrak{s} \in \mathscr{S} \Bigg\}.$$

The following theorem presents the weak form of robust duality, linking the primal problem $(\mathscr{RCBP})$ with its corresponding dual problem $(\mathscr{RCBD})$.

Theorem 4 (Weak Robust Duality): Suppose that $(\mathfrak{z}_0, \mathfrak{k}_0) \in \mathscr{G}$ and $(u, v, \mathfrak{T}^*) \in \mathscr{F}_{\mathscr{RCBD}}$. If Γ is a ∂_c-pseudoconvex at (u, v) on $\mathscr{F}_{\mathscr{RCBD}}$, that φ, $\mathscr{T}_{\mathfrak{r}}$, $\mathfrak{r} \in \mathscr{R}$ and $\zeta_{\mathfrak{s}}$, $\mathfrak{s} \in \mathscr{S}$ are ∂_c-quasiconvex at (u, v) on $\mathscr{F}_{\mathscr{RCBD}}$, then

$$\Gamma(\mathfrak{z}_0, \mathfrak{k}_0) \nleq \Gamma(u, v).$$

Proof. Assume, for the sake of contradiction, that

$$\Gamma(\mathfrak{z}_0, \mathfrak{k}_0) \leq \Gamma(u, v). \tag{25}$$

Since Γ is a ∂_c-pseudoconvex at (u, v) on $\mathscr{F}_{\mathscr{RCBD}}$, by Definition 4, the inequality above yields, for any $\vartheta^* \in \partial_c \Gamma(u, v)$,

$$\left\langle \vartheta^*, \; (\mathfrak{z}_0, \mathfrak{k}_0) - (u, v) \right\rangle < 0.$$

By $\mathfrak{x} > 0$, the following inequality

$$\left\langle \mathfrak{x}\, \vartheta^*, \; (\mathfrak{z}_0, \mathfrak{k}_0) - (u, v) \right\rangle < 0 \tag{26}$$

holds for any $\vartheta^* \in \partial_c \Gamma(u, v)$.

From $(u, v, \mathfrak{T}^*) \in \mathscr{F}_{\mathscr{RCBD}}$, it follows that $(u, v) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $\mathfrak{x} > 0$, $\mathfrak{y} \geq 0$, $\mathfrak{f} \geq 0$ and

$$(0, 0) \in \mathfrak{x}\partial_c \Gamma(u, v) + \mathfrak{x} a^* \partial_c \varphi(u, v) + \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i} \, \beta_{\mathfrak{r}i} \right) + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j} \, \lambda_{\mathfrak{s}j} \right), \tag{27}$$

and

$$\mathfrak{x}\varphi(u,\,v)\geq 0, \tag{28}$$

$$\mathfrak{y}_{\mathfrak{r}}\mathscr{T}_{\mathfrak{r}}(u,\,\xi_{\mathfrak{r}i})\geq 0,\;\;\mathfrak{r}\in\mathscr{R}, \tag{29}$$

$$\mathfrak{f}_{\mathfrak{s}}\zeta_{\mathfrak{s}}\big((u,\,v),\,\rho_{\mathfrak{s}j}\big)\geq 0,\;\;\mathfrak{s}\in\mathscr{S}. \tag{30}$$

By (27), there exist $\vartheta^*\in\partial_c\Gamma(u,\,v),\;\eta^*\in\partial_c\varphi(u,\,v),\;\beta^*_{\mathfrak{r}i}\in\partial_c\mathscr{T}_{\mathfrak{r}}(u,\,\xi_{\mathfrak{r}i}),\;\mathfrak{r}\in\mathscr{R}$ and $\lambda^*_{\mathfrak{s}j}\in\partial_c\zeta_{\mathfrak{s}}\big((u,\,v),\,\rho_{\mathfrak{s}j}\big),\;\mathfrak{s}\in\mathscr{S}$ such that

$$\mathfrak{x}\;\vartheta^*+\mathfrak{x}a^*\eta^*+\sum_{\mathfrak{r}=1}^{p}\mathfrak{y}_{\mathfrak{r}}\left(\sum_{i=1}^{n_{\mathfrak{r}}}\bar{u}_{\mathfrak{r}i}\;\beta^*_{\mathfrak{r}i}\right)+\sum_{\mathfrak{s}=1}^{q}\mathfrak{f}_{\mathfrak{s}}\;\left(\sum_{j=1}^{n_{\mathfrak{s}}}\bar{w}_{\mathfrak{s}j}\;\lambda^*_{\mathfrak{s}j}\right)=0. \tag{31}$$

By $(\mathfrak{z}_0,\,\mathfrak{k}_0)\in\mathscr{G}$ and $(u,\,v,\,\mathfrak{T}^*)\in\mathscr{F}_{\mathscr{RCBD}}$, it follows that

$$\begin{aligned}
&\mathfrak{x}\varphi(\mathfrak{z}_0,\,\mathfrak{k}_0)\leq\mathfrak{x}\varphi(u,\,v),\\
&\mathfrak{y}_{\mathfrak{r}}\mathscr{T}_{\mathfrak{r}}(\mathfrak{z}_0,\,\xi_{\mathfrak{r}i})\leq\mathfrak{y}_{\mathfrak{r}}\mathscr{T}_{\mathfrak{r}}(u,\,\xi_{\mathfrak{r}i}),\;\;\mathfrak{r}\in\mathscr{R},\\
&\mathfrak{f}_{\mathfrak{s}}\zeta_{\mathfrak{s}}\big((\mathfrak{z}_0,\,\mathfrak{k}_0),\,\rho_{\mathfrak{s}j}\big)\leq\mathfrak{f}_{\mathfrak{s}}\zeta_{\mathfrak{s}}\big((u,\,v),\,\rho_{\mathfrak{s}j}\big),\;\;\mathfrak{s}\in\mathscr{S}.
\end{aligned} \tag{32}$$

Since $\varphi,\;\mathscr{T}_{\mathfrak{r}}$ and $\zeta_{\mathfrak{s}}$ are ∂_c-quasiconvex at $(u,\,v)$ on $\mathscr{F}_{\mathscr{RCBD}}$, by Definition 6, for any $\eta^*\in\partial_c\varphi(u,\,v),\;\beta^*_{\mathfrak{r}i}\in\partial_c\mathscr{T}_{\mathfrak{r}}(u,\,\xi_{\mathfrak{r}i}),\;\mathfrak{r}\in\mathscr{R}$ and $\lambda^*_{\mathfrak{s}j}\in\partial_c\zeta_{\mathfrak{s}}\big((u,\,v),\,\rho_{\mathfrak{s}j}\big),\;\mathfrak{s}\in\mathscr{S}$

$$\begin{aligned}
&\Big\langle\eta^*,\;(\mathfrak{z}_0,\,\mathfrak{k}_0)-(u,\,v)\Big\rangle\leq 0,\\
&\Big\langle\beta^*_{\mathfrak{r}i},\;(\mathfrak{z}_0,\,\mathfrak{k}_0)-(u,\,v)\Big\rangle\leq 0,\;\forall\,\mathfrak{r}\in\mathscr{R},\\
&\Big\langle\lambda^*_{\mathfrak{s}j},\;(\mathfrak{z}_0,\,\mathfrak{k}_0)-(u,\,v)\Big\rangle\leq 0,\;\forall\,\mathfrak{s}\in\mathscr{S}.
\end{aligned} \tag{33}$$

Again using the feasibility of $(u,\,v,\,\mathfrak{T}^*)$ in $\mathscr{RCBD}$, we get that the following inequality

$$\left\langle\mathfrak{x}a^*\eta^*+\sum_{\mathfrak{r}=1}^{p}\Big(\sum_{i=1}^{n_{\mathfrak{r}}}\bar{u}_{\mathfrak{r}i}\mathfrak{y}_{\mathfrak{r}}\;\beta^*_{\mathfrak{r}i}\Big)+\sum_{\mathfrak{s}=1}^{q}\;\Big(\sum_{j=1}^{n_{\mathfrak{s}}}\bar{w}_{\mathfrak{s}j}\mathfrak{f}_{\mathfrak{s}}\;\lambda^*_{\mathfrak{s}j}\Big),\;(\mathfrak{z}_0,\,\mathfrak{k}_0)-(u,\,v)\right\rangle\leq 0 \tag{34}$$

holds for any $\eta^*\in\partial_c\varphi(u,\,v),\;\beta^*_{\mathfrak{r}i}\in\partial_c\mathscr{T}_{\mathfrak{r}}(u,\,\xi_{\mathfrak{r}i}),\;\mathfrak{r}\in\mathscr{R}$ and $\lambda^*_{\mathfrak{s}j}\in\partial_c\zeta_{\mathfrak{s}}\big((u,\,v),\,\rho_{\mathfrak{s}j}\big),\;\mathfrak{s}\in\mathscr{S}.$ Adding both sides of the inequalities (26) and (34), we get the following inequality

$$\left\langle \mathfrak{x}\, \vartheta^* + \mathfrak{x} a^* \eta^* + \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i}\, \beta^*_{\mathfrak{r}i} \right) + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j}\, \lambda^*_{\mathfrak{s}j} \right), (\mathfrak{z}_0, \mathfrak{k}_0) - (u, v) \right\rangle < 0$$

holds for any $\vartheta^* \in \partial_c \Gamma(u, v)$, $\eta^* \in \partial_c \varphi(u, v)$, $\beta^*_{\mathfrak{r}i} \in \partial_c \mathscr{T}_{\mathfrak{r}}(u, \xi_{\mathfrak{r}i})$, $\mathfrak{r} \in \mathscr{R}$ and $\lambda^*_{\mathfrak{s}j} \in \partial_c \zeta_{\mathfrak{s}}\big((u, v), \rho_{\mathfrak{s}j}\big)$, $\mathfrak{s} \in \mathscr{S}$, contradicting (31), thereby completing the proof. □

The next theorem demonstrates the strong robust duality connection between the primal problem $(\mathscr{BP})$ and its associated dual problem $(\mathscr{MH})$.

Theorem 5 (Strong Robust Duality): Let $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ be a robust optimal solution of $(\mathscr{BP})$, where (ENMFCQ) holds. Then there exists $\overline{\mathfrak{T}^*} = \{(\bar{\mathfrak{x}}, \bar{\mathfrak{y}}, \bar{\mathfrak{f}}) : \bar{\mathfrak{x}} > 0,\ \bar{\mathfrak{y}} \geq 0,\ \bar{\mathfrak{f}} \geq 0\}$, such that $((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \overline{\mathfrak{T}^*})$ is a feasible point of $(\mathscr{RCBD})$. Moreover, if Γ is a ∂_c-pseudoconvex at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$, and if φ, $\mathscr{T}_{\mathfrak{r}}$, $\mathfrak{r} \in \mathscr{R}$ and $\zeta_{\mathfrak{s}}$, $\mathfrak{s} \in \mathscr{S}$ are ∂_c-quasiconvex at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$, then $((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \overline{\mathfrak{T}^*})$ is a robust optimal solution of $(\mathscr{RCBD})$.

Proof. Let $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ be a robust optimal solution of $(\mathscr{BP})$, where (ENMFCQ) holds. Then, by Theorem 2, there exists $\mathfrak{x} > 0$, $\vartheta^* \in \partial_c \Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$, $a^* > 0$, $\eta^* \in \partial_c \varphi(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$, $\mathfrak{y}_{\mathfrak{r}} \geq 0$, $\mathfrak{r} \in \mathscr{R}$, $\bar{u}_{\mathfrak{r}i} \geq 0$, $\beta^*_{\mathfrak{r}i} \in \partial_c \mathscr{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i})$, $\xi_{\mathfrak{r}i} \in \Omega_{\mathfrak{r}}(\bar{\mathfrak{z}})$, $i \in I_{\mathfrak{r}} := \{1, \ldots, n_{\mathfrak{r}}\}$, $n_{\mathfrak{r}} \in \mathbb{N}$, and $\mathfrak{f}_{\mathfrak{s}} \geq 0$, $\mathfrak{s} \in \mathscr{S}$, $\bar{w}_{\mathfrak{s}j} \geq 0$, $\lambda^*_{\mathfrak{s}j} \in \partial_c \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}j}\big)$, $\rho_{\mathfrak{s}j} \in \Lambda_{\mathfrak{s}}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$, $j \in J_{\mathfrak{s}} := \{1, \ldots, n_{\mathfrak{s}}\}$, $n_{\mathfrak{s}} \in \mathbb{N}$, such that

$$\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i} = 1, \quad \sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j} = 1,$$

$$0 = \mathfrak{x}\, \vartheta^* + \mathfrak{x} a^* \eta^* + \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i}\, \beta^*_{\mathfrak{r}i} \right) + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j}\, \lambda^*_{\mathfrak{s}j} \right) \tag{35}$$

and

$$\mathfrak{y}_{\mathfrak{r}} \max_{\xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}} \mathscr{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}}) = 0, \quad \mathfrak{r} \in \mathscr{R}, \tag{36}$$

$$\mathfrak{f}_{\mathfrak{s}} \max_{\rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}} \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}}\big) = 0, \quad \mathfrak{s} \in \mathscr{S}. \tag{37}$$

Since $\xi_{\mathfrak{r}i} \in \Omega_{\mathfrak{r}}(\bar{\mathfrak{z}})$,

$$\mathscr{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i}) = \max_{\xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}} \mathscr{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}}), \quad \forall\, i \in I_{\mathfrak{r}}, \ \forall\, \mathfrak{r} \in \mathscr{R}.$$

Thus, it follows by (36) that

$$\mathfrak{y}_{\mathfrak{r}} \mathscr{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i}) = 0, \quad \forall\, i \in I_{\mathfrak{r}}, \ \forall\, \mathfrak{r} \in \mathscr{R}. \tag{38}$$

And since $\rho_{\mathfrak{s}j} \in \Lambda_{\mathfrak{s}}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$,

$$\zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\rho_{\mathfrak{s}j}\big)=\max_{\rho_{\mathfrak{s}}\in\Lambda_{\mathfrak{s}}}\zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\rho_{\mathfrak{s}}\big)\quad\forall\, j\in J_{\mathfrak{s}},\ \ \mathfrak{s}\in\mathscr{S}.$$

Thus, it follows by (37) that

$$\mathfrak{f}_{\mathfrak{s}}\ \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\rho_{\mathfrak{s}j}\big)=0\quad\forall\, j\in J_{\mathfrak{s}},\ \ \mathfrak{s}\in\mathscr{S}. \tag{39}$$

Since $\varphi(\bar{\mathfrak{z}},\bar{\mathfrak{k}})=0,$ letting $\bar{\mathfrak{y}}:=(\mathfrak{y}_{\mathfrak{r}},\bar{u}_{\mathfrak{r}i})$ and $\bar{\mathfrak{f}}:=(\mathfrak{f}_{\mathfrak{s}},\bar{w}_{\mathfrak{s}j}),$ one deduces that

$$((\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\overline{\mathfrak{T}^*})\in\mathscr{F}_{\mathscr{RCBD}}.$$

Let us demonstrate that $((\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\overline{\mathfrak{T}^*})$ is a robust optimal solution of $(\mathscr{RCBD})$. Conversely, suppose that there exists a point $((\mathfrak{z}_1,\mathfrak{k}_1),\mathfrak{T}^*{}_1)\in\mathscr{F}_{\mathscr{RCBD}}$ such that

$$\Gamma(\bar{\mathfrak{z}},\bar{\mathfrak{k}})-\Gamma(\mathfrak{z}_1,\mathfrak{k}_1)<0.$$

Since $\big((\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\overline{\mathfrak{T}^*}\big)\in\mathscr{F}_{\mathscr{RCBD}}$ and $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})\in\mathscr{G}$, we conclude from Theorem 4 that

$$\Gamma(\bar{\mathfrak{z}},\bar{\mathfrak{k}})-\Gamma(\mathfrak{z}_1,\mathfrak{k}_1)\geq 0.$$

Which leads to a contradiction. Hence, the proof is complete. □

Finally, we provide a numerical example to demonstrate the practical utility of our duality results.

Example 2 Revisiting the problem $(\mathscr{EH})$ discussed in Example 1, we now examine its Mond-Weir type dual problem

$$(\mathscr{MEH})\begin{cases}\max\limits_{u,\,v}\Gamma(u,v)=\frac{1}{2}|u|+\frac{1}{2}|v|+u+\frac{1}{2}v\\ \\ \text{subject to}\\ \qquad \mathfrak{x}\varphi(u,v)\geq 0,\ \ \mathfrak{y}_1\mathscr{T}_1(u,\xi_1)\geq 0,\ \ \mathfrak{f}_1\zeta_1\big((u,v),\rho_1\big)\geq 0,\\ \qquad (0,0)\in\mathfrak{x}\partial_c\Gamma(u,v)+\mathfrak{x}a^*\partial_c\varphi(u,v)+\mathfrak{y}_1\,\partial_c\mathscr{T}_1(u,\xi_1)+\mathfrak{f}_1\,\partial_c\zeta_1\big((u,v),\rho_1\big),\\ \\ \qquad \mathfrak{T}^*=\Big\{\big(\mathfrak{x},\ \mathfrak{y}_1,\ \mathfrak{f}_1\big):\ \mathfrak{x}>0,\ \mathfrak{y}_1\geq 0,\ \mathfrak{f}_1\geq 0\Big\},\\ \qquad (u,v)\in\mathbb{R}^2.\end{cases}$$

We assert that $(u,v)=(-1,0)$ is a feasible solution of $(\mathscr{MEH})$.

Remark that

$$\partial_c\Gamma(-1, 0) = \left\{ \left(\frac{1}{2}, 1\right), \left(\frac{1}{2}, 0\right) \right\}, \ \partial_c\varphi(-1, 0) = \{(0, 1)\},$$

$$\partial_c\mathscr{T}_1(\cdot, 0)(-1) = \left\{ \frac{-1}{\exp} \right\}, \text{ and } \partial_c\zeta_1\big((\cdot, \cdot), 0\big)(-1, 0) = \{(0, -1)\}.$$

Note that $\Gamma(\cdot, \cdot)$ is ∂_c-pseudoconvex at $(-1, 0)$, that $\varphi(\cdot, \cdot)$, $\mathscr{T}_1(\cdot, \xi_1)$ and $\zeta_1\big((\cdot, \cdot), \rho_1\big)$ are ∂_c-quasiconvex at $(-1, 0)$.

For $\mathfrak{T}^* = \left(\frac{1}{4}, \frac{\exp}{8}, \frac{3}{4}\right)$, we have

$$\frac{1}{4}\left(\frac{1}{2}, 1\right) + 2\frac{1}{4}\left(0, 1\right) + \frac{\exp}{8}\left(-\frac{1}{\exp}, 0\right) + \frac{3}{4}\left(0, -1\right) = \left(0, 0\right).$$

This simplifies

$$(0, 0) \in \mathfrak{x}\partial_c\Gamma(-1, 0) + \mathfrak{x}a^*\partial_c\varphi(-1, 0) + \mathfrak{y}_1\, \partial_c\mathscr{T}_1(-1, \xi_1) + \mathfrak{f}_1\, \partial_c\zeta_1\big((-1, 0), \rho_1\big).$$

Furthermore, we have

$$\mathfrak{x}\varphi(-1, 0) = 0 \geq 0, \ \ \mathfrak{y}_1\mathscr{T}_1(-1, \xi_1) = 1 - \exp(-1) \geq 0,$$

$$\mathfrak{f}_1\zeta_1\big((-1, 0), \rho_1\big) = 0 \geq 0.$$

$*$ Since $\mathscr{G} = \big\{(x, 0) \in \mathbb{R} \times \mathbb{R} \mid x \in \mathbb{R}^+\big\}$, for any feasible solution $(\mathfrak{z}, \mathfrak{k}) \in \mathscr{G}$ of $(\mathscr{E}\mathscr{H})$ and any feasible solution $(u, v, \mathfrak{T}^*) \in \mathscr{F}_{\mathscr{M}\mathscr{E}\mathscr{H}}$ of $(\mathscr{M}\mathscr{E}\mathscr{H})$, we have

$$\Gamma(\mathfrak{z}, \mathfrak{k}) - \Gamma(u, v) \geq 0.$$

Therefore, Theorem 4 is applicable to both $(\mathscr{E}\mathscr{H})$ and $(\mathscr{M}\mathscr{E}\mathscr{H})$.

$*$ It is known that $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) = (0, 0)$ is a robust optimal solution of $(\mathscr{E}\mathscr{H})$ where (ENMFCQ) holds and that (11) and (12) is satisfied at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}, \overline{\mathfrak{T}^*}) = \left((0, 0), \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$. Since $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}, \overline{\mathfrak{T}^*})$ is a feasible point of $(\mathscr{M}\mathscr{E}\mathscr{H})$, then for any feasible point $(u, v, \mathfrak{T}^*)$ of $(\mathscr{M}\mathscr{E}\mathscr{H})$, we have

$$\Gamma(u, v) - \Gamma(0, 0) = \frac{1}{2}|u| + \frac{1}{2}|v| + u + \frac{1}{2}v \leq 0.$$

Therefore, $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}, \overline{\mathfrak{T}^*})$ is robust optimal solution for $(\mathscr{M}\mathscr{E}\mathscr{H})$, which proves that Theorem 5 is valid.

5. Conclusion

This paper has examined nondifferentiable bilevel optimization problems that involve both upper- and lower-level constraints under uncertainty. For the considered nonsmooth framework, we have established sufficient optimality conditions by employing generalized convexity concepts defined via Clarke subdifferentials. In addition, we have formulated the associated Mond-Weir-type dual problem and derived several duality theorems between the primal and dual formulations under similar generalized convexity assumptions. These results not only extend but also generalize existing findings in the literature concerning nonsmooth bilevel optimization.

As far as we are aware, this is the first study to establish sufficient optimality conditions and Mond−Weir-type duality results for such a wide class of nonsmooth bilevel optimization problems within the framework of generalized convexity.

Nevertheless, several avenues remain open for future exploration. In particular, it would be worthwhile to develop analogous optimality and duality results for other variants of bilevel problems. We aim to address these directions in future research.

As ∂_c-pseudoconvexity and ∂_c-quasiconvexity are weaker than ∂_c-convexity, the sufficient optimality theorem using a ∂_c-pseudoconvex objective function and ∂_c-quasiconvex constraints generalizes the sufficient theorem requiring ∂_c-convexity for both the objective and constraints. Therefore, our results offer a broader and more refined framework compared to the results presented in [23].

Acknowledgement

We are deeply grateful to the anonymous referees and the editor for their invaluable feedback and constructive suggestions, which significantly improved the quality of this manuscript. The fourth author gratefully acknowledges the financial support received from a Seed Grant (F. No. NITMZ/RA-4/MA/2023/405) at the National Institute of Technology Mizoram.

Conflict of interest

The authors declare no competing financial interest.

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