

Research Article

Fractional-Order Iterative BVPs: Uniqueness and Hyers-Ulam Stability

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Abstract: This article investigates the existence and uniqueness of solutions to a fractional-order iterative boundary value problem characterized by Caputo derivatives. The analysis is conducted within the framework of Banach spaces, employing both the classical contraction principle and Schauder's fixed point approach to confirm the problem's solvability. Novel criteria ensuring the uniqueness of solutions are derived using Rus's contraction mapping theorem. The work also presents a detailed proof of the Hyers-Ulam stability for the proposed problem.

Keywords: Caputo differential operator, iterative, boundary value problem, Kernel, two metrics, Rus's contraction mapping theorem, Hyers-Ulam stability

MSC: 34A08, 34B15, 34K10

1. Introduction

Iterative Differential Equations (IDE) naturally emerge in various scientific domains such as infection modeling, charged particle dynamics, and delayed interactions [1, 2]. Functional Differential Equations (FDEs), in particular, serve as foundational models across biotechnology, physics, and engineering, as thoroughly examined in the authoritative treatises by Hale et al. [3], Kolmanovskii et al. [4], along with the extensive literature cited therein [5, 6]. The study of IDEs finds its early origins in the pioneering contributions of Eder [7].

A variety of investigations have explored this topic. For instance, Feckan [8] in 1993 analyzed the IDE

$$\kappa'(x) = \mathcal{F}(\kappa(\kappa(x))), \quad x \in \mathbb{R},$$

$$\kappa(0) = 0,$$

and proved the existence of solutions using a classical Fixed-Point Theorem (FPT). In [9], the authors addressed the Existence and Uniqueness (EU) of solutions to IDEs

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$$\kappa'(x) = \mathcal{F}(x, \kappa(\phi(x) + \psi(\kappa(x)))) , \quad x \in \mathbb{R},$$

$$\kappa(x_0) = x_0,$$

utilizing the iterative Picard approach. More recently, Kaufmann [10] examined the existence of solutions to the second-order IDE

$$\kappa''(x) = \mathcal{G}(x, \kappa(x), \kappa^{[2]}(x)), \quad x \in [\alpha, \beta],$$

$$\kappa(\alpha) = \alpha, \quad \kappa(\beta) = \beta,$$

via some FPT.

Fractional derivatives, however, provide a more refined tool for modeling memory and hereditary properties inherent in many physical processes. The field of fractional calculus has experienced rapid growth due to its applicability in numerous areas such as physics, biology, chemistry, and nuclear dynamics [11–18]. Key texts highlight both the theoretical framework and the interplay between classical and fractional-order systems.

For example, in the article [19], a certain refined form of Lyapunov-type inequalities is considered in the setting of fractional calculus, namely, in the context of FDEs complemented by fractional Boundary Conditions (BCs). Their approach is based on putting the differential equation in an equivalent integral formulation, allowing them to derive the Green function and inequality thereafter. The new inequality is a generalization of the known findings in the literature and a potent instrument to study the qualitative dynamics of fractional systems. Moreover, the obtained inequality is used to investigate the zero distribution of the Mittag-Leffler function, which is the main function in the theory of FDEs. Using the new Lyapunov-type inequality, the authors manage to find a real interval where the Mittag-Leffler function has no real zeros. The analytical framework and the estimates presented in this work are of a high level and which makes the work a useful addition to the body of research that exists in FDEs and applications.

More precisely, in [19], the authors derived a Lyapunov-type inequality for the Fractional Boundary Value Problem (FBVP)

$${}^c\mathcal{D}_a^\alpha \kappa(x) + Q(x)\kappa(x) = 0, \quad x \in (a, b),$$

$$\kappa(a) = 0, \quad {}^c\mathcal{D}_a^\beta \kappa(b) = 0,$$

where $\alpha \in (1, 2]$ and $\beta \in (0, 1]$. This formulation was subsequently utilized to determine zero-free intervals for a specific form of the Mittag-Leffler function.

In recent years, considerable research has emerged concerning the existence, stability, and optimal control of various classes of FDEs [20–28]. Notably, Zhou et al. [29] analyzed the Iterative Functional Differential Equations (IFDE) involving a parameter:

$${}^c\mathcal{D}_a^q \kappa(x) = \mathcal{F}(x, \kappa(x), \kappa(\kappa(x))) + \lambda, \quad x \in [a, b],$$

$$\kappa(x) = \phi(x), \quad x \in [a_1, a],$$

$$\kappa(x) = \psi(x), \quad x \in [b, b_1],$$

where $q \in (0, 1)$, $\kappa \in \mathbb{R} \setminus \{0\}$, and $\lambda \in \mathbb{R}$. The existence of a solution is established using the Schauder FPT.

The topic of Hyers-Ulam Stability (HUS) in the fractional setting has also gained significant momentum. The classical origin of this concept traces back to Ulam (1940) and Hyers (1941), with later generalizations by Rassias during the 1980s-1990s. Obloza [30] initiated the study of HUS for linear differential equations in 1997. Relevant developments can be found in [31–33]. In particular, Sousa et al. [34] obtained HUS for the FDE

$${}^H\mathcal{D}_{z_0^+}^{\kappa, \xi; \psi} \kappa(z) = \mathcal{F}(z, \kappa, \kappa(z-a)), \quad z \in [0, T],$$

$$\kappa(z) = \phi(z), \quad z \in [z_0 - a, z_0],$$

where $a > 0$, $0 < \kappa, \xi \leq 1$, and ${}^H\mathcal{D}_{z_0^+}^{\kappa, \xi; \psi}$ represents the ψ -Hilfer fractional derivative operator. Recent studies have highlighted the practical applications of Caputo-type fractional derivatives in modeling real-world systems. For example, Agarwal et al. [35] and Salim et al. [36] investigate iterative Caputo FBVPs in applied contexts, while Ma et al. [37] employ He's fractal calculus to capture complex nonlinear phenomena. Furthermore, Yao et al. [38] and Jung [39] study HUS, ensuring that approximate solutions remain close to exact ones under perturbations. Analytical techniques such as the Variational Iteration Method (VIM) and Homotopy Perturbation Method (HPM) offer efficient series solutions without linearization, providing both computational validation and theoretical insight [40]. Despite these advances, a unified framework addressing both solvability and HUS for nonlinear iterative FBVPs under generalized fractional operators is still limited, motivating the present work. FDEs have gained attention for modeling memory and hereditary effects in complex systems. VIM has been effectively applied to FDEs, as shown by Ibraheem et al. [41] and Turkyilmazoglu [42, 43], demonstrating efficient convergence and practical applicability.

In contemporary studies, Prasad et al. [44] presented new EU results for two-point FBVPs employing classical analytical techniques. Remarkably, the authors of [44] also expand the theoretical scenery by involving the so-called Hyers-Ulam stability, which is a measure of robustness that becomes especially relevant in modeling situations with uncertainty. This stability guarantees that the approximate solutions are around the exact solutions, improving the trustworthiness of the mathematical models in practical applications. The derived kernel estimates, together with their integral counterparts, exhibit a profound and precise understanding of fractional calculus and its inherent subtleties. The given illustrative example not only confirms the theory but also paves the way to its implementation in control systems, signal propagation, and nonlinear dynamics. The article is an interesting addition to the increasingly rich field of FDEs, and in particular to those of an iterative structure and stability in the pursuit of perturbations.

Motivated by the aforementioned developments, this article investigates the existence, uniqueness, and HUS of solutions for a class of IFDEs:

$${}^C\mathcal{D}_{0^+}^{\vartheta_1} v(z) = f(z, v(z), v^{[2]}(z)), \quad z \in [0, 1], \quad (1)$$

satisfying the conditions

$$v(0) = B_0, \quad {}^C\mathcal{D}_{0^+}^{\vartheta_2} v(1) = D_0, \quad (2)$$

where $v^{[2]}(z) = v(v(z))$, $\vartheta_1 \in (1, 2]$, $\vartheta_2 \in (0, 1]$, $0 \leq B_0 \leq D_0 \leq 1$, ${}^C\mathcal{D}_{0+}^\bullet$ is the Caputo derivative. FBVPs are useful for modeling systems with memory and hereditary effects. The Caputo derivative is particularly suitable because BCs can be expressed using both standard and fractional derivatives. In iterative problems, the repeated application of fractional operators models feedback and long-range dependencies, as observed in anomalous diffusion, viscoelastic materials, and population dynamics. In [44], the authors studied a two-point IFDE with simple BCs. In contrast, this work investigates a class of IFDEs with mixed fractional BCs, allowing for richer memory effects. Our analysis employs a two-metric fixed-point approach in Banach and metric spaces.

To the best of our knowledge, a systematic and unified treatment that simultaneously addresses the solvability and HUS for such IFDEs, particularly under generalized fractional operators and nonlinear iterative structures, remains underexplored in the current literature. This work aims to fill that gap by presenting new results supported by illustrative examples.

The layout of the work is as follows:

- **Sect. 2:** Provides fundamental definitions and lemmas that shed light on the behaviour of solutions to the FBVP (1)-(2).
- **Sect. 3:** Deals with the construction of the kernel and the derivation of its bounds.
- **Sect. 4:** Demonstrates the existence, uniqueness, and HUS of the FBVP (1)-(2).
- **Sect. 5:** Offers an example that validates the main results obtained.

2. Auxiliary results

Our main results rely on key concepts from fractional calculus and stability theory, which are summarized in this section.

Definition 1 [13, 16] Let $\varphi \in L^1((z_0, z_1); \mathbb{R})$, where $(z_0, z_1) \in \mathbb{R}^2$ with $z_0 < z_1$ and φ . Then

$$I_{z_0+}^{\vartheta_1} \varphi(z) = \frac{1}{\Gamma(\vartheta_1)} \int_{z_0}^z (z - \kappa)^{\vartheta_1 - 1} \varphi(\kappa) d\kappa, \quad \text{a.e. } z \in [z_0, z_1].$$

Definition 2 [45] The FBVP (1)-(2) is said to be HUS if there exists a constant $C_f > 0$ depending on function f s.t., for any $\varepsilon > 0$ and for every $v(z) \in C(J, \mathbb{R})$ satisfying

$$\left| {}^C\mathcal{D}_{0+}^{\vartheta_1} v(z) - f(z, v(z), v^{[2]}(z)) \right| < \varepsilon, \quad \forall z \in J,$$

there exists an exact solution $\widehat{v}(z) \in C(J, \mathbb{R})$ of (1)-(2) s.t.

$$|v(z) - \widehat{v}(z)| < C_f \varepsilon, \quad \forall z \in J.$$

Lemma 1 [13, 16] Let $\vartheta_1 \in (m - 1, m]$ for some $m \in \mathbb{N}$. Then the homogeneous Caputo-type fractional differential equation

$${}^C\mathcal{D}_{0+}^{\vartheta_1} v(z) = 0$$

has the general solution

$$v(z) = \sum_{j=0}^{m-1} D_j z^j,$$

where $D_j \in \mathbb{R}$ and $j = \overline{0, m-1}$.

Lemma 2 [13, 16] Let $\vartheta_1 \in (m-1, m]$ for some $m \in \mathbb{N}$. Then for any suitably smooth function φ , the identity

$$I_{0+}^{\vartheta_1} ({}^C \mathcal{D}_{0+}^{\vartheta_1} \varphi)(z) = \varphi(z) + \sum_{j=0}^{m-1} C_j z^j$$

holds, where $C_j \in \mathbb{R}$ and $j = \overline{0, m-1}$.

Lemma 3 [13] If $\vartheta_1 > 0$ and $\varphi \in C^m[0, z_1]$ for $m = \lceil \vartheta_1 \rceil$, then the Caputo derivative ${}^C \mathcal{D}_{0+}^{\vartheta_1} \varphi$ exists and satisfies

$${}^C \mathcal{D}_{0+}^{\vartheta_1} \varphi(z) = I_{0+}^{m-\vartheta_1} \left(\frac{d^m}{dz^m} \varphi(z) \right), \quad z \in (0, z_1].$$

Lemma 4 [12] Let $\varphi \in C[0, z_1]$. Then the Caputo fractional derivative satisfies the continuity estimate

$$\| {}^C \mathcal{D}_{0+}^{\vartheta_1} \varphi \|_{\infty} \leq \frac{1}{\Gamma(1-\vartheta_1)} \sup_{z \in [0, z_1]} \int_0^z (z-\kappa)^{-\vartheta_1} |\varphi'(\kappa)| d\kappa.$$

2.1 Supplementary note on alternative fractional derivatives

Although the present work focuses on the Caputo fractional derivative, which is particularly suited for problems involving initial conditions specified in terms of integer-order derivatives, it is worthwhile to briefly mention other fractional derivative formulations that extend the modeling flexibility of fractional calculus. These alternative operators broaden the scope of applications and provide complementary perspectives for analyzing complex systems.

Two-Scale Fractal Derivative. For $f: \mathbb{R}^+ \rightarrow \mathbb{R}$, the two-scale fractal derivative of order $\alpha \in (0, 1)$ is defined by

$$\mathcal{D}_{\lambda}^{\alpha} f(z) = \lim_{h \rightarrow 0^+} \frac{f(z+h^{\lambda}) - f(z)}{h^{\alpha}},$$

where $\lambda > 0$ denotes the fractal dimension parameter. This derivative is specifically crafted to model systems exhibiting fractal geometries, such as flow through porous media and heat transfer across rough surfaces, by incorporating scale-dependent fractal dimensions. It effectively captures spatial complexities where the Caputo derivative's integral memory effect, which accounts for the full process history, may fall short in representing local heterogeneities typical of fractal or heterogeneous media [37, 46, 47].

He's Fractal Derivative. For $f \in AC[0, z]$ and $0 < \alpha < 1$, He's fractal derivative is defined as

$$\mathcal{D}_{\mathbb{H}}^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \int_0^z (z-\xi)^{-\alpha} f'(\xi) d\xi,$$

which generalizes the Caputo derivative by relaxing smoothness requirements and is well-suited to nonsmooth or chaotic phenomena. Known for its capacity to handle nonsmooth, discontinuous, or chaotic systems, such as nonlinear oscillations in engineering, He's fractal derivative relaxes the stringent integrability and smoothness conditions of the Caputo derivative, making it particularly useful for iterative fractional boundary value problems in complex, non-ideal physical environments [48].

Remark 1 While our study focuses on Caputo-type derivatives, this brief introduction to alternative fractional derivatives highlights the broader landscape of fractional calculus and suggests potential avenues for both theoretical and applied research.

3. Kernel and bounds

The structure of the solution to the related Caputo-type FDE with sequential BCs is examined in this section, and kernel bounds are constructed.

Lemma 5 Let $h(z) \in C([0, 1], \mathbb{R})$. Then $v(z) \in C([0, 1], \mathbb{R})$ is a solution of the FDE

$${}^C\mathcal{D}_{0^+}^{\vartheta_1} v(z) = h(z), \quad z \in (0, 1), \quad (3)$$

satisfying the BCs (2) if and only if

$$v(z) = B_0 + D_0 \Gamma(2 - \vartheta_2)z + \int_0^1 G_{\vartheta_1}(z, \kappa) h(\kappa) d\kappa$$

where

$$G_{\vartheta_1}(z, \kappa) = \begin{cases} \frac{(z - \kappa)^{\vartheta_1 - 1}}{\Gamma(\vartheta_1)} - \frac{\Gamma(2 - \vartheta_2)z(1 - \kappa)^{\vartheta_1 - \vartheta_2 - 1}}{\Gamma(\vartheta_1 - \vartheta_2)}, & \kappa \leq z, \\ -\frac{\Gamma(2 - \vartheta_2)z(1 - \kappa)^{\vartheta_1 - \vartheta_2 - 1}}{\Gamma(\vartheta_1 - \vartheta_2)}, & z \leq \kappa. \end{cases} \quad (4)$$

Proof. For the FBVP (3), (2), let $v(z) \in C[0, 1]$ be the solution. Then, (3) can be written as follows in light of Lemma 2:

$$v(z) = \frac{1}{\Gamma(\vartheta_1)} \int_0^z (z - \kappa)^{\vartheta_1 - 1} h(\kappa) d\kappa - C_0 - C_1 z,$$

where $C_0, C_1 \in \mathbb{R}$. Using $v(0) = B_0$, we obtain $C_0 = -B_0$. Therefore

$$v(z) = \int_0^z \frac{(z - \kappa)^{\vartheta_1 - 1}}{\Gamma(\vartheta_1)} h(\kappa) d\kappa + B_0 - C_1 z. \quad (5)$$

Applying ${}^C\mathcal{D}_{0^+}^{\vartheta_2}$ on both sides of (5), we obtain

$${}^c \mathcal{D}_{0^+}^{\vartheta_2} v(z) = \int_a^z \frac{(z-\kappa)^{\vartheta_1-\vartheta_2-1}}{\Gamma(\vartheta_1-\vartheta_2)} h(\kappa) d\kappa - \frac{C_1}{\Gamma(2-\vartheta_2)} z^{1-\vartheta_2}. \quad (6)$$

Thus, ${}^c \mathcal{D}_{0^+}^{\vartheta_2} v(1) = D_0$ implies $C_1 = \Gamma(2-\vartheta_2) \left[\int_0^1 \frac{(1-\kappa)^{\vartheta_1-\vartheta_2-1}}{\Gamma(\vartheta_1-\vartheta_2)} h(\kappa) d\kappa - D_0 \right]$ and the unique solution of FBVP (3), (2) is given by

$$v(z) = \begin{cases} \int_0^z \left[\frac{(z-\kappa)^{\vartheta_1-1}}{\Gamma(\vartheta_1)} - \frac{\Gamma(2-\vartheta_2)z(1-\kappa)^{\vartheta_1-\vartheta_2-1}}{\Gamma(\vartheta_1-\vartheta_2)} \right] h(\kappa) d\kappa - \\ \int_z^1 \left[\frac{\Gamma(2-\vartheta_2)z(1-\kappa)^{\vartheta_1-\vartheta_2-1}}{\Gamma(\vartheta_1-\vartheta_2)} \right] h(\kappa) d\kappa + B_0 + D_0 \Gamma(2-\vartheta_2)z \\ = B_0 + D_0 \Gamma(2-\vartheta_2)z + \int_0^1 G_{\vartheta_1}(z, \kappa) h(\kappa) d\kappa \end{cases}$$

where $G_{\vartheta_1}(z, \kappa)$ is given in (4). The Figure 1 shows the diagrammatic view of kernel $G_{\vartheta_1}(z, \kappa)$ for $\vartheta_1 = 1.5$ and $\vartheta_2 = 0.5$. Direct computation yields the converse.

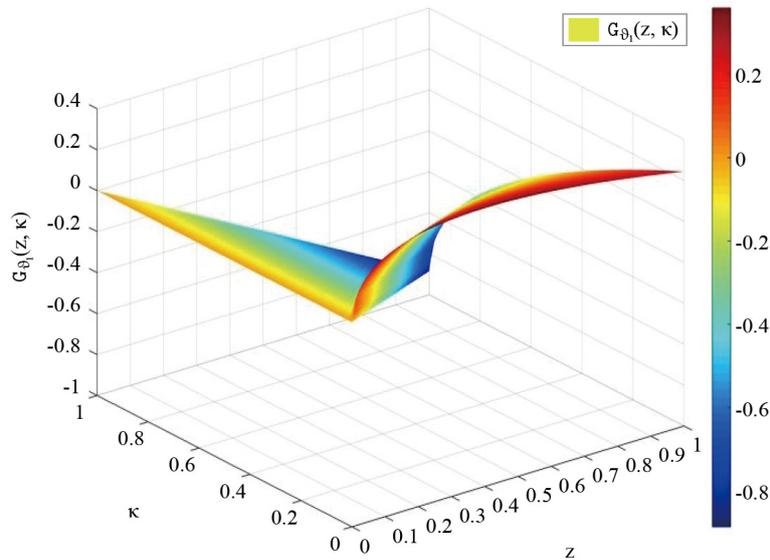


Figure 1. The kernel $G_{\vartheta_1}(z, \kappa)$ for $\vartheta_1 = 1.5, \vartheta_2 = 0.5$

□

Lemma 6 The kernel $G_{\vartheta_1}(z, \kappa)$ provided by (4) has the properties:

- (a) $|G_{\vartheta_1}(z, \kappa)| \leq |G_{\vartheta_1}(\kappa, \kappa)|, \quad \forall z, \kappa \in [0, 1],$
- (b) $\int_0^1 |G_{\vartheta_1}(\kappa, \kappa)| d\kappa = \frac{\Gamma(2-\vartheta_2)}{\Gamma(\vartheta_1-\vartheta_2+2)}.$

Proof. The Kernel $G_{\vartheta_1}(z, \kappa)$ is given in (4). Let $0 \leq z \leq \kappa \leq 1$. Then

$$\begin{aligned} |G_{\vartheta_1}(z, \kappa)| &= \left| \frac{(z - \kappa)^{\vartheta_1 - 1}}{\Gamma(\vartheta_1)} - \frac{\Gamma(2 - \vartheta_2)z(1 - \kappa)^{\vartheta_1 - \vartheta_2 - 1}}{\Gamma(\vartheta_1 - \vartheta_2)} \right| \\ &\leq \left| \frac{\Gamma(\vartheta_1 - \vartheta_2)(z - \kappa)^{\vartheta_1 - 1} - \Gamma(\vartheta_1)\Gamma(2 - \vartheta_2)z(1 - \kappa)^{\vartheta_1 - \vartheta_2 - 1}}{\Gamma(\vartheta_1)\Gamma(\vartheta_1 - \vartheta_2)} \right| \\ &\leq \frac{1}{\Gamma(\vartheta_1 - \vartheta_2)} \left| \Gamma(2 - \vartheta_2)\kappa(1 - \kappa)^{\vartheta_1 - \vartheta_2 - 1} \right| \\ &\leq |G_{\vartheta_1}(\kappa, \kappa)|. \end{aligned}$$

We can do the same for $0 \leq \kappa \leq z \leq 1$. Hence the inequality (a). To prove (b), we have

$$\begin{aligned} \int_0^1 |G_{\vartheta_1}(\kappa, \kappa)| d\kappa &= \frac{\Gamma(2 - \vartheta_2)}{\Gamma(\vartheta_1 - \vartheta_2)} \int_0^1 |-\kappa(1 - \kappa)^{\vartheta_1 - \vartheta_2 - 1}| d\kappa \\ &= \frac{\Gamma(2 - \vartheta_2)}{\Gamma(\vartheta_1 - \vartheta_2)} \left[\frac{1}{\vartheta_1^2 - 2\vartheta_1\vartheta_2 + \vartheta_1 + \vartheta_2(\vartheta_2 - 1)} \right] \\ &= \frac{\Gamma(2 - \vartheta_2)}{\Gamma(\vartheta_1 - \vartheta_2 + 2)}. \end{aligned}$$

This completes the proof. □

Remark 2 On $[0, 1] \times [0, 1]$, the kernel $G_{\vartheta_1}(z, \kappa)$ is continuous. The associated integral operator is compact and transforms bounded sets into equicontinuous sets in $C[0, 1]$, according to the estimate in Lemma 6.

We now define the operator associated with the FBVP:

$$(Tv)(z) = B_0 + D_0\Gamma(2 - \vartheta_2)z + \int_0^1 G_{\vartheta_1}(z, \kappa) f(\kappa, v(\kappa), v^{[2]}(\kappa)) d\kappa,$$

where f satisfies the appropriate continuity and growth conditions.

Note that this operator satisfies the BCs:

$$(Tv)(0) = B_0, \quad {}^C\mathcal{D}_{0^+}^{\vartheta_2}(Tv)(1) = D_0.$$

This formulation paves the way for applying fixed-point techniques in subsequent sections.

4. Main results

4.1 Hypotheses for the FBVP (1)-(2)

To ensure well-posedness and facilitate the application of fixed-point principles to the FBVP (1)-(2), we impose the following structural assumptions:

(H₁) $f: [0, 1]^3 \rightarrow \mathbb{R}$ is continuous. The image $v(z) \in [0, 1]$ is required for the solution to be well-defined, according to the iterative part $v^{[2]}(z)$. To achieve this, we require $0 \leq v(z) \leq 1$ for every $z \in [0, 1]$.

(H₂) There are positive constants Φ_1, Φ_2 s.t.

$$\left\{ \begin{array}{l} -\Phi_1 \leq f(z, a, b) \leq \Phi_2, \quad \forall z, a, b \in [0, 1], \text{ and} \\ \frac{D_0 \Gamma(\vartheta_1 - \vartheta_2 + 1)}{(\Phi_2 - 2\Phi_1) \Gamma(\vartheta_1) + \Gamma(\vartheta_1 - \vartheta_2 + 1) \Phi_1} > 1. \end{array} \right.$$

(H₃) There exist constants $\Phi_1 > 0$ and $\Phi_2 > 0$ s.t.

$$|f(z, a_1, b_1) - f(z, a_2, b_2)| \leq \Phi_1 |a_1 - a_2| + \Phi_2 |b_1 - b_2|,$$

$\forall z, a_1, a_2, b_1, b_2 \in [0, 1]$.

4.2 Existence of solution

Theorem 1 Assume (H₁)-(H₂) true. Then there exists a solution to the FBVP (1)-(2).

Proof. To demonstrate the findings, we use the Banach space $E = (C[0, 1], \|\cdot\|)$ with the norm specified as $E = \max_{z \in [0, 1]} |v(z)|$. For $R > 0$ define

$$\omega_R = \{v \in C[0, 1]: \|v\| \leq R\}.$$

Then ω_R is nonempty, convex, closed, and bounded in E .

By the Green representation for the problem (1)-(2), there exists a kernel $G_{\vartheta_1}(z, \kappa)$ such that any solution of the FBVP satisfies:

$$(Tv)(z) = B_0 + D_0 \Gamma(2 - \vartheta_2) z + \int_0^1 G_{\vartheta_1}(z, \kappa) f(\kappa, v(\kappa), v^{[2]}(\kappa)) d\kappa,$$

and we define $T: E \rightarrow E$ by the right-hand side above.

By hypothesis (H₂) there exists a constant $C_f > 0$ such that

$$|f(\kappa, x, y)| \leq C_f, \quad \text{for all } \kappa \in [0, 1], x, y \in [0, 1].$$

Using Lemma 6(b), we define

$$K_0 := \sup_{z \in [0, 1]} \int_0^1 |G_{\vartheta_1}(z, \kappa)| d\kappa \leq \int_0^1 |G_{\vartheta_1}(\kappa, \kappa)| d\kappa = \frac{\Gamma(2 - \vartheta_2)}{\Gamma(\vartheta_1 - \vartheta_2 + 2)} < \infty.$$

Choose

$$R := |B_0| + |D_0| \Gamma(2 - \vartheta_2) + C_f K_0,$$

and consider ω_R defined above with this R . Clearly ω_R is nonempty, convex, closed, and bounded.

For any $v \in \omega_R$ and every $z \in [0, 1]$ we have

$$\begin{aligned} |(Tv)(z)| &\leq |B_0| + |D_0| \Gamma(2 - \vartheta_2) |z| + \int_0^1 |G_{\vartheta_1}(z, \kappa)| |f(\kappa, v(\kappa), v^{[2]}(\kappa))| d\kappa \\ &\leq |B_0| + |D_0| \Gamma(2 - \vartheta_2) + C_f \int_0^1 |G_{\vartheta_1}(z, \kappa)| d\kappa \\ &\leq |B_0| + |D_0| \Gamma(2 - \vartheta_2) + C_f K_0 \leq R. \end{aligned}$$

Taking the supremum over $z \in [0, 1]$ yields $\|Tv\| \leq R$, so $T(\omega_R) \subseteq \omega_R$.

Hypothesis (H_1) gives continuity of f and the integral operator with continuous kernel G_{ϑ_1} is continuous on $C[0, 1]$.

Hence T is continuous on ω_R .

By Lemma 6(a), the kernel satisfies $|G_{\vartheta_1}(z, \kappa)| \leq |G_{\vartheta_1}(\kappa, \kappa)|$ for all $z, \kappa \in [0, 1]$, and G_{ϑ_1} is continuous on $[0, 1] \times [0, 1]$. For $v \in \omega_R$ the family $\{Tv: v \in \omega_R\}$ is uniformly bounded by R . Moreover, for $z_1, z_2 \in [0, 1]$,

$$\begin{aligned} |(Tv)(z_1) - (Tv)(z_2)| &\leq |D_0| \Gamma(2 - \vartheta_2) |z_1 - z_2| + \int_0^1 |G_{\vartheta_1}(z_1, \kappa) - G_{\vartheta_1}(z_2, \kappa)| |f(\kappa, v(\kappa), v^{[2]}(\kappa))| d\kappa \\ &\leq |D_0| \Gamma(2 - \vartheta_2) |z_1 - z_2| + C_f \int_0^1 |G_{\vartheta_1}(z_1, \kappa) - G_{\vartheta_1}(z_2, \kappa)| d\kappa. \end{aligned}$$

The right-hand side tends to 0 as $z_2 \rightarrow z_1$ uniformly in v , because G_{ϑ_1} is continuous on the compact set $[0, 1] \times [0, 1]$. Thus the family is equicontinuous. By Arzelà–Ascoli, $T(\omega_R)$ is relatively compact in E , so T is compact.

Applying Schauder's fixed-point theorem (see e.g. [49]) to the continuous, compact self-map $T: \omega_R \rightarrow \omega_R$ yields a fixed point $v \in \omega_R$. This fixed point is a solution of the FBVP (1)-(2). \square

4.3 Uniqueness of solution

Theorem 2 Suppose that conditions (H_1) – (H_3) are satisfied. Also, assume that

$$\left[\frac{\Gamma(2 - \vartheta_2)(\Phi_1 + \Phi_2)}{\Gamma(\vartheta_1 - \vartheta_2 + 2)} \right] < 1.$$

Then the FBVP (1)-(2) has a unique solution.

Proof. The fixed point of T is v since (H₁) holds. Let v₁ and v₂ be separated fixed points of T. Then, for any z ∈ [0, 1], one conclude

$$\begin{aligned}
 |v_2(z) - v_1(z)| &= |(Tv_2)(z) - (Tv_1)(z)| \\
 &= \left| \int_0^1 G_{\vartheta_1}(z, \kappa) \left[f(\kappa, v_2(\kappa), v_2^{[2]}(\kappa)) - f(\kappa, v_1(\kappa), v_1^{[2]}(\kappa)) \right] d\kappa \right| \\
 &\leq \int_0^1 |G_{\vartheta_1}(\kappa, \kappa)| \left[\Phi_1 |v_2(\kappa) - v_1(\kappa)| + \Phi_2 |v_2^{[2]}(\kappa) - v_1^{[2]}(\kappa)| \right] d\kappa \\
 &\leq \left[\frac{\Gamma(2 - \vartheta_2)(\Phi_1 + \Phi_2)}{\Gamma(\vartheta_1 - \vartheta_2 + 2)} \right] |v_2 - v_1| \\
 &< \|v_2 - v_1\|.
 \end{aligned}$$

This completes the proof. □

To prove the EU of a solution to the FBVP (1)-(2), we utilize two distinct metrics following the framework of Rus's FPT [50, 51]. For this purpose, let us consider the space of continuous real-valued functions on the interval [0, 1], denoted by E = C([0, 1]).

Take into consideration the two metrics on E for v₁, v₂ ∈ E,

$$\left. \begin{aligned}
 \text{(i)} \quad D(v_1, v_2) &= \max_{z \in [0, 1]} |v_2 - v_1|, \\
 \text{(ii)} \quad D^*(v_1, v_2) &= \left[\int_0^1 |v_2(z) - v_1(z)|^p \right]^{\frac{1}{p}}, \quad p > 1.
 \end{aligned} \right\} \quad (7)$$

The pair (C([0, 1]), D), where D is defined in (7)(i), forms a complete metric space. Similarly, the pair (C([0, 1]), D*), associated D* is provided in (7)(ii), constitutes a metric space.

The relationship between the two metrics on E:

$$D^*(v_1, v_2) \leq D(v_1, v_2), \quad \forall v_1, v_2 \in E. \quad (8)$$

Theorem 3 (Rus [52]). Let E be a non-empty set associated with two metrics, D and D*. Suppose that the metric space (E, D) is complete. For the D, let T: E → E be a mapping which is continuous. The following assertions are assumed to be true:

(i) For some constant Φ > 0, we have

$$D(Tv_1, Tv_2) \leq \Phi D^*(v_1, v_2) \quad \text{for all } v_1, v_2 \in E.$$

(ii) There exists a constant $\kappa \in (0, 1)$ such that

$$D^*(Tv_1, Tv_2) \leq \kappa D^*(v_1, v_2) \quad \text{for all } v_1, v_2 \in E.$$

Then, the mapping T has a unique fixed point $v^* \in E$ such that $Tv^* = v^*$.

Theorem 4 Assume that (H_1) and (H_3) are true. Moreover, let $p, q > 1$ be real constants s.t. $\frac{1}{p} + \frac{1}{q} = 1$. If the following inequality is satisfied:

$$(\Phi_1 + \Phi_2) \left(\int_0^1 |G_{\partial_1}(\kappa, \kappa)|^q d\kappa \right)^{\frac{1}{q}} < 1, \quad (9)$$

then the FBVP (1)-(2) has a unique (nontrivial) solution in E .

Proof. Let $z \in [0, 1]$ and $v_1, v_2 \in C([0, 1])$. In view of Hölder's inequality [53], we then have

$$\begin{aligned} & |(Tv_2)(z) - (Tv_1)(z)| \\ &= \left| \int_0^1 G_{\partial_1}(z, \kappa) \left[f(\kappa, v_2(\kappa), v_2^{[2]}(\kappa)) - f(\kappa, v_1(\kappa), v_1^{[2]}(\kappa)) \right] d\kappa \right| \\ &\leq \int_0^1 |G_{\partial_1}(z, \kappa)| \left| f(\kappa, v_2(\kappa), v_2^{[2]}(\kappa)) - f(\kappa, v_1(\kappa), v_1^{[2]}(\kappa)) \right| d\kappa \\ &\leq \int_0^1 |G_{\partial_1}(\kappa, \kappa)| \left[\Phi_1 |v_2(\kappa) - v_1(\kappa)| + \Phi_2 |v_2^{[2]}(\kappa) - v_1^{[2]}(\kappa)| \right] d\kappa \\ &\leq \left[\int_0^1 |G_{\partial_1}(\kappa, \kappa)|^q d\kappa \right]^{\frac{1}{q}} \times (\Phi_1 + \Phi_2) \left[\int_0^1 |v_2(\kappa) - v_1(\kappa)|^p d\kappa \right]^{\frac{1}{p}} \\ &\leq (\Phi_1 + \Phi_2) \left[\int_0^1 |G_{\partial_1}(\kappa, \kappa)|^q d\kappa \right]^{\frac{1}{q}} D^*(v_1, v_2). \end{aligned} \quad (10)$$

Thus, we define

$$\Phi = (\Phi_1 + \Phi_2) \left[\int_0^1 |G_{\partial_1}(\kappa, \kappa)|^q d\kappa \right]^{\frac{1}{q}},$$

we can observe that

$$D(Tv_1, Tv_2) \leq \Phi D^*(v_1, v_2) \quad (11)$$

condition (i) of Theorem 3 holds for all $v_1, v_2 \in E$ when $\Phi > 0$. Now, we can apply (8) to (11) and the inequality (i) of Theorem 3 to obtain for $v_1, v_2 \in E$,

$$D(Tv_1, Tv_2) \leq \Phi D^*(v_1, v_2) \leq \Phi D(v_1, v_2).$$

Consequently, given an arbitrary $\varepsilon > 0$, choosing $\delta = \frac{\varepsilon}{\Phi}$ guarantees that $D(v_1, v_2) < \varepsilon$ holds for all pairs satisfying $D(v_1, v_2) < \delta$. This establishes the continuity of T on E under the metric D . Furthermore, the analysis confirms that T acts as a contraction mapping on the set E when equipped with the distance function D^* . Consider the following for each $v_1, v_2 \in E$ from (10),

$$\begin{aligned} \left[\int_0^1 |(Tv_2)(z) - (Tv_1)(z)|^p dz \right]^{\frac{1}{p}} &\leq \left[\int_0^1 |(\Phi_1 + \Phi_2) \left[\int_0^1 |G_{\partial_1}(\kappa, \kappa)|^q d\kappa \right]^{\frac{1}{q}} D^*(v_1, v_2)|^p dz \right]^{\frac{1}{p}} \\ &\leq (\Phi_1 + \Phi_2) \left[\int_0^1 |G_{\partial_1}(\kappa, \kappa)|^q d\kappa \right]^{\frac{1}{q}} D^*(v_1, v_2). \end{aligned}$$

That is

$$D^*(Tv_1, Tv_2) \leq (\Phi_1 + \Phi_2) \left[\int_0^1 |G_{\partial_1}(\kappa, \kappa)|^q d\kappa \right]^{\frac{1}{q}} D^*(v_1, v_2).$$

From the assumption (9), we have

$$D^*(Tv_1, Tv_2) \leq \kappa D^*(v_1, v_2), \quad \text{for } \kappa < 1, \forall v_1, v_2 \in E.$$

Invoking Theorem 3, the operator T admits a unique fixed point in E . Consequently, the FBVP (1)-(2) possesses a unique (nontrivial) solution. \square

Previous Theorem 4 yields the novel result below for the possibilities $p = q = 2$.

Proposition 1 Assume that the hypotheses (H_1) and (H_3) are satisfied. Furthermore, suppose the inequality

$$(\Phi_1 + \Phi_2) \sqrt{\left(\int_0^1 |G_{\partial_1}(\kappa, \kappa)|^q d\kappa \right)^{\frac{1}{q}}} < 1 \tag{12}$$

holds. Under these conditions, the FBVP (1)-(2) admits a unique, nontrivial solution in E .

4.4 Hyers-Ulam stability

Theorem 5 Let the conditions (H_1) - (H_3) be satisfied. Then, the FBVP (1)-(2) exhibits HUS, provided that for every $\varepsilon > 0$, the solution $\hat{v}(z)$ of the problem (1)-(2) satisfies the inequality

$$\left| \widehat{v}(z) - B_0 - D_0 \Gamma(2 - \vartheta_2) z - \int_0^1 G_{\vartheta_1}(z, \kappa) f(\kappa, \widehat{v}(\kappa), \widehat{v}^{[2]}(\kappa)) d\kappa \right| < \varepsilon. \quad (13)$$

Proof. Let $\widehat{v}(z)$ be the unique solution of the FBVP (1)-(2). Then, we have

$$\begin{aligned} |\widehat{v}(z) - v(z)| &= \left\{ \begin{array}{l} \left| \widehat{v}(z) - B_0 - D_0 \Gamma(2 - \vartheta_2) z - \int_0^1 G_{\vartheta_1}(z, \kappa) f(\kappa, \widehat{v}(\kappa), \widehat{v}^{[2]}(\kappa)) d\kappa \right. \\ \left. + \int_0^1 G_{\vartheta_1}(z, \kappa) f(\kappa, \widehat{v}(\kappa), \widehat{v}^{[2]}(\kappa)) d\kappa \right. \\ \left. - \int_0^1 G_{\vartheta_1}(z, \kappa) f(\kappa, v(\kappa), v^{[2]}(\kappa)) d\kappa \right| \end{array} \right. \\ &\leq \left\{ \begin{array}{l} \left| \widehat{v}(z) - B_0 - D_0 \Gamma(2 - \vartheta_2) z - \int_0^1 G_{\vartheta_1}(z, \kappa) f(\kappa, \widehat{v}(\kappa), \widehat{v}^{[2]}(\kappa)) d\kappa \right| \\ \left. + \int_0^1 G_{\vartheta_1}(z, \kappa) \left| f(\kappa, \widehat{v}(\kappa), \widehat{v}^{[2]}(\kappa)) - f(\kappa, v(\kappa), v^{[2]}(\kappa)) \right| d\kappa \right. \end{array} \right. \\ &\leq \varepsilon + \int_0^1 G_{\vartheta_1}(z, \kappa) \left| f(\kappa, \widehat{v}(\kappa), \widehat{v}^{[2]}(\kappa)) - f(\kappa, v(\kappa), v^{[2]}(\kappa)) \right| d\kappa \\ &\leq \varepsilon + \int_0^1 |G_{\vartheta_1}(z, \kappa)| \left[\Phi_1 |\widehat{v}(\kappa) - v(\kappa)| + \Phi_2 |\widehat{v}^{[2]}(\kappa) - v^{[2]}(\kappa)| \right] d\kappa \\ &\leq \varepsilon + \left[\frac{\Gamma(2 - \vartheta_2)}{\Gamma(\vartheta_1 - \vartheta_2 + 2)} (\Phi_1 + \Phi_2) \right] \|\widehat{v} - v\| \\ &\leq \varepsilon + \|\widehat{v} - v\| \end{aligned}$$

that implies $\|\widehat{v} - v\| \leq C_f \varepsilon$, where $\left[1 - \frac{\Gamma(2 - \vartheta_2)}{\Gamma(\vartheta_1 - \vartheta_2 + 2)} (\Phi_1 + \Phi_2) \right]^{-1}$. The proof is now complete. \square

5. Illustrative example

Example 1 Consider the following iterative FBVP:

$${}^c \mathcal{D}_{0^+}^{\vartheta_1} v(z) = \lambda \cos(v^{[2]}(z)), \quad z \in [0, 1], \quad (14)$$

$$v(0) = \frac{\pi}{4}, \quad {}^c \mathcal{D}_{0^+}^{\vartheta_2} v(1) = \frac{\pi}{2}. \quad (15)$$

In this framework, the nonlinear term is

$$f(z, a, b) = \lambda \cos(v),$$

and since $\cos(\cdot)$ is bounded over \mathbb{R} , we have

$$-|\lambda| \leq \lambda \cos(v) \leq |\lambda|.$$

Thus, condition (H_2) is satisfied.

To ensure the existence of a solution to the FBVP (14)-(15), we invoke Theorem 1, which holds true for all values of λ satisfying

$$|\lambda| < \frac{\pi}{2} \left[\frac{2 - \vartheta_2}{\Gamma(\vartheta_1 - \vartheta_2)} \right].$$

Furthermore, by the Mean Value Theorem, there exists a point $\omega \in (0, 1)$ such that

$$\begin{aligned} |\lambda \cos(v_1) - \lambda \cos(v_2)| &= |\lambda| \cdot |\cos(v_1) - \cos(v_2)| \\ &= |\lambda| \cdot |\cos'(\omega)| \cdot |v_1 - v_2| \\ &= |\lambda| \cdot |-\sin(\omega)| \cdot |v_1 - v_2| \\ &\leq q|\lambda| \cdot |v_1 - v_2|. \end{aligned}$$

Thus, the nonlinearity f satisfies the Lipschitz-type condition given in hypothesis (H_3) . Hence, by Theorem 2, the FBVP (14)-(15) admits a unique solution whenever

$$|\lambda| < \frac{\Gamma(\vartheta_1 - \vartheta_2 + 2)}{\Gamma(2 - \vartheta_2)}.$$

Now, considering the structure of the kernel G_{ϑ_1} , the associated Lipschitz estimate is expressed as

$$(\Phi_1 + \Phi_2) \sqrt{\int_0^1 |G_{\vartheta_1}(\kappa, \kappa)|^2 d\kappa} = \frac{\Phi_1 + \Phi_2}{\sqrt{(2\vartheta_1 - 2\vartheta_2 - 1)(2\vartheta_1 - 2\vartheta_2 + 1)(\vartheta_1 - \vartheta_2)}}.$$

Hence, by Proposition 1, the FBVP (14)-(15) possesses a unique solution provided that

$$\frac{\Phi_1 + \Phi_2}{\sqrt{(2\vartheta_1 - 2\vartheta_2 - 1)(2\vartheta_1 - 2\vartheta_2 + 1)(\vartheta_1 - \vartheta_2)}} < 1.$$

To validate the theoretical findings numerically, consider the parameters

$$\vartheta_1 = 0.9, \quad \vartheta_2 = 0.5, \quad \lambda = 0.3.$$

For these values, all the above inequalities are satisfied, ensuring the existence and uniqueness of a positive solution. The numerical simulation (Figure 2) shows that the solution $v(z)$ increases smoothly from $v(0) = \frac{\pi}{4}$ to approximately $v(1) \approx 1.54$, in accordance with the prescribed BCs.

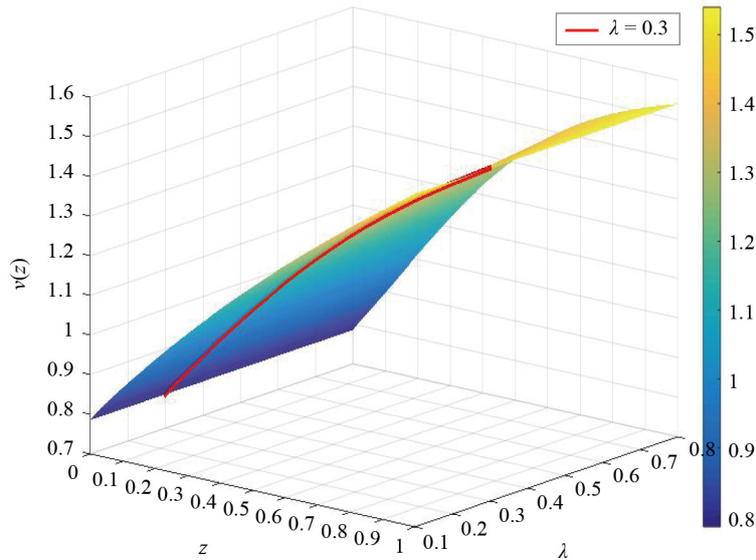


Figure 2. Solution behavior of the FBVP(14)-(15)

Figure 2 illustrates the solution curve for the given parameters, confirming the consistency of the numerical solution with the analytical bounds established in Theorems 1 and 2. This validates the theoretical model and highlights its applicability to fractional systems with trigonometric nonlinearities.

Finally, the HUS of the FBVP (14)-(15) is guaranteed by Theorem 5. Thus, this example demonstrates both the theoretical validity and practical applicability of the proposed framework.

6. Discussion

The stability constant C_f was observed to depend inversely on the kernel norm and the Lipschitz coefficients, which aligns with intuitive expectations: stronger nonlinearities or larger kernel values require tighter bounds to guarantee convergence. It is also important to recognize the limitations of the present study, particularly the restriction $0 \leq v(z) \leq 1$, which may limit the direct applicability of the results to certain classes of problems. Addressing these constraints in future work will allow the extension of the framework to more general fractional systems and broader functional settings, thereby enhancing both the theoretical relevance and practical applicability of the findings.

7. Conclusion

In this study, we investigated the EU of solutions for the FBVP (1)-(2), involving a differential operator interpreted in the Caputo sense. By applying Schauder's FPT along with the Banach contraction principle in the setting of a Banach space, we verified the existence of solutions under suitable hypotheses. Moreover, through the use of Rus's contraction theorem in a metric space framework, we formulated new conditions ensuring the uniqueness of the solution to the FBVP (1)-(2). We also established the HUS of the model, thereby confirming the solution's stability against minor perturbations. The findings of this work not only enhance the theoretical understanding of IFDEs but also lay the groundwork for future research on their applications in complex dynamical systems.

Beyond the theoretical analysis, it is also important to validate such models through explicit or approximate solutions. Well-established analytical techniques in the field of FDEs, such as VIM and HPM methods, can be effectively applied to nonlinear iterative systems. These approaches not only generate approximate solutions without linearization but also allow direct comparison with theoretical predictions, thereby reinforcing the stability and convergence properties demonstrated in this work. Future research will focus on integrating such methods into the current framework to strengthen the practical relevance of the results.

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Conflict of interest

The authors declare no competing financial interest.

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