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Research Article

Positive Solutions of Hybrid Nonlinear Integro-Fractional Differential Equations via Dhage's Fixed Point Theorem

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Abstract: This paper investigates the existence, uniqueness, and monotonicity of positive solutions for a new class of hybrid nonlinear fractional integro-differential equations involving the Caputo–Fabrizio derivative. Unlike the classical Caputo operator, the Caputo–Fabrizio derivative is characterized by a non-singular exponential kernel, which leads to improved analytical tractability and better modeling of memory effects. Using Dhage's fixed point theorem, sufficient conditions for the well-posedness of the problem are derived. A concrete example is provided to demonstrate the effectiveness of the theoretical results, showing the relevance of the Caputo–Fabrizio approach in fractional modeling.

Keywords: existence, uniqueness, monotonicity, Caputo-Fabrizio derivative, fixed point theorems, operator theory

MSC: 34A08, 39A30, 39A13, 34K20, 47H1

1. Introduction

Fractional calculus has emerged as a robust mathematical framework that generalises classical differential and integral operators to non-integer orders, allowing for the modeling of complex dynamical systems with memory and hereditary properties [1, 2]. This extended framework has gained significant attention due to its broad applications across scientific and engineering disciplines, including viscoelastic materials [3], anomalous diffusion processes [4], thermal systems [5], control theory [6], and biological modeling [7, 8].

A key advancement in fractional calculus is the Caputo–Fabrizio derivative, introduced as an alternative to traditional fractional derivatives [9]. Unlike the classical Riemann–Liouville and Caputo derivatives, the Caputo–Fabrizio derivative is characterized by a non-singular and exponential kernel, which simplifies numerical computations and enhances its suitability for modeling systems with memory effects [10, 11]. Several recent studies have extended and enriched the theory of this operator. For instance, Alqhtani et al. [12] introduced the Mittag Leffler–Caputo–Fabrizio fractional derivative together with a numerical approach, while Nosheen et al. [13] established new results on Caputo derivatives and Caputo–Fabrizio integral operators via (s, m)-convex functions. Benzahi et al. [14] investigated Caputo–Fabrizio type

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fractional differential equations with non-instantaneous impulses and proved existence and stability results. Moumen et al. [15] proposed a novel fractional integral associated with the Caputo–Fabrizio derivative, and Ali et al. [16] analyzed integro-differential equations involving the Caputo–Fabrizio operator together with the fractional q-integral of the Riemann–Liouville type.These contributions highlight the growing importance of the Caputo–Fabrizio derivative in the literature and its effectiveness in capturing complex dynamical behaviors, with successful applications already reported in diffusion models [17] and control systems [18].

Hybrid fractional differential equations which integrate integro-differential components provide a versatile framework for modeling systems that exhibit both local and non-local dynamics [19, 20]. These equations are particularly useful in representing phenomena with intricate interactions and time-dependent behaviors. The incorporation of the Caputo-Fabrizio derivative into hybrid fractional equations has opened new avenues for analyzing systems where classical methods may be inadequate, such as non-Markovian processes [21], fractal networks [22], and bioengineering applications [23, 24].

The study of positive solutions in hybrid fractional differential equations is of fundamental importance, particularly in modeling physical, biological, and engineering systems where positivity is a natural requirement for instance, in population dynamics and diffusion processes. Theoretical advancements in this area frequently rely on fixed-point theorems, which provide rigorous conditions ensuring the existence of positive solutions. Notably, Dhage's fixed-point theorem has been instrumental in analyzing hybrid equations by integrating the principles of fractional calculus with functional analysis techniques [25]. Recent research has extended these approaches to non-singular kernel derivatives, such as the Caputo-Fabrizio derivative, enabling the study of systems with memory effects and finite-time singularities [11, 26, 27]. Additionally, other fixed-point techniques, including Krasnoselskii's theorem and its variants, have been widely employed to establish the existence of positive solutions under various boundary conditions [28–32]. These methodologies offer insights into the behavior of nonlinear systems with hybrid dynamics, paving the way for practical applications in science and engineering [24]. Haoues et al. [33] analyzed the following mathematical problem:

$$\begin{cases} {}^{c}\mathfrak{D}_{t_0}^{\zeta}\left(\frac{w(t)}{h(t)+\frac{1}{\Gamma(\mu)}\int_{t_0}^{t}(t-\vartheta)^{\mu-1}\Phi(\vartheta,\, w(\vartheta))\mathrm{d}\vartheta}\right)=\Psi\left(t, w\left(t\right)\right), \quad t_0\le t\le T, \\ w\left(t_0\right)=h\left(t_0\right)\theta\ge 0, \end{cases} \tag{1}$$

where ${}^{c}\mathfrak{D}_{t_0}^{\zeta}$ represents the Caputo derivative of fractional order $0<\zeta\le 1$, $0<\mu\le 1$, $0\le t_0<T$. The functions Ψ, Φ: $[t_0,\, T]\times\mathbb{R}\to\mathbb{R}$ and h: $[t_0,\, T]\to\mathbb{R}$ are continuous. Their work established the existence, uniqueness, and monotonicity of positive solutions.

This research focuses on the hybrid nonlinear fractional integro-differential equation:

$$\begin{cases} {}^{CF}\mathfrak{D}^{\zeta}\left(\frac{w(t)}{h(t)+{}^{CF}\mathfrak{I}^{\mu}\Phi(\vartheta,\, w(\vartheta))}\right)=\Psi\left(t, w\left(t\right)\right), \quad 0\le t\le 1, \\ \left.\frac{w(t)}{h(t)+{}^{CF}\mathfrak{I}^{\mu}\Phi(\vartheta,\, w(\vartheta))}\right|_{t=0}=\theta\ge 0, \end{cases} \tag{2}$$

where ${}^{CF}\mathfrak{D}^{\zeta}$ denotes the Caputo–Fabrizio fractional derivative of order $0<\zeta\le 1$ and $\mathfrak{I}_{CF}^{\mu}$ is the corresponding fractional integral of order $0<\mu\le 1$, Ψ, Φ: $[0,\, 1]\times\mathbb{R}\to\mathbb{R}$ and h: $[0,\, 1]\to\mathbb{R}$ are given continuous functions. This paper builds upon the foundational work of Haoues et al. [33] and others [34], extending the theoretical framework to investigate the existence, uniqueness, and monotonicity of positive solutions in hybrid fractional differential equations. By employing Dhage's fixed-point theorem [25] alongside related analytical techniques [29], we establish rigorous conditions that ensure the existence of such solutions. These contributions significantly expand the applicability of hybrid fractional differential equations in modeling complex systems.

The importance of the present study lies in extending hybrid fractional models to operators with non-singular kernels, which provide a more accurate representation of processes with memory and hereditary effects while avoiding the limitations caused by singular kernels. Such models are particularly relevant in physics, bioengineering, and control theory, where positivity of solutions is a natural requirement. At the same time, the motivation for this study arises from the growing demand for mathematical tools capable of capturing the intricate dynamics of hybrid systems. By exploiting the distinctive properties of the Caputo–Fabrizio derivative, this work aims to deliver new insights into the modeling, analysis, and solution of hybrid fractional differential equations, particularly in domains where memory and hereditary effects play a central role [35, 36].

The paper is structured as follows: Section 2 introduces essential mathematical tools and definitions. Section 3 presents the main results, including theorems on the existence, uniqueness, and monotonicity of solutions. Section 4 illustrates the theoretical findings through a practical example. Finally, Section 5 concludes with a discussion on the study implications and potential directions for future research.

2. Essential materials

This section introduces the fundamental definitions, theorems, and mathematical tools essential for the analysis and discussion in this study.

Definition 1 ([2]) The fractional integral of order $\zeta > 0$ for a continuous function w: $[0, +\infty) \to \mathbb{R}$ is represented as:

$$\mathfrak{I}^{\zeta} w(t) = \frac{1}{\Gamma(\zeta)} \int_{t_0}^{t} (t-s)^{\zeta-1} w(s)\, ds, \tag{3}$$

where $\Gamma(\zeta)$ denotes the Gamma function.

Definition 2 ([2]) The Caputo fractional derivative of order $\zeta > 0$ for a continuous function w: $[0, +\infty) \to \mathbb{R}$ is given by:

$${}^{C}\mathfrak{D}^{\zeta} w(t) = \mathfrak{I}^{n-\zeta} w^{(n)}(t) = \frac{1}{\Gamma(n-\zeta)} \int_{0}^{t} (t-s)^{n-\zeta-1} w^{(n)}(s)\, ds, \tag{4}$$

where $n = [\zeta] + 1$, and $w^{(n)}(t)$ represents the n-th derivative of $w(t)$.

Definition 3 ([9]) The Caputo-Fabrizio fractional integral of order $0 < \zeta < 1$ for a function $w \in L^1 [0, 1]$ is expressed as:

$${}^{CF}\mathfrak{I}^{\zeta} w(t) = \frac{2(1-\zeta)}{\mathscr{M}(\zeta)(2-\zeta)} w(t) + \frac{2\zeta}{M(\zeta)(2-\zeta)} \int_{0}^{t} w(s) ds, \quad t \geq 0, \tag{5}$$

where $\mathscr{M}(\zeta)$ represents a normalization constant dependent on ζ. For instance, when $\mathscr{M}(\zeta) = \frac{2}{2-\zeta}$, this simplifies to:

$${}^{CF}\mathfrak{I}^{\zeta} w(t) = (1-\zeta) w(t) + \zeta \int_{0}^{t} w(s) ds, \quad t \geq 0, \tag{6}$$

Definition 4 ([9]) The Caputo-Fabrizio fractional derivative of order $0 < \zeta < 1$ for a function $w \in AC [0, 1]$ is defined as:

$$
{}^{CF}\mathfrak{D}^{\zeta} w(t) = \frac{(2-\zeta)\mathscr{M}(\zeta)}{2(1-\zeta)} \int_0^t \exp\left(-\frac{\zeta}{1-\zeta}(t-s)\right) w'(s) ds, \quad t \in [0,\, 1]. \tag{7}
$$

It is worth noting that ${}^{CF}\mathfrak{D}^{\zeta} w(t) = 0$ if and only if w is a constant function. When $\mathscr{M}(\zeta) = \frac{2}{2-\zeta}$, the derivative takes the form:

$$
{}^{CF}\mathfrak{D}^{\zeta} w(t) = \frac{1}{(1-\zeta)} \int_0^t \exp\left(-\frac{\zeta}{1-\zeta}(t-s)\right) w'(s) ds, \quad t \in [0,\, 1]. \tag{8}
$$

Lemma 1 Let $0 < \zeta \leq 1$ and suppose that $\frac{w}{l}$ is differentiable on $[0,\, 1]$ and $\Psi(t) \in L^1[0,\, 1]$. Then the equation

$$
\begin{cases} {}^{CF}\mathfrak{D}^{\zeta}\left(\frac{w(t)}{l(t)}\right) = \Psi(t), \quad 0 \leq t \leq 1, \\ w(0) = l(0)\,\theta \geq 0, \end{cases} \tag{9}
$$

is equivalent to

$$
w(t) = l(t)\left(c + a_{\zeta}\Psi(t) + b_{\zeta}\int_0^t \Psi(\vartheta)\,\mathrm{d}\vartheta\right), \quad 0 \leq t \leq 1, \tag{10}
$$

where

$$
a_{\zeta} = \frac{2(1-\zeta)}{(2-\zeta)M(\zeta)}, \quad b_{\zeta} = \frac{2\zeta}{(2-\zeta)M(\zeta)}, \quad c = \theta - a_{\zeta}\Psi(0). \tag{11}
$$

Proof. Assume $w(t)$ satisfies (9). From [27, Proposition 1], the equation

$$
{}^{CF}\mathfrak{D}^{\zeta}\left(\frac{w(t)}{l(t)}\right) = \Psi(t) \tag{12}
$$

implies

$$
\frac{w(t)}{l(t)} - \frac{w(0)}{l(0)} = a_{\zeta}\left(\Psi(t) - \Psi(0)\right) + b_{\zeta}\int_0^t \Psi(\vartheta)\,\mathrm{d}\vartheta. \tag{13}
$$

Thus from the initial condition $w(0) = l(0)\,\theta$, we obtain

$$
\frac{w(t)}{l(t)} = \theta - a_{\zeta}\Psi(0) + a_{\zeta}\Psi(t) + b_{\zeta}\int_0^t \Psi(\vartheta)\,\mathrm{d}\vartheta. \tag{14}
$$

Hence we get (10). The converse implication is easly.

The fixed point theorem established by Dhage [37] is an indispensable tool for proving our main results.

Theorem 1 ([25, 37]) Consider Q as a non-empty, bounded, closed, and convex subset of a Banach algebra X. Let $S\colon Q \to X$ and $R\colon Q \to X$ be two operators that satisfy the following conditions:

1) R is Lipschiz operator with constant ς,

2) S is completely continuous,

3) $SvRw \in Q$ for all $v, w \in Q$.

Then the equation $RwSw = w$ admits a solution, with $\varsigma M < 1$, where $M = \sup\{\|Rw\| : w \in Q\}$.

3. Main results

Let $C[0, 1]$ denote the Banach space of all real-valued continuous functions defined over the compact interval $[0, 1]$, equipped with the maximum norm. Define the subset

$$\mathscr{C}_\theta = \left\{w \in C[0, 1] : w(t) \geq \left(h(0) + a_\mu \Phi(0, w(0))\right)\theta, \quad 0 \leq t \leq 1\right\}, \tag{15}$$

of $C[0, 1]$.

Due to the lemma 1, $w(t)$ is a solution of problem (2) if and only if $w(t)$ satisfies the equation

$$w(t) = \left(h(t) + a_\mu \Phi(t, w(t)) + b_\mu \int_0^t \Phi(\vartheta, w(\vartheta))\,\mathrm{d}\vartheta\right)$$

$$\times \left(c + a_\zeta \Psi(t, w(t)) + b_\zeta \int_0^t \Psi(\vartheta, w(\vartheta))\,\mathrm{d}\vartheta\right). \tag{16}$$

In order to effectively make use of Dhage fixed point theorem 1, we define the operators $R, S\colon \mathscr{C}_\theta \to \mathscr{C}_\theta$ by

$$(Rw)(t) = h(t) + a_\mu \Phi(t, w(t)) + b_\mu \int_0^t \Phi(\vartheta, w(\vartheta))\,\mathrm{d}\vartheta, \tag{17}$$

and

$$(Sw)(t) = c + a_\zeta \Psi(t, w(t)) + b_\zeta \int_0^t \Psi(\vartheta, w(\vartheta))\,\mathrm{d}\vartheta, \tag{18}$$

for any $t \in [0, 1]$.

Definition 5 For any $w \in [a, b] \subset \mathbb{R}^+$, we define the upper-control function by

$$U(t, w) = \sup\{\Psi(t, \vartheta) : a \leq \vartheta \leq w\}, \tag{19}$$

and the lower-control function by

$$L(t, w) = \inf\{\Psi(t, \vartheta) : w \leq \vartheta \leq b\}. \tag{20}$$

It is clear that these functions are non-decreasing on $[a, b]$, i.e.

$$L(t, w) \leq \Psi(t, w) \leq U(t, w), \quad t \in [0, 1]. \tag{21}$$

The following assumptions will be outlined for use in the subsequent sections.

(C1) Let $t \in [0, 1]$ and $w \in C[0, 1]$, we have

$$h(t) + a_\mu \Phi(t, w(t)) + b_\mu \int_0^t \Phi(\vartheta, w(\vartheta)) \mathrm{d}\vartheta > 0, \tag{22}$$

and

$$c + a_\zeta \Psi(t, w(t)) + b_\zeta \int_0^t \Psi(\vartheta, w(\vartheta)) \mathrm{d}\vartheta \geq 0. \tag{23}$$

(C2) Let $\overline{w}$, $\underline{w} \in \mathscr{C}_\theta$ be the upper and lower solutions for (2) respectively, such that $\overline{w}(0) = \underline{w}(0) = (h(0) + a_\mu \Phi(0, w(0))\theta$ and $\left(h(0) + a_\mu \Phi(0, w(0))\right)\theta \leq \underline{w}(0) \leq \overline{w}(0) \leq b$, $t \in [0, 1]$. Moreover,

$$\begin{cases} {}^{CF}\mathfrak{D}_0^\zeta \left(\frac{\overline{w}(t)}{h(t) + a_\mu \Phi(t, \overline{w}(t)) + b_\mu \int_0^t \Phi(\vartheta, \overline{w}(\vartheta)) \mathrm{d}\vartheta} \right) \geq U(t, \overline{w}(t)), \\ {}^{CF}\mathfrak{D}_0^\zeta \left(\frac{\underline{w}(t)}{h(t) + a_\mu \Phi(t, \underline{w}(t)) + b_\mu \int_0^t \Phi(\vartheta, \underline{w}(\vartheta)) \mathrm{d}\vartheta} \right) \leq L(t, \underline{w}(t)), \end{cases} \tag{24}$$

for any $t \in [0, 1]$.

(C3) Let Φ be monotonic non-decreasing function with respect to w and there exists $L_2 > 0$ such that

$$|\Phi(t, w) - \Phi(t, w)| \leq L_2 |v - w|, \; v, w \in \mathbb{R}, \tag{25}$$

and for if $w \in \mathscr{C}_\theta$ there exists a positive constant c_1 such that

$$\max\{|\Psi(t, w(t))| : (t, w) \in [0, 1] \times \mathscr{C}_\theta\} \leq c_1, \tag{26}$$

where

$$0 < L_2 \left(a_\mu + b_\mu\right) \left(|c| + c_1 \left(a_\zeta + b_\zeta\right)\right) < 1. \tag{27}$$

(C4) There exists $L_1 > 0$ such that

$$|\Psi(t, v) - \Psi(t, w)| \leq L_1 |v - w|, v, w \in \mathbb{R}. \tag{28}$$

3.1 *Existence of positive solutions*

In this subsection, we will examine the existence of positive solutions for (2).

Theorem 2 Suppose that (C1)–(C3) are fulfilled. Then the problem (2) admits at last one positive bounded below solution that is bounded below by $w \in \mathscr{C}_\theta$ satisfying $\underline{w}(t) \leq w(t) \leq \overline{w}(t), t \in [0, 1]$.

Proof. Let $Q = \{w \in \mathscr{C}_\theta : \underline{w}(t) \leq w(t) \leq \overline{w}(t), t \in [t_0, T]\}$, endowed with the norm $\|w\| = \max_{t\in[0, 1]} |w(t)|$, then for any $w \in Q$, we have $\|w\| \leq b$. Hence, Q is a convex, bounded and closed subset of $\mathscr{C}_\theta$. Moreover, the continuity of Φ and Ψ implies the continuity of the operators R and S on Q. Now, if $w \in Q$ then by (C3) we have

$$\begin{aligned} |(Sw)(t)| &= \left| c + a_\zeta \Psi(t, w(t)) + b_\zeta \int_0^t \Psi(\vartheta, w(\vartheta))\, \mathrm{d}\vartheta \right| \\ &\leq |c| + a_\zeta c_1 + b_\zeta c_1 \int_0^t \mathrm{d}\vartheta \\ &\leq |c| + a_\zeta c_1 + b_\zeta c_1. \end{aligned} \tag{29}$$

Hence, $S(Q)$ is uniformly bounded. Next, we show the equicontinuity of S. Let $w \in Q$, then for any $t_1, t_2 \in [0, 1], t_2 > t_1$, we get

$$\begin{aligned} &|(Sw)(t_2) - (Sw)(t_1)| \\ =7 &\left| a_\zeta \Psi(t_2, w(t_2)) + b_\zeta \int_0^{t_2} \Psi(\vartheta, w(\vartheta))\, \mathrm{d}\vartheta - a_\zeta \Psi(t_1, w(t_1)) + b_\zeta \int_0^{t_1} \Psi(\vartheta, w(\vartheta))\, \mathrm{d}\vartheta \right| \\ \leq &a_\zeta |\Psi(t_2, w(t_2)) - \Psi(t_1, w(t_1))| + b_\zeta \int_{t_1}^{t_2} |\Psi(\vartheta, w(\vartheta))|\, \mathrm{d}\vartheta \\ = &a_\zeta |\Psi(t_2, w(t_2)) - \Psi(t_1, w(t_1))| + b_\zeta c_1 |t_2 - t_1|. \end{aligned} \tag{30}$$

As $t_1 \to t_2$ the right-hand side of the above inequality tends to zero. Consequently, $S(Q)$ is equicontinuous. The theorem of Arzela-Ascoli implies that S is compact. Hence S is completely continuous operator.

Now, by hypothesis (C3), for any $v, w \in Q$, we get

$$
\begin{aligned}
|(Rv)(t)-(Rw)(t)| &\leq a_\mu |\Phi(t, v(t))-\Phi(t, w(t))| + b_\mu \int_0^t |\Phi(\vartheta, v(\vartheta))-\Phi(\vartheta, w(\vartheta))| \mathrm{d}\vartheta \\
&\leq a_\mu L_2 \|v-w\| + b_\mu L_2 \int_0^t \mathrm{d}\vartheta \, \|v-w\| \\
&\leq \left(a_\mu + b_\mu\right) L_2 \|v-w\|. \qquad (31)
\end{aligned}
$$

Then R is Lipschiz mapping with Lipschitz constant $\varsigma = \left(a_\mu + b_\mu\right) L_2$, that satisfying $\varsigma \sup\{\|Sw\| : w \in Q\} < 1$. We need to prove that $RvSw \in Q$ for all $v, w \in Q$. Then, by Definition 5 and the hypothesis (C3), we have

$$
\begin{aligned}
(Rv)(t)(Sw)(t) &= \left(h(t) + a_\mu \Phi(t, w(t)) + b_\mu \int_0^t \Phi(\vartheta, w(\vartheta)) \mathrm{d}\vartheta\right) \\
&\quad \times \left(c + a_\zeta \Psi(t, w(t)) + b_\zeta \int_0^t \Psi(\vartheta, w(\vartheta)) \mathrm{d}\vartheta\right) \\
&\leq \left(h(t) + a_\mu \Phi(t, w(t)) + b_\mu \int_0^t \Phi(\vartheta, w(\vartheta)) \mathrm{d}\vartheta\right) \\
&\quad \times \left(c + a_\zeta U(t, w(t)) + b_\zeta \int_0^t U(\vartheta, w(\vartheta)) \mathrm{d}\vartheta\right) \\
&\leq \left(h(t) + a_\mu \Phi(t, \overline{w}(t)) + b_\mu \int_0^t \Phi(\vartheta, \overline{w}(\vartheta)) \mathrm{d}\vartheta\right) \\
&\quad \times \left(c + a_\zeta U(t, \overline{w}(t)) + b_\zeta \int_0^t U(\vartheta, \overline{w}(\vartheta)) \mathrm{d}\vartheta\right) \\
&\leq \overline{w}(t), \qquad (32)
\end{aligned}
$$

and

$$
\begin{aligned}
(Rv)(t)(Sw)(t) &= \left(h(t) + a_\mu \Phi(t, w(t)) + b_\mu \int_0^t \Phi(\vartheta, w(\vartheta)) \mathrm{d}\vartheta\right) \\
&\quad \times \left(c + a_\zeta \Psi(t, w(t)) + b_\zeta \int_0^t \Psi(\vartheta, w(\vartheta)) \mathrm{d}\vartheta\right)
\end{aligned}
$$

$$\geq \left(h(t)+a_{\mu}\Phi(t, w(t))+b_{\mu}\int_{0}^{t}\Phi(\vartheta, w(\vartheta))\,\mathrm{d}\vartheta\right)$$

$$\times\left(c+a_{\zeta}L(t, w(t))+b_{\zeta}\int_{0}^{t}L(\vartheta, w(\vartheta))\,\mathrm{d}\vartheta\right)$$

$$\geq \left(h(t)+a_{\mu}\Phi(t, \underline{w}(t))+b_{\mu}\int_{0}^{t}\Phi(\vartheta, \underline{w}(\vartheta))\,\mathrm{d}\vartheta\right)$$

$$\times\left(c+a_{\zeta}L(t, \underline{w}(t))+b_{\zeta}\int_{0}^{t}L(\vartheta, \underline{w}(\vartheta))\,\mathrm{d}\vartheta\right)$$

$$\geq \underline{w}(t). \tag{33}$$

Hence, $\underline{w}(t) \leq (Rv)(t)(Sw)(t) \leq \overline{w}(t)$, $t \in [0, 1]$, that is $(RvSw)(Q) \subseteq Q$. By Dhage's fixed point theorem, the equation $RwSw = w$ admits fixed point $w \in Q$. Then, the problem (2) admits positive bounded below solution $w \in \mathscr{C}_{\theta}$.

Next, the preceding theorem introduces the following specific cases.

Corollary 1 Suppose that (C3) holds and there exist $k_1, k_2, k_3, k_4 \in C[0, 1]$, such that

$$k_1(t) \leq \Phi(t, w(t)) \leq k_2(t), \tag{34}$$

and

$$k_3(t) \leq \Psi(t, w(t)) \leq k_4(t). \tag{35}$$

If

$$h(t)+a_{\mu}k_1(t)+b_{\mu}\int_{0}^{t}k_1(\vartheta)\,\mathrm{d}\vartheta > 0, \tag{36}$$

and

$$c+a_{\zeta}k_3(t)+b_{\zeta}\int_{0}^{t}k_3(\vartheta)\,\mathrm{d}\vartheta \geq 0, \tag{37}$$

then the problem (2) admits at least one positive solution that is bounded below. Moreover

$$\left(h(t)+a_\mu k_1(t)+b_\mu\int_0^t k_1(\vartheta)\,\mathrm{d}\vartheta\right)\left(c+a_\zeta k_3(t)+b_\zeta\int_0^t k_3(\vartheta)\,\mathrm{d}\vartheta\right)$$

$$\leq w(t)$$

$$\leq\left(h(t)+a_\mu k_2(t)+b_\mu\int_0^t k_2(\vartheta)\,\mathrm{d}\vartheta\right)\left(c+a_\zeta k_4(t)+b_\zeta\int_0^t k_4(\vartheta)\,\mathrm{d}\vartheta\right). \tag{38}$$

Proof. By the definition of control functions and the assumption (35), we obtain

$$k_3(t)\leq L(t,\,w(t))\leq U(t,\,w(t))\leq k_4(t),\quad \text{for all } t\in[0,\,1].$$

Now, we consider the fractional differential equations

$$\begin{cases} {}^{CF}\mathfrak{D}_0^{\zeta}\left(\frac{w(t)}{h(t)+a_\mu k_1(t)+b_\mu\int_0^t k_1(\vartheta)\mathrm{d}\vartheta}\right)=k_3(t),\quad w(0)=\big(h(0)+a_\mu k_1(0)\big)\,\theta \\ {}^{CF}\mathfrak{D}_0^{\zeta}\left(\frac{w(t)}{h(t)+a_\mu k_2(t)+b_\mu\int_0^t k_2(\vartheta)\mathrm{d}\vartheta}\right)=k_4(t),\quad w(0)=\big(h(0)+a_\mu k_2(0)\big)\,\theta \end{cases} \tag{39}$$

According to Lemma 1, the solutions of (39) are given by

$$w(t)=\left(h(t)+a_\mu k_1(t)+b_\mu\int_0^t k_1(\vartheta)\,\mathrm{d}\vartheta\right)\left(c+a_\zeta k_3(t)+b_\zeta\int_0^t k_3(\vartheta)\,\mathrm{d}\vartheta\right), \tag{40}$$

and

$$w(t)=\left(h(t)+a_\mu k_2(t)+b_\mu\int_0^t k_2(\vartheta)\,\mathrm{d}\vartheta\right)\left(c+a_\zeta k_4(t)+b_\zeta\int_0^t k_4(\vartheta)\,\mathrm{d}\vartheta\right). \tag{41}$$

Therefore,

$$w(t)=\left(h(t)+a_\mu k_1(t)+b_\mu\int_0^t k_1(\vartheta)\,\mathrm{d}\vartheta\right)\left(c+a_\zeta k_3(t)+b_\zeta\int_0^t k_3(\vartheta)\,\mathrm{d}\vartheta\right)$$

$$\leq\left(h(t)+a_\mu k_1(t)+b_\mu\int_0^t k_1(\vartheta)\,\mathrm{d}\vartheta\right)\left(c+a_\zeta L(t,\,w(t))+b_\zeta\int_0^t L(\vartheta,\,w(\vartheta))\,\mathrm{d}\vartheta\right), \tag{42}$$

and

$$w(t)=\left(h(t)+a_\mu k_2(t)+b_\mu\int_0^t k_2(\vartheta)\,\mathrm{d}\vartheta\right)\left(c+a_\zeta k_4(t)+b_\zeta\int_0^t k_4(\vartheta)\,\mathrm{d}\vartheta\right)$$

$$\geq\left(h(t)+a_\mu k_2(t)+b_\mu\int_0^t k_2(\vartheta)\,\mathrm{d}\vartheta\right)\left(c+a_\zeta U(t,\,w(t))+b_\zeta\int_0^t U(\vartheta,\,w(\vartheta))\,\mathrm{d}\vartheta\right). \tag{43}$$

Thus, we can characterize the upper and lower solutions accordingly.

$$\overline{w}(t)=\left(h(t)+a_\mu k_2(t)+b_\mu\int_0^t k_2(\vartheta)\,\mathrm{d}\vartheta\right)\left(c+a_\zeta U(t,\,w(t))+b_\zeta\int_0^t U(\vartheta,\,w(\vartheta))\,\mathrm{d}\vartheta\right), \tag{44}$$

and

$$\underline{w}(t)=\left(h(t)+a_\mu k_1(t)+b_\mu\int_0^t k_1(\vartheta)\,\mathrm{d}\vartheta\right)\left(c+a_\zeta L(t,\,w(t))+b_\zeta\int_0^t L(\vartheta,\,w(\vartheta))\,\mathrm{d}\vartheta\right). \tag{45}$$

Hence, due to Theorem 2, we infer that the problem (2) admits a positive $w\in\mathscr{C}_\theta$ that is bounded below.

Corollary 2 Suppose that (C3), (34) and (36) fulfilled, and

$$\lim_{w\to\theta}\frac{\Psi(t,\,w)}{w}=\alpha(t), \tag{46}$$

where $\alpha\in C[0,\,1]$, $t\in[0,\,1]$. Then the equation (2) admits a positive solution that is bounded below.

3.2 *Uniqueness positive solutions*

In this subsection, we will use Banach contraction mapping principle to show that (2) have unique positive solution that is bounded below.

Theorem 3 Suppose that hypotheses (C1)–(C4) are fulfilled. If

$$(a_\mu+b_\mu)L_2\left(|c|+(a_\zeta+b_\zeta)c_1\right)+(a_\zeta+b_\zeta)L_1\left(h_m+(a_\mu+b_\mu)c_2\right)<1, \tag{47}$$

then the problem (2) admits a unique positive solution that is bounded below.

Proof. In view of Theorem 2, the problem (2) admits a positive bounded below solution in Q. It is now necessary to demonstrate that the product operator $RwSw$ is a contraction on $C[0,\,1]$. Indeed, for any $v,\,w\in\mathscr{C}_\theta$ and $t\in[0,\,1]$, we get

$$|(RvSv)(t)-(RwSw)(t)|$$

$$=\left|\left(h(t)+a_\mu\Phi(t,\,v(t))+b_\mu\int_0^t\Phi(\vartheta,\,v(\vartheta))\,\mathrm{d}\vartheta\right)\right.$$

$$\times\left(c+a_{\zeta}\Psi(t,\,v(t))+b_{\zeta}\int_0^t\Psi(\vartheta,\,v(\vartheta))\,\mathrm{d}\vartheta\right)$$

$$-\left(h(t)+a_{\mu}\Phi(t,\,w(t))+b_{\mu}\int_0^t\Phi(\vartheta,\,w(\vartheta))\,\mathrm{d}\vartheta\right)$$

$$\times\left(c+a_{\zeta}\Psi(t,\,w(t))+b_{\zeta}\int_0^t\Psi(\vartheta,\,w(\vartheta))\,\mathrm{d}\vartheta\right)\Bigg|$$

$$\leq\left(a_{\mu}\left|\Phi(t,\,v(t))-\Phi(t,\,w(t))\right|+b_{\mu}\int_0^t\left|\Phi(\vartheta,\,v(\vartheta))-\Phi(\vartheta,\,w(\vartheta))\right|\mathrm{d}\vartheta\right)$$

$$\times\left(|c|+a_{\zeta}\left|\Psi(t,\,w(t))\right|+b_{\zeta}\int_0^t\left|\Psi(\vartheta,\,w(\vartheta))\right|\;\mathrm{d}\vartheta\right)$$

$$+\left(a_{\zeta}\left|\Psi(t,\,v(t))-\Psi(t,\,w(t))\right|+b_{\zeta}\int_0^t\left|\Psi(\vartheta,\,v(\vartheta))-\Psi(\vartheta,\,w(\vartheta))\right|\mathrm{d}\vartheta\right)$$

$$\times\left(|h(t)|+a_{\mu}\left|\Phi(t,\,v(t))\right|+b_{\mu}\int_0^t\left|\Phi(\vartheta,\,v(\vartheta))\right|\mathrm{d}\vartheta\right), \tag{48}$$

therefore,

$$\left|(RvSv)(t)-(RwSw)(t)\right|$$

$$\leq\left(a_{\mu}+b_{\mu}\right)L_2\left\|v-w\right\|\left(|c|+\left(a_{\zeta}+b_{\zeta}\right)c_1\right)$$

$$+\left(a_{\zeta}+b_{\zeta}\right)L_1\left\|v-w\right\|\left(h_m+\left(a_{\mu}+b_{\mu}\right)c_2\right)$$

$$=\left[\left(a_{\mu}+b_{\mu}\right)L_2\left(|c|+\left(a_{\zeta}+b_{\zeta}\right)c_1\right)+\left(a_{\zeta}+b_{\zeta}\right)L_1\left(h_m+\left(a_{\mu}+b_{\mu}\right)c_2\right)\right]\left\|v-w\right\|, \tag{49}$$

where

$$\max\left\{\left|\Psi(t,\,w(t))\right|:(t,\,w)\in[0,\,1]\times\mathscr{C}_{\theta}\right\}\leq c_1, \tag{50}$$

$$\max\left\{\left|\Phi(t,\,w(t))\right|:(t,\,w)\in[0,\,1]\times\mathscr{C}_{\theta}\right\}\leq c_2, \tag{51}$$

and

$$h_m = \max\{|h(t)| : t \in [0, 1]\}. \tag{52}$$

Hence, by (47) we can conclude that the product operator $RwSw$ is a contraction. Therefore, we have ensured that problem (2) admits a positive bounded below solution $w \in \mathscr{C}_\theta$ that is unique.

3.3 *Monotonicity of positive solutions*

Theorem 4 Suppose that h, Ψ, and Φ are non-decreasing functions in both of their arguments, and that $\Psi(0, w(0)) \geq 0$ as well as $\Phi(0, w(0)) \geq 0$. Further, if the conditions (C1) through (C4) are satisfied, then the problem (2) admits a positive, bounded below solution that is monotonic and non-decreasing.

Proof. Let us define the subset $Q_{nd} = \{w \in Q\colon w \text{ is non-decreasing on } [0, 1]\}$. This subset Q_{nd} is both closed and convex within Q. The operator S is uniformly bounded and compact, while the operator R is Lipschitz continuous with a Lipschitz constant ς, satisfying $\varsigma \sup\{\|Sw\| : w \in Q_{nd}\} < 1$. To use Dhage's theorem, it suffices to show that $RwSw \in Q_{nd}$ for all $w \in Q_{nd}$.

To establish this, consider $v, w \in Q_{nd}$ and let $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$. It remains to demonstrate that the operators preserve the monotonicity of the elements in Q_{nd}. From this setup, we observe the following

$$\begin{aligned}
&(RvSw)(t_2) - (RvSw)(t_1)\\
&= (Rv)(t_2)((Sw)(t_2) - (Sw)(t_2)) + ((Rv)(t_2) - (Rv)(t_1))(Sw)(t_1)\\
&= \left(h(t) + a_\mu \Phi(t_2, v(t_2)) + b_\mu \int_0^{t_2} \Phi(\vartheta, v(\vartheta))\, d\vartheta\right)\\
&\quad \times \left(c + a_\zeta [\Psi(t_2, w(t_2)) - \Psi(t_1, w(t_1))] + b_\zeta \int_{t_1}^{t_2} \Psi(\vartheta, w(\vartheta))\, d\vartheta\right)\\
&\quad + \left(h(t_2) - h(t_1) + a_\mu [\Phi(t_2, v(t_2)) - \Phi(t_1, v(t_1))] + b_\mu \int_{t_1}^{t_2} \Phi(\vartheta, v(\vartheta))\, d\vartheta\right)\\
&\quad \times \left(c + a_\zeta \Psi(t_1, w(t_1)) + b_\zeta \int_0^{t_1} \Psi(\vartheta, w(\vartheta))\, d\vartheta\right)\\
&\geq 0.
\end{aligned} \tag{53}$$

This follows from condition (C1) and the fact that h, Ψ, and Φ are non-decreasing functions.

Consequently, applying Dhage's fixed-point theorem ensures that the composite operator $RS\colon Q_{nd} \to Q_{nd}$ has a fixed point. This fixed point satisfies the required positivity and monotonicity properties, thereby serving as a solution to problem (2).

4. An example

Consider the following problem

$$\begin{cases} {}^{CF}\mathfrak{D}^{\zeta}\left(\frac{w(t)}{46+t+\frac{1}{2}\left(1+\vartheta(t)+\frac{w(t)}{w(t)+100}\right)+\frac{1}{2}\int_0^t\left(1+\vartheta(s)+\frac{w(s)}{w(s)+100}\right)ds}\right)=t^2+\frac{t^2w(t)}{w(t)+200}, \quad 0\le t\le 1, \\ w(0)=48, \end{cases} \tag{54}$$

$${}^{CF}\mathfrak{D}^{\zeta}w(t)=\frac{1}{(1-\zeta)}\int_0^t \exp\left(-\frac{\zeta}{1-\zeta}(t-s)\right)w'(s)ds, \quad t\in[0,\,1]. \tag{55}$$

where $\zeta=\frac{1}{4}$, $\mu=\frac{1}{2}$, $\theta=\frac{1}{3}$, $\Psi(t,\,w)=t^2+\frac{t^2w}{w+200}$, $\Phi(t,\,w)=1+t+\frac{w}{w+100}$ and $h(t)=46+t$. We can find Φ is non-decreasing on w. Then

$$0\le t^2\le\Psi(t,\,w)\le t^2+1\le 2, \text{ for } t\in[0,\,1],\, w\in[0,\,\infty), \tag{56}$$

$$1\le 1+t\le\Phi(t,\,w)\le 2+t\le 3, \text{ for } t\in[0,\,1],\, w\in[0,\,\infty), \tag{57}$$

$L_2=\frac{1}{100}$, $L_1=\frac{1}{200}$, $c_2=3$, $c_1=2$, $c=48$ and by taking $M(\zeta)=\frac{2}{2-\zeta}$, $M(\mu)=\frac{2}{2-\mu}$ we obtain

$$a_\zeta=1-\zeta=\frac{3}{4},\quad b_\zeta=\zeta=\frac{1}{4}, \tag{58}$$

$$a_\mu=1-\mu=\frac{1}{2},\quad b_\mu=\mu=\frac{1}{2}. \tag{59}$$

Furthermore, by substituting all the aforementioned calculations in (C3), we get

$$L_2\left(a_\mu+b_\mu\right)\left(|c|+c_1\left(a_\zeta+b_\zeta\right)\right)=0.5<1. \tag{60}$$

Hence, by Corollary 1, the problem (54) admits a positive solution.
From this and condition (47), this positive solution is unique. Indeed,

$$\left(a_\mu+b_\mu\right)L_2\left(|c|+\left(a_\zeta+b_\zeta\right)c_1\right)+\left(a_\zeta+b_\zeta\right)L_1\left(h_m+\left(a_\mu+b_\mu\right)c_2\right)=0.75<1. \tag{61}$$

Then, by Corollary 1, problem (54) has a positive solution which satisfies

$$
\left(46+t+\frac{1}{2}(0)+\frac{1}{2}(0)\right)\left(48+\frac{3}{4}1+\frac{1}{4}t\right)
$$

$$
\leq w(t)
$$

$$
\leq\left(46+t+\frac{1}{2}2+\frac{1}{2}2t\right)\left(48+\frac{3}{4}(3)+\frac{1}{4}(3t)\right), \tag{62}
$$

where the lower solution is

$$
\underline{w}(t)=\left(\frac{1}{4}t+\frac{195}{4}\right)(t+46), \tag{63}
$$

and the upper solution is

$$
\overline{w}(t)=\frac{3}{2}t^2+\frac{543}{4}t+\frac{9447}{4}. \tag{64}
$$

5. Conclusion

This study extends the theory of hybrid nonlinear integro-fractional differential equations to the setting of the Caputo–Fabrizio derivative. By proving existence, uniqueness, and monotonicity of positive solutions via Dhage's fixed point theorem, we highlight the new possibilities offered by non-singular kernels. Future research may address multidimensional systems, variable-order derivatives, and numerical implementations to strengthen the connection between theory and applications.

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Conflict of interest

The authors declare no competing financial interest.

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