Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Collocation Method for Functional 3-Kind Volterra Integral Equations

Dana Tahsin Abdul Rahman[1], Parviz Darania[1,2*]

[1] Department of Mathematics, Urmia University, Urmia, Iran
[2] Department of Mathematics, University of Bonab, Bonab, Iran
E-mail: p.darania@urmia.ac.ir

Received: 30 July 2025; **Revised:** 18 September 2025; **Accepted:** 23 September 2025

Abstract: In this work, we present a numerical approach based on the collocation method for solving weakly singular Volterra Integral Equations of the Third Kind (3K-VIEs) with proportional delays. We employ modified graded meshes to analyze the solvability and convergence of the proposed method. Furthermore, we rigorously establish the convergence rate in terms of the parameter p for the collocation scheme using m collocation parameters. To validate the theoretical findings, we provide numerical examples, demonstrating the reliability and effectiveness of the method.

Keywords: 3K-VIEs, proportional delays, collocation methods, solvability, convergence, noncompact operators

MSC: 45A05, 45D05, 45E99

1. Introduction

Volterra integral equations have diverse applications in fields like population dynamics, viscoelasticity, and engineering. They are particularly useful for modeling systems where the current state depends on the past history of the system, such as in the study of hereditary effects, material behavior, and population growth with memory effects. Also, the Volterra integral operator, with its weakly singular kernel, provides the essential mathematical structure that defines fractional derivatives and integrals. This isn't just a convenient tool for solving Fractional Differential Equations (FDEs), it is the language in which they are inherently written.

Considering that most phenomena in nature have nonlinear mathematical models and one of the powerful tools for describing and analyzing these models is nonlinear integral equations. In [1], Webb, studied some of these equations. Also, by using some linearization methods, nonlinear equations can be converted into linear equations. Therefore, we studied the linear equation in full detail. If necessary, the nonlinear equation can be linearized, and the results of this reserch can be used to analyze that equation.

We investigate the functional Volterra Integral Equation of the Third Kind (3K-VIE):

$$t^{\mu} y(t) = (\mathscr{K}_{\gamma} y)(t) + (\mathscr{K}_{\gamma,\, q} y)(t) + f(t), \qquad t \in \Delta, \tag{1}$$

where $\Delta = [t_0,\, T]$, and the Volterra integral operators $\mathscr{K}\,\gamma$ and $\mathscr{K}\,\gamma,\, q$ are defined as:

DOI: https://doi.org/10.37256/cm.6620258065

$$
\begin{aligned}
(\mathscr{K}_\gamma y)(t) &= \int_0^t (t-z)^{-\gamma} H(t,\, z) y(z) \mathrm{d}z, \\
(\mathscr{K}_{\gamma,\, q} y)(t) &= \int_0^{qt} (t-z)^{-\gamma} H_D(t,\, z) y(z) \mathrm{d}z,
\end{aligned}
\tag{2}
$$

with the following conditions:

1) Delay parameter: $0 < q < 1$.

2) Singularity parameter: $0 < \gamma \leq 1$.

3) Regularity parameter: $\mu > 0$.

4) Forcing term: $f \in C(\Delta)$.

5) Kernel functions: $H \in C(\Omega)$, and $H_D \in C(\Omega_D)$, where $\Omega = \{(t,\, z)\colon 0 \leq z \leq t \leq T\}$ and $\Omega_D = \{(t,\, z)\colon 0 \leq z \leq qt,\, t \in \Delta\}$.

Note that the term $(t-s)^{-\gamma}$ (with $0 < \gamma \leq 1$) in both integrals introduces a weakly singular kernel. This singularity is integrable but poses significant challenges for both analysis and numerical computation.

As it is discussed in papers [2] and [3], even in the case where there is no integral with proportional delay, the existence and uniqueness of the solution of problem (1) is subject to several conditions on the parameters μ and γ, as well as on the spectrum of the associated integral operator.

In their 2021 work, Song et al. [4] analyzed the specialized form of equation (1), with $\mu = 1$ and $\gamma = 0$:

$$
ty(t) = \int_0^t H(t,\, z) y(z),\, \mathrm{d}z + \int_0^{qt} H_D(t,\, z) y(z),\, \mathrm{d}z + f(t), \quad t \in \Delta,
\tag{3}
$$

demonstrating that the solution's existence and uniqueness properties are fundamentally governed by the delay parameter q.

Subsequently, in 2022, Song et al. [5] extended the analysis of [4] to a nonlinear 3K-VIE of the form

$$
ty(t) = \int_0^t H(t,\, z) K(y(z)),\, \mathrm{d}z + \int_0^{qt} H_D(t,\, z) K_D(y(z)),\, \mathrm{d}z + tf(t).
\tag{4}
$$

Song et al. [4, 5] developed a collocation method utilizing modified graded meshes to numerically solve equations (3) and (4). Their work demonstrated that:

I. Equation (3) with both compact and noncompact kernels is solvable and admits a unique solution when using this mesh approach.

II. They comprehensively analyzed the regularity, existence, and uniqueness properties of analytical solutions for both equations (3) and (4).

We observe that, when $H_D \equiv 0$ in equation (1), the equation simplifies to a weakly singular 3K-VIE:

$$
t^\mu y(t) = (K_\gamma y)(t) + f(t), \quad t \in \Delta.
\tag{5}
$$

The foundational analytical properties of solutions to equation (3), including existence, uniqueness, and regularity were rigorously established by Allaei et al. [2]. Their work derived explicit sufficient conditions guaranteeing the existence of unique continuous solutions on the interval I. Building on these theoretical results, Allaei et al. [3]

subsequently developed a comprehensive numerical analysis of the collocation method applied to (3), with particular focus on solvability and convergence analysis.

Collocation methods are widely recognized as one of the most effective approaches for solving integral equations [6–10]. The transition of collocation methods from a theoretical numerical analysis concept to a foundational tool in scientific and engineering applications has had a profound practical impact. The method's core principle involves approximating the solution to differential or integral equations with a polynomial. This polynomial is constrained to satisfy the equation exactly at a selected set of collocation points while still obeying the problem's boundary conditions.

The primary practical benefit of this approach is its ability to transform an intractable, infinite-dimensional problem into a finite-dimensional system of algebraic equations that computers can solve efficiently. Consequently, the significant advantages of collocation methods include their high degree of accuracy, capacity to manage complex path constraints, suitability for direct transcription in optimization problems, and the widespread availability of robust software implementations.

The practical impact of collocation methods is that they provide a computational "superpower". They enable scientists and engineers to translate complex, continuous dynamic problems from the real world into a form that computers can reliably and efficiently solve. This has been a key enabler for some of the most ambitious technological achievements of our time. They bridge the gap between theoretical models and practical, optimal solutions.

However, despite their success, the literature remains surprisingly sparse regarding their application to functional Volterra Integral Equations of the Third Kind (3K-VIEs) of the form (1). This work aims to bridge this gap by extending the collocation method to solve (1), offering several key advantages:

- Computational Efficiency: The method reduces the problem to a well-structured system of linear equations that naturally decomposes into smaller subsystems.
- Theoretical Foundation: We build upon and generalize earlier research to handle the more complex case of (1).

The paper is organized as follows: Section 2 introduces the collocation scheme and analyzes the solvability of the resulting system for (1). Section 3 presents a rigorous error analysis and establishes the convergence order. Section 4. provides numerical experiments validating the theoretical results and demonstrating the method's reliability.

2. Description of the method

Suppose that, collocation equation $u_n \in S_n^{(-1)}(\Delta_n)$ of the equation (1), given by

$$t^{\mu} u_n(t) = f(t) + (\mathscr{K}_{\gamma} u_n)(t) + (\mathscr{K}_{\gamma,\, q} u_n)(t), \quad t \in \Omega_n, \tag{6}$$

and, let Δ_n and Ω_n be given by

$$\Delta_n = \{t_n\colon 0 = t_0 < t_1 < \cdots < t_N = T,\ N > 0\},$$

and

$$\Omega_n\colon = \{t_{n,\, j} = t_n + c_j h_n\colon 0 < c_1 < \cdots < c_m \leq 1,\ 0 \leq n \leq N-1\},$$

where Δ_n the modified graded mesh that ensures the solvability of (6), given by

$$t_n = (T - t_1)(\tfrac{n-1}{N-1})^{\varsigma} + t_1, \quad \varsigma > 1, \quad n = 1, 2, ..., N, \tag{7}$$

where $t_0 = 0$, and $t_1 = \frac{T}{N}$, and $\{c_i\}_{i=1}^{m}$ are collocation parameters, $\{t_{n,\,j}\}$ are collocation points and $h_n = t_{n+1} - t_n$ are step-size. On each subinterval $\Gamma_n = (t_n,\, t_{n+1}]$, the approximations u_n, are the polynomials of degree m can be expressed in the form

$$u_n(t_n + sh_n) = \sum_{j=1}^{m} \Psi_j(s) U_{n,\,j}, \tag{8}$$

where

$$\Psi_j(s) = \prod_{\substack{i=1 \\ i \neq j}}^{m} \frac{s - c_i}{c_j - c_i}, \tag{9}$$

is the j-th Lagrange fundamental polynomial concerning the collocation parameters and

$$U_{n,\,j} = u_n(t_{n,\,j}) = u_n(t_n + c_j h_n).$$

For more detail, see [3].

The results of the investigations carried out to solve the delayed integral equations in comparison to the classical integral equations indicate that in the delayed equations with vanishing delay, the exact location of point $qt_{n,\,i}$ cannot be determined easily. Therefore, to determine the exact location of $qt_{n,\,i}$, according to [4], we have the following discussion.

Here, by regarding the proportional delay function qt, with $0 < q < 1$, for the collocation points $t_{n,\,i} := t_n + c_i h_n \in \Gamma_n$. The following is how we intend to define the generic ideas in terms of $q_{n,\,i}$ and $c_{n,\,i}$:

$$q_{n,\,i} = \lfloor q(n + c_i) \rfloor, \quad c_{n,\,i} := q(n + c_i) - q_{n,\,i} \in [0,\, 1).$$

The so-called collocation points are $c_i \in (0,\, 1]$, $i = 1, ..., m$. In addition, $\lfloor t \rfloor$ represents the largest integer less than or equal to the usual t. Second, as stated in [3], because the key idea for the location of the delay images is $qt_{n,\,i}$ and the defined collocation points $t_{n,\,i}$ are involved in Γ_n, we must discuss and come up with three basic phases as what follows. The notation $\lceil t \rceil$ is used to display the smallest integer that is bigger than and/or equal to t.

Phase I. If $0 \leq n < q^I$ is satisfied by some n, then $qt_{n,\,i} \in (t_n,\, t_{n+1}) = \lceil \frac{qc_1}{1-q} \rceil$ where the conditions $q_{n,\,i} = n$ and $i = 1, ..., m$ are met.

Phase II. As long as $q_I \leq n < q^{II} := \lceil \frac{qc_m}{1-q} \rceil$, we counter two cases so that there is a positive integer, namely v_n, for which we have $v_n < m$. More significantly, the two circumstances are as follows:

$$qt_{n,\,i} \in \begin{cases} (t_{n-1},\, t_n], & i = 1, 2, ..., v_n, \quad \text{Phase II-A}, \\ (t_n,\, t_{n+1}], & i = v_{n+1}, ..., v_m, \quad \text{Phase II-B}. \end{cases}$$

Phase III. There are two positive integers in terms of v_n and q_n such that $v_n \leq m$, $q_n < n-1$, and the following two instances are satisfied, provided that there exists n such that $q^{II} \leq n \leq N-1$:

$$qt_{n,\,i} \in \begin{cases} [t_{q_n},\, t_n], & i = 1,\, 2,\, \ldots,\, v_n, \\ (t_{q_{n+1}},\, t_n], & i = v_{n+1},\, \ldots,\, v_m. \end{cases}$$

For linear operators $(\mathscr{K}_\gamma u_h)(t_{n,\,i})$ and $(\mathscr{K}_{\gamma,\,q} u_h)(t_{n,\,i})$, we obtain

$$\begin{aligned} (\mathscr{K}_\gamma u_h)(t_{n,\,i}) &= \int_{t_0}^{t_{n,\,i}} (t_{n,\,i} - s)^{-\alpha} H(t_{n,\,i},\, s) u_h(s) \mathrm{d}s \\ &= \sum_{l=0}^{r-1} h_l \int_0^1 (t_{n,\,i} - t_l - sh_l)^{-\alpha} K(t_{n,\,i},\, t_l + sh_l) u_h(t_l + sh_l) \mathrm{d}s \\ &\quad + \sum_{l=r}^{n-1} h_l \int_0^1 (t_{n,\,i} - t_l - sh_l)^{-\alpha} K(t_{n,\,i},\, t_l + sh_l) u_h(t_l + sh_l) \mathrm{d}s \\ &\quad + h_n \int_0^{c_i} (t_{n,\,i} - t_n - sh_n)^{-\alpha} K(t_{n,\,i},\, t_n + sh_n) u_h(t_n + sh_n) \mathrm{d}s. \end{aligned} \tag{10}$$

By substituting the equation (8) in equation (12), we obtain

$$\begin{aligned} (\mathscr{K}_{\gamma,\,q} u_h)(t_{n,\,i}) &= \sum_{l=0}^{r-1} h_l \int_0^1 (t_{n,\,i} - t_l - sh_l)^{-\alpha} K(t_{n,\,i},\, t_l + sh_l) \left(\sum_{j=1}^{m} \Psi_j(s) U_{l,\,j} \right) \mathrm{d}s \\ &\quad + \sum_{l=r}^{n-1} h_l \int_0^1 (t_{n,\,i} - t_l - sh_l)^{-\alpha} K(t_{n,\,i},\, t_l + sh_l) \left(\sum_{j=1}^{m} \Psi_j(s) U_{l,\,j} \right) \mathrm{d}s \\ &\quad + h_n \int_0^{c_i} (t_{n,\,i} - t_n - sh_n)^{-\alpha} K(t_{n,\,i},\, t_n + sh_n) \left(\sum_{j=1}^{m} \Psi_j(s) U_{n,\,j} \right) \mathrm{d}s. \end{aligned} \tag{11}$$

In the same manner, we have

$$\begin{aligned} (\mathscr{K}_{\gamma,\,q} u_h)(t_{n,\,i}) = &\sum_{l=0}^{q_{n,\,i}-1} h_l \int_0^1 (t_{n,\,i} - t_l - sh_l)^{-\gamma} H_D(t_{n,\,i},\, t_l + sh_l) u_h(t_l + sh_l) \mathrm{d}s \\ &+ h_{q_{n,\,i}} \int_0^{c_{n,\,i}} (t_{n,\,i} - t_{q_{n,\,i}} - sh_{q_{n,\,i}})^{-\gamma} H_D(t_{n,\,i},\, t_{q_{n,\,i}} + sh_{q_{n,\,i}}) u_h(t_{q_{n,\,i}} + sh_{q_{n,\,i}}) \mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{q_{n,i}-1} h_l \int_0^1 (t_{n,i} - t_l - sh_l)^{-\gamma} H_D(t_{n,i}, t_l + sh_l) \left(\sum_{j=1}^{m} \Psi_j(s) U_{l,j} \right) \mathrm{d}s \\
&+ h_{q_{n,i}} \int_0^{c_{n,i}} (t_{n,i} - t_{q_{n,i}} - sh_{q_{n,i}})^{-\gamma} H_D(t_{n,i}, t_{q_{n,i}} + sh_{q_{n,i}}) \left(\sum_{j=1}^{m} \Psi_j(s) U_{q_{n,i},j} \right) \mathrm{d}s
\end{aligned} \tag{12}
$$

Upon substituting $t = t_{n,i}$ into equation (1) and making use of equation (8), it follows that

$$
\begin{aligned}
t_{n,i}^{\mu} u_h(t_{n,i}) =& f(t_{n,i}) + (\mathscr{K}_{\gamma} u_h)(t_{n,i}) + (\mathscr{K}_{\gamma,q} u_h)(t_{n,i}) \\
=& f(t_{n,i}) + \sum_{l=0}^{n-1} h_l^{1-\gamma} \sum_{j=1}^{m} \int_0^1 (\frac{t_{n,i} - t_l}{h_l} - s)^{-\gamma} H(t_{n,i}, t_l + sh_l) \Psi_j(s) \mathrm{d}s U_{l,j} \\
&+ h_n^{1-\gamma} \sum_{j=1}^{m} \int_0^{c_i} (\frac{t_{n,i} - t_n}{h_n} - s)^{-\gamma} H(t_{n,i}, t_n + sh_n) \Psi_j(s) \mathrm{d}s U_{n,j} \\
&+ \sum_{l=0}^{q_{n,i}-1} h_l^{1-\gamma} \sum_{j=1}^{m} \int_0^1 (\frac{t_{n,i} - t_l}{h_l} - s)^{-\gamma} H_D(t_{n,i}, t_l + sh_l) \Psi_j(s) \mathrm{d}s U_{l,j} \\
&+ h_{q_{n,i}}^{1-\gamma} \int_0^{c_{n,i}} (\frac{t_{n,i} - t_{q_{n,i}}}{h_{q_{n,i}}} - s)^{-\gamma} H_D(t_{n,i}, t_{q_{n,i}} + sh_{q_{n,i}}) \Psi_j(s) \mathrm{d}s U_{q_{n,i},j}.
\end{aligned} \tag{13}
$$

The general matrix form of the equation (13) can be written as follows

$$
\begin{aligned}
A_n^{\mu} U_n =& B_n + \sum_{l=0}^{n-1} h_l^{1-\gamma} \mathscr{H}_{\gamma,n,l}^{1,\Psi} U_l + h_n^{1-\gamma} \mathscr{H}_{\gamma,n,n}^{c_i,\Psi} U_n \\
&+ \sum_{l=0}^{q_{n,i}-1} h_l^{1-\gamma} \mathscr{H}_{D(q),\gamma,n,l}^{1,\Psi} U_l + h_{q_{n,i}}^{1-\gamma} \mathscr{H}_{D(q),\gamma,n,q_{n,i}}^{c_{n,i},\Psi} U_{q_{n,i}},
\end{aligned} \tag{14}
$$

where

$$
\boldsymbol{A}_n = \operatorname{diag}\left(t_{n,1}, t_{n,2}, \ldots, t_{n,m} \right),
$$

$$
\boldsymbol{B}_n = \left(f(t_{n,1}), f(t_{n,2}), \ldots, f(t_{n,m}) \right)^T,
$$

$$\boldsymbol{U}_n = \Big(U_{n,1}, U_{n,2}, \ldots, U_{n,m}\Big)^T, \tag{15}$$

and

$$\begin{aligned}
\left(\mathscr{H}^{\kappa, w, \rho}_{\gamma, \tau, \lambda}\right)_{i, r} &= \left(\int_0^{\kappa} (\frac{t_{\tau, i} - t_\lambda}{h_\lambda} - s)^{-\gamma} H(t_{\tau, i}, t_\lambda + \rho h_\lambda) w_r(\rho) \mathrm{d}\rho\right), \\
\left(\mathscr{H}^{\kappa, w, \rho}_{D(q), \gamma, \tau, \lambda}\right)_{i, r} &= \left(\int_0^{\kappa} (\frac{t_{\tau, i} - t_\lambda}{h_\lambda} - s)^{-\gamma} H_D(t_{\tau, i}, t_\lambda + \rho h_\lambda) w_r(\rho) \mathrm{d}\rho\right).
\end{aligned} \tag{16}$$

Discrete numerical descriptions of the collocation equations for the three phases are given, respectively. To do this, in the three phases I, II, and III, we first calculate the values of integral operators $(\mathscr{K}_\gamma u_n)(t_{n,i})$ and $(\mathscr{K}_{\gamma, q} u_n)(t_{n,i})$. Therefore, the integral operator $(\mathscr{K}_\gamma u_n)(t_{n,i})$ is written as follows after performing some calculations

$$\begin{aligned}
(\mathscr{K}_\gamma u_n)(t_{n,i}) = &\sum_{l=0}^{n-1} h_l^{1-\gamma} \sum_{j=1}^{m} \int_0^1 (\frac{t_{n,i} - t_l}{h_l} - s)^{-\gamma} H(t_{n,i}, t_l + s h_l) \Psi_j(s) \mathrm{d}s U_{l,j} \\
&+ h_n^{1-\gamma} \sum_{j=1}^{m} \int_0^{c_i} (\frac{t_{n,i} - t_n}{h_n} - s)^{-\gamma} H(t_{n,i}, t_n + s h_n) \Psi_j(s) \mathrm{d}s U_{n,j}.
\end{aligned} \tag{17}$$

Also, for delay integral operator $(\mathscr{K}_{\gamma, q} u_n)(t_{n,i})$, we have the following phases:
Phase I: The delay integral operator $(\mathscr{K}_{\gamma, q} u_n)(t_{n,i})$ at $t_{n,i}$ has the form

$$\begin{aligned}
(\mathscr{K}_{\gamma, q} u_n)(t_{n,i}) = &\sum_{l=0}^{n-1} h_l^{1-\gamma} \sum_{j=1}^{m} \int_0^1 (\frac{t_{n,i} - t_l}{h_l} - s)^{-\gamma} H_D(t_{n,i}, t_l + s h_l) \Psi_j(s) U_{l,j} \mathrm{d}s \\
&+ h_n^{1-\gamma} \sum_{j=1}^{m} \int_0^{c_{n,i}} (\frac{t_{n,i} - t_n}{h_n} - s)^{-\gamma} H_D(t_{n,i}, t_n + s h_n) \Psi_j(s) U_{n,j} \mathrm{d}s.
\end{aligned} \tag{18}$$

Phase II: In this phase, there exists $v_n < m$ such that

$$q t_{n,i} \in \begin{cases} (t_{n-1}, t_n], & i = 1, 2, \ldots, v_n, \\ (t_n, t_{n+1}], & i = v_{n+1}, \ldots, v_m. \end{cases}$$

If $i \leq v_n$ (Phase II-A). Then the delay integral operator $(\mathscr{K}_{\gamma, q} u_n)(t_{n,i})$ at $t_{n,i}$ has the form

$$
\begin{aligned}
(\mathscr{K}_{\gamma,\,q}u_n)(t_{n,\,i}) =& \sum_{l=0}^{n-2} h_l^{1-\gamma} \sum_{j=1}^{m} \int_0^1 (\frac{t_{n,\,i}-t_l}{h_l}-s)^{-\gamma} H_D(t_{n,\,i},\, t_l+sh_l)\Psi_j(s)\mathrm{d}sU_{l,\,j} \\
&+ h_{n-1}^{1-\gamma} \sum_{j=1}^{m} \int_0^{c_{n,\,i}} (\frac{t_{n,\,i}-t_{n-1}}{h_{n-1}}-s)^{-\gamma} H_D(t_{n,\,i},\, t_{n-1}+sh_{n-1})\Psi_j(s)\mathrm{d}sU_{n-1,\,j}.
\end{aligned} \tag{19}
$$

Also, if $i > v_n$ (Phase II-B). Then, the delay integral operator $(\mathscr{K}_{\gamma,\,q}u_n)(t_{n,\,i})$ at $t_{n,\,i}$ has the form

$$
\begin{aligned}
(\mathscr{K}_{\gamma,\,q}u_n)(t_{n,\,i}) =& \sum_{l=0}^{n-1} h_l^{1-\gamma} \sum_{j=1}^{m} \int_0^1 (\frac{t_{n,\,i}-t_l}{h_l}-s)^{-\gamma} H_D(t_{n,\,i},\, t_l+sh_l)\Psi_j(s)U_{l,\,j}\mathrm{d}s \\
&+ h_n^{1-\gamma} \sum_{j=1}^{m} \int_0^{c_{n,\,i}} (\frac{t_{n,\,i}-t_n}{h_n}-s)^{-\gamma} H_D(t_{n,\,i},\, t_n+sh_n)\Psi_j(s)U_{n,\,j}\mathrm{d}s.
\end{aligned} \tag{20}
$$

Phase III: For this phase, there exist positive integers $v_n < m$ and $q_n < n-1,$ such that

$$
qt_{n,\,i} \in \begin{cases} [t_{q_n},\, t_n], & i = 1,\, 2,\, \ldots,\, v_n, \\ (t_{q_{n+1}},\, t_n], & i = v_{n+1},\, \ldots,\, v_m. \end{cases}
$$

Thus, we have

$$
\begin{aligned}
(\mathscr{K}_{\gamma,\,q}u_n)(t_{n,\,i}) =& \sum_{l=0}^{q_{n,\,i}-1} h_l^{1-\gamma} \sum_{j=0}^{m} \int_0^1 (\frac{t_{n,\,i}-t_l}{h_l}-s)^{-\gamma} H_D(t_{n,\,i},\, t_l+sh_l)\Psi_j(s)\mathrm{d}sU_{l,\,j} \\
&+ h_{q_{n,\,i}}^{1-\gamma} \sum_{j=0}^{m} \int_0^{c_{n,\,i}} (\frac{t_{n,\,i}-t_{q_{n,\,i}}}{h_{q_{n,\,i}}}-s)^{-\gamma} H_D(t_{n,\,i},\, t_{q_{n,\,i}}+sh_{q_{n,\,i}})\Psi_j(s)\mathrm{d}sU_{q_{n,\,i},\,j}.
\end{aligned} \tag{21}
$$

Now, for simplicity, we define the matrices

$$
\begin{aligned}
\mathscr{S}_L &= \left[\begin{array}{c|c} \mathbf{0}_{v_n\times v_n} & \mathbf{0}_{(m-v_n)\times(m-v_n)} \\ \hline \mathbf{0}_{(m-v_n)\times v_n} & \boldsymbol{I}_{(m-v_n)\times(m-v_n)} \end{array}\right]_{m\times m}, \\
\mathscr{S}_U &= \left[\begin{array}{c|c} \boldsymbol{I}_{v_n\times v_n} & \mathbf{0}_{(m-v_n)\times(m-v_n)} \\ \hline \mathbf{0}_{(m-v_n)\times v_n} & \mathbf{0}_{(m-v_n)\times(m-v_n)} \end{array}\right]_{m\times m},
\end{aligned} \tag{22}
$$

and

$$
\begin{aligned}
&\mathscr{W}_n^{II}(q) = \mathscr{S}_U \mathscr{H}_{D(q),\,\gamma,\,n,\,n}^{c_{n,\,i},\,\Psi},\ \mathscr{Z}_{n-1}^{II}(q) = \mathscr{S}_U \mathscr{H}_{D(q),\,\gamma,\,n,\,n-1}^{1,\,\Psi},\\
&\hat{\mathscr{Z}}_{n-1}^{II}(q) = \mathscr{S}_L \mathscr{H}_{D(q),\,\gamma,\,n,\,n-1}^{c_{n,\,i},\,\Psi},\ \mathscr{Z}_{q_n+1}^{III}(q) = \mathscr{S}_U \mathscr{H}_{D(q),\,\gamma,\,n,\,q_n+1}^{c_{n,\,i},\,\Psi},\\
&\hat{\mathscr{Z}}_{q_n}^{III}(q) = \mathscr{S}_L \mathscr{H}_{D(q),\,\gamma,\,n,\,q_n}^{c_{n,\,i},\,\Psi}.
\end{aligned} \tag{23}
$$

Therefore, for the unknown vector U_n, the collocation phase I system is

$$
\left(I - h_n^{1-\gamma} A_n^{-\mu}\left(\mathscr{H}_{\gamma,\,n,\,n}^{c_i,\,\Psi} + \mathscr{H}_{D(q),\,\gamma,\,n,\,n}^{c_{n,\,i},\,\Psi}\right)\right) U_n = A_n^{-\mu} B_n + \sum_{l=0}^{n-1} h_l^{1-\gamma} A_n^{-\mu}\left(\mathscr{H}_{\gamma,\,n,\,l}^{1,\,\Psi} + \mathscr{H}_{D(q),\,\gamma,\,n,\,l}^{1,\,\Psi}\right) U_l. \tag{24}
$$

Moreover, the collocation equation's matrix form in phase II is then shown:

$$
\begin{aligned}
\left(I - h_n^{1-\gamma} A_n^{-\mu}\left(\mathscr{H}_{\gamma,\,n,\,n}^{c_i,\,\Psi} + \mathscr{W}_n^{II}(q)\right)\right) U_n =& A_n^{-\mu} B_n + \sum_{l=0}^{n-1} h_l^{1-\gamma} \mathscr{H}_{\gamma,\,n,\,l}^{1,\,\Psi} U_l\\
&+ \sum_{l=0}^{n-2} h_l^{1-\gamma} A_n^{-\mu} \mathscr{H}_{D(q),\,\gamma,\,n,\,l}^{1,\,\Psi} U_l\\
&+ h_{n-1}^{1-\gamma} A_n^{-\mu}\left(\hat{\mathscr{Z}}_{n-1}^{II}(q) + \mathscr{Z}_{n-1}^{II}(q)\right) U_{n-1}.
\end{aligned} \tag{25}
$$

The collocation system in phase III has been inferred as follows, by the defined matrices, just like in the earlier phases:

$$
\begin{aligned}
\left(I - h_n^{1-\gamma} A_n^{-\mu} \mathscr{H}_{\gamma,\,n,\,n}^{c_i,\,\Psi}\right) U_n =& A_n^{-\mu} B_n + \sum_{l=0}^{n-1} h_l^{1-\gamma} A_n^{-\mu} \mathscr{H}_{\gamma,\,n,\,l}^{1,\,\Psi} U_l\\
&+ \sum_{l=0}^{q_n-1} h_l^{1-\gamma} A_n^{-\mu} \mathscr{H}_{D(q),\,\gamma,\,n,\,l}^{1,\,\Psi} U_l\\
&+ h_{q_n}^{1-\gamma} A_n^{-\mu}\left(\hat{\mathscr{Z}}_{q_n}^{III}(q) + \mathscr{H}_{D(q),\,\gamma,\,n,\,q_n}^{1,\,\Psi}(q)\right) U_{q_n}\\
&+ h_{q_n}^{1-\gamma} A_n^{-\mu} \mathscr{Z}_{q_n+1}^{III}(q) U_{q_n+1}.
\end{aligned} \tag{26}
$$

Now, we show that, the system (24)-(26) has a unique solution. To do this, we set

$$\Theta_n(q)=\begin{cases}\Theta_n^I(q), & \text{Phase I},\\ \Theta_n^{II}(q), & \text{Phase II},\\ \Theta_n^{III}, & \text{Phase III},\end{cases} \tag{27}$$

where

$$\Theta_n^I(q)=\left(I-h_n^{1-\gamma}A_n^{-\mu}\left(\mathscr{H}_{\gamma,n,n}^{c_i,\Psi}+\mathscr{H}_{D(q),\gamma,n,n}^{c_{n,i},\Psi}\right)\right),$$

$$\Theta_n^{II}(q)=\left(I-h_n^{1-\gamma}A_n^{-\mu}\left(\mathscr{H}_{\gamma,n,n}^{c_i,\Psi}+\mathscr{W}_n^{II}(q)\right)\right), \tag{28}$$

$$\Theta_n^{III}=\left(I-h_n^{1-\gamma}A_n^{-\mu}\mathscr{H}_{\gamma,n,n}^{c_i,\Psi}\right).$$

Theorem 2.1 Assume that $H\in C(\Omega)$, $H_D\in C(\Omega_D)$ and $0<\mu+\gamma<1$. Then for any collocation parameters $0<c_1<\cdots<c_m\leq 1$ there exists $\tilde{h}^{(I)}>0$, $\tilde{h}^{(II)}>0$, $\tilde{h}^{(III)}>0$ and an $\tilde{h}>0$ such that the matrices $\Theta_n(q)$, are invertible whenever $h=\max_n h_n\leq\tilde{h}=\min\{\tilde{h}^{(I)},\tilde{h}^{(II)},\tilde{h}^{(III)}\}$.

Proof. For given collocation parameters $\{c_i\}_{i=1}^m$, we obtain following phases.

A1. In phase I, we obtain

$$\begin{aligned}\|\mathscr{H}_{\gamma,n,n}^{c_i,\Psi}\|_\infty= & \max_{i=1,\ldots,m}\sum_{j=1}^m\left|\int_0^{c_i}\left(\frac{t_{n,i}-t_n}{h_n}-s\right)^{-\gamma}H(t_{n,i},t_n+sh_n)\Psi_j(s)\mathrm{d}s\right|\\ \leq & \tfrac{1}{1-\gamma}\|H\|_\infty\Phi_m(c)\max_i c_i^{1-\gamma}\\ \leq & \tfrac{c_m^{1-\gamma}}{1-\gamma}\|H\|_\infty\Phi_m(c),\end{aligned} \tag{29}$$

where the Lebesgue constant $\Phi_m(c)$ is defined by

$$\Phi_m(c)=\max_{s\in[0,c_m]}\sum_{j=1}^m|\Psi_j(s)|,$$

and the constant $c=c(c_i)$ is dependent on the c_i.

Also

$$
\begin{aligned}
\|\mathscr{H}^{c_{n,i},\Psi}_{D(q),\gamma,n,n}\|_\infty &= \max_{i=1,\ldots,m}\sum_{j=1}^{m}\left|\int_0^{c_{n,i}}\left(\frac{t_{n,i}-t_n}{h_n}-s\right)^{-\gamma}H_D,(t_{n,i},t_n+sh_n)\Psi_j(s)\mathrm{d}s\right| \\
&\le \frac{1}{1-\gamma}\parallel H_D\parallel_\infty \Phi_m(c^*)\max_i c_{n,i}^{1-\gamma} \\
&\le \frac{c_{n,m}^{1-\gamma}}{1-\gamma}\parallel H_D\parallel_\infty \Phi_m(c^*),
\end{aligned}
\tag{30}
$$

where $c^* = c^*(c_{n,i})$ and

$$
\Phi_m(c^*) = \max_{s\in[0,c_{n,m}]}\sum_{j=1}^{m}|\Psi_j(s)|.
$$

Since $c_1 > 0,$ and $\nu+\gamma\in(0,1)$, the constant

$$
\tilde{h}^{(I)} = \left(\frac{2c_1^{-\mu}}{1-\gamma}\left(c_m^{1-\gamma}\parallel H\parallel_\infty \Phi_m(c)+c_{n,m}^{1-\gamma}\parallel H_D\parallel_\infty \Phi_m(c^*)\right)\right)^{-\frac{1}{1-\nu-\gamma}},
\tag{31}
$$

shows that $\parallel A_n^{-\gamma}\parallel_\infty \le (c_1h_n)^{-\gamma}$, and that the matrix A_n is invertible. Additionally, $\nu+\gamma<1$ implies that for $h_n\le\tilde{h}^{(I)}$,

$$
\begin{aligned}
&\parallel h_n^{1-\gamma}A_n^{-\mu}\left(\mathscr{H}^{c_i,\Psi}_{\gamma,n,n}+\mathscr{H}^{c_{n,i},\Psi}_{D(q),\gamma,n,n}\right)\parallel_\infty \\
\le& h_n^{1-\nu-\gamma}c_1^{-\mu}\left(\frac{c_m^{1-\gamma}}{1-\gamma}\parallel H\parallel_\infty \Phi_m(c)+\frac{c_{n,m}^{1-\gamma}}{1-\gamma}\parallel H_D\parallel_\infty \Phi_m(c^*)\right) \\
\le& (\tilde{h}^{(I)})^{1-\nu-\gamma}c_1^{-\mu}\left(\frac{c_m^{1-\gamma}}{1-\gamma}\parallel H\parallel_\infty \Phi_m(c)+\frac{c_{n,m}^{1-\gamma}}{1-\gamma}\parallel H_D\parallel_\infty \Phi_m(c^*)\right) \\
\le& \frac{1}{2}.
\end{aligned}
\tag{32}
$$

Therefore, the matrices $\Theta_n^{(I)}(q),\ n=0,1,\ldots,N-1,$ are invertible.

A2. In phase II, we have

$$
\parallel \mathscr{S}_U\parallel_\infty = 1,
\tag{33}
$$

and

$$\begin{aligned}\| \mathscr{W}_n^{II}(q) \|_\infty &= \| \mathscr{S}_U \mathscr{H}_{D(q),\,\gamma,\,n,\,n}^{c_{n,\,i},\,\Psi} \|_\infty \\ &\leq \| \mathscr{S}_U \|_\infty \| \mathscr{H}_{D(q),\,\gamma,\,n,\,n}^{c_{n,\,i},\,\Psi} \|_\infty \\ &= \| \mathscr{H}_{D(q),\,\gamma,\,n,\,n}^{c_{n,\,i},\,\Psi} \|_\infty \\ &\leq \frac{c_{n,\,m}^{1-\gamma}}{1-\gamma} \| H_D \|_\infty \Phi_m(c^*). \end{aligned} \tag{34}$$

Since $\nu+\gamma \in (0,\, 1)$ and $c_1 > 0$, the constant

$$\tilde{h}^{(II)} = \left(\frac{2c_1^{-\mu}}{1-\gamma} \left(c_m^{1-\gamma} \| H \|_\infty \Phi_m(c) + c_{n,\,m}^{1-\gamma} \| H_D \|_\infty \Phi_m(c^*) \right) \right)^{-\frac{1}{1-\nu-\gamma}}, \tag{35}$$

is well defined, therefore, it follows from the $\nu+\gamma < 1$, that for $h_n \leq \tilde{h}^{(II)}$, we get

$$\| h_n^{1-\gamma} A_n^{-\mu} \left(\mathscr{H}_{\gamma,\,n,\,n}^{c_i,\,\Psi} + \mathscr{W}_n^{II}(q) \right) \|_\infty \leq \frac{1}{2}. \tag{36}$$

Thus, the matrices $\Theta_n^{(II)}(q),\ n = 0,\, 1,\, ...,\, N-1$ are invertible .

A3. In phase III, in the same manner, we have

$$\begin{aligned} &\| h_n^{1-\gamma} A_n^{-\mu} \mathscr{H}_{\gamma,\,n,\,n}^{c_i,\,\Psi} \|_\infty \\ &\leq h_n^{1-\nu-\gamma} c_1^{-\mu} \left(\frac{c_m^{1-\gamma}}{1-\gamma} \| H \|_\infty \Phi_m(c) \right) \\ &\leq (\tilde{h}^{(III)})^{1-\nu-\gamma} c_1^{-\mu} \left(\frac{c_m^{1-\gamma}}{1-\gamma} \| H \|_\infty \Phi_m(c) \right) \\ &\leq \frac{1}{2}, \end{aligned} \tag{37}$$

where

$$\tilde{h}^{(III)} = \left(\frac{2c_1^{-\mu} c_m^{1-\gamma}}{1-\gamma} \| H \|_\infty \Phi_m(c) \right)^{-\frac{1}{1-\nu-\gamma}}. \tag{38}$$

Then, the matrices $\Theta_n^{(III)}$, $n = 0, 1, ..., N-1$, are invertible.

Now, acording the cases A1-A3, the matrices $\Theta_n(q)$, $n = 0, 1, ..., N-1$, are invertible, and the proof is complete.

3. Convergence

The main result of the convergence is presented in this section as a theorem in the following, which theoretically validates the collocation method's applicability.

Here, we rewrite functional 3K-VIEs (1), in the form

$$
\begin{aligned}
y(t) =& t^{-\mu} f(t) + \int_0^t t^{-\mu}(t-s)^{-\gamma} H(t, s) y(s) \mathrm{d}s + \int_0^{qt} t^{-\mu}(t-s)^{-\gamma} H_D(t, s) y(s) \mathrm{d}s, \\
=& g(t) + (\mathscr{Z}_\gamma y)(t) + (\mathscr{Z}_{\gamma, q} y)(t),
\end{aligned} \tag{39}
$$

where $t \in \Delta$, $\mu > 0$, $f(t) = t^\mu g(t)$ and g is a continuous function.

Theorem 3.1 In equation (39), assume that $\mu + \gamma \in (0, 1)$. Then the operators $(\mathscr{Z}_\gamma y)(t)$ and $(\mathscr{Z}_{\gamma, q} y)(t)$ are compact and bounded on $C[0, T]$ and $C[0, qt]$, respectively.

Proof. By rewriting operators $(\mathscr{Z}_\gamma y)(t)$ and $(\mathscr{Z}_{\gamma, q} y)(t)$, we have

$$
\begin{aligned}
(\mathscr{Z}_\gamma y)(t) &= \int_{t_0}^t t^{-1} t^{1-\mu-\gamma}(1 - \frac{s}{t})^{-\gamma} H(t, s) y(s) \mathrm{d}s, \\
(\mathscr{Z}_{\gamma, q} y)(t) &= \int_0^{qt} t^{-1} t^{1-\mu-\gamma}(1 - \frac{s}{t})^{-\gamma} H_D(t, s) y(s) \mathrm{d}s.
\end{aligned} \tag{40}
$$

Now, if we set $\mu + \gamma \in (0, 1)$ then, $1 - \mu - \gamma > 0$. Hence, the operators $(\mathscr{Z}_\gamma y)(t)$ and $(\mathscr{Z}_{\gamma, q} y)(t)$, are compact and the core are

$$
\psi_H(s) = \psi_{H_D}(s) = (1-s)^{-\gamma}.
$$

The proof is complete. For more details see [4].

Let equation

$$
t^\mu y(t) = f(t) + \int_{t_0}^t (t-s)^{-\gamma} H(t, s) y(s) \mathrm{d}s + \int_0^{qt} (t-s)^{-\gamma} H_D(t, s) y(s) \mathrm{d}s, \quad t \in \Delta, \tag{41}
$$

has a exact solution $y(t)$. Then, for any collocation parameters $0 < c_1 < \cdots < c_m \leq 1$, the exact solution $y(t)$ satisfies in

$$
y(t_n + sh_n) = \sum_{j=1}^m \Psi_j(s) y(t_{n, j}) + h_n^m R_{m, n}(s), \tag{42}
$$

where

$$R_{m,\,n}(s) = \int_0^1 H_m(s,\,\tau) y^{(m)}(t_n + \tau h_n) d\tau,$$

$$H_m(s,\,\tau) = \frac{1}{(m-1)!}\left((s-\tau)_+^{m-1} - \sum_{j=1}^{m} \Psi_j(s)(c_j - \tau)_+^{m-1} \right), \quad \tau \in [0,\,1], \tag{43}$$

$$(s-\tau)_+ = \begin{cases} s-\tau, & s-\tau \geq 0, \\ 0, & s-\tau < 0. \end{cases}$$

In order to make the underlying theorem more understandable, we first assume that

$$P^{III}_{q_{n,\,i}+1}(q) = \mathscr{S}_L\Big(P_n^{(q_{n,\,i}-1)}(q) + P_{q_{n,\,i}+1}(q)\Big),\, \hat{P}^{III}_{q_{n,\,i}}(q) = \mathscr{S}_U P^{I}_{q_{n,\,i}}(q),$$

$$P^{II}_{q_{n,\,i}}(q) = \mathscr{S}_L\Big(P_n^{(q_{n,\,i}-1)}(q) + P^{I}_{q_{n,\,i}}(q)\Big),\, \hat{P}^{II}_{q_{n,\,i}-1}(q) = \mathscr{S}_U P^{I}_{q_{n,\,i}-1}(q), \tag{44}$$

$$P_n^{(l)}(q) = \left(\int_0^1 \Big(\frac{t_{n,\,i} - t_l}{h_l} - s\Big)^{-\gamma} H_D(t_{n,\,i},\, t_l + s h_l) R_{m,\,l}(s) \mathrm{d}s \right)^T,\ i = 1,\,...,\,m,\ l < n,$$

$$P^{I}_{q_{n,\,i}}(q) = \left(\int_0^{c_{n,\,i}} \Big(\frac{t_{n,\,i} - t_{q_{n,\,i}}}{h_{q_{n,\,i}}} - s\Big)^{-\gamma} H_D(t_{n,\,i},\, t_{q_{n,\,i}} + s h_{q_{n,\,i}}) R_{m,\,q_{n,\,i}}(s) \mathrm{d}s \right)^T,\ i = 1,\,...,\,m, \tag{45}$$

$$P_l = \begin{cases} \displaystyle\int_0^1 \Big(\frac{t_{n,\,i} - t_l}{h_l} - s\Big)^{-\gamma} H(t_{n,\,i},\, t_l + s h_l) R_{m,\,l}(s) \mathrm{d}s, & l = 0,\,1,\,...,\,n-1, \\ \displaystyle\int_0^{c_i} \Big(\frac{t_{n,\,i} - t_l}{h_l} - s\Big)^{-\gamma} H(t_{n,\,i},\, t_l + s h_l) R_{m,\,l}(s) \mathrm{d}s, & l = n, \end{cases} \tag{46}$$

and

$$\mathscr{E}_l = \Big[\mathscr{E}_{l,\,1},\,...,\,\mathscr{E}_{l,\,m}\Big]^T. \tag{47}$$

The meaning of the matrices $\mathscr{S}_U$ and $\mathscr{S}_L$ are clear.

Theorem 3.2 In equation (1), assume that $f \in C^m(I),\ H \in C^m(\Omega),$ and $H_D \in C^m(\Omega_D)$ and

$$H(0,\,0),\, H_D(0,\,0) \neq \begin{cases} \frac{\Gamma(\mu+l)}{\Gamma(l)\Gamma(1-\gamma)}, & \mu+\gamma = 1, \quad l = 1,\,2,\,...,\,m, \\ \frac{\Gamma(\mu+\gamma+l)}{\Gamma(\mu+\gamma+l-1)\Gamma(1-\gamma)}, & \mu+\gamma \geq 1, \quad l = 1,\,2,\,...,\,m. \end{cases}$$

Then, for any collocation parameters with $0 < c_1 < c_2 < \cdots < c_m \leq 1$ and $\varsigma > 1$, in modified graded meshes, the collocation solutions with order p converges to the exact solution of equation (1). Thus we have

$$\| \mathscr{E} \|_\infty = \mathscr{O}(h^p), \tag{48}$$

where $p = m - \gamma + 1$.

Proof. The collocation error $\mathscr{E}(t) := y(t) - u_n(t)$ possesses the local representation

$$\mathscr{E}(t_n + sh_n) = \sum_{j=1}^{m} \Psi_j(s) \mathscr{E}_{n,\, j} + h_n^m R_{m,\, n}(s), \tag{49}$$

with $\mathscr{E}_{n-k} = \mathscr{E}(t_{n-k})$, $\mathscr{E}_{n,\, j} = \mathscr{E}(t_{n,\, j})$.

By substituting this equation and $t = t_{n,\, i}$, in collocation error $\mathscr{E}(t)$, we have

$$\begin{aligned}
t_{n,\, i}^{\mu} \mathscr{E}(t_{n,\, i}) =& (\mathscr{K}_\gamma \mathscr{E})(t_{n,\, i}) + (\mathscr{K}_{\gamma,\, q} \mathscr{E})(t_{n,\, i}) \\
=& \sum_{l=0}^{n-1} h_l^{1-\gamma} \sum_{j=1}^{m} \int_0^1 (\frac{t_{n,\, i} - t_l}{h_l} - s)^{-\gamma} H(t_{n,\, i},\, t_l + sh_l) \Psi_j(s) \mathrm{d}s \mathscr{E}_{l,\, j} \\
&+ \sum_{l=0}^{n-1} h_l^{m+1-\gamma} \int_0^1 (\frac{t_{n,\, i} - t_l}{h_l} - s)^{-\gamma} H(t_{n,\, i},\, t_l + sh_l) R_{m,\, l}(s) \mathrm{d}s \\
&+ h_n^{1-\gamma} \sum_{j=1}^{m} \int_0^{c_i} (\frac{t_{n,\, i} - t_n}{h_n} - s)^{-\gamma} H(t_{n,\, i},\, t_n + sh_n) \Psi_j(s) \mathrm{d}s \mathscr{E}_{n,\, j} \\
&+ h_n^{m+1-\gamma} \int_0^{c_i} (\frac{t_{n,\, i} - t_n}{h_n} - s)^{-\gamma} H(t_{n,\, i},\, t_n + sh_n) R_{m,\, n}(s) \mathrm{d}s \\
&+ \sum_{l=0}^{q_{n,\, i}-1} h_l^{1-\gamma} \sum_{j=1}^{m} \int_0^1 (\frac{t_{n,\, i} - t_l}{h_l} - s)^{-\gamma} H_D(t_{n,\, i},\, t_l + sh_l) \Psi_j(s) \mathrm{d}s \mathscr{E}_{l,\, j} \\
&+ \sum_{l=0}^{q_{n,\, i}-1} h_l^{m+1-\gamma} \int_0^1 (\frac{t_{n,\, i} - t_l}{h_l} - s)^{-\gamma} H_D(t_{n,\, i},\, t_l + sh_l) R_{m,\, l}(s) \mathrm{d}s \\
&+ h_{q_{n,\, i}}^{1-\gamma} \int_0^{c_{n,\, i}} (\frac{t_{n,\, i} - t_{q_{n,\, i}}}{h_{q_{n,\, i}}} - s)^{-\gamma} H_D(t_{n,\, i},\, t_{q_{n,\, i}} + sh_{q_{n,\, i}}) \Psi_j(s) \mathrm{d}s \mathscr{E}_{q_{n,\, i},\, j} \\
&+ h_{q_{n,\, i}}^{m+1-\gamma} \int_0^{c_{n,\, i}} (\frac{t_{n,\, i} - t_{q_{n,\, i}}}{h_{q_{n,\, i}}} - s)^{-\gamma} H_D(t_{n,\, i},\, t_{q_{n,\, i}} + sh_{q_{n,\, i}}) R_{m,\, q_{n,\, i}}(s) \mathrm{d}s.
\end{aligned} \tag{50}$$

As a general overview, subsequently, using developing (50) into three phases, namely I, II, and III, we shall demonstrate the error matrices. Eventually, We will demonstrate that the best feasible convergence rates are $\mathscr{O}(h^p)$, $(p = m-\gamma+1)$.

Now, according to section 2, the general matrix form of equation (50) in phase I, after some computation, can be written as:

$$\begin{aligned}\left(I-h_n^{1-\gamma}A_n^{-\mu}\left(\mathscr{H}_{\gamma,n,n}^{c_i,\Psi}+\mathscr{H}_{D(q),\gamma,n,n}^{c_{n,i},\Psi}\right)\right)\mathscr{E}_n &= \sum_{l=0}^{n} h_l^p A_n^{-\mu} P_l+\sum_{l=0}^{n-1} h_l^p A_n^{-\mu} P_n^{(l)}(q)+h_n^p A_n^{-\mu} P_n^I(q),\\ &+\sum_{l=0}^{n-1} h_l^{1-\gamma}A_n^{-\mu}\left(\mathscr{H}_{\gamma,n,l}^{1,\Psi}+\mathscr{H}_{D(q),\gamma,n,l}^{1,\Psi}\right)\mathscr{E}_l.\end{aligned} \tag{51}$$

Furthermore the phase II error system's matrix form is provided as follows:

$$\begin{aligned}\left(I-h_n^{1-\gamma}A_n^{-\mu}\left(\mathscr{H}_{\gamma,n,n}^{c_i,\Psi}+\mathscr{W}_n^{II}(q)\right)\right)\mathscr{E}_n &= \sum_{l=0}^{n} h_l^p A_n^{-\mu} P_l+\sum_{l=0}^{n-2} h_l^p A_n^{-\mu} P_n^{(l)}(q)\\ &+h_n^p A_n^{-\mu}\hat{P}_{n-1}^{II}(q)+h_n^p A_n^{-\mu} P_n^{II}(q)\\ &+\sum_{l=0}^{n-1} h_l^{1-\gamma}\mathscr{H}_{\gamma,n,l}^{1,\Psi}\mathscr{E}_l+\sum_{l=0}^{n-2} h_l^{1-\gamma}A_n^{-\mu}\mathscr{H}_{D(q),\gamma,n,l}^{1,\Psi}\mathscr{E}_l\\ &+h_{n-1}^{1-\gamma}A_n^{-\mu}\left(\hat{\mathscr{Z}}_{n-1}^{II}(q)+\mathscr{Z}_{n-1}^{II}(q)\right)\mathscr{E}_{n-1}.\end{aligned} \tag{52}$$

Also, in the same manner for phase III, we have

$$\begin{aligned}\left(I-h_n^{1-\gamma}A_n^{-\mu}\mathscr{H}_{\gamma,n,n}^{c_i,\Psi}\right)\mathscr{E}_n &= \sum_{l=0}^{n} h_l^p A_n^{-\mu} P_n+\sum_{l=0}^{q_n-1} h_l^p A_n^{-\mu} P_n^{(l)}(q)\\ &+h_n^p A_n^{-\mu}\hat{P}_{q_n}^{III}(q)+h_n^p A_n^{-\mu} P_{q_n+1}^{III}(q)+\sum_{l=0}^{n-1} h_l^{1-\gamma}A_n^{-\mu}\mathscr{H}_{\gamma,n,l}^{1,\Psi}\mathscr{E}_l\\ &+\sum_{l=0}^{q_n-1} h_l^{1-\gamma}A_n^{-\mu}\mathscr{H}_{D(q),\gamma,n,l}^{1,\Psi}\mathscr{E}_l+h_{q_n}^{1-\gamma}A_n^{-\mu}\left(\hat{\mathscr{Z}}_{q_n}^{III}(q)+\mathscr{H}_{D(q),\gamma,n,q_n}^{1,\Psi}(q)\right)\mathscr{E}_{q_n}\\ &+h_{q_n}^{1-\gamma}A_n^{-\mu}\mathscr{Z}_{q_n+1}^{III}(q)\mathscr{E}_{q_n+1}.\end{aligned} \tag{53}$$

Now, let $h = \max_n h_n$, then by this assumption, and according to Theorem 2.1, modified graded meshes with $0 < h < \tilde{h}$ have a unique solution for each of the aforementioned linear algebraic systems in triple phases. Therefore, in the same manner as the proof of Theorems 4.1 and 4.2, in [3], we have

$$\| \mathscr{E} \|_\infty = \mathscr{O}(h^p), \tag{54}$$

and this completes the proof.

4. Numerical illustration

In this section, we present the numerical results of a few test issues that were resolved using the article's suggested approach. The following test problems are taken into consideration. We have examined two test issues that are resolved using collocation techniques in order to test and then validate the analytical conclusions. The following examples show the benefits of the collocation approach by tabulating some numerical findings in terms of maximum errors and convergence orders for a proportional vanishing delay parameter, i.e., $q = 0.25$ and $q = 0.5$. For convenience, we will demonstrate the end point L_∞-errors and convergence orders in our examples using, respectively, $\mathscr{E}$ and p such that

$$\mathscr{E}_N = \| y - u_N \|_\infty,$$

and

$$p := \log_2\left(\frac{\mathscr{E}_N}{\mathscr{E}_{2N}}\right),$$

are satisfied. As shown in Tables 1-3 and Figures 1-6, the maximum errors for $m = 2$ and 3 are quantified for Examples 4.1-4.3.

Example 4.1 The object of our study is the following third-kind Volterra Integral Equation (VIE) with a weakly singular kernel and a proportional delay:

$$t^{\frac{1}{4}} y(t) = f(t) + \frac{1}{2}\int_0^t (t-s)^{-\frac{1}{2}} y(s)\mathrm{d}s + \frac{3}{2}\int_0^{\frac{1}{4}t} (t-s)^{-\frac{1}{2}} y(s)\mathrm{d}s, \tag{55}$$

where the source term $f(t)$ is constructed to ensure the existence of the exact solution $y(t) = t^{\frac{5}{2}}$.

Table 1. Maximum errors $\| \mathscr{E} \|_\infty$ and the order of convergence for $m = 2, 3$ in Example 4.1

N	$c = (0.7, 1)$	p	$c = (\frac{4-\sqrt{6}}{10}, \frac{4+\sqrt{6}}{10}, 1)$	p
2^2	1.074×10^{-1}	–	1.619×10^{-3}	–
2^3	2.392×10^{-1}	2.167	2.033×10^{-4}	2.993
2^4	6.079×10^{-3}	1.976	2.233×10^{-5}	3.186
2^5	1.576×10^{-3}	1.947	2.300×10^{-6}	3.279
2^6	4.077×10^{-4}	1.950	2.264×10^{-7}	3.344

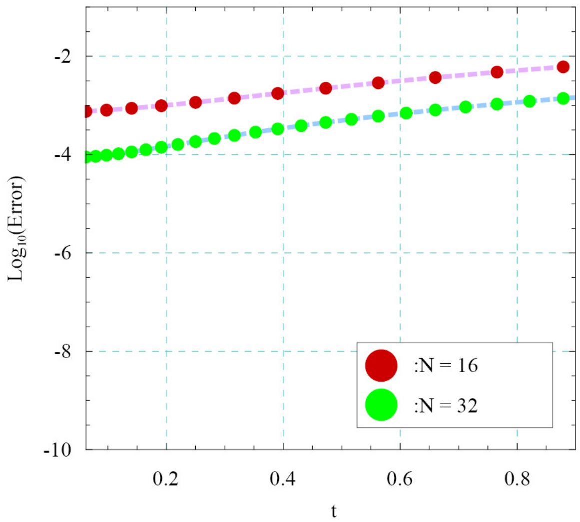

Figure 1. Comparison between collocation method with $m=2$ and exact solution, in Example 4.1

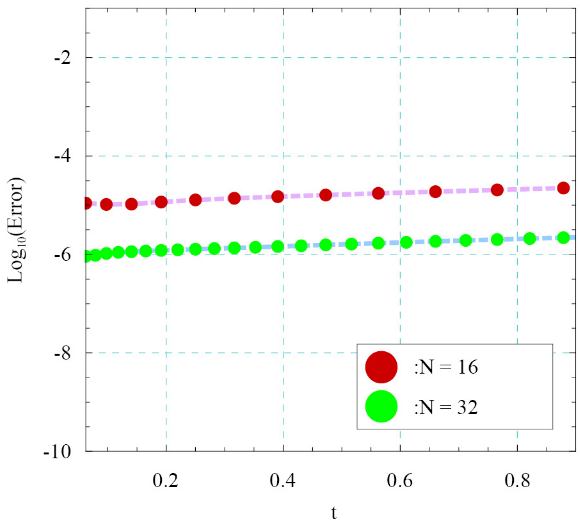

Figure 2. Comparison between collocation method with $m=3$ and exact solution, in Example 4.1

Example 4.2 The object of our study is the following third-kind VIE with a weakly singular kernel and a proportional delay:

$$ty(t) = \quad f(t)+\tfrac{1}{2}\int_0^t y(s)\mathrm{d}s+\frac{1}{2}\int_0^{\frac{1}{2}t} y(s)\mathrm{d}s, \tag{56}$$

where the source term $f(t)$ is constructed to ensure the existence of the exact solution $y(t)=t^{\frac{5}{2}}$.

Table 2. Maximum errors $\| \mathscr{E} \|_\infty$ and the order of convergence for $m = 2, 3$ in Example 4.2

N	$c = (0.7, 1)$	p	$c = (\frac{4-\sqrt{6}}{10}, \frac{4+\sqrt{6}}{10}, 1)$	p
2^2	5.78×10^{-2}	–	2.03×10^{-3}	–
2^3	1.06×10^{-2}	2.43	3.60×10^{-4}	2.50
2^4	2.34×10^{-3}	2.18	6.36×10^{-5}	2.50
2^5	5.24×10^{-4}	2.16	1.12×10^{-5}	2.50
2^6	1.21×10^{-4}	2.01	1.98×10^{-6}	2.50

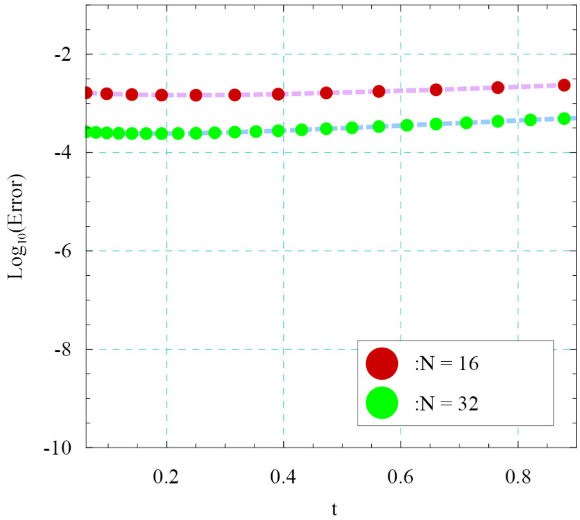

Figure 3. Comparison between collocation method with $m = 2$ and exact solution, in Example 4.2

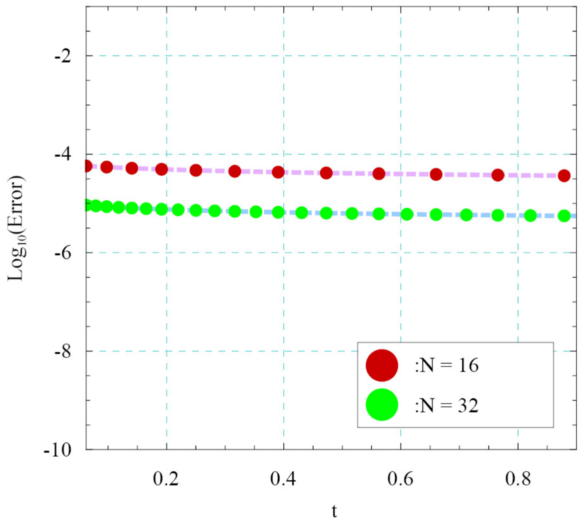

Figure 4. Comparison between collocation method with $m = 3$ and exact solution, in Example 4.2

Example 4.3 The object of our study is the following third-kind VIE with a weakly singular kernel and a proportional delay:

$$t^{\frac{2}{3}}y(t) = \quad f(t) + \frac{\sqrt{3}}{3\pi}\int_0^t (t-s)^{-\frac{2}{3}}s^{\frac{1}{3}}y(s)\mathrm{d}s + \int_0^{\frac{1}{2}t}(t-s)^{-\frac{2}{3}}s^{\frac{1}{3}}y(s)\mathrm{d}s, \tag{57}$$

where the source term $f(t)$ is constructed to ensure the existence of the exact solution $y(t) = t^{\frac{13}{4}}$.

Table 3. Maximum errors $\|\mathscr{E}\|_\infty$ and the order of convergence for $m = 2,\ 3$ in Example 4.3

N	$c = (0.7,\ 1)$	p	$c = (\frac{4-\sqrt{6}}{10},\ \frac{4+\sqrt{6}}{10},\ 1)$	p
2^2	4.92×10^{-2}	–	3.38×10^{-3}	–
2^3	1.01×10^{-2}	2.27	3.55×10^{-4}	3.25
2^4	3.15×10^{-3}	1.68	3.74×10^{-5}	3.25
2^5	9.26×10^{-4}	1.77	3.93×10^{-6}	3.25
2^6	2.66×10^{-4}	1.79	4.13×10^{-7}	3.25

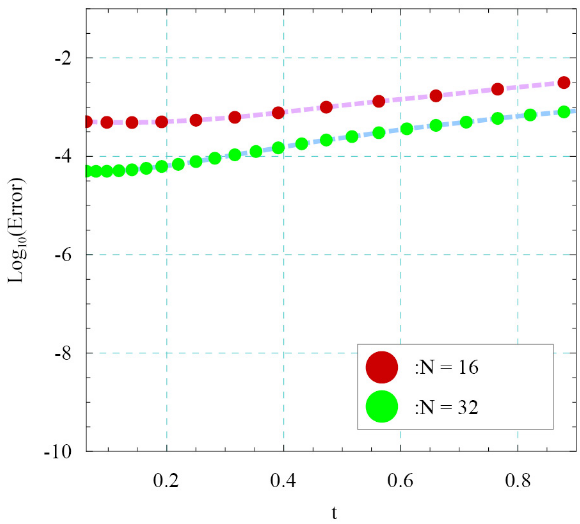

Figure 5. Comparison between collocation method with $m = 2$ and exact solution, in Example 4.3

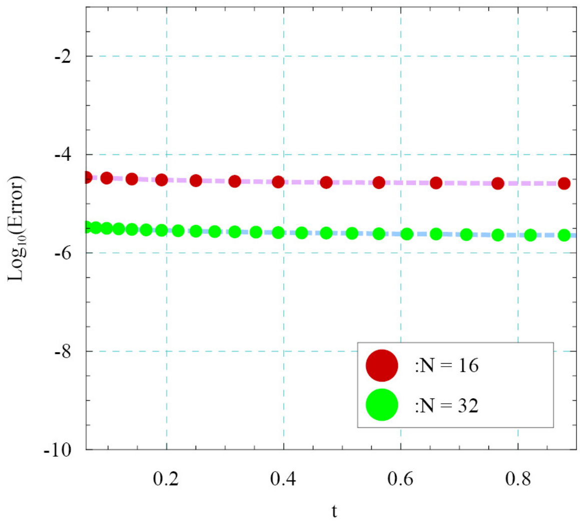

Figure 6. Comparison between collocation method with $m = 3$ and exact solution, in Example 4.3

Example 4.4 The object of our study is the following third-kind VIE with a weakly singular kernel and a proportional delay:

$$t^{\frac{1}{2}}y(t) = \quad f(t) + \int_0^t (t-s)^{-\frac{1}{2}}(s+1)y(s)\mathrm{d}s + \int_0^{\frac{1}{4}t} (t-s)^{-\frac{1}{2}}(s+2)y(s)\mathrm{d}s, \tag{58}$$

where the source term $f(t)$ is constructed to ensure the existence of the exact solution $y(t) = t^{\frac{1}{2}}$. The numerical result in Table 4 show that the maximum errors and Central Processing Unit (CPU) time for $m = 2$, 3 in Example 4.4.

Table 4. Maximum errors $\| \mathscr{E} \|_\infty$ and CPU time for $m = 2$, 3 in Example 4.4

N	$m = 2$	CPU time	$m = 3$	CPU time
2^3	1.25×10^{-1}	0.890	2.14×10^{-2}	2.47
2^4	5.71×10^{-2}	2.62	8.95×10^{-3}	5.62
2^5	2.45×10^{-2}	9.53	8.03×10^{-3}	21.9
2^6	9.92×10^{-3}	36.1	2.88×10^{-3}	84.9

5. Conclusion

This paper examines a class of vanishing delay Volterra Integral Equations (VIEs) of the third kind. A collocation approximation technique is used to obtain numerical solutions. We cover the unique solvability, existence, and regularity of the precise solution. The primary theoretical contribution is a thorough demonstration of a convergence rate of order m (in terms of parameter p) for the method with m collocation parameters. This theoretical result is validated by numerical experiments. Future research directions include the application of multistep collocation techniques to weakly singular versions of these equations.

Acknowledgement

The authors gratefully acknowledge the anonymous referees for their careful reading of the manuscript and constructive comments.

Disclosure

Validation, Parviz Darania; Writing–review and editing, Dana Tahsin Abdul Rahman. All authors have read and agreed to the published version of the manuscript.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Webb JRL. Initial value problems for Caputo fractional equations with singular nonlinerities. *Electron Journal of Differential Equation*. 2019; 117(1): 1-32.

[2] Allaei S, Yang Z, Brunner H. Existence, uniqueness and regularity of solutions to a class of third-kind Volterra integral equations. *Journal of Integral Equations and Applications*. 2015; 27(3): 325-342.

[3] Allaei S, Yang Z, Brunner H. Collocation methods for third-kind VIEs. *IMA Journal of Numerical Analysis*. 2017; 37(3): 1104-1124.

[4] Song H, Xiao Y, Chen M. Collocation methods for third-kind Volterra integral equations with proportional delays. *Applied Mathematics and Computation*. 2021; 388(1): 1-11.

[5] Song H, Yang Z, Xiao Y. Iterated collocation methods for nonlinear third-kind Volterra integral equations with proportional delays. *Computational and Applied Mathematics*. 2022; 41(191): 1-23.

[6] Teimoori P, Pishbin S. Error analysis of piecewise collocation method for delay integro-differential-algebraic equations. *Journal of Computational and Applied Mathematics*. 2025; 472(1): 116750.

[7] Kamil S, Pishbin S, Darania P. An effective computational approach based on new multi-step collocation method for solving integral-algebraic equations. *Numerical Algorithms*. 2025; 100(1): 1333-1363.

[8] Pishbin S, Darania P. Piecewise polynomial numerical method for Volterra integral equations of the fourth-kind with constant delay. *Hacettepe Journal of Mathematics and Statistics*. 2024; 53(1): 74-87.

[9] Alsahlanee M, Darania P, Pishbin S. A class of higher-order collocation scheme for the integral equation of the convolution type. *Journal of Mathematics*. 2025; 2025(1): 1-15.

[10] Aourir E, Dastjerdi L. Numerical investigation of the mesh-free collocation approach for solving third kind VIEs with nonlinear vanishing delays. *International Journal of Computer Mathematics*. 2025; 102(1): 112-129.