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Research Article

# A Structural Exploration of Demi-Complementation on ADLs

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**Abstract:** On an Almost Distributive Lattice (ADL), the concept of demi-complementation is introduced. From demi-complemented ADLs, we derived some important and interesting properties. Every demi-complemented ADL is given set equivalent conditions that yields a pseudo-complemented ADL, an *MS*-ADL, and a pseudo-complemented lattice.

***Keywords***: Almost Distributive Lattice (ADL), demi-complementation, pseudo-complementation, stone ADL, *MS*-ADL

**MSC:** 06D99, 06D15

## 1. Introduction

Swamy et al. introduced the notion of an Almost Distributive Lattice (ADL) as a unifying framework that encompasses several ring-theoretic extensions of Boolean algebras and distributive lattices [1]. In [2], the idea of "pseudo-complementation" was proposed by the authors on an ADL with a value of 0, and it was shown that it can be defined by an equation. The paper also revealed that in an ADL possessing a least element, the existence of a pseudo-complementation leads to a bijective relationship, where each maximal element in $\Sigma$ uniquely determines a pseudo-complementation.

In the context of an ADL $\Sigma$ with a least element, it was established that any pseudo-complementation gives rise to a subset $\Sigma^* = \{\eta^* \mid \eta \in \Sigma\}$, which forms a Boolean algebra. Notably, this Boolean structure remains essentially the same (up to isomorphism), regardless of the specific pseudo-complementation chosen. Separately, the work in [3] concentrated on the investigation of Stone Almost Distributive Lattices (ADLs). In [4], Gezahagne Mulat Addis introduced the class of *MS*-ADLs-an equational variety that serves as a general framework encompassing De Morgan ADLs. The study presented in [5] explored ADLs equipped with weak pseudo-complementation, highlighting various algebraic characteristics of such structures. Moreover, in Rao [6], examined a related notion termed demi-complementation in the setting of distributive lattices, offering insights into its foundational Characteristics.

The present paper introduces the idea of demi-complementation on ADLs. It shows that the axioms defining a demi-complementation are logically independent. Several key forms of demi-complements in ADLs are examined. Furthermore, the paper establishes a collection of equivalent conditions under which a demi-complementation on an


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ADL can be extended to a pseudo-complementation. It is also demonstrated that, for an ADL equipped with a demi-complementation, there exist specific equivalent criteria that ensure the structure satisfies the requirements of both a pseudo-complemented ADL and an $MS$-ADL.

## 2. Preliminaries

This section aggregates pivotal definitions and principal results as derived from [1], which will function as foundational instruments throughout the remainder of this study.

**Definition 1** [1] An algebra, as characterized in $(\Sigma, \vee, \wedge, 0)$, of the specified type $(2, 2, 0)$, is termed an ADL with zero if it adheres to the following fundamental principles:

(1) $(\eta \vee \xi) \wedge \varsigma = (\eta \wedge \varsigma) \vee (\xi \wedge \varsigma)$;

(2) $\eta \wedge (\xi \vee \varsigma) = (\eta \wedge \xi) \vee (\eta \wedge \varsigma)$;

(3) $(\eta \vee \xi) \wedge \xi = \xi$;

(4) $(\eta \vee \xi) \wedge \eta = \eta$;

(5) $\eta \vee (\eta \wedge \xi) = \eta$;

(6) $0 \wedge \eta = 0$, for any $\eta, \xi, \in \Sigma$.

Let us define a partial order $\leq$ on the set $\Sigma$ by the condition: for all $\eta, \xi \in \Sigma$, we write $\eta \leq \xi$ if $\eta = \eta \wedge \xi$, or equivalently, $\eta \vee \xi = \xi$. This relation clearly satisfies the requirements of a partial order on $\Sigma$. We refer to an element $m \in \Sigma$ as maximal when no other element in $\Sigma$ exceeds it under the partial ordering. The set of all elements possessing this property is denoted by $\mathscr{M}_{\text{Max.elts}}$.

According to Swamy's findings in [1], ADL possesses nearly all the properties of a distributive lattice, except the right distributive law of join over meet, and the commutative laws for both join $(\vee)$ and meet $(\wedge)$. If any one of these omitted conditions holds in an ADL $\Sigma$, then $\Sigma$ becomes a distributive lattice. A subset $\mathscr{J}$ of $\Sigma$ is defined to be an ideal if it is nonempty and satisfies the following: for all $\eta, \xi \in \mathscr{J}$ and $\in \Sigma$, the elements $\eta \vee \xi$ and $\eta \wedge$ belong to $\mathscr{J}$. For any subset $\mathscr{G}$ of $\Sigma$ the smallest ideal containing $\mathscr{G}$ is given by $(\mathscr{G}] := \{(\bigvee_{i=1}^{n} \eta_i) \wedge \mid \eta_i \in \mathscr{G}, \in \Sigma, n \in \mathbb{N}\}$. Let $\mathscr{G} = \{\eta\}$ be a singleton subset of $\Sigma$. The ideal generated by $\eta$ is denoted as $(\eta]$, which is referred to as a principal ideal. For elements $\eta, \xi \in \Sigma$, it holds that $(\eta] \vee (\xi] = (\eta \vee \xi]$ and $(\eta] \cap (\xi] = (\eta \wedge \xi]$. Therefore, the structure $(\mathscr{PI}(\Sigma), \vee, \cap)$, comprising all principal ideals, forms a sublattice of the complete lattice $(\mathfrak{I}(\Sigma), \vee, \cap)$ of all ideals.

**Definition 2** [2] A unary operation $\eta \mapsto \eta^*$ on an ADL $\Sigma$ is said to define a pseudo-complementation if, for any elements $\eta, \xi \in \Sigma$, the following conditions hold:

(1) $\eta \wedge \xi = 0$ implies $\eta^* \wedge \xi = \xi$,

(2) $\eta \wedge \eta^* = 0$,

(3) $(\eta \vee \xi)^* = \eta^* \wedge \xi^*$.

We refer to an ADL $\Sigma$ as a Pseudo-Complemented ADL (PCADL) when each of its elements admits a pseudo-complement. If $m \in \mathscr{M}_{\text{Max.elts}}$, then an $\Sigma$ is called a Stone ADL provided it is pseudo-complemented and satisfies the identity $\eta^* \vee \eta = 0$ for every $\eta \in \Sigma$.

**Theorem 1** [2] Consider an ADL $\Sigma$ where a pseudo-complementation is defined. Then, for all elements $\eta$ and $\xi$ in $\Sigma$, the identities below are fulfilled:

(1) $0^* \in \mathscr{M}_{\text{Max.elts}}$,

(2) $\eta \in \mathscr{M}_{\text{Max.elts}}$ implies $\eta^* = 0$,

(3) $0^{**} = 0$,

(4) $\eta^{**} \wedge \eta = \eta$,

(5) $\eta^{***} = \eta^*$,

(6) $\eta \leq \xi$ implies $\xi^* \leq \eta^*$,

(7) $\eta^* \wedge \xi^* = \xi^* \wedge \eta^*$,

(8) $(\eta \wedge \xi)^{**} = \eta^{**} \wedge \xi^{**}$,

(9) $\eta^* \wedge \xi = (\eta \wedge \xi)^* \wedge \xi^*$.

**Definition 3** [4] An $MS$-ADL is an algebra $(\Sigma,\ \vee,\ \wedge,\ ^{\circ},\ 0)$ characterized by the type (2, 2, 1, 0), wherein $\Sigma$ constitutes an ADL with maximal elements, and $\mapsto x^{\circ}$ denotes a unary operation on $\Sigma$, adhering to the following conditions:

(1) $\varepsilon^{\circ\circ} \wedge \varepsilon = \varepsilon$,

(2) $(\varepsilon \vee \pi)^{\circ} = \varepsilon^{\circ} \wedge \pi^{\circ}$,

(3) $(\varepsilon \wedge \pi)^{\circ} = \varepsilon^{\circ} \vee \pi^{\circ}$,

(4) $m^{\circ} = 0$ for all $m \in \mathscr{M}_{\text{Max.elts}}$ for all $\varepsilon,\ \pi \in \Sigma$.

**Definition 4** [5] A function $\eta \mapsto \eta$ from an ADL $\Sigma$ to itself is said to define a weak pseudo-complementation on $\Sigma$ if, for all $\eta,\ \xi \in \Sigma$, $\eta \wedge \xi = 0$ if and only if $\eta^{*} \wedge \xi = \xi$.

## 3. Demi-complementation on ADLs

The notion of demi-complementation is explored within the framework of ADLs. Several fundamental properties arising from demi-complemented ADLs are established. Furthermore, necessary and sufficient conditions are formulated under which a demi-complemented ADL becomes a PCADL or an $MS$-ADL.

Now we begin with the following definition.

**Definition 5** Consider an ADL $\Sigma$ that contains maximal elements. A unary operation $\varepsilon \mapsto \varepsilon^{\circ}$, defined from $\Sigma$ to itself, is referred to as demi-complementation on $\Sigma$ if it fulfills the conditions listed below for every $\eta,\ \xi \in \Sigma$:

($\mathscr{C}1$) $0^{\circ} \in \mathscr{M}_{\text{Max.elts}}$,

($\mathscr{C}2$) $\eta \wedge \eta^{\circ} = 0$,

($\mathscr{C}3$) $(\eta \vee \xi)^{\circ} = \eta^{\circ} \wedge \xi^{\circ}$.

An ADL $\Sigma$ containing $\mathscr{M}_{\text{Max.elts}}$ is called demi-complemented if there exists a unary operation $\eta \mapsto \eta^{\circ}$ on $\Sigma$ that qualifies as a demi-complementation. To begin with, we explore the mutual independence of the defining conditions.

**Example 1** Consider the set $\mathscr{B} = \{0,\ 1,\ 2\}$, a 3-element discrete ADL with $(0,\ 0,\ 0)$ designated as the least element. Let $\mathscr{B}^3 = \mathscr{B} \times \mathscr{B} \times \mathscr{B}$ denote the product ADL, where the operations are applied component-wise. For any element $\eta \in \mathscr{B}^3$, let $|\eta|$ represent the number of coordinates in $\eta$ that are non-zero. If $x = (x_1,\ x_2,\ x_3) \in \mathscr{B}^3$ is a non-zero element, define its demi-complement by $\varepsilon^{\circ} = (\varepsilon_1^{\circ},\ \varepsilon_2^{\circ},\ \varepsilon_3^{\circ})$.

$$\varepsilon_i^{\circ} = \begin{cases} 0 & \text{if } \varepsilon_i \neq 0 \\ 1 & \text{if } \varepsilon_i = 0 \text{ and } |\varepsilon| = 1 \\ 2 & \text{if } \varepsilon_i = 0 \text{ and } |\varepsilon| > 1 \end{cases} \tag{1}$$

and $0^{\circ} = (2,\ 2,\ 2)$. For example,

$$(1,\ 0,\ 0)^{\circ} = (0,\ 1,\ 1),\ (1,\ 2,\ 0)^{\circ} = (0,\ 0,\ 2),\ \text{and}\ (2,\ 0,\ 1)^{\circ} = (0,\ 2,\ 0).$$

It is straightforward to verify that the mapping $\varepsilon \mapsto \varepsilon^{\circ}$ satisfies the conditions $\mathscr{C}1$ and $\mathscr{C}2$. However, it does not satisfy $\mathscr{C}3$. To see this, consider the elements $\eta = (1,\ 0,\ 0)$ and $\xi = (0,\ 1,\ 0)$. Their join is

$$\eta \vee \xi = (1,\ 1,\ 0), \tag{2}$$

so that

$$(\eta \vee \xi)^{\circ} = (0,\ 0,\ 2). \tag{3}$$

On the other hand,

$$\eta^{\circ} = (0,\ 1,\ 1) \text{ and } \xi^{\circ} = (1,\ 0,\ 1), \tag{4}$$

which gives

$$\eta^{\circ} \wedge \xi^{\circ} = (0,\ 0,\ 1) \neq (\eta \vee \xi)^{\circ}. \tag{5}$$

**Example 2** Consider an infinite Almost Distributive Lattice $\Sigma$ with the least element 0. Let $x_0 \in \Sigma$ be a fixed element such that $x_0 \neq 0$ and $x_0 \notin \mathscr{M}_{\text{Max.elts}}$. Consider a unary operation $\varepsilon \mapsto \varepsilon^{\circ}$ on $\Sigma$ mapping into itself as described below:

$$\eta^{\circ} = \begin{cases} 0 & \text{if } \ \eta \neq 0 \\ \varepsilon_0 & \text{if } \ \eta = 0 \end{cases} \tag{6}$$

One can readily check that the operation $\varepsilon \mapsto \varepsilon^{\circ}$ satisfies conditions $\mathscr{C}2$ and $\mathscr{C}3$ but does not fulfill $\mathscr{C}1$.

**Example 3** Define a unary operation on a bounded distributive lattice $\Sigma$ by setting $\eta^{\circ} = 1$ for all $\eta \in \Sigma$. Under this definition, the operation $\varepsilon \mapsto \varepsilon^{\circ}$ satisfies the conditions $\mathscr{C}1$ and $\mathscr{C}3$, but it fails to satisfy $\mathscr{C}2$. For instance, taking $\eta = 1$, we find

$$1 \wedge 1^{\circ} = 1 \wedge 1 = 1 \neq 0, \tag{7}$$

which shows that $\mathscr{C}2$ does not hold.

The following construction illustrates a demi-complemented ADL that is neither a lattice in the usual sense nor a discrete ADL.

**Example 4** Let $\Sigma = \{0,\ \eta,\ \xi,\ \varsigma\}$. The operations $\vee$ and $\wedge$ are defined on $\Sigma$ as given in Figure 1.

| $\wedge$ | 0 | $\eta$ | $\xi$ | $\varsigma$ |
|---|---|---|---|---|
| 0 | 0 | $\eta$ | $\eta$ | $\eta$ |
| $\eta$ | $\eta$ | $\eta$ | $\eta$ | $\eta$ |
| $\xi$ | $\xi$ | $\xi$ | $\xi$ | $\xi$ |
| $\varsigma$ | $\varsigma$ | $\eta$ | $\xi$ | $\varsigma$ |

| $\wedge$ | 0 | $\eta$ | $\xi$ | $\varsigma$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $\eta$ | 0 | $\eta$ | $\xi$ | $\varsigma$ |
| $\xi$ | 0 | $\eta$ | $\xi$ | $\varsigma$ |
| $\varsigma$ | 0 | $\varsigma$ | $\varsigma$ | $\varsigma$ |

**Figure 1.** The set $\Sigma$ with $\vee$ and $\wedge$

In this case, the algebra $(\Sigma,\ \vee,\ \wedge)$ forms an ADL that is neither a lattice nor a discrete ADL. Consider a unary operation $\varepsilon \mapsto \varepsilon^{\circ}$ on $\Sigma$ by setting $\varepsilon^{\circ} = 0$ for all $\varepsilon \neq 0$, and $0^{\circ} = \eta$. With this definition, the map $\varepsilon \mapsto \varepsilon^{\circ}$ satisfies the properties of a demi-complementation on $\Sigma$.

**Lemma 1** Assume that $\Sigma$ is an ADL and that $\varepsilon \mapsto \varepsilon^\circ$ defines a demi-complementation on $\Sigma$. Then, for all elements $\eta, \xi \in \Sigma$, the following statements hold:

(1) $\eta \in \mathscr{M}_{\text{Max.elts}}$ imples $\eta^\circ = 0$,

(2) $0^{\circ\circ} = 0$,

(3) $\eta^\circ = 0$ implies $\eta^{\circ\circ} \in \mathscr{M}_{\text{Max.elts}}$,

(4) $\eta^\circ \leq 0^\circ$,

(5) $\eta^\circ \wedge \xi^\circ = \xi^\circ \wedge \eta^\circ$,

(6) $\eta \leq \xi$ implies $\xi^\circ \leq \eta^\circ$,

(7) $\eta^\circ \leq (\eta \wedge \xi)^\circ$ and $\xi^\circ \leq (\eta \wedge \xi)^\circ$,

(8) $(\eta \vee \xi)^\circ = (\xi \vee \eta)^\circ$,

(9) $\eta^{\circ\circ} \wedge \eta = \eta$ implies $\eta^{\circ\circ\circ} \leq \eta^\circ$,

(10) $\eta = 0$ implies $\eta^{\circ\circ} = 0$.

**Proof.** (1) Let $\eta$ be a maximal element in $\Sigma$. Since $\eta \leq \eta \vee \eta^\circ$, it follows that $\eta = \eta \vee \eta^\circ$. Applying the demi-complementation operation yields

$$\eta^\circ = (\eta \vee \eta^\circ)^\circ = \eta^\circ \wedge \eta^{\circ\circ}. \tag{8}$$

By condition $\mathscr{C}2$, this implies $\eta^\circ = 0$. Hence, the complement of any maximal element is zero.

(2) As $0^\circ$ is a maximal element, it follows from (1) that $0^{\circ\circ} = 0$.

(3) Suppose $\eta^\circ = 0$. Then, by condition $\mathscr{C}1$, it follows that $\eta^{\circ\circ} = 0^\circ$, which is a maximal element.

(4) Let $\eta \in \Sigma$. Since $a \vee 0 = \eta$, applying the demi-complementation gives

$$\eta^\circ = (\eta \vee 0)^\circ. \tag{9}$$

By property $\mathscr{C}3$, we also have

$$(\eta \vee 0)^\circ = \eta^\circ \wedge 0^\circ. \tag{10}$$

Therefore,

$$\eta^\circ = \eta^\circ \wedge 0^\circ, \tag{11}$$

which implies $\eta^\circ \leq 0^\circ$.

(5) Consider any $\eta,\ \xi \in \Sigma$. From condition (4), it follows that both $\eta^\circ$ and $\xi^\circ$ are less than or equal to $0^\circ$. This means that $0^\circ$ is a common upper bound for $\eta^\circ$ and $\xi^\circ$. Therefore, their meet exists and satisfies the identity

$$\eta^\circ \wedge \xi^\circ = \xi^\circ \wedge \eta^\circ. \tag{12}$$

(6) Let $\eta,\ \xi \in \Sigma$ be such that $\eta \leq \xi$. Then, we get $\xi = \eta \vee \xi$. By (6), we get $\xi^\circ = (\eta \vee \xi)^\circ = \eta^\circ \wedge \xi^\circ = \xi^\circ \wedge \eta^\circ$. Therefore $\xi^\circ \leq \eta^\circ$.

(7) Let $\eta,\ \xi \in \Sigma$. Since $\eta = \eta \vee (\eta \wedge \xi)$, applying the demi-complementation yields

$$\eta^{\circ} = (\eta \vee (\eta \wedge \xi))^{\circ} = \eta^{\circ} \wedge (\eta \wedge \xi)^{\circ}. \tag{13}$$

This implies that $\eta^{\circ} \leq (\eta \wedge \xi)^{\circ}$. Moreover, since $\eta \wedge \xi \leq \xi$, it follows by property (6) that

$$\xi^{\circ} \leq (\eta \wedge \xi)^{\circ}. \tag{14}$$

(8) From (5), it is follows.
(9) Let $\eta^{\circ\circ} \wedge \eta = \eta$. Then,

$$\eta^{\circ\circ} \vee \eta = \eta^{\circ\circ} \vee (\eta^{\circ\circ} \wedge \eta) = \eta^{\circ\circ}. \tag{15}$$

Applying the demi-complementation to both sides gives

$$\eta^{\circ\circ\circ} = (\eta^{\circ\circ} \vee \eta)^{\circ} = \eta^{\circ\circ\circ} \wedge \eta^{\circ}. \tag{16}$$

Thus $\eta^{\circ\circ\circ} \leq \eta^{\circ}$.
(10) Let $\eta = 0$. Using condition (2), we obtain

$$\eta^{\circ\circ} = 0^{\circ\circ} = 0. \tag{17}$$

**Proposition** If an ADL admits a pseudo-complementation, then it is also demi-complemented.
**Proof.** This result follows directly from Definition 2 and Theorem 1.

The reverse implication of the above result is not valid; in other words, not every demi-complemented ADL is pseudo-complemented. Imagine, for instance, an intriguing ADL $\Sigma$ harboring a minimum of two distinct elements. Craft a unary operation $\varepsilon \mapsto \varepsilon^{\circ}$ upon $\Sigma$ by ingeniously designating $0^{\circ}$ to ascend as a maximal element of $\Sigma$, while astutely maintaining $\varepsilon^{\circ} = 0$ for each and every $\varepsilon \neq 0$. It is evident that this operation defines a valid demi-complementation. However, it does not qualify as a pseudo-complementation. To see why, take elements $\eta, \xi \in \Sigma$ such that $\eta \wedge \xi = 0$. Then

$$\eta^{\circ} \wedge \xi = 0 \wedge \xi = 0 \neq \xi, \tag{18}$$

which violates the condition required for pseudo-complementation. In [5], authors demonstrated that such a weak structure can be used to define a proper pseudo complementation. Although demi complementations do not always yield pseudo complementations, the following result characterizes exactly when a demi complementation gives rise to a pseudo complementation.

**Theorem 2** A unary operation on an ADL that defines a demi-complementation will also define a pseudo-complementation if and only if it constitutes a weak pseudo-complementation.

**Proof.** It follows directly.

**Theorem 3** Consider an almost distributive lattice $(\Sigma, \vee, \wedge, 0)$. A demi-complementation $\varepsilon \mapsto \varepsilon^{\circ}$ on $\Sigma$ is characterized as a pseudo-complementation if it satisfies the following conditions:

(1) $\eta^{\circ\circ} \wedge \eta = \eta$,

(2) $(\eta \wedge \xi)^{\circ\circ} = \eta^{\circ\circ} \wedge \xi^{\circ\circ}$.

**Proof.** Suppose $\varepsilon \mapsto \varepsilon^{\circ}$ is a demi-complementation on $\Sigma$, and consider that this operation fulfills the specified conditions. To demonstrate that $^{\circ}$ indeed defines a pseudo-complementation, we need only verify the defining property of a pseudo-complement. Let $\eta,\ \xi \in \Sigma$ be such that their meet is zero, i.e., $\eta \wedge \xi = 0$. Under this assumption, we proceed to examine the required implication.

$$\begin{aligned}
\xi &= \xi^{\circ\circ} \wedge \xi && \text{by (1)} \\
&= 0^{\circ} \wedge \xi^{\circ\circ} \wedge \xi && \text{since } 0^{\circ} \text{ is maximal} \\
&= (\eta^{\circ} \wedge \eta^{\circ\circ})^{\circ} \wedge \xi^{\circ\circ} \wedge \xi && \text{by } (\mathscr{C}2) \\
&= (\eta \vee \eta^{\circ})^{\circ\circ} \wedge \xi^{\circ\circ} \wedge \xi && \text{by } (\mathscr{C}3) \\
&= ((\eta \vee \eta^{\circ}) \wedge \xi)^{\circ\circ} \wedge \xi && \text{by (2)} \\
&= ((\eta \wedge \xi) \vee (\eta^{\circ} \wedge \xi))^{\circ\circ} \wedge \xi && \\
&= (\eta^{\circ} \wedge \xi)^{\circ\circ} \wedge \xi && \text{since } \eta \wedge \xi = 0 \\
&= \eta^{\circ\circ\circ} \wedge \xi^{\circ\circ} \wedge \xi && \text{by (2)} \\
&= \eta^{\circ\circ\circ} \wedge \xi && \text{by (1)} \\
&= \eta^{\circ} \wedge \xi && \text{by (1) and Lemma 1(9)}
\end{aligned} \tag{19}$$

It follows that $^{\circ}$ defines a pseudo-complementation on $\Sigma$. The reverse direction is established by Theorem 1.

**Theorem 4** A demi-complementation $\mapsto^{\circ}$ on an ADL $\Sigma$ is a pseudo-complementation iff it satisfies the following condition:

$$\eta^{\circ} \wedge \xi = (\eta \wedge \xi)^{\circ} \wedge \xi. \tag{20}$$

**Proof.** Suppose that the mapping $\varepsilon \mapsto \varepsilon^{\circ}$ defines a demi-complementation on the ADL $\Sigma$, and let us consider that the operation $^{\circ}$ satisfies a given condition. To establish that $^{\circ}$ acts as a pseudo-complementation on $\Sigma$, it suffices to verify the necessary properties. Take any elements $\eta,\ \xi \in \Sigma$ such that $\eta \wedge \xi = 0$. According to the assumed condition, it follows that

$$\eta^{\circ} \wedge \xi = (\eta \wedge \xi)^{\circ} \wedge \xi = 0^{\circ} \wedge \xi = \xi \tag{21}$$

because of $0^\circ$ is maximal. Thus $^\circ$ is a pseudo-complementation on $\Sigma$.

On the other hand, suppose that $^\circ$ functions as a pseudo-complementation on $\Sigma$. For every $\eta,\ \xi \in \Sigma$, we have

$$\begin{aligned} \eta \wedge \xi \wedge (\eta \wedge \xi)^\circ = 0 &\Rightarrow \eta^\circ \wedge \xi \wedge (\eta \wedge \xi)^\circ = \xi \wedge (\eta \wedge \xi)^\circ \\ &\Rightarrow \eta^\circ \wedge \xi \wedge (\eta \wedge \xi)^\circ \wedge \xi = \xi \wedge (\eta \wedge \xi)^\circ \wedge \xi \\ &\Rightarrow \eta^\circ \wedge (\eta \wedge \xi)^\circ \wedge \xi = (\eta \wedge \xi)^\circ \wedge \xi \\ &\Rightarrow (\eta \wedge \xi)^\circ \wedge \eta^\circ \wedge \xi = (\eta \wedge \xi)^\circ \wedge \xi \\ &\Rightarrow \eta^\circ \wedge \xi = (\eta \wedge \xi)^\circ \wedge \xi \qquad \text{since } (\eta \wedge \xi) \wedge \eta^\circ \wedge \xi = 0 \end{aligned} \tag{22}$$

Hence the operation $^\circ$ is satisfies the given condition.

The following theorem is a generalization of Theorem 1.7 of U.M. Swamy et al. [1] offering new equivalent conditions that determine when a demi-complemented ADL attains the structure of a demi-complemented lattice.

**Theorem 5** The following assertions hold equivalently within any demi-complemented ADL $\Sigma$.

(1) $\Sigma$ is a demi-complemented lattice;

(2) the poset $(\Sigma,\ \leq)$ is directed above;

(3) $(\Sigma,\ \vee,\ \wedge)$ is a distributive lattice;

(4) $\vee$ is commutative;

(5) $\wedge$ is commutative;

(6) $\vee$ is right distributive over $\wedge$;

(7) the relation $\theta = \{(\varepsilon,\ \pi) \in \Sigma \times \Sigma \mid \xi \wedge \eta = \xi\}$ is antisymmetric;

(8) for each $\eta \in \Sigma$, the relation $\phi_\eta = \{(\eta,\ \varepsilon) \in \Sigma \times \Sigma \mid \varepsilon \vee \eta = \pi \vee \eta \text{ and } \varepsilon^\circ \vee \eta = \pi^\circ \vee \eta\}$ is a congruence on $\Sigma$.

In the following, we establish a connection between the class of all demi-complemented ADLs and *MS*-ADLs.

**Theorem 6** A demi-complemented ADL is an *MS*-ADL iff it satisfies the following:

(1) $\varepsilon^{\circ\circ} \wedge \varepsilon = \varepsilon$,

(2) $(\varepsilon \wedge \pi)^\circ = \varepsilon^\circ \vee \pi^\circ$.

**Proof.** This conclusion is derived using Definition 3 along with the notion of demi-complementation.

According to [1], it was demonstrated that the collection $\mathscr{I}(\Sigma)$ comprising all ideals of an ADL $\Sigma$ constitutes a distributive lattice, while the collection $\mathscr{P}\mathscr{I}(\Sigma)$, consisting of all principal ideals of the same ADL $\Sigma$, represents a sublattice of $\mathscr{I}(\Sigma)$. The subsequent result explores the connection between the demi-complementations of these classes of ideals within an ADL.

**Theorem 7** An ADL $\Sigma$ with $m \in \mathscr{M}_{\text{Max.elts}}$ is demi-complemented if and only if $\mathscr{P}\mathscr{I}(\Sigma)$ is a demi-complemented lattice.

**Proof.** Suppose $\Sigma$ is a demi-complemented ADL with $^\circ$ as its demi-complementation. For any $(\eta] \in \mathscr{P}\mathscr{I}(\Sigma)$, set $(\eta]^\perp = (\eta^\circ]$. We need to demonstrate that $\perp$ acts as a demi-complementation on $\mathscr{P}\mathscr{I}(\Sigma)$. Observing that $(0]^\perp = (0^\circ] = \Sigma$, we note that $0^\circ$ is maximal in $\Sigma$. Therefore, $(0]^\perp$ is also maximal in $\mathscr{P}\mathscr{I}(\Sigma)$. Now, $(\eta] \cap (\eta]^\perp = (\eta] \cap (\eta^\circ] = (\eta \wedge \eta^\circ] = (0]$. For any $(\eta],\ (\xi] \in \mathscr{P}\mathscr{I}(\Sigma)$, we get $((\eta] \vee (\xi])^\perp = (\eta \vee \xi]^\perp = ((\eta \vee \xi)^\circ] = (\eta^\circ \wedge \xi^\circ] = (\eta^\circ] \cap (\xi^\circ] = (\eta]^\perp \cap (\xi]^\perp$. Therefore, $(\eta]^\perp$ is a demi-complement of $(\eta]$ in $\mathscr{P}\mathscr{I}(\Sigma)$. Hence, $\mathscr{P}\mathscr{I}(\Sigma)$ is a demi-complemented ADL.

On the other hand, suppose that $\mathscr{P}\mathscr{I}(\Sigma)$ constitutes a demi-complemented lattice with $^\circ$ serving as its demi-complementation. For each $(\eta] \in \mathscr{P}\mathscr{I}(\Sigma)$, there is an $\eta_1 \in \Sigma$ satisfying

$$(\eta]^\circ = \begin{cases} (m] & \text{if } \eta = 0 \\ (\eta_1] & \text{if } \eta \neq 0 \end{cases} \tag{23}$$

the demi-complement of $(\eta]$ in $\mathscr{P}\mathscr{I}(\Sigma)$. Clearly $(0]^\circ = \Sigma$ is the unique maximal element in $\mathscr{P}\mathscr{I}(\Sigma)$. For any $\eta \in \Sigma$, define $\eta^\perp = \eta_1 \wedge m$. It is now required to prove that $\perp$ is a demi-complementation on $\Sigma$. Suppose there exists $\eta_1,\ \eta_2$ such that $(\eta]^\circ = (\eta_1] = (\eta_2]$. Then $\eta_1 \wedge m = \eta_2 \wedge \eta_1 \wedge m = \eta_1 \wedge \eta_2 \wedge m = \eta_2 \wedge m$. Hence $\perp$ is well-defined. Clearly $0^\perp = m \wedge m$ is maximal in $\Sigma$. Since $^\circ$ is demi-complementation on $\mathscr{P}\mathscr{I}(\Sigma)$, we get $(\eta \wedge \eta_1] = (\eta] \cap (\eta_1] = (\eta] \cap (\eta]^\circ = (0]$. Hence $\eta \wedge \eta^\perp = \eta \wedge \eta_1 \wedge m = 0 \wedge m = 0$. Now, suppose that $(\eta]^\circ = (\eta_1]$ and $(\xi]^\circ = (1]$ for some $\eta_1,\ 1 \in \Sigma$. Then, we get

$$(\eta \vee \xi]^\circ = ((\eta] \vee (\xi])^\circ = (\eta]^\circ \cap (\xi]^\circ = (\eta_1] \cap (1] = (\eta_1 \wedge 1]. \tag{24}$$

Hence, by definition, $(\eta \vee \xi)^\perp = \eta_1 \wedge 1 \wedge m = \eta_1 \wedge 1 \wedge m \wedge m = \eta_1 \wedge m \wedge 1 \wedge m = \eta^\perp \wedge \xi^\perp$. Therefore $\perp$ is a demi-complementation on $\Sigma$.

**Theorem 8** For demi-complementation $^\circ$ on an ADL $\Sigma$ with $m \in \mathscr{M}_{\text{Max.elts}}$, define $\circ_m : \Sigma \longrightarrow \Sigma$ by $\eta^{\circ_m} = \eta^\circ \wedge m$ for any $\eta \in \Sigma$. Then $\circ_m$ is a demi-complementation on $\Sigma$ in which $0^{\circ_m} = m$.

**Proof.** Clearly $\circ_m$ is well-defined. Since $0^\circ$ is maximal, we get that $0^{\circ_m} = 0^\circ \wedge m = m$. Hence $0^{\circ_m}$ is maximal. Let $\eta \in \Sigma$. Then, we get $\eta \wedge \eta^{\circ_m} = \eta \wedge \eta^\circ \wedge m = 0 \wedge m = 0$. Now, let $\eta,\ \xi \in \Sigma$. Then $(\eta \vee \xi)^{\circ_m} = (\eta \vee \xi)^\circ \wedge m = \eta^\circ \wedge \xi^\circ \wedge m = \eta^\circ \wedge \xi^\circ \wedge m \wedge m = \eta^\circ \wedge m \wedge \xi^\circ \wedge m = \eta^{\circ_m} \wedge xi^{\circ_m}$. Therefore $\circ_m$ is a demi-complementation on $\Sigma$.

**Theorem 9** There exists an injection map from $\mathscr{M}_{\text{Max.elts}}$ into $\mathscr{D}\mathscr{C}(\Sigma)$, where $\Sigma$ and $\mathscr{D}\mathscr{C}(\Sigma)$ is the set of all demi-complementations on $\Sigma$.

**Proof.** Define a mapping $f : \mathscr{M}_{\text{Max.elts}} \longrightarrow \mathscr{D}\mathscr{C}(\Sigma)$ by $f(m) = \circ_m$ for all $m \in \mathscr{M}_{\text{Max.elts}}$. By Theorem 8, $f$ is well-defined. Let $m,\ n \in \mathscr{M}_{\text{Max.elts}}$ be such that $f(m) = f(n)$. Then, we get $\circ_m = \circ_n$. That is $\eta^{\circ_m} = \eta^{\circ_n}$ for all $\eta \in \Sigma$. In particular, for $0 \in \Sigma$, we get

$$m = 0^\circ \wedge m = 0^{\circ_m} = 0^{\circ_n} = 0^\circ \wedge n = n. \tag{25}$$

Thus $f$ is an injection of $\mathscr{M}_{\text{Max.elts}}$ into $\mathscr{D}\mathscr{C}(\Sigma)$. Therefore $f$ is injective.

**Theorem 10** Let $\mathscr{T}$ be an ideal in an ADL $\Sigma$ and $\varepsilon \mapsto \varepsilon^\circ$ is a demi-complementation on $\Sigma$. For any $\varepsilon,\ \pi \in \mathscr{T}$, define a binary relation $\theta_{\mathscr{T}}$ on $\Sigma$ by

$$(\varepsilon,\ \pi) \in \theta_{\mathscr{T}} \text{ if and only if } \varepsilon \wedge \eta^\circ = \pi \wedge \eta^\circ \tag{26}$$

for some $\eta \in \mathscr{T}$. Then $\theta_{\mathscr{T}}$ is a congruence on $\Sigma$ such that $\mathscr{T} \subseteq ker\ \theta_{\mathscr{T}}$.

**Proof.** Clearly $\theta_{\mathscr{T}}$ is reflexive and symmetric. To prove the transitivity, let us take $(\varepsilon,\ \pi) \in \theta_{\mathscr{T}}$ and $(\pi,\ \chi) \in \theta_{\mathscr{T}}$. Then $\varepsilon \wedge \eta^\circ = \pi \wedge \eta^\circ$ and $\pi \wedge \xi^\circ = \chi \wedge \xi^\circ$ for some $\eta,\ \xi \in \mathscr{T}$. Hence $\varepsilon \wedge (\eta \vee \xi)^\circ = \varepsilon \wedge \eta^\circ \wedge \xi^\circ = \pi \wedge \eta^\circ \wedge \xi^\circ = \eta^\circ \wedge \pi \wedge \xi^\circ = \eta^\circ \wedge \chi \wedge \xi^\circ = \chi \wedge \eta^\circ \wedge \xi^\circ = \chi \wedge (\eta \vee \xi)^\circ$. Since $\eta \vee \xi \in \mathscr{T}$, we get $(\varepsilon,\ \chi) \in \theta_{\mathscr{T}}$. Thus $\theta_{\mathscr{T}}$ is transitive. Now, let $(\varepsilon,\ \pi) \in \theta_{\mathscr{T}}$ and $(\chi,\ \lambda) \in \theta_{\mathscr{T}}$. Then $\varepsilon \wedge \eta^\circ = \pi \wedge \eta^\circ$ and $\chi \wedge \xi^\circ = \lambda \wedge \xi^\circ$ for some $\eta,\ \xi \in \mathscr{T}$. Then

$$\varepsilon \wedge \chi \wedge (\eta \vee \xi)^\circ = \varepsilon \wedge \eta^\circ \wedge \chi \wedge \xi^\circ = \pi \wedge \eta^\circ \wedge \lambda \wedge \xi^\circ = \pi \wedge \lambda \wedge (\eta \vee \xi)^\circ \tag{27}$$

which gives that $(\varepsilon \wedge \chi,\ \pi \wedge \lambda) \in \theta_{\mathscr{T}}$. Again, we observe that

$$
\begin{aligned}
(\varepsilon \vee \chi) \wedge (\eta \vee \xi)^\circ &= (\varepsilon \vee \chi) \wedge (\eta^\circ \wedge \xi^\circ) \\
&= (\varepsilon \wedge \eta^\circ \wedge \xi^\circ) \vee (\chi \wedge \eta^\circ \wedge \xi^\circ) \\
&= (\varepsilon \wedge \eta^\circ \wedge \xi^\circ) \vee (\eta^\circ \wedge \chi \wedge \xi^\circ) \\
&= (\pi \wedge \eta^\circ \wedge \xi^\circ) \vee (\eta^\circ \wedge \lambda \wedge \xi^\circ) \qquad (28) \\
&= (\pi \wedge \eta^\circ \wedge \xi^\circ) \vee (\lambda \wedge \eta^\circ \wedge \xi^\circ) \\
&= (\pi \vee \lambda) \wedge (\eta^\circ \wedge \xi^\circ) \\
&= (\pi \vee \lambda) \wedge (\eta \vee \xi)^\circ
\end{aligned}
$$

which yields that $(\varepsilon \vee \chi,\ \pi \vee \lambda) \in \theta_{\mathscr{T}}$. Therefore $\theta_{\mathscr{T}}$ is a congruence on $\Sigma$. Let $\eta \in \mathscr{T}$. Since $\eta \wedge \eta^\circ = 0 = 0 \wedge \eta^\circ$, we get $(\eta,\ 0) \in \theta_{\mathscr{T}}$. Hence $\eta \in ker\ \theta_{\mathscr{T}}$. Therefore $\mathscr{T} \subseteq ker\ \theta_{\mathscr{T}}$.

Since $\theta_{\mathscr{T}}$ is a congruence on $\Sigma$, it is a well-known fact that the set $\Sigma/\theta_{\mathscr{T}}$ of all congruence classes given as $\{[\varepsilon]_{\theta_{\mathscr{T}}} \mid \varepsilon \in \Sigma\}$ is a demi-complemented ADL in which the demi-complementation is given by $[\varepsilon]^\circ_{\theta_{\mathscr{T}}} = [\varepsilon^\circ]_{\theta_{\mathscr{T}}}$ for all $\varepsilon \in \Sigma$.

**Corollary 1** For any ideal $\mathscr{T}$ of demi-complemented ADL $\Sigma$, the quotient algebra $\Sigma/\theta_{\mathscr{T}} = \{[\varepsilon]_{\theta_{\mathscr{T}}} \mid \varepsilon \in \Sigma\}$ is a demi-complemented lattice where $[\varepsilon]_{\theta_{\mathscr{T}}}$ is the congruence class of $\varepsilon$ modulo $\theta_{\mathscr{T}}$.

**Proof.** To prove that $\Sigma/\theta_{\mathscr{T}}$ is a lattice, let us take $[\varepsilon]_{\theta_{\mathscr{T}}},\ [\varepsilon]_{\theta_{\mathscr{T}}} \in \Sigma/\theta_{\mathscr{T}}$. For any $\zeta \in \Sigma$, suppose that $\zeta \in [\varepsilon \vee \pi]_{\theta_{\mathscr{T}}}$. Then, we get the following consequence:

$$
\begin{aligned}
(\zeta, \varepsilon \vee \pi) \in \theta_{\mathscr{T}} &\Leftrightarrow \zeta \wedge \eta^\circ = (\varepsilon \vee \pi) \wedge \eta^\circ \qquad \text{for some } \eta \in \mathscr{T} \\
&\Leftrightarrow \zeta \wedge \eta^\circ = (\pi \vee \varepsilon) \wedge \eta^\circ \qquad (29) \\
&\Leftrightarrow \zeta \in [\pi \vee \varepsilon]_\theta
\end{aligned}
$$

which gives that $[\varepsilon]_{\theta_{\mathscr{T}}} \vee [\varepsilon]_{\theta_{\mathscr{T}}} = [\varepsilon \vee \pi]_{\theta_{\mathscr{T}}} = [\pi \vee \varepsilon]_{\theta_{\mathscr{T}}} = [\pi]_{\theta_{\mathscr{T}}} \vee [\varepsilon]_{\theta_{\mathscr{T}}}$. By Theorem 5 (4), it concludes that $\Sigma/\theta_{\mathscr{T}}$ is a lattice and hence a distributive lattice.

**Theorem 11** Let $\mathscr{T}$ be an ideal in an ADL $\Sigma$ and $\varepsilon \mapsto \varepsilon^\circ$ is a demi-complementation on $\Sigma$. For any $\varepsilon,\ \pi \in \mathscr{T}$, define a binary relation $\hat{\theta}_{\mathscr{T}}$ on $\Sigma$ by

$$
(\varepsilon,\ \pi) \in \hat{\theta}_{\mathscr{T}} \ \text{ if and only if } \ \eta^\circ \wedge \varepsilon = \eta^\circ \wedge \pi \qquad (30)
$$

for some $\eta \in \mathscr{T}$. Then $\hat{\theta}_{\mathscr{T}}$ is a congruence on $\Sigma$ such that $\mathscr{T} \subseteq ker\ \hat{\theta}_{\mathscr{T}}$.

**Proof.** It follows in similar lines of Theorem 10.

**Theorem 12** Let $\Sigma$ be an ADL with $m \in \mathscr{M}_{\text{Max.elts}}$, the following statements are thus equivalent:
(1) $\Sigma$ is demi-complemented;
(2) for any ideal $\mathscr{T}$, $\hat{\theta}_{\mathscr{T}}$ is demi-complemented;
(3) $\hat{\theta}_{\{0\}}$ is demi-complemented.
**Proof.** (1)$\Rightarrow$ (2): It is clear by Corollary 1.
(2)$\Rightarrow$ (3): It is clear.
(3)$\Rightarrow$ (1): Assume condition (3).

Let $\varepsilon,\ \pi \in \Sigma$ and choose $(\varepsilon,\ \pi) \in \hat{\theta}_{\{0\}}$. Since $0^{\circ}$ is maximal, we get $\varepsilon = 0^{\circ} \wedge \varepsilon = 0^{\circ} \wedge \pi = \pi$. Hence $\hat{\theta}_{\{0\}}$ (here after, we represent it simply $\hat{\theta}$) is the smallest congruence on $\Sigma$. For any $[\eta]_{\hat{\theta}} \in \Sigma/\hat{\theta}$, there exists some $\eta_1 \in \Sigma$ such that $[\eta]^{\circ}_{\hat{\theta}} = [m]_{\hat{\theta}}$ if $\eta = 0$ otherwise $[\eta]^{\circ}_{\hat{\theta}} = [\eta_1]_{\hat{\theta}}$ which is the demi-complement of $[\eta]_{\hat{\theta}}$ in $\Sigma/\hat{\theta}$. For any $\eta \in \Sigma$, define $\eta^{\perp} = \eta_1 \wedge m$. It is now required to derive that $\perp$ is a demi-complementation on $\Sigma$. Suppose there exists $\eta_1,\ \eta_2$ such that $[\eta]^{\circ}_{\hat{\theta}} = [\eta_1]_{\hat{\theta}} = [\eta_2]_{\hat{\theta}}$. As $\hat{\theta}$ is the smallest congruence, we get $\eta_1 = \eta_2$. Hence $\perp$ is well-defined. Clearly $0^{\perp} = m \wedge m$ is maximal in $\Sigma$. Since $^{\circ}$ is demi-complementation on $\Sigma/\hat{\theta}$, we get

$$[\eta \wedge \eta_1]_{\hat{\theta}} = [\eta]_{\hat{\theta}} \cap [\eta_1]_{\hat{\theta}} = [\eta]_{\hat{\theta}} \cap [\eta]^{\circ}_{\hat{\theta}} = [0]_{\hat{\theta}}. \tag{31}$$

Since $\hat{\theta}$ is the smallest congruence, we get $\eta \wedge \eta_1 = 0$. Hence $\eta \wedge \eta^{\perp} = \eta \wedge \eta_1 \wedge m = 0 \wedge m = 0$. Now, suppose that $[\eta]^{\circ}_{\hat{\theta}} = [\eta_1]_{\hat{\theta}}$ and $[\xi]^{\circ}_{\hat{\theta}} = [1]_{\hat{\theta}}$ for some $\eta_1,\ 1 \in \Sigma$. Then, we get

$$[\eta \vee \xi]^{\circ}_{\hat{\theta}} = ([\eta]_{\hat{\theta}} \vee [\xi])^{\circ}_{\hat{\theta}} = [\eta]^{\circ}_{\hat{\theta}} \cap [\xi]^{\circ}_{\hat{\theta}} = [\eta_1]_{\hat{\theta}} \cap [1]_{\hat{\theta}} = [\eta_1 \wedge 1]_{\hat{\theta}}. \tag{32}$$

Hence, by definition, $(\eta \vee \xi)^{\perp} = \eta_1 \wedge 1 \wedge m = \eta_1 \wedge 1 \wedge m \wedge m = \eta_1 \wedge m \wedge 1 \wedge m = \eta^{\perp} \wedge \xi^{\perp}$. Therefore $\perp$ is a demi-complementation on $\Sigma$.

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## Conflict of interest

All authors declare that they have no conflicts of interest with respect to the publication of this article.

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