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Research Article

Transformation Semigroups Which Are Disjoint Union of Symmetric Groups

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Abstract: Let $T(X)$ be the full transformation semigroup on a set X. For an equivalence relation E on X, define a subsemigroup $T_{E^*}(X)$ of $T(X)$ by

$$T_{E^*}(X) = \{\alpha \in T(X) : \text{ for all } x, y \in X, (x, y) \in E \Leftrightarrow (x\alpha, y\alpha) \in E\}.$$

Let $Q_{E^*}(X)$ be the subset of $T_{E^*}(X)$ consisting of all transformations that each E-class contains exactly one element of its image. Then $Q_{E^*}(X)$ forms a right group. In addition, for a nonempty subset Y of X, define $S_Y(X)$ as a subset of $T(X)$ consisting of all transformations mapping X onto Y such that the restriction on Y is a permutation. Then $S_Y(X)$ is a left group. Furthermore, $Q_{E^*}(X)$ and $S_Y(X)$ can be expressed as a union of symmetric groups. This paper investigates some algebraic properties of $Q_{E^*}(X)$ and $S_Y(X)$, calculates their ranks when X is finite, and establishes conditions for isomorphism. We also characterize and enumerate all maximal subsemigroups when X is finite. Finally, we address the problem of embedding arbitrary left groups into $S_Y(X)$. Our results provide a complete algebraic classification of these transformation semigroups and demonstrate their significance as representations for right and left groups, thereby contributing to the broader understanding of transformation semigroups that decompose as unions of symmetric groups.

***Keywords*:** transformation semigroup, equivalence relation, symmetric group, right group, left group

MSC: 20M20, 20M17, 20M19

1. Introduction

The set of all functions from a set X into itself, denoted as $T(X)$, forms a regular semigroup under the composition of functions. This semigroup is known as the *full transformation semigroup on* X and is important in algebraic semigroup theory. Similar to Cayley's Theorem for groups, it can be shown that any semigroup S can be embedded in the full transformation semigroup $T(S^1)$ where S^1 is the monoid obtained from S by adjoining an identity if necessary.

For an equivalence relation E on a set X, in 2010, Deng et al. [1] introduced a new semigroup called the *transformation semigroup that preserves double direction equivalence*, denoted by $T_{E^*}(X)$, defined by

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$$T_{E^*}(X) = \{\alpha \in T(X) : \forall x, y \in X,\ (x, y) \in E \Leftrightarrow (x\alpha, y\alpha) \in E\}.$$

This semigroup is a generalization of the full transformation semigroup $T(X)$ and is defined as a subsemigroup of $T(X)$. In the field of topology, $T_{E^*}(X)$ is a semigroup consisting of continuous self-maps of the topological space X, where the E-classes form a basis. In [1], the authors investigated the regularity of elements and Green's relations in $T_{E^*}(X)$. Further research is available in references [2–4].

In a recent study, Sangkhanan [5] investigated the regular part, denoted by $\mathrm{Reg}(T)$, of the transformation semigroup $T_{E^*}(X)$ and showed that it is the largest regular subsemigroup of $T_{E^*}(X)$. He also described Green's relations and ideals of $\mathrm{Reg}(T)$. If the set X is partitioned by the equivalence relation E into subsets A_i for all i in the index set I, the author defined the subsemigroup now denoted $Q_{E^*}(X)$ (originally written as $Q(\mathbf{2})$) of $\mathrm{Reg}(T)$ as follows:

$$Q_{E^*}(X) = \{\alpha \in \mathrm{Reg}(T) : |A_i\alpha| < 2 \ \text{ for all } \ i \in I\},$$

or, equivalently,

$$Q_{E^*}(X) = \{\alpha \in T_{E^*}(X) : |A_i\alpha| = 1 \ \text{ and } \ A_i \cap X\alpha \neq \emptyset \ \text{ for all } \ i \in I\}.$$

He also proved that for each $\alpha \in Q_{E^*}(X)$, $|X\alpha \cap A_i| = 1$ for all $i \in I$, meaning that $X\alpha$ is a cross section of the partition X/E induced by the equivalence relation E; i.e., each E-class contains exactly one element of $X\alpha$. He then showed that $Q_{E^*}(X)$ is the (unique) minimal ideal of $\mathrm{Reg}(T)$, which is referred to as the *kernel* of $\mathrm{Reg}(T)$ (see [6] for details). Finally, the author demonstrated that the kernel $Q_{E^*}(X)$ of $\mathrm{Reg}(T)$ is a right group and can be expressed as a union of symmetric groups, and that every right group can be embedded in the kernel $Q_{E^*}(X)$.

To improve clarity, we unify the notation by using $Q_{E^*}(X)$ throughout, where E represents the same equivalence relation as in $T_{E^*}(X)$.

Let Y be a nonempty subset of a set X. In [7], Laysirikul defined a subsemigroup $PG_Y(X)$ of the full transformation semigroup $T(X)$ by

$$PG_Y(X) = \{\alpha \in T(X) : \alpha|_Y \in \mathrm{Sym}(Y)\},$$

where $\mathrm{Sym}(Y)$ is the symmetric group on Y. The author demonstrated that $PG_Y(X)$ is regular, and provided characterizations of left regularity, right regularity, and complete regularity of elements in $PG_Y(X)$. Subsequent research on this semigroup has been conducted in [8–10].

Consider a nonempty subset Y of X and a subsemigroup $\mathbb{S}(Y)$ of $T(Y)$. In 2022, Konieczny [11] introduced $T_{\mathbb{S}(Y)}(X)$ as the collection of all transformations $\alpha \in T(X)$ that satisfy $\alpha|_Y \in \mathbb{S}(Y)$. We note that $T_{\mathbb{S}(Y)}(X)$ is a generalization of $PG_Y(X)$. In [11], the author provided a characterization of regular elements within $T_{\mathbb{S}(Y)}(X)$ and established conditions under which $T_{\mathbb{S}(Y)}(X)$ constitutes a regular semigroup (with further specifications regarding inverse semigroups and completely regular semigroups). Under the assumption that $\mathbb{S}(Y)$ includes the identity mapping id_Y on Y, Green's relations in $T_{\mathbb{S}(Y)}(X)$ were expressed through the corresponding Green's relations in $\mathbb{S}(Y)$. These broader findings were subsequently applied to derive specific results for the semigroup $T_{\Gamma(Y)}(X)$, where $\Gamma(Y)$ represents the semigroup of all full injective transformations on Y. The work concluded with discussions on generalizations and extensions of $T_{\mathbb{S}(Y)}(X)$.

Recently, Sangkhanan [12] examined the transformation semigroup $T_G(X)$, which consists of all transformations on a set X that, when restricted to a subset Y, belong to a permutation group G on Y. More precisely, this semigroup is defined as:

$$T_G(X) = \{\alpha \in T(X) : \alpha|_Y \in G\}$$

where G is a subgroup of $\mathrm{Sym}(Y)$. Obviously, $T_G(X)$ is a subsemigroup of $PG_Y(X)$. It is worth noting that $T_G(X)$ represents a special case of $T_{\mathbb{S}(Y)}(X)$. At the same time, it generalizes $PG_Y(X)$ in the sense that when G equals the symmetric group $\mathrm{Sym}(Y)$, we have $T_G(X) = PG_Y(X)$. In [12], the author investigated this semigroup using an approach similar to that employed in [9].

By [12, Corollary 3.8 and Theorem 3.10], the minimal ideal of $T_G(X)$ is given by

$$Q = \{\alpha \in T_G(X) : |X\alpha \setminus Y| = 0\} = \{\alpha \in T_G(X) : X\alpha = Y\},$$

which forms a left group. Furthermore, [12, Theorem 3.13] establishes that Q can be expressed as a union of permutation groups. Taking $G = \mathrm{Sym}(Y)$ yields $T_G(X) = PG_Y(X)$, and in this case the minimal ideal of $PG_Y(X)$ is

$$S_Y(X) = \{\alpha \in PG_Y(X) : |X\alpha \setminus Y| = 0\} = \{\alpha \in PG_Y(X) : X\alpha = Y\},$$

which is a left group and can be represented as a union of symmetric groups.

In this paper, we will examine the semigroups $Q_{E^*}(X)$ and $S_Y(X)$ more thoroughly. Our study will include proving certain algebraic properties of these semigroups, determining their ranks when X is finite, and exploring isomorphism conditions. Furthermore, we will identify and count all maximal subsemigroups for finite X. Finally, we will study the problem of embedding arbitrary left groups into $S_Y(X)$.

2. Basic properties

We begin by establishing some notations and results that will be used throughout this work. For all undefined notions, the reader is referred to [6, 13, 14].

For each $\alpha \in T(X)$ and $A \subseteq X$, let $A\alpha = \{a\alpha : a \in A\}$. Evidently, by this notation, $X\alpha$ means the range or image of α. Let S be a subsemigroup of $T(X)$. The *partition* of a member α in S, denoted $\pi(\alpha)$, is the family of all inverse images of elements in the range of α, that is

$$\pi(\alpha) = \{x\alpha^{-1} : x \in X\alpha\}.$$

It is easy to see that $\pi(\alpha)$ is a partition of X induced by α. In addition, due to [1], define a mapping α_* from $\pi(\alpha)$ onto $X\alpha$ by

$$(x\alpha^{-1})\alpha_* = x \;\text{ for each }\; x \in X\alpha.$$

In accordance with the notation established in [13, p.241], we represent any transformation $\alpha \in T(X)$ using the following matrix form:

$$\alpha = \begin{pmatrix} X_i \\ a_i \end{pmatrix}.$$

In this representation, the index i ranges over some unspecified index set I. We use $\{a_i\}$ as shorthand for the set $\{a_i : i \in I\}$. Additionally, we employ the notational conventions where $X\alpha$ represents the image set $\{a_i\}$, and $a_i\alpha^{-1}$ denotes the preimage X_i. If $X_i = \{x_i\}$ is a singleton, then we also write $\begin{pmatrix} X_i \\ a_i \end{pmatrix}$ as $\begin{pmatrix} x_i \\ a_i \end{pmatrix}$.

We now recall some key definitions used throughout this work. For an equivalence relation E on a nonempty set X, we define the semigroup $T_{E^*}(X)$ as

$$T_{E^*}(X) = \{\alpha \in T(X) : \forall x, y \in X,\ (x, y) \in E \Leftrightarrow (x\alpha, y\alpha) \in E\}.$$

When the equivalence relation E partitions X into equivalence classes A_i indexed by $i \in I$, we define the right group

$$Q_{E^*}(X) = \{\alpha \in T_{E^*}(X) : |A_i\alpha| = 1 \text{ and } A_i \cap X\alpha \neq \emptyset \text{ for all } i \in I\},$$

which can be expressed equivalently as

$$Q_{E^*}(X) = \{\alpha \in T_{E^*}(X) : \pi(\alpha) = X/E \text{ and } A \cap X\alpha \neq \emptyset \text{ for all } A \in X/E\}.$$

Furthermore, for any nonempty subset Y of X, we define the left group

$$S_Y(X) = \{\alpha \in T(X) : X\alpha = Y \text{ and } \alpha|_Y \in \mathrm{Sym}(Y)\}.$$

To illustrate these definitions concretely, consider the following example.

Example 1 Let $X = \{1, 2, 3\}$, $Y = \{2, 3\}$ and $X/E = \{\{1, 2\}, \{3\}\}$. Then

$$Q_{E^*}(X) = \left\{ \begin{pmatrix} \{1,2\} & 3 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} \{1,2\} & 3 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} \{1,2\} & 3 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} \{1,2\} & 3 \\ 3 & 2 \end{pmatrix} \right\}$$

and

$$S_Y(X) = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix} \right\}.$$

We recall that the relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$, and $\mathscr{J}$ are *Green's relations* on a semigroup S. For each $a \in S$, we denote the $\mathscr{L}$-class, $\mathscr{R}$-class, $\mathscr{H}$-class, $\mathscr{D}$-class, and $\mathscr{J}$-class containing a in a semigroup S by L_a, R_a, H_a, D_a, and J_a, respectively.

A semigroup in which every element acts as a right zero is called a *right zero semigroup*; equivalently, a semigroup S is right zero if $xy = y$ for all $x, y \in S$. Dually, a *left zero semigroup* satisfies $xy = x$ for all $x, y \in S$.

A semigroup S is called a *right simple semigroup* when it contains no proper right ideals, and similarly, a *left simple semigroup* when it contains no proper left ideals. A semigroup is classified as a *right group* if it satisfies two conditions: being right simple and left cancellative. Likewise, a *left group* is both left simple and right cancellative. An equivalent characterization states that S is a right group if and only if for any two elements a and b in S, there exists exactly one element x in S that satisfies the equation $ax = b$. Similarly, S is a left group if and only if for any a and b in S, there exists exactly one x in S such that $xa = b$. As a consequence, the $\mathscr{R}$-relation on a right group S is trivial, while the $\mathscr{L}$-relation is trivial on a left group. Further details about these properties can be found in the lemma that follows, which incorporates dual statements from [6, Exercises 2 and 4 in §1.11].

Lemma 2 Let S be a semigroup. The following statements are equivalent.

1. S is a right [left] group.
2. S is a union of disjoint groups such that the set of identity elements of the groups is a right [left] zero subsemigroup of S.
3. S is regular and left [right] cancellative.

Refer to [6, Exercise 3 for §1.11] and [14, Exercise 6 for §2.6], every right group S can be written as a union of disjoint subgroups, each of which is isomorphic to one another. These subgroups are given by Se, where e is an idempotent element of S and serves as the group identity for Se. If e and f are distinct idempotent elements of S, then the map $x \mapsto xf$ is an isomorphism between the subgroups Se and Sf. Additionally, the $\mathscr{H}$-class H_e and the subgroup Se are equal for all idempotent elements e of S. This allows us to express a right group S as the (disjoint) union of all of its subgroups:

$$S = \bigcup_{e \in E(S)} Se = \bigcup_{e \in E(S)} H_e.$$

Moreover, we can use the dual statements of the above results to apply for a left group (e.g. change Se to eS). It should be observed that the number of $\mathscr{H}$-classes and idempotents in $E(S)$ are identical.

For the right group $Q_{E^*}(X)$, several important structural properties were established in [5]. Specifically, for any element α in $Q_{E^*}(X)$, the set $H_\alpha = \{\beta \in Q_{E^*}(X) : X\alpha = X\beta\}$ forms a subgroup of $Q_{E^*}(X)$ as shown in [5, Corollary 3.8]. Furthermore, [5, Theorem 3.9] demonstrates that $Q_{E^*}(X)$ can be represented as a union of symmetric groups. From the proof of this theorem, we can observe that for any $\alpha \in Q_{E^*}(X)$, the subgroup H_α is isomorphic to the symmetric group on the set $X\alpha$. For clarity, we provide a brief explanation of this isomorphism here.

For each $\beta \in H_\alpha$, by the definition of $Q_{E^*}(X)$, we have $\pi(\beta) = \pi(\alpha) = X/E$. Thus, the mapping $\beta_* : \pi(\alpha) \to X\alpha$ is a bijection from X/E to $X\alpha$. Moreover, we note that $A \cap X\alpha$ is a singleton for all $A \in X/E$. It follows that the set H_α can be identified with the group of all bijections from $X\alpha$ to itself, which is isomorphic to the symmetric group on the set $X\alpha$, denoted by $\mathrm{Sym}(X\alpha)$. Since $Q_{E^*}(X)$ can be expressed as the union of all such subgroups H_α for $\alpha \in Q_{E^*}(X)$, we conclude that $Q_{E^*}(X)$ is a union of symmetric groups.

Let S be a subsemigroup of the transformation semigroup $T(X)$, where X is a finite set. The classical characterization from [6, Exercise 8 of §2.2] provides necessary and sufficient conditions for S to be a right group or a left group. For convenience, we restate these fundamental results below.

Proposition 3 [6, Exercise 8 of §2.2] Let S be a subsemigroup of the semigroup $T(X)$ of all transformations on a finite set X. S is a left group if and only if all its members have the same range; S is a right group if and only if all its members have the same partition of X.

In what follows, we will generalize this characterization to an arbitrary (possibly infinite) set X and establish its connection to the right group $Q_{E^*}(X)$. This extension relies on a key result from [6, Lemma 2.6], which states that for any two transformations $\alpha, \beta \in T(X)$, they are $\mathscr{R}$-related if and only if their partitions are identical, that is, $\pi(\alpha) = \pi(\beta)$.

Theorem 4 Let X be a nonempty set and S be a subsemigroup of $T(X)$. The following statements are equivalent.

1. S is a right group.
2. S is regular and all members of S have the same partition.
3. S is a regular subsemigroup of $Q_{E^*}(X)$ for some equivalence relation E on X.

Proof. (1) $\Rightarrow$ (2). Assume that S is a right group. Then S is regular. By a direct consequence of [6, Lemma 2.6], all members of S have the same partition since the $\mathscr{R}$-relation on the right group S is trivial.

(2) $\Rightarrow$ (3). Suppose that all members of S have the same partition $\mathscr{P}$ of X. Let $\mathscr{P} = \{P_i : i \in I\}$ where I is an index set. Let $\alpha \in S$. Clearly, $|P_i\alpha| = 1$ for all $i \in I$. Next, we will show that $X\alpha$ is a cross section of $\mathscr{P}$ for all $\alpha \in S$.

First, we claim that $|P_i \cap X\alpha| \leq 1$ for all $i \in I$. Suppose to the contrary that $|P_0 \cap X\alpha| > 1$ for some $P_0 \in \mathscr{P}$. Then there exist two distinct elements $a, b \in P_0 \cap X\alpha$. Note that $a\alpha^{-1}, b\alpha^{-1} \in \mathscr{P}$ such that $a\alpha^{-1} \neq b\alpha^{-1}$. Let $x \in a\alpha^{-1}$ and $y \in b\alpha^{-1}$. Since $|P_0\alpha| = 1$, we obtain $a\alpha = b\alpha$ which implies that

$$x\alpha^2 = (x\alpha)\alpha = a\alpha = b\alpha = (y\alpha)\alpha = y\alpha^2.$$

Then $x, y \in P$ for some $P \in \pi(\alpha^2)$. Since $\alpha^2 \in S$, we have $\pi(\alpha^2) = \mathscr{P}$ from which it follows that x and y are in the same class in $\mathscr{P}$ which is a contradiction. Thus $|P_i \cap X\alpha| \leq 1$ for all $i \in I$. Now, suppose that $P_1 \cap X\alpha = \emptyset$ for some $P_1 \in \mathscr{P}$. Observe that since S is regular, there exists some element β in S such that the relation $\alpha = \alpha\beta\alpha$ holds. Assume that

$$P_1\beta_* \in P_2 \text{ and } P_2\alpha_* \in P_3$$

for some $P_2, P_3 \in \mathscr{P}$. Observe that $P_3 \neq P_1$ since $X\alpha \cap P_1 = \emptyset$. From $\alpha = \alpha\beta\alpha$, we obtain that

$$(P_3\beta_*)\alpha = (P_2\alpha_*)\beta\alpha = P_2\alpha_* \in P_3 \text{ and } (P_1\beta_*)\alpha = P_2\alpha_* \in P_3.$$

Hence $(P_3\beta_*)\alpha = (P_1\beta_*)\alpha$ since $|P_3 \cap X\alpha| \leq 1$. Because $\beta\alpha \in S$, it follows that P_3 and P_1 are the same class in $\mathscr{P}$, a contradiction. Thus $|P_i \cap X\alpha| = 1$ for all $i \in I$. It follows that $X\alpha$ is a cross section of $\mathscr{P}$ for all $\alpha \in S$.

Let E be the equivalence relation on X induced by $\mathscr{P}$. Then by the definition of $Q_{E^*}(X)$, S is a regular subsemigroup of $Q_{E^*}(X)$.

(3) $\Rightarrow$ (1). Suppose that (3) holds. Then S is regular and left cancellative since $Q_{E^*}(X)$ is a right group. Hence S is a right group. □

Next, we present an additional property of finite right groups that will be instrumental in our subsequent discussions. We begin by recalling a fundamental structure theorem for right groups which can be found in [6, Theorem 1.27].

Theorem 5 [6, Theorem 1.27] The following assertions concerning a semigroup S are equivalent.

1. S is a right group.
2. S is right simple and contains an idempotent.
3. S is the direct product $G \times E$ of a group G and a right zero semigroup E.

In the proof of [6, Theorem 1.27], the group G is identified with Se for a fixed idempotent e, while the right zero semigroup E is identified with $E(S)$, the set of all idempotents in S. By combining these identifications with the dual results for left groups, we derive the following characterization.

Theorem 6 A right [left] group S is isomorphic to the direct product $G \times E$ of a group G and a right [left] zero semigroup E where E is the set $E(S)$ of all idempotents in S and G is the group Se $[eS]$ $(= H_e)$ where $e \in E(S)$.

Theorem 7 Every subsemigroup of a finite right group is also a right group.

Proof. Let T be a subsemigroup of a finite right group S. Then S is isomorphic to the direct product $G \times E$ of a group G and a right zero semigroup E. Thus, T is isomorphic to a subset $\overline{T}$ of $G \times E$. Let π_1 and π_2 be the projection maps of $\overline{T}$ into G and E, respectively. It is clear that $\overline{T}\pi_2$ is a right zero subsemigroup of E. To verify that $\overline{T}\pi_1$ is a subgroup of G, take $g, h \in \overline{T}\pi_1$. Then there exist $e, f \in \overline{T}\pi_2$ with $(g, e), (h, f) \in \overline{T}$. Because $\overline{T} \subseteq G \times E$ is a subsemigroup, we get $(g, e)(h, f) = (gh, ef) \in \overline{T}$, hence $gh \in \overline{T}\pi_1$. Since G is finite, closure under multiplication already implies that $\overline{T}\pi_1$ is a subgroup of G. We claim that $\overline{T}$ is the direct product of $\overline{T}\pi_1$ and $\overline{T}\pi_2$. For, let $(g, e) \in \overline{T}\pi_1 \times \overline{T}\pi_2$. Then there are $h \in \overline{T}\pi_1$ and $i \in \overline{T}\pi_2$ such that $(h, e) \in \overline{T}$ and $(g, i) \in \overline{T}$. Moreover, since $\overline{T}\pi_1$ is a group, there is $h^{-1} \in \overline{T}\pi_1$ and $f \in \overline{T}\pi_2$ such that $(h^{-1}, f) \in \overline{T}$. Hence $(g, e) = (gh^{-1}h, ife) = (g, i)(h^{-1}, f)(h, e) \in \overline{T}$ from which it follows that $\overline{T} = \overline{T}\pi_1 \times \overline{T}\pi_2$. Therefore, T is isomorphic to the direct product of the group $\overline{T}\pi_1$ and the right zero semigroup $\overline{T}\pi_2$, and so T is a right group. □

In the remainder of this section, we present a property of the semigroup $S_Y(X)$.

Based on [6, Exercise 8 of §2.2], for a finite set X, a subsemigroup S of $T(X)$ is a left group if and only if all its elements have an identical image. We now extend this characterization to encompass an arbitrary (potentially infinite) set X and demonstrate its relationship to the left group $S_Y(X)$. This generalization builds upon an essential result from [6, Lemma 2.5], which establishes that two transformations $\alpha, \beta \in T(X)$ are $\mathscr{L}$-related exactly when their images coincide, specifically when $X\alpha = X\beta$. The proof of the subsequent theorem closely mirrors that of Theorem 4.

Theorem 8 Let X be a nonempty set and S a subsemigroup of $T(X)$. The following statements are equivalent.

1. S is a left group.
2. S is regular and all members of S have the same image.
3. S is a regular subsemigroup of $S_Y(X)$ for some subset Y of X.

3. Isomorphism conditions

Having established the fundamental characterizations of right groups and left groups as subsemigroups of $T(X)$ in the preceding section, we now shift our focus to the question of when two such semigroups are isomorphic. Specifically, we examine the structural conditions under which $Q_{E^*}(X)$ is isomorphic to $Q_{F^*}(Y)$ for equivalence relations E and F on sets X and Y, respectively, and similarly for left groups $S_Y(X)$ and $S_Z(W)$. The characterization results from the previous section provide the foundation for determining these isomorphism conditions.

To establish a criterion for isomorphism, we present a useful result in the following theorem.

Theorem 9 Let S and T be right [left] groups which can be written as direct products of groups and right [left] zero semigroups $G_1 \times E_1$ and $G_2 \times E_2$, respectively. Then $S \cong T$ if and only if $G_1 \cong G_2$ and $|E_1| = |E_2|$.

Proof. We first establish our result in the context of a right group. Following analogous reasoning, we then derive the corresponding result for a left group. Assume that $S \cong T$ via an isomorphism $\phi : S \to T$. As we mentioned above, $G_1 \cong Se$ for some idempotent $e \in E(S)$. It is straightforward to verify that $Se \cong T(e\phi)$. Hence $G_1 \cong Se \cong T(e\phi) = Tf \cong G_2$ for some $f = e\phi \in E(T)$. Moreover, $E_1 \cong E(S) \cong E(T) \cong E_2$ which implies $|E_1| = |E_2|$ since any two right zero semigroups of the same cardinality are isomorphic. The converse is clear. □

For an indexed collection of sets $\{A_i\}_{i\in I}$, we denote their product as $\prod_{i\in I} A_i$ and an element in $\prod_{i\in I} A_i$ with i-coordinate a_i is designated as $(a_i)_{i\in I}$.

Let I be an index set (finite or infinite), let $\{a_i\}_{i\in I}$ be a family of cardinal numbers, and let $\{A_i\}_{i\in I}$ be a family of sets such that $|A_i| = a_i$ for all $i \in I$. Then the product of the cardinals $\{a_i\}_{i\in I}$ is the cardinal defined by

$$\prod_{i\in I} a_i := \left| \prod_{i\in I} A_i \right|.$$

By combining [15, Exercise 10, p.40 and Exercise 8, p.151], we obtain an isomorphism condition for two symmetric groups, as follows.

Lemma 10 Let A and B be any nonempty sets. Then

$$\mathrm{Sym}(A) \cong \mathrm{Sym}(B) \text{ if and only if } |A| = |B|.$$

To establish an isomorphism theorem for $Q_{E^*}(X)$, we require the following two lemmas.

Lemma 11 [5, Lemma 4.1] Let $\alpha \in Q_{E^*}(X)$. Then α is an idempotent if and only if $A\alpha \subseteq A$ for all $A \in X/E$.

By Lemma 11, for each idempotent $\varepsilon \in E(Q_{E^*}(X))$, we can express ε in the form

$$\varepsilon = \begin{pmatrix} A_i \\ a_i \end{pmatrix}$$

where $a_i \in A_i$ for all $i \in I$ and $X/E = \{A_i : i \in I\}$.

Lemma 12 Let $X/E = \{A_i : i \in I\}$. Then $|E(Q_{E^*}(X))| = \prod_{i \in I} |A_i|$.

Proof. For each $\varepsilon \in E(Q_{E^*}(X))$, we have $A_i\varepsilon \subseteq A_i$ for all $i \in I$. So we can write $\varepsilon = \begin{pmatrix} A_i \\ a_i \end{pmatrix}$ where $a_i \in A_i$. Define a function $\varphi : E(Q_{E^*}(X)) \to \prod_{i \in I} A_i$ by

$$\varepsilon\varphi = (A_i\varepsilon_*)_{i \in I} = (a_i)_{i \in I} \in \prod_{i \in I} A_i.$$

To show that φ is an injection, let $\varepsilon_1 = \begin{pmatrix} A_i \\ a_i \end{pmatrix}$ and $\varepsilon_2 = \begin{pmatrix} A_i \\ b_i \end{pmatrix}$ in $E(Q_{E^*}(X))$ such that $\varepsilon_1\varphi = \varepsilon_2\varphi$. Then $a_i = A_i\varepsilon_{1*} = A_i\varepsilon_{2*} = b_i$ for all $i \in I$. Hence $\varepsilon_1 = \varepsilon_2$, which implies that φ is an injection. Finally, let $(a_i)_{i \in I} \in \prod_{i \in I} A_i$. Define $\varepsilon \in E(Q_{E^*}(X))$ by $\varepsilon = \begin{pmatrix} A_i \\ a_i \end{pmatrix}$. We obtain $(a_i)_{i \in I} = (A_i\varepsilon_*)_{i \in I} = \varepsilon\varphi$. Hence φ is a surjective map, implying that φ is also a bijection. Therefore, $|E(Q_{E^*}(X))| = \left|\prod_{i \in I} A_i\right| = \prod_{i \in I} |A_i|$. □

By [5, Theorem 3.8], the author proved that the subgroup H_α of $Q_{E^*}(X)$ is isomorphic to the symmetric group on the set $X\alpha$. Since $X\alpha$ is a cross section of the partition X/E, it is obvious that the symmetric group on the set $X\alpha$ is isomorphic to the symmetric group on X/E. To sum it up with Theorem 6, we state the useful proposition as follows.

Proposition 13 $Q_{E^*}(X)$ is isomorphic to the direct product $\mathrm{Sym}(X/E) \times E(Q_{E^*}(X))$.

We can prove the proposition directly by constructing an explicit isomorphism between $Q_{E^*}(X)$ and $\mathrm{Sym}(X/E) \times E(Q_{E^*}(X))$. For completeness, we present the construction. For convenience, denote by $[x]$ the equivalence class of $x \in X$ in X/E.

Given $\alpha \in Q_{E^*}(X)$, define $\hat{\alpha} : X/E \to X/E$ by

$$A\hat{\alpha} = [A\alpha_*] \text{ for all } A \in X/E,$$

where $\alpha_* : \pi(\alpha) \to X\alpha$ is the map introduced in Section 2.

Since $\alpha \in Q_{E^*}(X)$, we have $\pi(\alpha) = X/E$, and thus $\hat{\alpha} : X/E \to X/E$ is a bijection; hence $\hat{\alpha} \in \mathrm{Sym}(X/E)$.

Next, for $\alpha \in Q_{E^*}(X)$, define $\bar{\alpha} : X \to X$ by sending each $x \in X$ to the unique element of $[x] \cap X\alpha$. It is straightforward to verify that $\bar{\alpha} \in E(Q_{E^*}(X))$.

Define

$$\varphi : Q_{E^*}(X) \to \mathrm{Sym}(X/E) \times E(Q_{E^*}(X)), \qquad \alpha \mapsto (\hat{\alpha}, \bar{\alpha}).$$

We claim that φ is an isomorphism. Let $\alpha, \beta \in Q_{E^*}(X)$. For the first coordinate, for any $A \in X/E$,

$$A\widehat{\alpha\beta} = [A(\alpha\beta)_*], \qquad A\hat{\alpha} = [A\alpha_*], \qquad A\hat{\alpha}\hat{\beta} = [[A\alpha_*]\beta_*].$$

Using $(\alpha\beta)_* = \alpha_*\beta$, we obtain $[A(\alpha\beta)_*] = [(A\alpha_*)\beta] = [[A\alpha_*]\beta_*]$, whence $\widehat{\alpha\beta} = \hat{\alpha}\hat{\beta}$.

For the second coordinate, if $x \in X$, then $x\overline{\alpha\beta}$ is the unique element of $[x] \cap X(\alpha\beta) = [x] \cap X\beta$. Writing $x\bar{\alpha} = b \in [x] \cap X\alpha$, we have $b\bar{\beta} = c \in [b] \cap X\beta = [x] \cap X\beta$, and uniqueness yields $x\overline{\alpha\beta} = c = b\bar{\beta} = x\bar{\alpha}\bar{\beta}$. Hence $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$.

Thus φ is a homomorphism. It is routine to verify that φ is bijective. Therefore, φ is an isomorphism.

Through the integration of results established in Theorem 9, Lemma 10, and Proposition 13, we can derive the following theorem.

Theorem 14 Let E and F be equivalence relations on nonempty sets X and Y, respectively. Then $Q_{E^*}(X) \cong Q_{F^*}(Y)$ if and only if $|X/E| = |Y/F|$ and $\prod_{A \in X/E} |A| = \prod_{B \in Y/F} |B|$.

Next, we will calculate the cardinality of the set $Q_{E^*}(X)$. Let X be a finite set and E an equivalence relation on X such that $X/E = \{A_1, A_2, \ldots, A_n\}$ and $|A_1||A_2|\cdots|A_n| = m$. By Proposition 13 and Lemma 12, we have

$$|Q_{E^*}(X)| = |\mathrm{Sym}(X/E)||E(Q_{E^*}(X))| = n! \cdot m.$$

If A and B are sets, the set of all functions from A to B is denoted by B^A. The number of such functions is the cardinal number of this set, defined as:

$$|B^A| = b^a$$

where $a = |A|$ and $b = |B|$. For finite sets, if A has m elements and B has n elements, this becomes n^m.

Now, we establish a theorem that characterizes when two transformation semigroups $S_{Y_1}(X_1)$ and $S_{Y_2}(X_2)$ are isomorphic. To accomplish this, we require the following preliminary results.

Lemma 15 Let Y be a nonempty subset of X. Then $|E(S_Y(X))| = |Y|^{|X \setminus Y|}$.

Proof. For each $\alpha \in E(S_Y(X))$, we have $\alpha|_Y$ as the identity map. By the definition of $S_Y(X)$, we see that $|E(S_Y(X))|$ is the number of all functions from $X \setminus Y$ to Y. Hence $|E(S_Y(X))| = |Y|^{|X \setminus Y|}$. □

For the subset Q of the semigroup $T_G(X)$ defined previously, Sangkhanan [12, Corollary 3.12] established that, for every $\alpha \in Q$, the $\mathscr{H}$-class H_α constitutes a subgroup of Q. Furthermore, this subgroup is isomorphic to the permutation group G on the set Y. In the context of $S_Y(X)$, we can employ a similar approach to demonstrate that, for each $\alpha \in S_Y(X)$, the $\mathscr{H}$-class H_α forms a subgroup of $S_Y(X)$ isomorphic to the symmetric group on the set Y. Merging this insight with Theorem 6, we present this significant finding in the subsequent proposition.

Proposition 16 $S_Y(X)$ is isomorphic to the direct product $\mathrm{Sym}(Y) \times E(S_Y(X))$.

By combining Theorem 9, Lemma 10, and Proposition 16, we obtain the following theorem.

Theorem 17 Let Y_1 and Y_2 be nonempty subsets of X_1 and X_2, respectively. Then

$$S_{Y_1}(X_1) \cong S_{Y_2}(X_2) \text{ if and only if } |Y_1| = |Y_2| \text{ and } |Y_1|^{|X_1 \setminus Y_1|} = |Y_2|^{|X_2 \setminus Y_2|}.$$

We conclude this section by calculating the cardinality of the set $S_Y(X)$. Let Y be a nonempty subset of a finite set X such that $|X| = n$ and $|Y| = m$. By Proposition 16 and Lemma 15, we have

$$|S_Y(X)| = |\operatorname{Sym}(Y)||E(S_Y(X))| = m! \cdot m^{n-m}.$$

4. Ranks

In this section, we will find a generating set and compute the rank of $Q_{E^*}(X)$ and $S_Y(X)$. Recall that a generating set of a semigroup S that has the smallest possible number of elements is called a *minimal generating set*. The rank of S, represented as rank(S), refers to the number of elements in a minimal generating set of S. In other words, the rank of a semigroup S indicates the minimum number of elements needed to generate S; that is

$$\operatorname{rank}(S) = \min\{|T| : T \subseteq S, \langle T \rangle = S\}.$$

To compute the rank of $Q_{E^*}(X)$, we consider a generating set of any right groups. The following lemmas will be necessary in order to prove the main theorem of this section.

Lemma 18 Let G be a generating set of a right group S. Then $G \cap H_e$ is nonempty for all idempotents e in S.

Proof. Let $e \in E(S)$. Then there are $g_1, g_2, \ldots, g_m \in G$ such that $g_1 g_2 \cdots g_m = e$. Let $g_m \in H_f$ for some idempotent f in S. Then $e = g_1 g_2 \cdots g_m f = ef = f$ since f is the identity in H_f and $E(S)$ is right zero. Hence $g_m \in G \cap H_e \neq \emptyset$. □

By the above lemma, any generating set G of a right group S contains at least one element in each $\mathscr{H}$-class. Hence rank$(S) \geq |E(S)|$.

At this point, we can acquire a generating set for any right group.

Theorem 19 Let S be a right group and $e \in E(S)$. If G is a generating set of the group H_e, then $G \cup E(S)$ is a generating set of S.

Proof. Assume that G is a generating set of the group H_e. Let $x \in S$. Then $x \in H_f$ for some idempotent f in S. We have $xe \in Se = H_e$ and $xe = g_1 g_2 \cdots g_m$ for some $g_1, g_2, \ldots, g_m \in G$. Hence $x = xf = xef = g_1 g_2 \cdots g_m f$, which implies that $x \in \langle G \cup E(S) \rangle$. □

Theorem 20 Let S be a finite right group and $e \in E(S)$. If G is a minimal generating set of the group H_e, then

$$\operatorname{rank}(S) = \max\{|G|, |E(S)|\}.$$

Proof. Suppose that G is a minimal generating set of the group H_e. If $|G| \leq |E(S)|$, then there is an injection $\phi: G \to E(S)$. By [6, Theorem 1.27], we have $S \cong H_e \times E(S)$. Let

$$H = \{(g, g\phi) : g \in G\} \quad \text{and} \quad K = \{(e, f) : f \in E(S) \setminus G\phi\}.$$

We assert that $H \cup K$ is a generating set of $H_e \times E(S)$. For, let $(x, h) \in H_e \times E(S)$. If $h = g\phi$ for some $g \in G$, then $xg^{-1} = g_1 g_2 \cdots g_m$ for some $g_1, g_2, \ldots, g_m \in G$. Hence,

$$(x, h) = (xg^{-1}g, h) = (g_1g_2\cdots g_mg, g\phi).$$

Since $E(S)$ is right zero, we obtain

$$(g_1g_2\cdots g_mg, g\phi) = (g_1, g_1\phi)(g_2, g_2\phi)\cdots(g_m, g_m\phi)(g, g\phi),$$

which implies that $(x, h) \in \langle H\rangle \subseteq \langle H\cup K\rangle$. If $h \in E(S)\setminus G\phi$, then $x = g_1g_2\cdots g_m$ for some $g_1, g_2, \ldots, g_m \in G$ and so

$$(x, h) = (xe, h) = (g_1g_2\cdots g_me, h) = (g_1, g_1\phi)(g_2, g_2\phi)\cdots(g_m, g_m\phi)(e, h).$$

Thus $(x, h) \in \langle H\cup K\rangle$. We conclude that

$$\operatorname{rank}(S) = \operatorname{rank}(H_e \times E(S)) \leq |H\cup K| = |H| + |K| = |G| + (|E(S)| - |G|) = |E(S)|.$$

As mentioned before, we have $\operatorname{rank}(S) \geq |E(S)|$. Therefore,

$$\operatorname{rank}(S) = |E(S)| = \max\{|G|, |E(S)|\}.$$

On the other hand, assume that $|G| > |E(S)|$. Then there is a surjection $\phi : G \to E(S)$. Let $H = \{(g, g\phi) : g \in G\}$. By the same argument as above, we can show that H is a generating set of S (up to isomorphism). Hence $\operatorname{rank}(S) \leq |H| = |G|$. Since G is a minimal generating set of the group H_e, we obtain

$$\operatorname{rank}(S) = |G| = \max\{|G|, |E(S)|\}. \qquad \square$$

We observe that, by applying similar arguments, all the results established above can be analogously derived for the case of a left group.

Let $X/E = \{A_i : i \in I\}$. We have $|E(Q_{E^*}(X))| = \prod_{i\in I}|A_i|$. By Proposition 13, $Q_{E^*}(X)$ is isomorphic to the direct product of $\operatorname{Sym}(X/E)$ and $E(Q_{E^*}(X))$. Moreover, it is well-known that the symmetric group on a set Y has rank 2 when $|Y| \geq 2$. By using Theorem 20, we obtain the rank of $Q_{E^*}(X)$ as follows.

Corollary 21 Let E be a nontrivial equivalence relation on a nonempty finite set X. Let $X/E = \{A_i : i \in I\}$ such that $\prod_{i\in I}|A_i| = m$. Then $\operatorname{rank}(Q_{E^*}(X)) = \max\{2, m\}$.

We note that if $|X| = 1$, then $Q_{E^*}(X) = T(X)$ which is a trivial semigroup with rank 1. If $|X| > 1$ and E is the universal relation $X \times X$, then $Q_{E^*}(X)$ is the set of all constant maps on X, which is a right zero semigroup with order $|X|$. It is straightforward to verify that the minimal generating set of any right zero semigroup with cardinality n has size n. Hence, $\operatorname{rank}(Q_{E^*}(X)) = |X|$, which is consistent with Corollary 21.

We illustrate $Q_{E^*}(X)$ with a finite example and determine its rank along with a minimal generating set, connecting to the results of this section. For convenience, we use the abbreviated notation

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \equiv (a_1, a_2, \ldots, a_n).$$

The same convention applies when some A_i are singletons.

Example 22 Let $X = \{1, 2, 3, 4, 5, 6\}$ and let E be an equivalence relation on X with

$$X/E = \{A_1, A_2, A_3\}, \quad A_1 = \{1, 2, 3\}, \quad A_2 = \{4, 5\}, \quad A_3 = \{6\}.$$

Then $Q_{E^*}(X) = \{\alpha_1, \alpha_2, \ldots, \alpha_{36}\}$, where

$$\alpha_1 = (1, 4, 6), \quad \alpha_2 = (2, 4, 6), \quad \alpha_3 = (3, 4, 6), \quad \alpha_4 = (1, 5, 6), \quad \alpha_5 = (2, 5, 6),$$

$$\alpha_6 = (3, 5, 6), \quad \alpha_7 = (4, 1, 6), \quad \alpha_8 = (4, 2, 6), \quad \alpha_9 = (4, 3, 6), \quad \alpha_{10} = (5, 1, 6),$$

$$\alpha_{11} = (5, 2, 6), \quad \alpha_{12} = (5, 3, 6), \quad \alpha_{13} = (4, 6, 1), \quad \alpha_{14} = (4, 6, 2), \quad \alpha_{15} = (4, 6, 3),$$

$$\alpha_{16} = (5, 6, 1), \quad \alpha_{17} = (5, 6, 2), \quad \alpha_{18} = (5, 6, 3), \quad \alpha_{19} = (6, 1, 4), \quad \alpha_{20} = (6, 2, 4),$$

$$\alpha_{21} = (6, 3, 4), \quad \alpha_{22} = (6, 1, 5), \quad \alpha_{23} = (6, 2, 5), \quad \alpha_{24} = (6, 3, 5), \quad \alpha_{25} = (1, 6, 4),$$

$$\alpha_{26} = (2, 6, 4), \quad \alpha_{27} = (3, 6, 4), \quad \alpha_{28} = (1, 6, 5), \quad \alpha_{29} = (2, 6, 5), \quad \alpha_{30} = (3, 6, 5),$$

$$\alpha_{31} = (6, 4, 1), \quad \alpha_{32} = (6, 4, 2), \quad \alpha_{33} = (6, 4, 3), \quad \alpha_{34} = (6, 5, 1), \quad \alpha_{35} = (6, 5, 2),$$

$$\alpha_{36} = (6, 5, 3).$$

By Lemma 11, the idempotent set is

$$E(Q_{E^*}(X)) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}.$$

To construct a minimal generating set for $Q_{E^*}(X)$, select the idempotent $e = \alpha_1 = (1, 4, 6)$. Its $\mathscr{H}$-class is

$$H_e = \{(1, 4, 6), (4, 1, 6), (4, 6, 1), (6, 1, 4), (1, 6, 4), (6, 4, 1)\} = \{\alpha_1, \alpha_7, \alpha_{13}, \alpha_{19}, \alpha_{25}, \alpha_{31}\}.$$

Since $H_e \cong \mathrm{Sym}(\{1, 4, 6\})$, the set $G = \{(1, 4, 6), (4, 1, 6)\} = \{\alpha_1, \alpha_7\}$ generates H_e. By Theorem 19, the set

$$G \cup E(Q_{E^*}(X)) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$$

generates $Q_{E^*}(X)$. Observing that $\alpha_7^2 = (4, 1, 6)^2 = (1, 4, 6) = \alpha_1$ yields

$$Q_{E^*}(X) = \langle \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \rangle.$$

Since $|\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}| = 6 = \max\{2, m\}$, where $m = |A_1||A_2||A_3| = 3 \cdot 2 \cdot 1 = 6$, Corollary 21 confirms that $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ is a minimal generating set of $Q_{E^*}(X)$ and $\operatorname{rank}(Q_{E^*}(X)) = 6$.

By Proposition 16, $S_Y(X)$ is isomorphic to the direct product of $\operatorname{Sym}(Y)$ and $E(S_Y(X))$. By using the dual statements of Theorem 20 and Lemma 15, we obtain the rank of $S_Y(X)$ as follows.

Corollary 23 Let X be a nonempty finite set and let Y be a subset of X such that $|Y| \geq 2$. Then $\operatorname{rank}(S_Y(X)) = \max\{2, |Y|^{|X \setminus Y|}\}$.

5. Maximal subsemigroups

We now return to examining maximal subsemigroups. A subsemigroup of a semigroup S is called *maximal* if it is proper (i.e., strictly contained in S) and not properly contained in any other proper subsemigroup of S. Similarly, a maximal proper subgroup of a group G is a subgroup not contained in any other proper subgroup of G. It is a standard result that, in the case of a finite group G, every subsemigroup must in fact be a subgroup.

In this section, we will characterize and enumerate all maximal subsemigroups that exist within right [left] groups expressible as direct products between finite groups and right [left] zero semigroups. The section concludes by extending our analysis to the specific semigroups $Q_{E^*}(X)$ and $S_Y(X)$, where X is a finite set.

Lemma 24 Let S be a finite right group which can be written as the direct product of a group G and a right zero semigroup E. Let T be a subsemigroup of S and $e \in E(S)$. If $Te = Se$ and $E(T) = E(S)$, then $T = S$.

Proof. We note by Theorem 7 that T is a right group. Assume that $Te = Se$ and $E(T) = E(S)$. Then for all $f \in E(S)$, we have $Tf = Tef = Sef = Sf$. Hence,

$$T = \bigcup_{f \in E(T)} Tf = \bigcup_{f \in E(S)} Tf = \bigcup_{f \in E(S)} Sf = S. \qquad \square$$

To characterize a maximal subsemigroup of a finite right group, we need the following lemma, which appeared in [16].

Lemma 25 [16, Lemma 4.3] Let S be a semigroup and let M be a subsemigroup of S such that $|S \setminus M| = 1$. Then M is a maximal subsemigroup of S.

By Lemma 25 and the dual statement of [16, Example 4.4], we have the following proposition.

Proposition 26 Let S be a right zero semigroup. Then M is a maximal subsemigroup of S if and only if $M = S \setminus \{x\}$ for some $x \in S$.

By the above proposition, we conclude that every nontrivial right zero semigroup has a maximal subsemigroup. In addition, note that if a finite right zero semigroup S has m elements, then the number of its maximal subsemigroups is also m.

Now, we provide a characterization of maximal subsemigroups of any finite right group.

Theorem 27 Let S be a finite right group which can be written as the direct product of a group G and a nontrivial right zero semigroup $E(S)$, and let T be a subsemigroup of S. Then T is a maximal subsemigroup of S if and only if T can be written as a direct product $H \times E(S)$ or $G \times F$ where H is a maximal subgroup of G and F is a maximal subsemigroup of $E(S)$.

Proof. Assume that T is a maximal subsemigroup of S. Then, by Theorem 7, T is a right group which can be written as a direct product $Te \times E(T)$ where e is an idempotent in T. We have Te is a subgroup of Se and $E(T)$ is a right zero subsemigroup of $E(S)$.

If Te is a proper subgroup of Se, then there is a maximal subgroup H of Se such that $Te \subseteq H \subsetneq Se$. Clearly, $H \times E(S)$ is a subsemigroup of $Se \times E(S)$ and

$$Te \times E(T) \subseteq H \times E(S) \subsetneq Se \times E(S).$$

Since $Te \times E(T)$ is maximal, we obtain $Te \times E(T) = H \times E(S)$ and so $Te = H$ and $E(T) = E(S)$.

If $E(T)$ is a proper subsemigroup of $E(S)$, then there is a maximal subsemigroup F of $E(S)$ such that $E(T) \subseteq F \subsetneq E(S)$. We have

$$Te \times E(T) \subseteq Se \times F \subsetneq Se \times E(S).$$

Since $Te \times E(T)$ is maximal, we obtain $Te \times E(T) = Se \times F$, and so $Te = Se$ and $E(T) = F$.

Conversely, suppose that T can be written as a direct product $Te \times E(S)$ where Te (with $e \in E(T)$) is a maximal subgroup of Se. To show that T is maximal, let U be a subsemigroup of S such that $T \subseteq U \subseteq S$. Again, by Theorem 7, U is a right group which can be written as the direct product $Ue \times E(U)$ where Ue is a subgroup of Se and $E(U)$ is a right zero subsemigroup of $E(S)$. Clearly,

$$Te \times E(S) \subseteq Ue \times E(U) \subseteq Se \times E(S).$$

Hence $E(U) = E(S)$. By maximality of Te, we obtain $Te = Ue$ or $Ue = Se$, which implies by Lemma 24 that $T = U$ or $U = S$. It is concluded that T is maximal.

Finally, assume that T can be written as a direct product $Se \times E(T)$ where $Te = Se$ and $E(T)$ is a maximal subsemigroup of $E(S)$. To show that T is maximal, let U be a subsemigroup of S such that $T \subseteq U \subseteq S$. By the same argument as above, we can write

$$Se \times E(T) \subseteq Ue \times E(U) \subseteq Se \times E(S).$$

Hence $Te = Se = Ue$. By maximality of $E(T)$, we obtain $E(T) = E(U)$ or $E(U) = E(S)$. Again by Lemma 24, $T = U$ or $U = S$, and so T is maximal. □

As a direct consequence of Theorem 27 and Proposition 26, we obtain the following corollary.

Corollary 28 Let S be a finite right group which can be written as the direct product of a group G and a nontrivial right zero semigroup $E(S)$, and let T be a subsemigroup of S. Then T is a maximal subsemigroup of S if and only if T can be written as the direct product $H \times E(S)$ or $G \times F$ where H is a maximal subgroup of G and $F = E(S) \setminus \{e\}$ for some $e \in E(S)$.

Let S be a finite right group which can be written as the direct product of a group G and a nontrivial right zero semigroup $E(S)$, where the number of maximal subgroups of G is n and $|E(S)| = m > 1$. We also note by the above corollary that the number of maximal subsemigroups of S is $n + m$. Furthermore, let X be a finite set and E an equivalence relation on X that is not the identity relation. If $X/E = \{A_1, A_2, \ldots, A_n\}$ and $|A_1||A_2| \cdots |A_n| = m$, then the number of maximal subsemigroups of $Q_{E^*}(X)$ is $s_n + m$ where s_n is the number of maximal subgroups of the symmetric group of order n (see [17] Section 8.5 and [18] A290138 for details).

By applying duality principles to our preceding results, we can conclude that, when $|Y| = n$ and $|X| = m$, the total number of maximal subsemigroups in $S_Y(X)$ equals $s_n + n^{m-n}$.

6. Embeddability

For right groups, Sangkhanan established in [5, Theorem 4.3] that any right group S can be embedded within the subsemigroup $Q_{E^*}(S)$ for some appropriately chosen equivalence relation E on S. In this section, we extend our investigation to address the analogous embedding problem for left groups.

Recall that, for any semigroup S, the *inner right translation* associated with an element $a \in S$ is the mapping $\rho_a : S \to S$ defined by $x \mapsto xa$. It is straightforward to verify that, for any $a, b \in S$, we have $\rho_a \rho_b = \rho_{ab}$, which demonstrates that the collection $\{\rho_a : a \in S\}$ forms a subsemigroup of $T(S)$ under function composition. We refer to the mapping $a \mapsto \rho_a$ as the *regular representation* of the semigroup S.

We note that, according to the dual formulation presented in [6, Lemma 1.26], within any left group S, each idempotent element is a right identity element of S.

We now have the necessary tools to establish the embedding theorem for arbitrary left groups.

Theorem 29 A left group S can be embedded in the semigroup $S_Y(X)$ for some set Y and X.

Proof. Let $E(S)$ be the set of all idempotents in S. As previously established, S can be represented as the disjoint union of isomorphic groups, specifically the collection $\{eS : e \in E(S)\}$. If S is a group, then S can be embedded in $\mathrm{Sym}(S)$ by Cayley's theorem. We have $\mathrm{Sym}(S) = S_Y(X)$ where $Y = X = S$. Assume that S is not a group. Then the set $E(S)$ has at least two elements. Let e_0 be a fixed idempotent in S. Define the sets

$$Y = E(S) \cup e_0 S \cup \{b\} \quad \text{and} \quad X = Y \cup \{a\},$$

constructed by adjoining the elements b and a to $E(S) \cup e_0 S$ and Y, respectively. Write $E(S) \setminus \{e_0\} = \{e_i : i \in I\}$ and $e_0 S = \{e_0 s : s \in S\}$. For each $t \in S$, we have $t \in e_t S$ for some $e_t \in E(S)$. Define $\omega_t : X \to X$ by

$$\omega_t = \begin{cases} \begin{pmatrix} a & b & e_i & e_0 s \\ e_t & b & e_i & e_0 s \rho_t \end{pmatrix}_{i \in I,\, s \in S} & \text{if } e_t \neq e_0, \\[2em] \begin{pmatrix} a & b & e_i & e_0 s \\ b & b & e_i & e_0 s \rho_t \end{pmatrix}_{i \in I,\, s \in S} & \text{if } e_t = e_0. \end{cases}$$

It is straightforward to verify that $X\omega_t \subseteq Y$.

We claim that, for each $t \in S$, $\omega_t|_Y$ is a bijection. Let $u, v \in Y$ be such that $u\omega_t = v\omega_t$. If $u = b$ or $v = b$, it is easy to verify that $u = b = v$. Now, we suppose that $u \neq b$ and $v \neq b$. If $u = e_i$ and $v = e_0 s$ for some $i \in I$ and $s \in S$, then $e_i = u\omega_t = v\omega_t = e_0 s \rho_t = e_0 st$, which implies that $e_i \in e_0 S$. It contradicts the fact that $e_i S \cap e_0 S = \emptyset$. Without loss of generality, we can consider the following two cases. If $u = e_0 s$ and $v = e_0 s'$ for some $s, s' \in S$, then

$$e_0 st = e_0 s \rho_t = u\omega_t = v\omega_t = e_0 s' \rho_t = e_0 s' t.$$

By right cancellation, we have $u = e_0 s = e_0 s' = v$. For the case when $u = e_i$ and $v = e_{i'}$ for some $i, i' \in I$, it is obvious that $u = v$. Hence $\omega_t|_Y$ is injective. To show that $\omega_t|_Y$ is surjective, let $y \in Y$. If $y = e_i$ for some $i \in I$, then $y\omega_t = e_i = y$. If $y = e_0 s$ for some $s \in S$, then $e_0 st^{-1}\omega_t = e_0 st^{-1}\rho_t = e_0 st^{-1}t = e_0 se$, where t^{-1} is the inverse of t in the group eS if $t \in eS$ for some $e \in E(S)$. We obtain $e_0 st^{-1}\omega_t = e_0 s = y$ since e is a right identity element in S. Therefore, $\omega_t|_Y$ is a bijection. It is concluded that $\omega_t \in S_Y(X)$ for all $t \in S$.

Finally, we show that the mapping $\varphi : S \to S_Y(X)$ defined by $t \mapsto \omega_t$ is an embedding of S into $S_Y(X)$. Let $t, u \in S$. If $e_t = e_0$, then $tu = e_0 tu \in e_0 S$, which implies that $e_{tu} = e_0$. We obtain

$$(tu)\varphi = \omega_{tu} = \begin{pmatrix} a & b & e_i & e_0 s \\ b & b & e_i & e_0 s\rho_{tu} \end{pmatrix}_{i\in I,\, s\in S} = \begin{pmatrix} a & b & e_i & e_0 s \\ b & b & e_i & e_0 s\rho_t\rho_u \end{pmatrix}_{i\in I,\, s\in S}.$$

Moreover,

$$(t\varphi)(u\varphi) = \begin{pmatrix} a & b & e_i & e_0 s \\ b & b & e_i & e_0 s\rho_t \end{pmatrix}_{i\in I,\, s\in S} \begin{pmatrix} a & b & e_i & e_0 s \\ b & b & e_i & e_0 s\rho_u \end{pmatrix}_{i\in I,\, s\in S} \quad \text{if} \quad e_u = e_0,$$

and

$$(t\varphi)(u\varphi) = \begin{pmatrix} a & b & e_i & e_0 s \\ b & b & e_i & e_0 s\rho_t \end{pmatrix}_{i\in I,\, s\in S} \begin{pmatrix} a & b & e_i & e_0 s \\ e_u & b & e_i & e_0 s\rho_u \end{pmatrix}_{i\in I,\, s\in S} \quad \text{if} \quad e_u \neq e_0.$$

In both cases, we have

$$(t\varphi)(u\varphi) = \begin{pmatrix} a & b & e_i & e_0 s \\ b & b & e_i & e_0 s\rho_t\rho_u \end{pmatrix}_{i\in I,\, s\in S} = \omega_{tu} = (tu)\varphi.$$

If $e_t \neq e_0$, then $tu \in e_t S$, which implies that $e_{tu} = e_t$. We obtain

$$(tu)\varphi = \omega_{tu} = \begin{pmatrix} a & b & e_i & e_0 s \\ e_t & b & e_i & e_0 s\rho_{tu} \end{pmatrix}_{i\in I,\, s\in S} = \begin{pmatrix} a & b & e_i & e_0 s \\ e_t & b & e_i & e_0 s\rho_t\rho_u \end{pmatrix}_{i\in I,\, s\in S}.$$

Moreover,

$$(t\varphi)(u\varphi) = \begin{pmatrix} a & b & e_i & e_0 s \\ e_t & b & e_i & e_0 s\rho_t \end{pmatrix}_{i\in I,\, s\in S} \begin{pmatrix} a & b & e_i & e_0 s \\ b & b & e_i & e_0 s\rho_u \end{pmatrix}_{i\in I,\, s\in S} \quad \text{if} \quad e_u = e_0,$$

and

$$(t\varphi)(u\varphi) = \begin{pmatrix} a & b & e_i & e_0 s \\ e_t & b & e_i & e_0 s\rho_t \end{pmatrix}_{i\in I,\, s\in S} \begin{pmatrix} a & b & e_i & e_0 s \\ e_u & b & e_i & e_0 s\rho_u \end{pmatrix}_{i\in I,\, s\in S} \quad \text{if} \quad e_u \neq e_0.$$

In both cases, we have

$$(t\varphi)(u\varphi) = \begin{pmatrix} a & b & e_i & e_0 s \\ e_t & b & e_i & e_0 s\rho_t\rho_u \end{pmatrix}_{i\in I,\, s\in S} = \omega_{tu} = (tu)\varphi.$$

To show that φ is injective, let $t,\, u \in S$ be such that $t\varphi = u\varphi$. Then $\omega_t = \omega_u$. We consider the following two cases.
Case 1: $e_t = e_0$. Then $a\omega_u = a\omega_t = b$, which implies that $e_u = e_0$. We can write

$$\begin{pmatrix} a & b & e_i & e_0 s \\ b & b & e_i & e_0 s\rho_t \end{pmatrix}_{i\in I,\, s\in S} = \omega_t = \omega_u = \begin{pmatrix} a & b & e_i & e_0 s \\ b & b & e_i & e_0 s\rho_u \end{pmatrix}_{i\in I,\, s\in S}.$$

We obtain

$$e_0 t = e_0 e_0 t = e_0 e_0 \rho_t = (e_0 e_0)\omega_t = (e_0 e_0)\omega_u = e_0 e_0 \rho_u = e_0 e_0 u = e_0 u.$$

Moreover, since $t,\, u \in e_0 S$ and e_0 is the identity of the group $e_0 S$, we have $t = e_0 t = e_0 u = u$.
Case 2: $e_t \neq e_0$. Then $a\omega_u = a\omega_t = e_t$, which implies that $e_u = e_t$. Let $e_t = e_u = e$. We can write

$$\begin{pmatrix} a & b & e_i & e_0 s \\ e & b & e_i & e_0 s\rho_t \end{pmatrix}_{i\in I,\, s\in S} = \omega_t = \omega_u = \begin{pmatrix} a & b & e_i & e_0 s \\ e & b & e_i & e_0 s\rho_u \end{pmatrix}_{i\in I,\, s\in S}.$$

Since e is the identity of the group eS and $E(S)$ is a left zero semigroup, we have

$$t = et = e(e_0 e_0)t = e(e_0 e_0 \rho_t) = e(e_0 e_0)\omega_t = e(e_0 e_0)\omega_u = e(e_0 e_0 \rho_u) = e(e_0 e_0)u = eu = u.$$

By combining both cases, we conclude that $t = u$, and so φ is injective. It is concluded that φ is an embedding of S into $S_Y(X)$. □

7. Conclusion

In this paper, we have conducted a comprehensive study of two important classes of transformation semigroups that can be expressed as disjoint union of symmetric groups: the right group $Q_{E^*}(X)$ and the left group $S_Y(X)$. We established complete characterizations of these semigroups, determined isomorphism conditions, calculated their ranks, characterized maximal subsemigroups, and established embedding results. These results contribute significantly to our understanding of transformation semigroups that decompose as unions of symmetric groups, providing both theoretical insights and practical tools for their analysis.

It is important to note that all our findings inherently depend on the assumption of finiteness. Considering S as a right group (with analogous considerations applying to left groups), the computation of the rank of S becomes significantly challenging when S is infinite. Additionally, when X is infinite, determining the rank of $\mathrm{Sym}(X/E)$ presents considerable difficulties. Moreover, in the case of an infinite S, the validity of Theorem 27 cannot be guaranteed, and determining the structure of maximal subsemigroups of S remains an open problem. These constraints delineate the scope of our current methodology and point toward promising avenues for future investigation.

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Conflict of interest

The authors declare no competing financial interest.

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