Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

UNIVERSAL WISER PUBLISHER

Research Article

Optimizing Healthcare Facility Allocation Using Fuzzy N-Bipolar Soft Expert Decision Approach

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Received: 31 July 2025; **Revised:** 4 September 2025; **Accepted:** 11 September 2025

Abstract: Efficient planning in the healthcare sector requires selecting the most suitable facility location based on multiple, often conflicting criteria and expert evaluations. This study proposes Fuzzy N-Bipolar Soft Expert (FN-BSE) set model, which integrates expert input, bipolar assessments, and multinary evaluation within a fuzzy framework. Theoretical structure of the model is developed through fundamental operations and their algebraic properties. Applying FN-BSE set model to a case study of seven potential healthcare locations, expert judgments were aggregated, and alternatives were ranked. The results indicate that location ω_3 is the optimal choice. Analysis of positive and negative evaluations shows the model effectively balances conflicting opinions, and comparative evaluation demonstrates its superiority over existing approaches in handling uncertainty and supporting robust Multi-Attribute Group Decision-Making (MAGDM).

Keywords: fuzzy N-bipolar soft expert sets, N-soft sets, soft expert sets, Multi-Attribute Group Decision-Making (MAGDM), healthcare

MSC: 03E72, 03E75, 90B50

1. Introduction

Healthcare is a critical sector characterized by complex and high-stakes environments where decisions directly impact patient well-being and operational efficiency. Within this context, effective Decision-Making (DM) is crucial, as choices such as allocating medical resources [1] or selecting optimal facility locations [2] directly impact public health and patient safety [3–5]. These decisions often involve multiple criteria and stakeholders [6], making group DM indispensable [7]. However, uncertainty, ambiguity, and differing expert opinions frequently complicate the process. Advanced mathematical models that can integrate diverse viewpoints and handle imprecise information are therefore essential for supporting reliable and effective healthcare decisions. Healthcare facilities, such as hospitals and clinics, often operate under uncertain conditions where decision-makers must balance patient needs, financial constraints, and staff availability. Uncertainty theories have been widely adopted in healthcare to better represent imprecise information,

DOI: https://doi.org/10.37256/cm.6520258086

such as expert judgments, patient demand, and evolving clinical requirements. These approaches allow more reliable and informed choices compared to classical methods that assume complete and precise data. The main advantage of these uncertainty-based frameworks is their ability to model real-world complexity, but a limitation is that many cannot simultaneously consider multiple experts, conflicting criteria, and graded evaluations.

Managing uncertainty and vagueness in DM has been extensively supported by fuzzy set theory, first introduced by Zadeh [8] in 1965. Unlike classical set theory, which strictly classifies elements as either belonging or not belonging to a set, fuzzy set allows elements to have degrees of membership ranging continuously between 0 and 1. This flexibility has made fuzzy sets foundational in applications such as medical diagnosis [9]. Numerous extensions-including type-2 fuzzy sets [10], intuitionistic fuzzy sets [11], bipolar fuzzy sets [12], and $Q[\varepsilon]$-fuzzy sets [13]-have further enhanced the ability to model complex and dynamic decision problems. While these extensions improve representational power and allow handling of graded uncertainty, they still often neglect the integration of multiple experts and bipolar assessments simultaneously.

In healthcare, decisions rarely rely on a single criterion, and conflicting priorities often require consideration of multiple factors simultaneously. Fuzzy sets provide a way to quantify uncertainty in each factor, making them particularly suitable for MAGDM problems. This perspective motivates the use of advanced frameworks that can handle multiple criteria, expert judgments, and uncertain information in a unified manner. The limitation of classical fuzzy approaches is that they typically focus on single-criterion uncertainty, failing to fully capture expert conflicts or opposing evaluations.

Soft set (S-set) theory [14] has been recognized as a powerful mathematical tool for handling uncertainties that arise in practical DM. Several works have expanded its scope, including studies on incomplete S-sets [15, 16], Fuzzy Soft (FS) sets [17], intuitionistic FS sets [18], general reviews on S-sets [19], and other advanced S-set variants [20]. Beyond theoretical developments, applications of S-sets span a wide range of fields such as data mining [21], forecasting [22], and medical diagnosis [23]. More recent studies have explored healthcare-specific contexts, such as vocal risk diagnosis [24], expert systems for coronary heart disease [25], healthcare decision support [26], and healthcare claims analysis [27]. S-sets provide an advantage in modeling parameterized uncertainty, but their main limitation is the reliance on binary evaluations, which cannot capture graded or conflicting opinions from multiple experts.

Building upon this foundation, Bipolar Soft (BS) sets were proposed to extend the expressive capacity of S-sets by incorporating dual perspectives in evaluations [28]. This line of work further led to Fuzzy BS (FBS) sets [29], bipolar hypersoft sets [30], and fuzzy bipolar hypersoft sets [31]. Such frameworks have been particularly useful in domains where conflicting or opposing information must be modeled. For example, they have been successfully applied to healthcare-related DM tasks such as cardiovascular treatment planning [32] and autism severity assessment [33]. While BS-based frameworks provide richer dual-perspective evaluations, they often lack multilevel grading and do not fully integrate expert input, limiting their applicability in group DM scenarios.

In another direction, N-Soft (N-S) sets were developed to enhance S-set theory by allowing multinary evaluations [34]. Subsequent advancements include Fuzzy N-S (FN-S) sets [35], N-hypersoft sets [36, 37], M-parametrized N-S sets [38], and bipolar M-parametrized N-S sets [39]. Applications of these models have addressed pressing challenges in healthcare, such as tumor cell detection [40] and infectious disease identification [41]. These frameworks can capture richer evaluation structures, but without expert involvement or bipolar assessment, they remain insufficient for fully nuanced healthcare DM.

Soft Expert (SE) sets [42] extend S-sets by incorporating expert knowledge into DM, allowing more informed and context-sensitive evaluations. They have been further developed into Fuzzy SE (FSE) sets [43], N-Soft Expert (N-SE) sets and Fuzzy N-SE (FN-SE) sets [44], Bipolar SE (BSE) sets [45], and Fuzzy BSE (FBSE) sets [46]. Additional extensions include intuitionistic fuzzy SE sets [47], generalized neutrosophic SE sets [48], and bipolar N-SE sets [49]. The advantage of these approaches is the integration of expert input, multilevel evaluation, or bipolarity in isolation; however, no single framework integrates all these aspects with fuzzy uncertainty, which limits their comprehensiveness.

Despite these advances, most existing models focus on only one or a few aspects of uncertainty or evaluation, which limits their ability to capture multiple dimensions simultaneously. In particular, many frameworks cannot fully integrate expert involvement, bipolar perspectives (positive and negative assessments), and multilevel evaluations in healthcare DM under fuzzy uncertainty. Previous studies have addressed parts of this problem: N-bipolar Soft (N-BS) sets [50] allow multilevel bipolar evaluations, while Fuzzy N-BS (FN-BS) sets [51] and Pythagorean FN-BS sets [52] incorporate fuzzy uncertainty into these evaluations. The N-BS Expert (N-BSE) sets [53] additionally include expert judgments. Related developments on N-bipolar hypersoft sets [54, 55] further extend the representational capabilities of these frameworks, for instance by handling hierarchical or multilevel parameterization. These studies provide a solid theoretical and methodological basis that can be adapted to complex DM problems, including healthcare, where multiple criteria, expert opinions, and conflicting evaluations must be considered simultaneously. However, a unified framework that simultaneously combines all these aspects remains lacking. To address this gap, the present study applies the FN-BSE set model to healthcare, focusing on the selection of optimal facility locations under uncertain and conflicting conditions.

1.1 *Research gap and motivations*

Allocating new healthcare facilities in a metropolitan region is a critical decision with direct consequences for patient access, treatment quality, operational costs, staff availability, and overall public health outcomes. For example, a city planning to open three new primary healthcare clinics must balance multiple criteria: ensuring accessibility for underserved populations, minimizing operational costs, managing limited staff resources, and accounting for uncertain patient demand. Experts from different departments may provide conflicting opinions on which criteria are most important, making the decision process highly complex. Traditional methods, which often rely on binary or single-criteria evaluations, cannot adequately capture these nuances.

Existing DM frameworks such as basic S-set models, SE sets, BSE sets, and N-SE sets address certain aspects of the problem but remain limited. A basic S-set model provides only a binary yes/no evaluation of potential locations, ignoring the subtleties of accessibility, cost, and demand. An SE set incorporates expert input, but still treats each criterion in a binary fashion, failing to capture graded evaluations or conflicts among experts. BSE sets model positive and negative aspects of each location, such as advantages versus disadvantages, but still rely on binary evaluations. N-SE sets allow multilevel evaluations and expert input, yet they cannot distinguish strong advantages from strong disadvantages due to the lack of bipolar assessment.

These limitations demonstrate the need for a comprehensive approach capable of integrating multiple dimensions simultaneously. FN-BSE sets provide such a solution by combining expert judgments, multilevel evaluations, bipolar assessments, and fuzzy uncertainty. In the context of healthcare facility allocation, FN-BSE sets enable decision-makers to evaluate and rank potential clinic locations more effectively, considering not only experts' opinions and conflicting criteria but also the degree of positive or negative impact of each alternative under uncertain conditions. By providing a richer, more nuanced representation of real-world decision scenarios, FN-BSE sets support informed and reliable DM in complex healthcare environments.

1.2 *Main contributions*

This study makes several theoretical and methodological contributions to MAGDM under uncertainty. First, we develop the FN-BSE set model, extending existing S-set frameworks by simultaneously integrating expert judgments, multilevel evaluations, bipolar assessments, and fuzzy uncertainty. Unlike previous models that address only some of these aspects, the FN-BSE set provides a unified mathematical structure for capturing complex decision scenarios, establishing a foundation for further extensions in S-set theory. Second, we define a comprehensive set of algebraic operations for FN-BSE sets, including subset, equality, union, intersection, and complement. The algebraic properties of these operations-such as commutativity, associativity, and distributivity-are rigorously formulated and proven. These definitions generalize operations from prior S-set and fuzzy frameworks and provide a formal methodology for systematically manipulating FN-BSE sets, supporting consistent handling of multiple experts' conflicting evaluations and multilevel bipolar assessments. Third, we establish formal measures for evaluating and ranking alternatives within the FN-BSE framework. Score functions and comparison mechanisms account for both positive and negative evaluations as well as multilevel memberships, enabling robust synthesis of expert opinions under uncertainty. This methodological framework enhances the interpretability, consistency, and comparability of decision outcomes, addressing limitations of previous models. Finally, although illustrated through a healthcare-focused case study, the primary contribution is the generalizable theoretical and methodological structure, applicable across diverse domains requiring MAGDM under uncertainty. The study advances both the theory of FN-BSE sets and the methodology for systematic aggregation and evaluation of complex expert assessments.

1.3 *Outline of the paper*

This paper is structured as follows: Section 2 reviews the essential concepts and notations that form the foundation of this study. Section 3 introduces the novel FN-BSE set model, detailing its essential operations and algebraic properties through illustrative examples. Section 4 discusses the application of FN-BSE set framework in MAGDM, presenting a case study on selecting the optimal location for a new healthcare facility, thereby demonstrating the model's utility in real-world DM. A thorough examination of the suggested model is given in Section 5, which also highlights its benefits and drawbacks in relation to DM and contrasts it with other methods. Finally, Section 6 summarizes the key findings and discusses some future research directions, including the extension of the FN-BSE set model to other related models.

2. Preliminary concepts

This section reviews the basic concepts and definitions relevant to this study. The discussion is organized into two parts: (i) Basic S-Sets models and their fuzzy and extended forms, and (ii) Bipolar and NS Set models that incorporate multiple experts, bipolarity, and graded evaluations. Throughout, the universal set of alternatives (or objects) is denoted by W, and the set of characteristics (or parameters) by B. The set of ordered grades is $G = \{0,\ 1,\ \ldots,\ N-1\}$ where $N \in \{2,\ 3,\ \ldots\}$. Experts are represented by the set E, while the set of opinions is $\mathcal{O} = \{0 = \text{disagree},\ 1 = \text{agree}\}$. Additionally, $\mathcal{K} \subset \mathcal{Z}$ where $\mathcal{Z} = B \times E \times \mathcal{O}$.

2.1 *Basic soft sets*

The concept of S-sets was first introduced by Molodtsov [14] as a parameterized tool to handle uncertainty without requiring additional membership functions or probability distributions. This framework has since been widely applied in DM problems, particularly when qualitative attributes dominate the evaluation process.

Definition 1 [14] An S-set is defined to be the pair $(\eta,\ B)$, where $\eta : B \to 2^W$, and 2^W denotes the collection of all crisp subsets of W.

The classical S-set provides a simple binary characterization of alternatives, but it does not incorporate the notion of partial membership. To overcome this limitation, Maji et al. [17] introduced FS sets.

Definition 2 [17] An FS set is an ordered pair $(\zeta,\ B)$, where $\zeta : B \to I^W$, and I^W denotes the collection of all fuzzy subsets of W.

FS sets allow each alternative to be associated with a degree of membership in $[0,\ 1]$, thus enabling more nuanced modeling of real-world uncertainty compared to crisp S-sets.

To model multi-expert settings, Alkhazaleh and Salleh [42, 43] extended S-sets into the so-called SE sets and their fuzzy counterpart (FSE).

Definition 3 [42] An SE set is defined as a pair $(\gamma,\ \mathcal{K})$, where $\gamma : \mathcal{K} \to 2^W$.

Definition 4 [43] An FSE set is defined as $(\rho,\ \mathcal{K})$ such that $\rho : \mathcal{K} \to I^W$.

The SE and FSE frameworks explicitly consider expert opinions, making them particularly suitable for group DM contexts. Furthermore, the introduction of a NOT operation [42] allows the modeling of complementary attributes.

Definition 5 [42] The NOT set of a set $\mathcal{K}$, denoted as $\neg\mathcal{K}$, is given by $\neg\mathcal{K} = \{\neg\kappa \mid \kappa \in \mathcal{K}\}$, where $\neg\kappa = (\neg\flat,\ \varepsilon,\ o)$ means the negation of $\kappa = (\flat,\ \varepsilon,\ o)$.

2.2 *Bipolar and N-soft sets*

While classical and FS sets model uncertainty, they do not capture bipolar information (simultaneous positive and negative evaluations). To address this, Shabir and Naz [28] proposed BS sets.

Definition 6 [28] A BS set is a structure $(\alpha,\ \beta,\ B)$, where $\alpha: B \to 2^W$ and $\beta: \neg B \to 2^W$. For each $\flat \in B$, it holds that $\alpha(\flat) \cap \beta(\neg\flat) = \emptyset$, where $\alpha(\flat),\ \beta(\neg\flat) \subseteq W$.

BS sets distinguish between positive and negative attributes, a property useful in contexts such as risk-benefit analysis. Their fuzzy counterpart, FBS sets, allow graded bipolar evaluations [29].

Definition 7 [29] An FBS set is a structure $(\delta,\ \lambda,\ B)$, where $\delta: B \to I^W$ and $\lambda: \neg B \to I^W$. For each $\flat \in B$ and $\omega \in W$, it holds that $0 \le \delta(\flat)(\omega) + \lambda(\neg\flat)(\omega) \le 1$, where $\delta(\flat)(\omega),\ \lambda(\neg\flat)(\omega) \in [0,\ 1]$.

For group-based bipolar evaluations, Dalkılıç et al. [45] introduced the BSE set, later extended by Ali et al. [46] to FBSE sets.

Definition 8 [45] A BSE set is represented as a triple $(\Delta,\ \Lambda,\ \mathcal{K})$, where $\Delta: \mathcal{K} \to 2^W$ and $\Lambda: \neg\mathcal{K} \to 2^W$. These mappings satisfy the condition that for every $\kappa \in \mathcal{K}$, $\Delta(\kappa) \cap \Lambda(\neg\kappa) = \emptyset$, with $\Delta(\kappa),\ \Lambda(\neg\kappa) \subseteq W$.

Definition 9 [46] An FBSE set is represented as a triple $(\Omega,\ \Gamma,\ \mathcal{K})$, where $\Omega:\ \mathcal{K} \to I^W$ and $\Gamma: \neg\mathcal{K} \to I^W$. These mappings satisfy the condition that for every $\kappa \in \mathcal{K}$ and $\omega \in W$, $0 \le \Omega(\kappa)(\omega) + \Gamma(\neg\kappa)(\omega) \le 1$, where $\Omega(\kappa)(\omega),\ \Gamma(\neg\kappa)(\omega) \in [0,\ 1]$.

Another important development was the introduction of N-S sets by Fatimah et al. [34], which allow multigraded evaluations beyond binary or fuzzy degrees.

Definition 10 [34] An N-S set is characterized as a triple $(\Psi,\ B,\ N)$, where $\Psi:\ B \to 2^{W\times G}$. For each $\flat \in B$, there is a distinct pair $(\omega,\ g_\flat) \in W \times G$ such that $(\omega,\ g_\flat) \in \Psi(\flat)$, or equivalently, $\Psi(\flat)(\omega) = g_\flat$, with $\omega \in W$ and $g_\flat \in G$.

Their fuzzy extension, FN-S sets [35], generalizes the idea by associating a membership degree with each graded evaluation.

Definition 11 [35] An FN-S set is represented as a trio $(\chi,\ B,\ N)$, where $\chi: B \to I^{W\times G}$. For every $\flat \in B$, there is a unique pair $(\omega,\ g_\flat) \in W \times G$ such that $\langle(\omega,\ g_\flat),\ \chi_{g_\flat}\rangle \in \chi(\flat)$, or equivalently, $\chi(\flat)(\omega) = \langle g_\flat,\ \chi_{g_\flat}\rangle$, where $\omega \in W$, $g_\flat \in G$, and $\chi_{g_\flat} \in [0,\ 1]$.

To model group DM with multigraded opinions, Ali and Akram [44] extended these concepts into N-SE and FN-SE sets.

Definition 12 [44] An N-SE set is represented as a triple $(\Theta,\ \mathcal{K},\ N)$, where $\Theta:\ \mathcal{K} \to 2^{W\times G}$. For every $\kappa \in \mathcal{K}$, there exists a distinct pair $(\omega,\ g_\kappa) \in W \times G$ such that $(\omega,\ g_\kappa) \in \Theta(\kappa)$, or equivalently, $\Theta(\kappa)(\omega) = g_\kappa$, with $\omega \in W$ and $g_\kappa \in G$.

Definition 13 [44] An FN-SE set is represented as a triple $(\Upsilon,\ \mathcal{K},\ N)$, with $\Upsilon: \mathcal{K} \to I^{W\times G}$. For each $\kappa \in \mathcal{K}$, there is a unique pair $(\omega,\ g_\kappa) \in W \times G$ such that $\langle(\omega,\ g_\kappa),\ \Upsilon_{g_\kappa}\rangle \in \Upsilon(\kappa)$, or equivalently, $\Upsilon(\kappa)(\omega) = \langle g_\kappa,\ \Upsilon_{g_\kappa}\rangle$, where $\omega \in W$, $g_\kappa \in G$, and $\Upsilon_{g_\kappa} \in [0,\ 1]$.

Finally, Musa et al. [53] introduced the N-BSE set, which integrates bipolarity, multigraded evaluations, and expert opinions into a unified framework.

Definition 14 [53] An N-BSE set is represented by a quadruple $(\pi,\ \tau,\ \mathcal{K},\ N)$, where $\pi: \mathcal{K} \to 2^{W\times G}$ and $\tau: \neg\mathcal{K} \to 2^{W\times G}$. These mappings adhere to the following criteria: For every $\kappa \in \mathcal{K}$, there is a unique $(\omega,\ g_\kappa) \in W \times G$ such that $(\omega,\ g_\kappa) \in \pi(\kappa)$, or equivalently, $\pi(\kappa)(\omega) = g_\kappa$. Likewise, for every $\neg\kappa \in \neg\mathcal{K}$, there is a unique pair $(\omega,\ g_{\neg\kappa}) \in W \times G$ such that $(\omega,\ g_{\neg\kappa}) \in \tau(\neg\kappa)$, or equivalently, $\tau(\neg\kappa)(\omega) = g_{\neg\kappa}$. Furthermore, it is necessary that $g_\kappa + g_{\neg\kappa} \le N - 1$, where $\omega \in W$ and $g_\kappa,\ g_{\neg\kappa} \in G$.

Taken together, these progressive extensions-from classical S-sets to fuzzy, bipolar, N-S, and finally N-BSE sets-demonstrate a systematic effort to capture increasing levels of complexity in DM environments. The N-BSE set framework is particularly powerful for healthcare facility location problems, where evaluations often involve conflicting expert opinions, simultaneous pros and cons, and graded levels of suitability under uncertainty.

3. Fuzzy N-bipolar soft expert sets

In this section, we introduce our proposed model, FN-BSE set. The motivation behind FN-BSE set is to handle DM scenarios in a fuzzy environment, where both the evaluations of multiple experts and the graded assessments of alternatives are considered simultaneously. We first define the FN-BSE set structure, then detail its key operations using illustrative examples, and finally explore its algebraic properties.

3.1 *Foundational structure*

This subsection formally defines the FN-BSE set model and provides an illustrative example to demonstrate its structure.

Definition 15 A quadruple $(\psi,\ \mu,\ \mathcal{K},\ N)$ is called an FN-BSE set, where $\psi : \mathcal{K} \to I^{W\times G}$ and $\mu : \neg\mathcal{K} \to I^{W\times G}$, such that for each $\kappa \in \mathcal{K}$, there is a unique $(\omega,\ g_\kappa) \in W \times G$ such that $\langle(\omega,\ g_\kappa),\ \psi_{g_\kappa}\rangle \in \psi(\kappa)$, and for each $\neg\kappa \in \neg\mathcal{K}$, there is a unique $(\omega,\ g_{\neg\kappa}) \in W \times G$ such that $\langle(\omega,\ g_{\neg\kappa}),\ \mu_{g_{\neg\kappa}}\rangle \in \mu(\neg\kappa)$.

Table 1. Tabulated illustration of the FN-BSE set $(\psi,\ \mu,\ \mathcal{K},\ N)$

$(\psi,\ \mu,\ \mathcal{K},\ N)$	ω_1	ω_2	$\cdots$	ω_n
$(\flat_1,\ \varepsilon_1,\ 1)$	$\psi((\flat_1,\ \varepsilon_1,\ 1))(\omega_1)$	$\psi((\flat_1,\ \varepsilon_1,\ 1))(\omega_2)$	$\cdots$	$\psi((\flat_1,\ \varepsilon_1,\ 1))(\omega_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\flat_1,\ \varepsilon_t,\ 1)$	$\psi((\flat_1,\ \varepsilon_t,\ 1))(\omega_1)$	$\psi((\flat_1,\ \varepsilon_t,\ 1))(\omega_2)$	$\cdots$	$\psi((\flat_1,\ \varepsilon_t,\ 1))(\omega_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\flat_m,\ \varepsilon_1,\ 1)$	$\psi((\flat_m,\ \varepsilon_1,\ 1))(\omega_1)$	$\psi((\flat_m,\ \varepsilon_1,\ 1))(\omega_2)$	$\cdots$	$\psi((\flat_m,\ \varepsilon_1,\ 1))(\omega_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\flat_m,\ \varepsilon_t,\ 1)$	$\psi((\flat_m,\ \varepsilon_t,\ 1))(\omega_1)$	$\psi((\flat_m,\ \varepsilon_t,\ 1))(\omega_2)$	$\cdots$	$\psi((\flat_m,\ \varepsilon_t,\ 1))(\omega_n)$
$(\flat_1,\ \varepsilon_1,\ 0)$	$\psi((\flat_1,\ \varepsilon_1,\ 0))(\omega_1)$	$\psi((\flat_1,\ \varepsilon_1,\ 0))(\omega_2)$	$\cdots$	$\psi((\flat_1,\ \varepsilon_1,\ 0))(\omega_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\flat_1,\ \varepsilon_t,\ 0)$	$\psi((\flat_1,\ \varepsilon_t,\ 0))(\omega_1)$	$\psi((\flat_1,\ \varepsilon_t,\ 0))(\omega_2)$	$\cdots$	$\psi((\flat_1,\ \varepsilon_t,\ 0))(\omega_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\flat_m,\ \varepsilon_1,\ 0)$	$\psi((\flat_m,\ \varepsilon_1,\ 0))(\omega_1)$	$\psi((\flat_m,\ \varepsilon_1,\ 0))(\omega_2)$	$\cdots$	$\psi((\flat_m,\ \varepsilon_1,\ 0))(\omega_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\flat_m,\ \varepsilon_t,\ 0)$	$\psi((\flat_m,\ \varepsilon_t,\ 0))(\omega_1)$	$\psi((\flat_m,\ \varepsilon_t,\ 0))(\omega_2)$	$\cdots$	$\psi((\flat_m,\ \varepsilon_t,\ 0))(\omega_n)$
$(\neg\flat_1,\ \varepsilon_1,\ 1)$	$\mu((\neg\flat_1,\ \varepsilon_1,\ 1))(\omega_1)$	$\mu((\neg\flat_1,\ \varepsilon_1,\ 1))(\omega_2)$	$\cdots$	$\mu((\neg\flat_1,\ \varepsilon_1,\ 1))(\omega_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\neg\flat_1,\ \varepsilon_t,\ 1)$	$\mu((\neg\flat_1,\ \varepsilon_t,\ 1))(\omega_1)$	$\mu((\neg\flat_1,\ \varepsilon_t,\ 1))(\omega_2)$	$\cdots$	$\mu((\neg\flat_1,\ \varepsilon_t,\ 1))(\omega_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\neg\flat_m,\ \varepsilon_1,\ 1)$	$\mu((\neg\flat_m,\ \varepsilon_1,\ 1))(\omega_1)$	$\mu((\neg\flat_m,\ \varepsilon_1,\ 1))(\omega_2)$	$\cdots$	$\mu((\neg\flat_m,\ \varepsilon_1,\ 1))(\omega_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\neg\flat_m,\ \varepsilon_t,\ 1)$	$\mu((\neg\flat_m,\ \varepsilon_t,\ 1))(\omega_1)$	$\mu((\neg\flat_m,\ \varepsilon_t,\ 1))(\omega_2)$	$\cdots$	$\mu((\neg\flat_m,\ \varepsilon_t,\ 1))(\omega_n)$
$(\neg\flat_1,\ \varepsilon_1,\ 0)$	$\mu((\neg\flat_1,\ \varepsilon_1,\ 0))(\omega_1)$	$\mu((\neg\flat_1,\ \varepsilon_1,\ 0))(\omega_2)$	$\cdots$	$\mu((\neg\flat_1,\ \varepsilon_1,\ 0))(\omega_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\neg\flat_1,\ \varepsilon_t,\ 0)$	$\mu((\neg\flat_1,\ \varepsilon_t,\ 0))(\omega_1)$	$\mu((\neg\flat_1,\ \varepsilon_t,\ 0))(\omega_2)$	$\cdots$	$\mu((\neg\flat_1,\ \varepsilon_t,\ 0))(\omega_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\neg\flat_m,\ \varepsilon_1,\ 0)$	$\mu((\neg\flat_m,\ \varepsilon_1,\ 0))(\omega_1)$	$\mu((\neg\flat_m,\ \varepsilon_1,\ 0))(\omega_2)$	$\cdots$	$\mu((\neg\flat_m,\ \varepsilon_1,\ 0))(\omega_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\neg\flat_m,\ \varepsilon_t,\ 0)$	$\mu((\neg\flat_m,\ \varepsilon_t,\ 0))(\omega_1)$	$\mu((\neg\flat_m,\ \varepsilon_t,\ 0))(\omega_2)$	$\cdots$	$\mu((\neg\flat_m,\ \varepsilon_t,\ 0))(\omega_n)$

For each $\kappa \in \mathcal{K}$ and $\omega \in W$, there is a unique assessment from the examination space G, denoted by g_κ, such that $\langle(\omega,\ g_\kappa),\ \psi_{g_\kappa}\rangle \in \psi(\kappa)$, or equivalently, $\psi(\kappa)(\omega) = \langle g_\kappa,\ \psi_{g_\kappa}\rangle$. Similarly, for each $\neg\kappa \in \neg\mathcal{K}$ and $\omega \in W$, there is a unique assessment from the examination space G, denoted by $g_{\neg\kappa}$, such that $\langle(\omega,\ g_{\neg\kappa}),\ \mu_{g_{\neg\kappa}}\rangle \in \mu(\neg\kappa)$, or equivalently, $\mu(\neg\kappa)(\omega) = \langle g_{\neg\kappa},\ \mu_{g_{\neg\kappa}}\rangle$.

Additionally, the following conditions hold:

$$g_{\kappa} + g_{\neg\kappa} \leq N - 1, \tag{1}$$

$$0 \leq \psi_{g_{\kappa}} + \mu_{g_{\neg\kappa}} \leq 1, \tag{2}$$

where $g_{\kappa},\ g_{\neg\kappa} \in G$, and $\psi_{g_{\kappa}},\ \mu_{g_{\neg\kappa}} \in [0,\ 1]$.

The FN-BSE set $(\psi,\ \mu,\ \mathcal{K},\ N)$ can be illustrated in a table format where $W = \{\omega_1,\ \omega_2,\ ...,\ \omega_n\}$, $B = \{\flat_1,\ \flat_2,\ ...,\ \flat_m\}$, and $E = \{\varepsilon_1,\ \varepsilon_2,\ ...,\ \varepsilon_t\}$ are all finite unless stated otherwise. This tabular illustration is presented in Table 1.

Remark 1 In the framework of FN-BSE sets, it is crucial to maintain logical consistency in the evaluation of attributes by experts. The following conditions must be satisfied:

1. An expert ε cannot give high levels of agreement (or disagreement) to both an attribute $\flat$ and its negation $\neg\flat$ for the same option. In formal terms, the assessments must comply with:

$$g_{(\flat,\ \varepsilon,\ o)} + g_{(\neg\flat,\ \varepsilon,\ o)} \leq N - 1, \quad \text{for any } o \in \mathcal{O}. \tag{3}$$

2. It is logically inconsistent for an expert ε to agree (or disagree) that an object is highly present with respect to both an attribute $\flat$ and its negation $\neg\flat$. The evaluations must adhere to the constraint:

$$0 \leq \psi_{g_{(\flat,\ \varepsilon,\ o)}} + \mu_{g_{(\neg\flat,\ \varepsilon,\ o)}} \leq 1, \text{ for any } o \in \mathcal{O}. \tag{4}$$

3. A specialist ε cannot concurrently attribute both high levels of agreement and disagreement to the same characteristic $\flat$ for a specific alternative. The assessments need to comply with:

$$g_{(\flat,\ \varepsilon,\ o_1)} + g_{(\flat,\ \varepsilon,\ o_2)} \leq N - 1, \quad \text{where } o_1 \neq o_2 \text{ and } o_1,\ o_2 \in \mathcal{O}. \tag{5}$$

This condition also applies to the negation $\neg\flat$.

4. It is contradictory for an expert ε to acknowledge that an object is significantly present concerning an attribute $\flat$, while also giving an equally high level of disagreement regarding the same attribute $\flat$. The assessments must be consistent:

$$0 \leq \psi_{g_{(\flat,\ \varepsilon,\ o_1)}} + \psi_{g_{(\flat,\ \varepsilon,\ o_2)}} \leq 1, \quad \text{where } o_1 \neq o_2 \text{ and } o_1,\ o_2 \in \mathcal{O}. \tag{6}$$

This condition similarly applies to the negation $\neg\flat$.

In order to enhance our comprehension of the essential characteristics of our latest model, let's analyze the example provided below.

Example 1 Suppose an educational institution is tasked with selecting the most effective teaching method from five available options: $W = \{\omega_1,\ \omega_2,\ \omega_3,\ \omega_4,\ \omega_5\}$. The institution establishes a panel of seasoned educators to develop a comprehensive report evaluating teaching techniques based on various criteria: $B = \{\flat_1 =$ student engagement, $\flat_2 =$ learning effectiveness$\}$, along with their associated negative characteristics: $\neg B = \{\neg\flat_1 =$ low engagement, $\neg\flat_2 =$ poor learning outcomes$\}$. To ensure a comprehensive evaluation, the organization distributes the report to two educational specialists $E = \{\varepsilon_1,\ \varepsilon_2\}$, who are experts in teaching methodologies, and takes into account their feedback $O = \{0 = \text{disagree},\ 1 = \text{agree}\}$ as demonstrated in Table 2, where

- "$\circ$" represents poor performance.
- "$\ast$" represents slightly poor performance.
- "$\ast\ast$" represent moderate performance.
- "$\ast\ast\ast$" represent good prformance.
- "$\ast\ast\ast\ast$" represent excellent prformance.

Table 2. Information table

$\mathcal{K} \backslash W$	ω_1	ω_2	ω_3	ω_4	ω_5
$(\flat_1,\ \varepsilon_1,\ 1)$	$\ast\ast\ast\ast$	$\ast\ast$	$\ast\ast\ast\ast$	$\ast$	$\circ$
$(\flat_1,\ \varepsilon_2,\ 1)$	$\ast$	$\ast\ast$	$\ast\ast$	$\ast\ast\ast$	$\circ$
$(\flat_2,\ \varepsilon_1,\ 1)$	$\ast\ast$	$\ast\ast$	$\ast\ast\ast$	$\ast$	$\circ$
$(\flat_2,\ \varepsilon_2,\ 1)$	$\ast\ast\ast$	$\ast\ast$	$\ast\ast$	$\ast\ast$	$\circ$
$(\flat_1,\ \varepsilon_1,\ 0)$	$\circ$	$\ast\ast$	$\circ$	$\circ$	$\ast$
$(\flat_1,\ \varepsilon_2,\ 0)$	$\ast$	$\ast\ast$	$\ast\ast$	$\circ$	$\ast\ast$
$(\flat_2,\ \varepsilon_1,\ 0)$	$\ast$	$\circ$	$\circ$	$\ast\ast$	$\ast\ast$
$(\flat_2,\ \varepsilon_2,\ 0)$	$\ast$	$\ast\ast$	$\ast$	$\ast$	$\circ$
$(\neg\flat_1,\ \varepsilon_1,\ 1)$	$\circ$	$\ast\ast$	$\circ$	$\ast\ast$	$\circ$
$(\neg\flat_1,\ \varepsilon_2,\ 1)$	$\ast\ast$	$\ast\ast$	$\ast$	$\circ$	$\circ$
$(\neg\flat_2,\ \varepsilon_1,\ 1)$	$\circ$	$\ast$	$\circ$	$\ast\ast$	$\circ$
$(\neg\flat_2,\ \varepsilon_2,\ 1)$	$\ast$	$\ast\ast$	$\circ$	$\ast\ast$	$\circ$
$(\neg\flat_1,\ \varepsilon_1,\ 0)$	$\ast\ast$	$\ast$	$\circ$	$\circ$	$\ast\ast$
$(\neg\flat_1,\ \varepsilon_2,\ 0)$	$\circ$	$\ast$	$\ast$	$\ast\ast\ast$	$\ast$
$(\neg\flat_2,\ \varepsilon_1,\ 0)$	$\ast\ast$	$\circ$	$\ast\ast\ast$	$\ast$	$\ast$
$(\neg\flat_2,\ \varepsilon_2,\ 0)$	$\ast\ast\ast$	$\ast$	$\ast\ast\ast$	$\ast$	$\ast\ast\ast\ast$

This assessment utilizing symbols can be conveniently converted into numerical values, such as $G = \{0,\ 1,\ 2,\ 3,\ 4\}$, where

- 0 is equivalent to $\circ$,
- 1 is equivalent to $\ast$,
- 2 is equivalent to $\ast\ast$,
- 3 is equivalent to $\ast\ast\ast$,
- 4 is equivalent to $\ast\ast\ast\ast$.

Membership values are assigned based on the evaluation grades of the attributes as follows:

For positive membership values (ψ_{g_κ}):

$$\psi_{g_\kappa} = \begin{cases} 0 \le \psi_{g_\kappa} < 0.2, & \text{when } g_\kappa = 0, \\ 0.2 \le \psi_{g_\kappa} < 0.4, & \text{when } g_\kappa = 1, \\ 0.4 \le \psi_{g_\kappa} < 0.6, & \text{when } g_\kappa = 2, \\ 0.6 \le \psi_{g_\kappa} < 0.8, & \text{when } g_\kappa = 3, \\ 0.8 \le \psi_{g_\kappa} \le 1, & \text{when } g_\kappa = 4. \end{cases} \tag{7}$$

For negative membership values ($\mu_{g_{\neg\kappa}}$):

$$\mu_{g_{\neg\kappa}} = \begin{cases} 0 \le \mu_{g_{\neg\kappa}} < 0.2, & \text{when } g_{\neg\kappa} = 0, \\ 0.2 \le \mu_{g_{\neg\kappa}} < 0.4, & \text{when } g_{\neg\kappa} = 1, \\ 0.4 \le \mu_{g_{\neg\kappa}} < 0.6, & \text{when } g_{\neg\kappa} = 2, \\ 0.6 \le \mu_{g_{\neg\kappa}} < 0.8, & \text{when } g_{\neg\kappa} = 3, \\ 0.8 \le \mu_{g_{\neg\kappa}} \le 1, & \text{when } g_{\neg\kappa} = 4. \end{cases} \tag{8}$$

Table 3. Tabulated illustration of the F5-BSE set $(\psi, \mu, \mathcal{K}, 5)$

$(\psi, \mu, \mathcal{K}, 5)$	ω_1	ω_2	ω_3	ω_4	ω_5
$(\flat_1, \varepsilon_1, 1)$	$\langle 4, 0.9\rangle$	$\langle 2, 0.5\rangle$	$\langle 4, 0.8\rangle$	$\langle 1, 0.2\rangle$	$\langle 0, 0\rangle$
$(\flat_1, \varepsilon_2, 1)$	$\langle 1, 0.2\rangle$	$\langle 2, 0.4\rangle$	$\langle 2, 0.5\rangle$	$\langle 3, 0.7\rangle$	$\langle 0, 0.1\rangle$
$(\flat_2, \varepsilon_1, 1)$	$\langle 2, 0.4\rangle$	$\langle 2, 0.5\rangle$	$\langle 3, 0.6\rangle$	$\langle 1, 0.2\rangle$	$\langle 0, 0.1\rangle$
$(\flat_2, \varepsilon_2, 1)$	$\langle 3, 0.7\rangle$	$\langle 2, 0.4\rangle$	$\langle 2, 0.4\rangle$	$\langle 2, 0.4\rangle$	$\langle 0, 0.1\rangle$
$(\flat_1, \varepsilon_1, 0)$	$\langle 0, 0\rangle$	$\langle 2, 0.4\rangle$	$\langle 0, 0\rangle$	$\langle 0, 0\rangle$	$\langle 1, 0.2\rangle$
$(\flat_1, \varepsilon_2, 0)$	$\langle 1, 0.2\rangle$	$\langle 2, 0.5\rangle$	$\langle 2, 0.4\rangle$	$\langle 0, 0\rangle$	$\langle 2, 0.4\rangle$
$(\flat_2, \varepsilon_1, 0)$	$\langle 1, 0.2\rangle$	$\langle 0, 0\rangle$	$\langle 0, 0\rangle$	$\langle 2, 0.4\rangle$	$\langle 2, 0.4\rangle$
$(\flat_2, \varepsilon_2, 0)$	$\langle 1, 0.2\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.3\rangle$	$\langle 1, 0.3\rangle$	$\langle 0, 0\rangle$
$(\neg\flat_1, \varepsilon_1, 1)$	$\langle 0, 0\rangle$	$\langle 2, 0.4\rangle$	$\langle 0, 0\rangle$	$\langle 2, 0.4\rangle$	$\langle 0, 0\rangle$
$(\neg\flat_1, \varepsilon_2, 1)$	$\langle 2, 0.4\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.2\rangle$	$\langle 0, 0\rangle$	$\langle 0, 0\rangle$
$(\neg\flat_2, \varepsilon_1, 1)$	$\langle 0, 0\rangle$	$\langle 1, 0.2\rangle$	$\langle 0, 0\rangle$	$\langle 2, 0.4\rangle$	$\langle 0, 0\rangle$
$(\neg\flat_2, \varepsilon_2, 1)$	$\langle 1, 0.3\rangle$	$\langle 2, 0.5\rangle$	$\langle 0, 0\rangle$	$\langle 2, 0.5\rangle$	$\langle 0, 0\rangle$
$(\neg\flat_1, \varepsilon_1, 0)$	$\langle 2, 0.4\rangle$	$\langle 1, 0.3\rangle$	$\langle 0, 0\rangle$	$\langle 0, 0\rangle$	$\langle 2, 0.5\rangle$
$(\neg\flat_1, \varepsilon_2, 0)$	$\langle 0, 0\rangle$	$\langle 1, 0.2\rangle$	$\langle 1, 0.2\rangle$	$\langle 3, 0.7\rangle$	$\langle 1, 0.2\rangle$
$(\neg\flat_2, \varepsilon_1, 0)$	$\langle 2, 0.4\rangle$	$\langle 0, 0\rangle$	$\langle 3, 0.6\rangle$	$\langle 1, 0.2\rangle$	$\langle 1, 0.2\rangle$
$(\neg\flat_2, \varepsilon_2, 0)$	$\langle 3, 0.6\rangle$	$\langle 1, 0.2\rangle$	$\langle 3, 0.6\rangle$	$\langle 1, 0.2\rangle$	$\langle 4, 0.8\rangle$

Therefore, the F5-BSE set $(\psi,\ \mu,\ \mathcal{K},\ 5)$ can be obtained via the information table (cf. Table 2) with corresponding membership values and represented in tabular form as shown in Table 3.

Notice that expert ε_1 agrees that method ω_1 performs excellently with respect to the attribute of student engagement $\flat_1$, i.e., $g_{(\flat_1,\ \varepsilon_1,\ 1)} = 4$ with the membership value $\psi_{g_{(\flat_1,\ \varepsilon_1,\ 1)}} = 0.9$, quantifying its strong alignment. Conversely, expert ε_1 disagrees that method ω_1 demonstrates poor performance with respect to student engagement $\flat_1$, i.e., $g_{(\flat_1,\ \varepsilon_1,\ 0)} = 0$ with the membership value $\psi_{g_{(\flat_1,\ \varepsilon_1,\ 0)}} = 0$, reflecting strong alignment. On the other hand, expert ε_1 agrees that method ω_1 exhibits poor performance with respect to the corresponding negative attribute, low engagement $\neg\flat_1$, i.e., $g_{(\neg\flat_1,\ \varepsilon_1,\ 1)} = 0$ with the membership value $\mu_{g_{(\neg\flat_1,\ \varepsilon_1,\ 1)}} = 0$, quantifying its strong alignment. However, expert ε_1 disagrees that method ω_1 has moderate performance with respect to the negative attribute of low engagement $\neg\flat_1$, i.e., $g_{(\neg\flat_1,\ \varepsilon_1,\ 1)} = 2$ with the membership value $\mu_{g_{(\neg\flat_1,\ \varepsilon_1,\ 1)}} = 0.4$, quantifying its alignment, and so on.

3.2 *Basic operations*

Here, we introduce the fundamental operations that can be performed on FN-BSE set.

Definition 16 An FN-BSE set $(\psi^{\mathbb{N}},\ \mu^{\mathbb{N}},\ \mathcal{K},\ N)$ is termed a relative null FN-BSE set if, for each $\kappa \in \mathcal{K}$ and $\omega \in W$, $\psi^{\mathbb{N}}(\kappa)(\omega) = \langle 0,\ 0\rangle$, and for each $\neg\kappa \in \neg\mathcal{K}$ and $\omega \in W$, $\mu^{\mathbb{N}}(\neg\kappa)(\omega) = \langle N-1,\ 1\rangle$.

Definition 17 An FN-BSE set $(\psi^{\mathbb{U}},\ \mu^{\mathbb{U}},\ \mathcal{K},\ N)$ is termed a relative whole FN-BSE set if, for every $\kappa \in \mathcal{K}$ and $\omega \in W$, $\psi^{\mathbb{U}}(\kappa)(\omega) = \langle N-1,\ 1\rangle$, and for every $\neg\kappa \in \neg\mathcal{K}$ and $\omega \in W$, $\mu^{\mathbb{U}}(\neg\kappa)(\omega) = \langle 0,\ 0\rangle$.

Definition 18 An FN-BSE set $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)$ is said to be a subset of $(\psi^2,\ \mu^2,\ \mathcal{K}^2,\ N)$, written as $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)$ $\tilde{\subseteq}$ $(\psi^2,\ \mu^2,\ \mathcal{K}^2,\ N)$, if:

1. $\mathcal{K}^1 \subseteq \mathcal{K}^2$.
2. For all $\kappa \in \mathcal{K}^1$ and $\omega \in W$, $g^1_\kappa \leq g^2_\kappa$ and $\psi^1_{g^1_\kappa} \leq \psi^2_{g^2_\kappa}$, where $\langle g^1_\kappa,\ \psi^1_{g^1_\kappa}\rangle = \psi^1(\kappa)(\omega)$ and $\langle g^2_\kappa,\ \psi^2_{g^2_\kappa}\rangle = \psi^2(\kappa)(\omega)$.
3. For all $\neg\kappa \in \neg\mathcal{K}^1$ and $\omega \in W$, $g^2_{\neg\kappa} \leq g^1_{\neg\kappa}$ and $\mu^2_{g^2_{\neg\kappa}} \leq \mu^1_{g^1_{\neg\kappa}}$, where $\langle g^1_{\neg\kappa},\ \mu^1_{g^1_{\neg\kappa}}\rangle = \mu^1(\neg\kappa)(\omega)$ and $\langle g^2_{\neg\kappa},\ \mu^2_{g^2_{\neg\kappa}}\rangle = \mu^2(\neg\kappa)(\omega)$.

Definition 19 Two FN-BSE sets $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)$ and $(\psi^2,\ \mu^2,\ \mathcal{K}^2,\ N)$ are considered equal if both $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)$ $\tilde{\subseteq}$ $(\psi^2,\ \mu^2,\ \mathcal{K}^2,\ N)$ and $(\psi^2,\ \mu^2,\ \mathcal{K}^2,\ N)$ $\tilde{\subseteq}$ $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)$ hold true.

Definition 20 Given an FN-BSE set $(\psi,\ \mu,\ \mathcal{K},\ N)$, the positive agree FN-BSE set, symbolized by $(\psi,\ \mu,\ \mathcal{K},\ N)^{\oplus 1}$, is an FN-BSE subset of $(\psi,\ \mu,\ \mathcal{K},\ N)$ defined by:

$$(\psi,\ \mu,\ \mathcal{K},\ N)^{\oplus 1} = \{\psi^{\oplus 1}(\kappa)(\omega) \mid \kappa \in B \times E \times \{1\}\}. \tag{9}$$

Definition 21 Given an FN-BSE set $(\psi,\ \mu,\ \mathcal{K},\ N)$, the positive disagree FN-BSE set, symbolized by $(\psi,\ \mu,\ \mathcal{K},\ N)^{\oplus 0}$, is an FN-BSE subset of $(\psi,\ \mu,\ \mathcal{K},\ N)$ defined by:

$$(\psi,\ \mu,\ \mathcal{K},\ N)^{\oplus 0} = \{\psi^{\oplus 0}(\kappa)(\omega) \mid \kappa \in B \times E \times \{0\}\}. \tag{10}$$

Definition 22 Given an FN-BSE set $(\psi,\ \mu,\ \mathcal{K},\ N)$, the negative agree FN-BSE set, symbolized by $(\psi,\ \mu,\ \mathcal{K},\ N)^{\ominus 1}$, is an FN-BSE subset of $(\psi,\ \mu,\ \mathcal{K},\ N)$ defined by:

$$(\psi,\ \mu,\ \mathcal{K},\ N)^{\ominus 1} = \{\mu^{\ominus 1}(\neg\kappa)(\omega) \mid \neg\kappa \in B \times E \times \{1\}\}. \tag{11}$$

Definition 23 Given an FN-BSE set $(\psi,\ \mu,\ \mathcal{K},\ N)$, the negative disagree FN-BSE set, symbolized by $(\psi,\ \mu,\ \mathcal{K},\ N)^{\ominus 0}$, is an FN-BSE subset of $(\psi,\ \mu,\ \mathcal{K},\ N)$ defined by:

$$(\psi,\ \mu,\ \mathcal{K},\ N)^{\ominus 0} = \{\mu^{\ominus 0}(\neg\kappa)(\omega) \mid \neg\kappa \in B \times E \times \{0\}\}. \tag{12}$$

Example 2 Consider the F5-BSE set $(\psi,\ \mu,\ \mathcal{K},\ 5)$ presented in Table 3 of Example 1. The corresponding positive agree, positive disagree, negative agree, and negative disagree F5-BSE subsets are displayed in Tables 4-7.

Table 4. Positive agree F5-BSE set $(\psi,\ \mu,\ \mathcal{K},\ 5)^{\oplus 1}$ obtained via the F5-BSE set $(\psi,\ \mu,\ \mathcal{K},\ 5)$ in Example 2

$(\psi,\ \mu,\ \mathcal{K},\ 5)^{\oplus 1}$	ω_1	ω_2	ω_3	ω_4	ω_5
$(\flat_1,\ \varepsilon_1,\ 1)$	$\langle 4,\ 0.9\rangle$	$\langle 2,\ 0.5\rangle$	$\langle 4,\ 0.8\rangle$	$\langle 1,\ 0.2\rangle$	$\langle 0,\ 0\rangle$
$(\flat_1,\ \varepsilon_2,\ 1)$	$\langle 1,\ 0.2\rangle$	$\langle 2,\ 0.4\rangle$	$\langle 2,\ 0.5\rangle$	$\langle 3,\ 0.7\rangle$	$\langle 0,\ 0.1\rangle$
$(\flat_2,\ \varepsilon_1,\ 1)$	$\langle 2,\ 0.4\rangle$	$\langle 2,\ 0.5\rangle$	$\langle 3,\ 0.6\rangle$	$\langle 1,\ 0.2\rangle$	$\langle 0,\ 0.1\rangle$
$(\flat_2,\ \varepsilon_2,\ 1)$	$\langle 3,\ 0.7\rangle$	$\langle 2,\ 0.4\rangle$	$\langle 2,\ 0.4\rangle$	$\langle 2,\ 0.4\rangle$	$\langle 0,\ 0.1\rangle$

Table 5. Positive disagree F5-BSE set $(\psi,\ \mu,\ \mathcal{K},\ 5)^{\oplus 0}$ obtained via the F5-BSE set $(\psi,\ \mu,\ \mathcal{K},\ 5)$ in Example 2

$(\psi,\ \mu,\ \mathcal{K},\ 5)^{\oplus 0}$	ω_1	ω_2	ω_3	ω_4	ω_5
$(\flat_1,\ \varepsilon_1,\ 0)$	$\langle 0,\ 0\rangle$	$\langle 2,\ 0.4\rangle$	$\langle 0,\ 0\rangle$	$\langle 0,\ 0\rangle$	$\langle 1,\ 0.2\rangle$
$(\flat_1,\ \varepsilon_2,\ 0)$	$\langle 1,\ 0.2\rangle$	$\langle 2,\ 0.5\rangle$	$\langle 2,\ 0.4\rangle$	$\langle 0,\ 0\rangle$	$\langle 2,\ 0.4\rangle$
$(\flat_2,\ \varepsilon_1,\ 0)$	$\langle 1,\ 0.2\rangle$	$\langle 0,\ 0\rangle$	$\langle 0,\ 0\rangle$	$\langle 2,\ 0.4\rangle$	$\langle 2,\ 0.4\rangle$
$(\flat_2,\ \varepsilon_2,\ 0)$	$\langle 1,\ 0.2\rangle$	$\langle 2,\ 0.4\rangle$	$\langle 1,\ 0.3\rangle$	$\langle 1,\ 0.3\rangle$	$\langle 0,\ 0\rangle$

Table 6. Negative agree F5-BSE set $(\psi,\ \mu,\ \mathcal{K},\ 5)^{\ominus 1}$ obtained via the F5-BSE set $(\psi,\ \mu,\ \mathcal{K},\ 5)$ in Example 2

$(\psi,\ \mu,\ \mathcal{K},\ 5)^{\ominus 1}$	ω_1	ω_2	ω_3	ω_4	ω_5
$(\neg\flat_1,\ \varepsilon_1,\ 1)$	$\langle 0,\ 0\rangle$	$\langle 2,\ 0.4\rangle$	$\langle 0,\ 0\rangle$	$\langle 2,\ 0.4\rangle$	$\langle 0,\ 0\rangle$
$(\neg\flat_1,\ \varepsilon_2,\ 1)$	$\langle 2,\ 0.4\rangle$	$\langle 2,\ 0.4\rangle$	$\langle 1,\ 0.2\rangle$	$\langle 0,\ 0\rangle$	$\langle 0,\ 0\rangle$
$(\neg\flat_2,\ \varepsilon_1,\ 1)$	$\langle 0,\ 0\rangle$	$\langle 1,\ 0.2\rangle$	$\langle 0,\ 0\rangle$	$\langle 2,\ 0.4\rangle$	$\langle 0,\ 0\rangle$
$(\neg\flat_2,\ \varepsilon_2,\ 1)$	$\langle 1,\ 0.3\rangle$	$\langle 2,\ 0.5\rangle$	$\langle 0,\ 0\rangle$	$\langle 2,\ 0.5\rangle$	$\langle 0,\ 0\rangle$

Table 7. Negative disagree F5-BSE set $(\psi,\ \mu,\ \mathcal{K},\ 5)^{\ominus 0}$ obtained via the F5-BSE set $(\psi,\ \mu,\ \mathcal{K},\ 5)$ in Example 2

$(\psi,\ \mu,\ \mathcal{K},\ 5)^{\ominus 0}$	ω_1	ω_2	ω_3	ω_4	ω_5
$(\neg\flat_1,\ \varepsilon_1,\ 0)$	$\langle 2,\ 0.4\rangle$	$\langle 1,\ 0.3\rangle$	$\langle 0,\ 0\rangle$	$\langle 0,\ 0\rangle$	$\langle 2,\ 0.5\rangle$
$(\neg\flat_1,\ \varepsilon_2,\ 0)$	$\langle 0,\ 0\rangle$	$\langle 1,\ 0.2\rangle$	$\langle 1,\ 0.2\rangle$	$\langle 3,\ 0.7\rangle$	$\langle 1,\ 0.2\rangle$
$(\neg\flat_2,\ \varepsilon_1,\ 0)$	$\langle 2,\ 0.4\rangle$	$\langle 0,\ 0\rangle$	$\langle 3,\ 0.6\rangle$	$\langle 1,\ 0.2\rangle$	$\langle 1,\ 0.2\rangle$
$(\neg\flat_2,\ \varepsilon_2,\ 0)$	$\langle 3,\ 0.6\rangle$	$\langle 1,\ 0.2\rangle$	$\langle 3,\ 0.6\rangle$	$\langle 1,\ 0.2\rangle$	$\langle 4,\ 0.8\rangle$

Definition 24 The FN-BSE complement of $(\psi,\ \mu,\ \mathcal{K},\ N)$, denoted as $(\psi,\ \mu,\ \mathcal{K},\ N)^{\tilde{c}}$, is defined by $(\psi,\ \mu,\ \mathcal{K},\ N)^{\tilde{c}} = (\psi^{\tilde{c}},\ \mu^{\tilde{c}},\ \mathcal{K},\ N)$, such that for every $\kappa \in \mathcal{K}$ and $\omega \in W$, $g_{\kappa}^{\tilde{c}} = g_{\neg\kappa}$ and $\psi^{\tilde{c}}_{g_{\kappa}^{\tilde{c}}} = \mu_{g_{\neg\kappa}}$, and for each $\neg\kappa \in \neg\mathcal{K}$ and $\omega \in W$, $g_{\neg\kappa}^{\tilde{c}} = g_{\kappa}$ and $\mu^{\tilde{c}}_{g_{\neg\kappa}^{\tilde{c}}} = \psi_{g_{\kappa}}$, where $\langle g_{\kappa}^{\tilde{c}},\ \psi^{\tilde{c}}_{g_{\kappa}^{\tilde{c}}}\rangle = \psi^{\tilde{c}}(\kappa)(\omega)$ and $\langle g_{\neg\kappa}^{\tilde{c}},\ \mu^{\tilde{c}}_{g_{\neg\kappa}^{\tilde{c}}}\rangle = \mu^{\tilde{c}}(\neg\kappa)(\omega)$.

Definition 25 The FN-BSE extended union of $(\psi^1, \mu^1, \mathcal{K}^1, N^1)$ and $(\psi^2, \mu^2, \mathcal{K}^2, N^2)$ is defined by $(\psi^1, \mu^1, \mathcal{K}^1, N^1)$ $\tilde{\cup}_\varepsilon$ $(\psi^2, \mu^2, \mathcal{K}^2, N^2) = (\psi, \mu, \mathcal{K}^1 \cup \mathcal{K}^2, \max(N^1, N^2))$, where for all $\kappa \in \mathcal{K}^1 \cup \mathcal{K}^2$ and $\omega \in W$:

$$\psi(\kappa)(\omega) = \begin{cases} \psi^1(\kappa)(\omega), & \text{if } \kappa \in \mathcal{K}^1 \setminus \mathcal{K}^2 \\ \psi^2(\kappa)(\omega), & \text{if } \kappa \in \mathcal{K}^2 \setminus \mathcal{K}^1 \\ \langle \max\{g^1_\kappa, g^2_\kappa\}, \max\{\psi^1_{g^1_\kappa}, \psi^2_{g^2_\kappa}\}\rangle, & \text{if } \kappa \in \mathcal{K}^1 \cap \mathcal{K}^2 \end{cases} \tag{13}$$

where $\langle g^1_\kappa, \psi^1_{g^1_\kappa}\rangle = \psi^1(\kappa)(\omega)$ and $\langle g^2_\kappa, \psi^2_{g^2_\kappa}\rangle = \psi^2(\kappa)(\omega)$, and for all $\neg\kappa \in \neg\mathcal{K}^1 \cup \neg\mathcal{K}^2$ and $\omega \in W$:

$$\mu(\neg\kappa)(\omega) = \begin{cases} \mu^1(\neg\kappa)(\omega), & \text{if } \neg\kappa \in \neg\mathcal{K}^1 \setminus \neg\mathcal{K}^2 \\ \mu^2(\neg\kappa)(\omega), & \text{if } \neg\kappa \in \neg\mathcal{K}^2 \setminus \neg\mathcal{K}^1 \\ \langle \min\{g^1_{\neg\kappa}, g^2_{\neg\kappa}\}, \min\{\mu^1_{g^1_{\neg\kappa}}, \mu^2_{g^2_{\neg\kappa}}\}\rangle, & \text{if } \neg\kappa \in \neg\mathcal{K}^1 \cap \neg\mathcal{K}^2 \end{cases} \tag{14}$$

where $\langle g^1_{\neg\kappa}, \mu^1_{g^1_{\neg\kappa}}\rangle = \mu^1(\neg\kappa)(\omega)$ and $\langle g^2_{\neg\kappa}, \mu^2_{g^2_{\neg\kappa}}\rangle = \mu^2(\neg\kappa)(\omega)$.

Definition 26 The FN-BSE extended intersecion of $(\psi^1, \mu^1, \mathcal{K}^1, N^1)$ and $(\psi^2, \mu^2, \mathcal{K}^2, N^2)$ is defined by $(\psi^1, \mu^1, \mathcal{K}^1, N^1)$ $\tilde{\cap}_\varepsilon$ $(\psi^2, \mu^2, \mathcal{K}^2, N^2) = (\psi, \mu, \mathcal{K}^1 \cup \mathcal{K}^2, \max(N^1, N^2))$, where for all $\kappa \in \mathcal{K}^1 \cup \mathcal{K}^2$ and $\omega \in W$:

$$\psi(\kappa)(\omega) = \begin{cases} \psi^1(\kappa)(\omega), & \text{if } \kappa \in \mathcal{K}^1 \setminus \mathcal{K}^2 \\ \psi^2(\kappa)(\omega), & \text{if } \kappa \in \mathcal{K}^2 \setminus \mathcal{K}^1 \\ \langle \min\{g^1_\kappa, g^2_\kappa\}, \min\{\psi^1_{g^1_\kappa}, \psi^2_{g^2_\kappa}\}\rangle, & \text{if } \kappa \in \mathcal{K}^1 \cap \mathcal{K}^2 \end{cases} \tag{15}$$

where $\langle g^1_\kappa, \psi^1_{g^1_\kappa}\rangle = \psi^1(\kappa)(\omega)$ and $\langle g^2_\kappa, \psi^2_{g^2_\kappa}\rangle = \psi^2(\kappa)(\omega)$, and for all $\neg\kappa \in \neg\mathcal{K}^1 \cup \neg\mathcal{K}^2$ and $\omega \in W$:

$$\mu(\neg\kappa)(\omega) = \begin{cases} \mu^1(\neg\kappa)(\omega), & \text{if } \neg\kappa \in \neg\mathcal{K}^1 \setminus \neg\mathcal{K}^2 \\ \mu^2(\neg\kappa)(\omega), & \text{if } \neg\kappa \in \neg\mathcal{K}^2 \setminus \neg\mathcal{K}^1 \\ \langle \max\{g^1_{\neg\kappa}, g^2_{\neg\kappa}\}, \max\{\mu^1_{g^1_{\neg\kappa}}, \mu^2_{g^2_{\neg\kappa}}\}\rangle, & \text{if } \neg\kappa \in \neg\mathcal{K}^1 \cap \neg\mathcal{K}^2 \end{cases} \tag{16}$$

where $\langle g^1_{\neg\kappa}, \mu^1_{g^1_{\neg\kappa}}\rangle = \mu^1(\neg\kappa)(\omega)$ and $\langle g^2_{\neg\kappa}, \mu^2_{g^2_{\neg\kappa}}\rangle = \mu^2(\neg\kappa)(\omega)$.

Definition 27 The FN-BSE restricted union of $(\psi^1, \mu^1, \mathcal{K}^1, N^1)$ and $(\psi^2, \mu^2, \mathcal{K}^2, N^2)$ is defined by $(\psi^1, \mu^1, \mathcal{K}^1, N^1)$ $\tilde{\cup}_r$ $(\psi^2, \mu^2, \mathcal{K}^2, N^2) = (\psi, \mu, \mathcal{K}^1 \cap \mathcal{K}^2, \max(N^1, N^2))$, where for all $\kappa \in \mathcal{K}^1 \cap \mathcal{K}^2 \neq \emptyset$ and $\omega \in W$:

$$\psi(\kappa)(\omega) = \langle \max\{g^1_\kappa, g^2_\kappa\}, \max\{\psi^1_{g^1_\kappa}, \psi^2_{g^2_\kappa}\}\rangle, \tag{17}$$

where $\langle g^1_\kappa,\ \psi^1_{g^1_\kappa}\rangle = \psi^1(\kappa)(\omega)$ and $\langle g^2_\kappa,\ \psi^2_{g^2_\kappa}\rangle = \psi^2(\kappa)(\omega)$, and for all $\neg\kappa \in \neg\mathcal{K}^1 \cap \neg\mathcal{K}^2 \neq \emptyset$ and $\omega \in W$:

$$\mu(\neg\kappa)(\omega) = \langle \min\{g^1_{\neg\kappa},\ g^2_{\neg\kappa}\},\ \min\{\mu^1_{g^1_{\neg\kappa}},\ \mu^2_{g^2_{\neg\kappa}}\}\rangle, \tag{18}$$

where $\langle g^1_{\neg\kappa},\ \mu^1_{g^1_{\neg\kappa}}\rangle = \mu^1(\neg\kappa)(\omega)$ and $\langle g^2_{\neg\kappa},\ \mu^2_{g^2_{\neg\kappa}}\rangle = \mu^2(\neg\kappa)(\omega)$.

Definition 28 The FN-BSE restricted intersection of $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N^1)$ and $(\psi^2,\ \mu^2,\ \mathcal{K}^2,\ N^2)$ is defined by $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N^1)\ \tilde{\cap}_r\ (\psi^2,\ \mu^2,\ \mathcal{K}^2,\ N^2) = (\psi,\ \mu,\ \mathcal{K}^1 \cap \mathcal{K}^2,\ \max(N^1,\ N^2))$, where for all $\kappa \in \mathcal{K}^1 \cap \mathcal{K}^2 \neq \emptyset$ and $\omega \in W$:

$$\psi(\kappa)(\omega) = \langle \min\{g^1_\kappa,\ g^2_\kappa\},\ \min\{\psi^1_{g^1_\kappa},\ \psi^2_{g^2_\kappa}\}\rangle, \tag{19}$$

where $\langle g^1_\kappa,\ \psi^1_{g^1_\kappa}\rangle = \psi^1(\kappa)(\omega)$ and $\langle g^2_\kappa,\ \psi^2_{g^2_\kappa}\rangle = \psi^2(\kappa)(\omega)$, and for all $\neg\kappa \in \neg\mathcal{K}^1 \cap \neg\mathcal{K}^2 \neq \emptyset$ and $\omega \in W$:

$$\mu(\neg\kappa)(\omega) = \langle \max\{g^1_{\neg\kappa},\ g^2_{\neg\kappa}\},\ \max\{\mu^1_{g^1_{\neg\kappa}},\ \mu^2_{g^2_{\neg\kappa}}\}\rangle, \tag{20}$$

where $\langle g^1_{\neg\kappa},\ \mu^1_{g^1_{\neg\kappa}}\rangle = \mu^1(\neg\kappa)(\omega)$ and $\langle g^2_{\neg\kappa},\ \mu^2_{g^2_{\neg\kappa}}\rangle = \mu^2(\neg\kappa)(\omega)$.

Definition 29 The OR-operation between two FN-BSE sets $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)$ and $(\psi^2,\ \mu^2,\ \mathcal{K}^2,\ N)$ is defined by $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\vee}\ (\psi^2,\ \mu^2,\ \mathcal{K}^2,\ N) = (\psi,\ \mu,\ \mathcal{K}^1 \times \mathcal{K}^2,\ N)$, where, for all $(\kappa^1,\ \kappa^2) \in \mathcal{K}^1 \times \mathcal{K}^2$, $\kappa^1 \in \mathcal{K}^1$, $\kappa^2 \in \mathcal{K}^2$, and $\omega \in W$:

$$\psi((\kappa^1,\ \kappa^2))(\omega) = \langle \max\{g^1_{\kappa^1},\ g^2_{\kappa^2}\},\ \max\{\psi^1_{g^1_{\kappa^1}},\ \psi^2_{g^2_{\kappa^2}}\}\rangle, \tag{21}$$

where $\langle g^1_{\kappa^1},\ \psi^1_{g^1_{\kappa^1}}\rangle = \psi^1(\kappa^1)(\omega)$ and $\langle g^2_{\kappa^2},\ \psi^2_{g^2_{\kappa^2}}\rangle = \psi^2(\kappa^2)(\omega)$, and for all $(\neg\kappa^1,\ \neg\kappa^2) \in \neg\mathcal{K}^1 \times \neg\mathcal{K}^2$, $\neg\kappa^1 \in \neg\mathcal{K}^1$, $\neg\kappa^2 \in \neg\mathcal{K}^2$, and $\omega \in W$:

$$\mu((\neg\kappa^1,\ \neg\kappa^2))(\omega) = \langle \min\{g^1_{\neg\kappa^1},\ g^2_{\neg\kappa^2}\},\ \min\{\mu^1_{g^1_{\neg\kappa^1}},\ \mu^2_{g^2_{\neg\kappa^2}}\}\rangle, \tag{22}$$

where $\langle g^1_{\neg\kappa^1},\ \mu^1_{g^1_{\neg\kappa^1}}\rangle = \mu^1(\neg\kappa^1)(\omega)$ and $\langle g^2_{\neg\kappa^2},\ \mu^2_{g^2_{\neg\kappa^2}}\rangle = \mu^2(\neg\kappa^2)(\omega)$.

Definition 30 The AND-operation between two FN-BSE sets $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)$ and $(\psi^2,\ \mu^2,\ \mathcal{K}^2,\ N)$ is defined by $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\wedge}\ (\psi^2,\ \mu^2,\ \mathcal{K}^2,\ N) = (\psi,\ \mu,\ \mathcal{K}^1 \times \mathcal{K}^2,\ N)$, where, for all $(\kappa^1,\ \kappa^2) \in \mathcal{K}^1 \times \mathcal{K}^2$, $\kappa^1 \in \mathcal{K}^1$, $\kappa^2 \in \mathcal{K}^2$, and $\omega \in W$:

$$\psi((\kappa^1,\ \kappa^2))(\omega) = \langle \min\{g^1_{\kappa^1},\ g^2_{\kappa^2}\},\ \min\{\psi^1_{g^1_{\kappa^1}},\ \psi^2_{g^2_{\kappa^2}}\}\rangle, \tag{23}$$

where $\langle g^1_{\kappa^1},\ \psi^1_{g^1_{\kappa^1}}\rangle = \psi^1(\kappa^1)(\omega)$ and $\langle g^2_{\kappa^2},\ \psi^2_{g^2_{\kappa^2}}\rangle = \psi^2(\kappa^2)(\omega)$, and for all $(\neg\kappa^1,\ \neg\kappa^2) \in \neg\mathcal{K}^1 \times \neg\mathcal{K}^2$, $\neg\kappa^1 \in \neg\mathcal{K}^1$, $\neg\kappa^2 \in \neg\mathcal{K}^2$, and $\omega \in W$:

$$\mu((\neg\kappa^1, \neg\kappa^2))(\omega) = \langle \max\{g^1_{\neg\kappa^1}, g^2_{\neg\kappa^2}\}, \max\{\mu^1_{g^1_{\neg\kappa^1}}, \mu^2_{g^2_{\neg\kappa^2}}\}\rangle, \tag{24}$$

where $\langle g^1_{\neg\kappa^1}, \mu^1_{g^1_{\neg\kappa^1}}\rangle = \mu^1(\neg\kappa^1)(\omega)$ and $\langle g^2_{\neg\kappa^2}, \mu^2_{g^2_{\neg\kappa^2}}\rangle = \mu^2(\neg\kappa^2)(\omega)$.

3.3 *Algebraic properties*

This subsection examines the essential algebraic properties of FN-BSE set.

Proposition 1 Let $(\psi^1, \mu^1, \mathcal{K}, N)$, $(\psi^2, \mu^2, \mathcal{K}, N)$, and $(\psi^3, \mu^3, \mathcal{K}, N)$ be three FN-BSE sets. Then

1. $(\psi^{\mathbb{N}}, \mu^{\mathbb{N}}, \mathcal{K}, N) \tilde{\subseteq} (\psi^1, \mu^1, \mathcal{K}, N)$.
2. $(\psi^1, \mu^1, \mathcal{K}, N) \tilde{\subseteq} (\psi^{\mathbb{U}}, \mu^{\mathbb{U}}, \mathcal{K}, N)$.
3. If $(\psi^1, \mu^1, \mathcal{K}, N) \tilde{\subseteq} (\psi^2, \mu^2, \mathcal{K}, N)$ and $(\psi^2, \mu^2, \mathcal{K}, N) \tilde{\subseteq} (\psi^3, \mu^3, \mathcal{K}, N)$, then $(\psi^1, \mu^1, \mathcal{K}, N) \tilde{\subseteq} (\psi^3, \mu^3, \mathcal{K}, N)$.

Proof. The proofs are apparent. □

Proposition 1 Let $(\psi^1, \mu^1, \mathcal{K}^1, N)$ and $(\psi^2, \mu^2, \mathcal{K}^2, N)$ be two FN-BSE sets. Then

1. $(\psi^1, \mu^1, \mathcal{K}^1, N) \tilde{\cup}_{\varepsilon} (\psi^2, \mu^2, \mathcal{K}^2, N)$ is the smallest FN-BSE set which contains both $(\psi^1, \mu^1, \mathcal{K}^1, N)$ and $(\psi^2, \mu^2, \mathcal{K}^2, N)$.
2. $(\psi^1, \mu^1, \mathcal{K}^1, N) \tilde{\cap}_r (\psi^2, \mu^2, \mathcal{K}^2, N)$ is the largest FN-BSE set which is contained in both $(\psi^1, \mu^1, \mathcal{K}^1, N)$ and $(\psi^2, \mu^2, \mathcal{K}^2, N)$.

Proof. The proofs are apparent. □

Proposition 3 Let $(\psi^1, \mu^1, \mathcal{K}, N)$ and $(\psi^2, \mu^2, \mathcal{K}, N)$ be two FN-BSE sets. Then

1. $(\psi^{\mathbb{N}}, \mu^{\mathbb{N}}, \mathcal{K}, N)^{\tilde{c}} = (\psi^{\mathbb{U}}, \mu^{\mathbb{U}}, \mathcal{K}, N)$.
2. $(\psi^{\mathbb{U}}, \mu^{\mathbb{U}}, \mathcal{K}, N)^{\tilde{c}} = (\psi^{\mathbb{N}}, \mu^{\mathbb{N}}, \mathcal{K}, N)$.
3. $((\psi^1, \mu^1, \mathcal{K}, N)^{\tilde{c}})^{\tilde{c}} = (\psi^1, \mu^1, \mathcal{K}, N)$.
4. If $(\psi^1, \mu^1, \mathcal{K}, N) \tilde{\subseteq} (\psi^2, \mu^2, \mathcal{K}, N)$, then $(\psi^2, \mu^2, \mathcal{K}, N)^{\tilde{c}} \tilde{\subseteq} (\psi^1, \mu^1, \mathcal{K}, N)^{\tilde{c}}$.
5. $(\psi^{\mathbb{N}}, \mu^{\mathbb{N}}, \mathcal{K}, N) \tilde{\subseteq} (\psi^1, \mu^1, \mathcal{K}, N) \tilde{\cap}_r (\psi^1, \mu^1, \mathcal{K}, N)^{\tilde{c}} \tilde{\subseteq} (\psi^1, \mu^1, \mathcal{K}, N) \tilde{\cup}_r (\psi^1, \mu^1, \mathcal{K}, N)^{\tilde{c}} \tilde{\subseteq} (\psi^{\mathbb{U}}, \mu^{\mathbb{U}}, \mathcal{K}, N)$.
6. If $(\psi^1, \mu^1, \mathcal{K}, N) \tilde{\subseteq} (\psi^2, \mu^2, \mathcal{K}, N)$, then $(\psi^1, \mu^1, \mathcal{K}, N) \tilde{\cap}_r (\psi^2, \mu^2, \mathcal{K}, N) = (\psi^1, \mu^1, \mathcal{K}, N)$.
7. If $(\psi^1, \mu^1, \mathcal{K}, N) \tilde{\subseteq} (\psi^2, \mu^2, \mathcal{K}, N)$, then $(\psi^1, \mu^1, \mathcal{K}, N) \tilde{\cup}_r (\psi^2, \mu^2, \mathcal{K}, N) = (\psi^2, \mu^2, \mathcal{K}, N)$.

Proof. The proofs are apparent. □

Proposition 4 Let $(\psi^1, \mu^1, \mathcal{K}^1, N)$ and $(\psi^2, \mu^2, \mathcal{K}^2, N)$ be two FN-BSE sets. Then

1. $\left((\psi^1, \mu^1, \mathcal{K}^1, N) \tilde{\cup}_{\varepsilon} (\psi^2, \mu^2, \mathcal{K}^2, N)\right)^{\tilde{c}} = (\psi^1, \mu^1, \mathcal{K}^1, N)^{\tilde{c}} \tilde{\cap}_{\varepsilon} (\psi^2, \mu^2, \mathcal{K}^2, N)^{\tilde{c}}$.
2. $\left((\psi^1, \mu^1, \mathcal{K}^1, N) \tilde{\cap}_{\varepsilon} (\psi^2, \mu^2, \mathcal{K}^2, N)\right)^{\tilde{c}} = (\psi^1, \mu^1, \mathcal{K}^1, N)^{\tilde{c}} \tilde{\cup}_{\varepsilon} (\psi^2, \mu^2, \mathcal{K}^2, N)^{\tilde{c}}$.
3. $\left((\psi^1, \mu^1, \mathcal{K}^1, N) \tilde{\cup}_r (\psi^2, \mu^2, \mathcal{K}^2, N)\right)^{\tilde{c}} = (\psi^1, \mu^1, \mathcal{K}^1, N)^{\tilde{c}} \tilde{\cap}_r (\psi^2, \mu^2, \mathcal{K}^2, N)^{\tilde{c}}$.
4. $\left((\psi^1, \mu^1, \mathcal{K}^1, N) \tilde{\cap}_r (\psi^2, \mu^2, \mathcal{K}^2, N)\right)^{\tilde{c}} = (\psi^1, \mu^1, \mathcal{K}^1, N)^{\tilde{c}} \tilde{\cup}_r (\psi^2, \mu^2, \mathcal{K}^2, N)^{\tilde{c}}$.
5. $\left((\psi^1, \mu^1, \mathcal{K}^1, N) \tilde{\vee} (\psi^2, \mu^2, \mathcal{K}^2, N)\right)^{\tilde{c}} = (\psi^1, \mu^1, \mathcal{K}^1, N)^{\tilde{c}} \tilde{\wedge} (\psi^2, \mu^2, \mathcal{K}^2, N)^{\tilde{c}}$.
6. $\left((\psi^1, \mu^1, \mathcal{K}^1, N) \tilde{\wedge} (\psi^2, \mu^2, \mathcal{K}^2, N)\right)^{\tilde{c}} = (\psi^1, \mu^1, \mathcal{K}^1, N)^{\tilde{c}} \tilde{\vee} (\psi^2, \mu^2, \mathcal{K}^2, N)^{\tilde{c}}$.

Proof. (1) Let $(\psi^1, \mu^1, \mathcal{K}^1, N) \tilde{\cup}_{\varepsilon} (\psi^2, \mu^2, \mathcal{K}^2, N) = (\psi^3, \mu^3, \mathcal{K}^1 \cup \mathcal{K}^2, N)$. Then, $\left((\psi^1, \mu^1, \mathcal{K}^1, N) \tilde{\cup}_{\varepsilon} (\psi^2, \mu^2, \mathcal{K}^2, N)\right)^{\tilde{c}} = (\psi^3, \mu^3, \mathcal{K}^1 \cup \mathcal{K}^2, N)^{\tilde{c}} = (\psi^{3\tilde{c}}, \mu^{3\tilde{c}}, \mathcal{K}^1 \cup \mathcal{K}^2, N)$. For all $\kappa \in \mathcal{K}^1 \cup \mathcal{K}^2$ and $\omega \in W$:

$$\psi^3(\kappa)(\omega) = \begin{cases} \psi^1(\kappa)(\omega), & \text{if } \kappa \in \mathcal{K}^1 \setminus \mathcal{K}^2 \\ \psi^2(\kappa)(\omega), & \text{if } \kappa \in \mathcal{K}^2 \setminus \mathcal{K}^1 \\ \langle \max\{g^1_\kappa, g^2_\kappa\}, \max\{\psi^1_{g^1_\kappa}, \psi^2_{g^2_\kappa}\}\rangle, & \text{if } \kappa \in \mathcal{K}^1 \cap \mathcal{K}^2 \end{cases} \tag{25}$$

where $\langle g^1_\kappa,\ \psi^1_{g^1_\kappa}\rangle = \psi^1(\kappa)(\omega)$ and $\langle g^2_\kappa,\ \psi^2_{g^2_\kappa}\rangle = \psi^2(\kappa)(\omega)$, and for all $\neg\kappa \in \neg\mathcal{K}^1 \cup \neg\mathcal{K}^2$ and $\omega \in W$:

$$\mu^3(\neg\kappa)(\omega) = \begin{cases} \mu^1(\neg\kappa)(\omega), & \text{if } \neg\kappa \in \neg\mathcal{K}^1 \setminus \neg\mathcal{K}^2 \\ \mu^2(\neg\kappa)(\omega), & \text{if } \neg\kappa \in \neg\mathcal{K}^2 \setminus \neg\mathcal{K}^1 \\ \langle \min\{g^1_{\neg\kappa},\ g^2_{\neg\kappa}\},\ \min\{\mu^1_{g^1_{\neg\kappa}},\ \mu^2_{g^2_{\neg\kappa}}\}\rangle, & \text{if } \neg\kappa \in \neg\mathcal{K}^1 \cap \neg\mathcal{K}^2, \end{cases} \tag{26}$$

where $\langle g^1_{\neg\kappa},\ \mu^1_{g^1_{\neg\kappa}}\rangle = \mu^1(\neg\kappa)(\omega)$ and $\langle g^2_{\neg\kappa},\ \mu^2_{g^2_{\neg\kappa}}\rangle = \mu^2(\neg\kappa)(\omega)$.

Then, for all $\kappa \in \mathcal{K}^1 \cup \mathcal{K}^2$ and $\omega \in W$:

$$\psi^{3^{\tilde{c}}}(\kappa)(\omega) = \mu^3(\neg\kappa)(\omega) = \begin{cases} \mu^1(\neg\kappa)(\omega), & \text{if } \kappa \in \mathcal{K}^1 \setminus \mathcal{K}^2 \\ \mu^2(\neg\kappa)(\omega), & \text{if } \kappa \in \mathcal{K}^2 \setminus \mathcal{K}^1 \\ \langle \min\{g^1_{\neg\kappa},\ g^2_{\neg\kappa}\},\ \min\{\mu^1_{g^1_{\neg\kappa}},\ \mu^2_{g^2_{\neg\kappa}}\}\rangle, & \text{if } \kappa \in \mathcal{K}^1 \cap \mathcal{K}^2 \end{cases} \tag{27}$$

and for all $\neg\kappa \in \neg\mathcal{K}^1 \cup \neg\mathcal{K}^2$ and $\omega \in W$

$$\mu^{3^{\tilde{c}}}(\neg\kappa)(\omega) = \psi^3(\kappa)(\omega) = \begin{cases} \psi^1(\kappa)(\omega), & \text{if } \neg\kappa \in \neg\mathcal{K}^1 \setminus \neg\mathcal{K}^2 \\ \psi^2(\kappa)(\omega), & \text{if } \neg\kappa \in \neg\mathcal{K}^2 \setminus \neg\mathcal{K}^1 \\ \langle \max\{g^1_\kappa,\ g^2_\kappa\},\ \max\{\psi^1_{g^1_\kappa},\ \psi^2_{g^2_\kappa}\}\rangle, & \text{if } \neg\kappa \in \neg\mathcal{K}^1 \cap \neg\mathcal{K}^2. \end{cases} \tag{28}$$

On the other hand, let $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)^{\tilde{c}}\ \tilde{\cap}_\varepsilon\ (\psi^2,\ \mu^2,\ \mathcal{K}^2,\ N)^{\tilde{c}} = (\psi^4,\ \mu^4,\ \mathcal{K}^1 \cup \mathcal{K}^2,\ N)$. For all $\kappa \in \mathcal{K}^1 \cup \mathcal{K}^2$ and $\omega \in W$:

$$\psi^4(\kappa)(\omega) = \begin{cases} \psi^{1^{\tilde{c}}}(\kappa)(\omega) = \mu^1(\neg\kappa)(\omega), & \text{if } \kappa \in \mathcal{K}^1 \setminus \mathcal{K}^2, \\ \psi^{2^{\tilde{c}}}(\kappa)(\omega) = \mu^2(\neg\kappa)(\omega), & \text{if } \kappa \in \mathcal{K}^2 \setminus \mathcal{K}^1, \\ \langle \min\{g^{1^{\tilde{c}}}_\kappa,\ g^{2^{\tilde{c}}}_\kappa\},\ \min\{\psi^{1^{\tilde{c}}}_{g^{1^{\tilde{c}}}_\kappa},\ \psi^{2^{\tilde{c}}}_{g^{2^{\tilde{c}}}_\kappa}\}\rangle & \\ = \langle \min\{g^1_{\neg\kappa},\ g^2_{\neg\kappa}\},\ \min\{\mu^1_{g^1_{\neg\kappa}},\ \mu^2_{g^2_{\neg\kappa}}\}\rangle, & \text{if } \kappa \in \mathcal{K}^1 \cap \mathcal{K}^2, \end{cases} \tag{29}$$

and for all $\neg\kappa \in \neg\mathcal{K}^1 \cup \neg\mathcal{K}^2$ and $\omega \in W$:

$$\mu^4(\neg\kappa)(\omega) = \begin{cases} \mu^{1^{\tilde{c}}}(\neg\kappa)(\omega) = \psi^1(\kappa)(\omega), & \text{if } \neg\kappa \in \neg\mathcal{K}^1 \setminus \neg\mathcal{K}^2, \\ \mu^{2^{\tilde{c}}}(\neg\kappa)(\omega) = \psi^2(\kappa)(\omega), & \text{if } \neg\kappa \in \neg\mathcal{K}^2 \setminus \neg\mathcal{K}^1, \\ \langle \max\{g^{1\tilde{c}}_{\neg\kappa},\ g^{2\tilde{c}}_{\neg\kappa}\},\ \max\{\mu^{1\tilde{c}}_{g^{1\tilde{c}}_{\neg\kappa}},\ \mu^{2\tilde{c}}_{g^{2\tilde{c}}_{\neg\kappa}}\}\rangle & \\ = \langle \max\{g^1_\kappa,\ g^2_\kappa\},\ \max\{\psi^1_{g^1_\kappa},\ \psi^2_{g^2_\kappa}\}\rangle, & \text{if } \neg\kappa \in \neg\mathcal{K}^1 \cap \neg\mathcal{K}^2. \end{cases} \tag{30}$$

Since $(\psi^3,\ \mu^3,\ \mathcal{K}^1 \cup \mathcal{K}^2,\ N)^{\tilde{c}}$ and $(\psi^4,\ \mu^4,\ \mathcal{K}^1 \cup \mathcal{K}^2,\ N)$ are equivalent for all $\kappa \in \mathcal{K}^1 \cup \mathcal{K}^2$, $\neg\kappa \in \neg\mathcal{K}^1 \cup \neg\mathcal{K}^2$, and $\omega \in W$, the proof follows.

The other components can be shown in a similar fashion. □

Proposition 5 Let $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)$ and $(\psi^2,\ \mu^2,\ \mathcal{K}^1,\ N)$ be two FN-BSE sets. Then

1. $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cup}_\varepsilon\ (\psi^2,\ \mu^2,\ \mathcal{K}^1,\ N) = (\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cup}_r\ (\psi^2,\ \mu^2,\ \mathcal{K}^1,\ N)$.
2. $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cap}_\varepsilon\ (\psi^2,\ \mu^2,\ \mathcal{K}^1,\ N) = (\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cap}_r\ (\psi^2,\ \mu^2,\ \mathcal{K}^1,\ N)$.
3. $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cup}_r\ (\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N) = (\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)$ and $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cap}_r\ (\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N) = (\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)$.
4. $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cup}_r\ (\psi^{\mathbb{N}},\ \mu^{\mathbb{N}},\ \mathcal{K}^1,\ N) = (\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)$ and $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cap}_r\ (\psi^{\mathbb{N}},\ \mu^{\mathbb{N}},\ \mathcal{K}^1,\ N) = (\psi^{\mathbb{N}},\ \mu^{\mathbb{N}},\ \mathcal{K}^1,\ N)$.
5. $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cup}_r\ (\psi^{\mathbb{U}},\ \mu^{\mathbb{U}},\ \mathcal{K}^1,\ N) = (\psi^{\mathbb{U}},\ \mu^{\mathbb{U}},\ \mathcal{K}^1,\ N)$ and $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cap}_r\ (\psi^{\mathbb{U}},\ \mu^{\mathbb{U}},\ \mathcal{K}^1,\ N) = (\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)$.

Proof. The proofs are apparent. □

Proposition 6 Let $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)$ and $(\psi^2,\ \mu^2,\ \mathcal{K}^2,\ N)$ be two FN-BSE sets. Then

1. $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cup}_\varepsilon\ \big((\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cap}_r\ (\psi^2,\ \mu^2,\ \mathcal{K}^2,\ N)\big) = (\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)$.
2. $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cap}_\varepsilon\ \big((\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cup}_r\ (\psi^2,\ \mu^2,\ \mathcal{K}^2,\ N)\big) = (\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)$.
3. $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cup}_r\ \big((\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cap}_\varepsilon\ (\psi^2,\ \mu^2,\ \mathcal{K}^2,\ N)\big) = (\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)$.
4. $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cap}_r\ \big((\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cup}_\varepsilon\ (\psi^2,\ \mu^2,\ \mathcal{K}^2,\ N)\big) = (\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)$.

Proof. (1) Suppose that $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cap}_r\ (\psi^2,\ \mu^2,\ \mathcal{K}^2,\ N) = (\psi^3,\ \mu^3,\ \mathcal{K}^1 \cap \mathcal{K}^2,\ N)$. Then, for all $\kappa \in \mathcal{K}^1 \cap \mathcal{K}^2$ and $\omega \in W$:

$$\psi^3(\kappa)(\omega) = \langle \min\{g^1_\kappa,\ g^2_\kappa\},\ \min\{\psi^1_{g^1_\kappa},\ \psi^2_{g^2_\kappa}\}\rangle, \tag{31}$$

where $\langle g^1_\kappa,\ \psi^1_{g^1_\kappa}\rangle = \psi^1(\kappa)(\omega)$ and $\langle g^2_\kappa,\ \psi^2_{g^2_\kappa}\rangle = \psi^2(\kappa)(\omega)$, and for all $\neg\kappa \in \neg\mathcal{K}^1 \cap \neg\mathcal{K}^2$ and $\omega \in W$:

$$\mu^3(\neg\kappa)(\omega) = \langle \max\{g^1_{\neg\kappa},\ g^2_{\neg\kappa}\},\ \max\{\mu^1_{g^1_{\neg\kappa}},\ \mu^2_{g^2_{\neg\kappa}}\}\rangle \tag{32}$$

where $\langle g^1_{\neg\kappa},\ \mu^1_{g^1_{\neg\kappa}}\rangle = \mu^1(\neg\kappa)(\omega)$ and $\langle g^2_{\neg\kappa},\ \mu^2_{g^2_{\neg\kappa}}\rangle = \mu^2(\neg\kappa)(\omega)$.

Now, let $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N)\ \tilde{\cup}_\varepsilon\ (\psi^3,\ \mu^3,\ \mathcal{K}^1 \cap \mathcal{K}^2,\ N) = (\psi^4,\ \mu^4,\ \mathcal{K}^1 \cup (\mathcal{K}^1 \cap \mathcal{K}^2),\ N) = (\psi^4,\ \mu^4,\ \mathcal{K}^1,\ N)$. Then, for all $\kappa \in \mathcal{K}^1 \cup (\mathcal{K}^1 \cap \mathcal{K}^2)$ and $\omega \in W$:

$$\psi^4(\kappa)(\omega) = \begin{cases} \psi^1(\kappa)(\omega), & \text{if } \kappa \in \mathcal{K}^1 \setminus (\mathcal{K}^1 \cap \mathcal{K}^2) \\ \psi^3(\kappa)(\omega), & \text{if } \kappa \in (\mathcal{K}^1 \cap \mathcal{K}^2) \setminus \mathcal{K}^1 = \emptyset \\ \langle \max\{g^1_\kappa,\ g^3_\kappa\},\ \max\{\psi^1_{g^1_\kappa},\ \psi^3_{g^3_\kappa}\}\rangle, & \text{if } \kappa \in \mathcal{K}^1 \cap (\mathcal{K}^1 \cap \mathcal{K}^2), \end{cases} \tag{33}$$

where $\langle g^3_\kappa,\ \psi^3_{g^3_\kappa}\rangle = \psi^3(\kappa)(\omega)$, and hence

$$\psi^4(\kappa)(\omega) = \begin{cases} \psi^1(\kappa)(\omega), & \text{if } \kappa \in \mathcal{K}^1 \setminus (\mathcal{K}^1 \cap \mathcal{K}^2) \\ \langle \max\{g^1_\kappa,\ \min\{g^1_\kappa,\ g^2_\kappa\}\},\ \max\{\psi^1_{g^1_\kappa},\ \min\{\psi^1_{g^1_\kappa},\ \psi^2_{g^2_\kappa}\}\}\rangle, & \text{if } \kappa \in \mathcal{K}^1 \cap (\mathcal{K}^1 \cap \mathcal{K}^2), \end{cases} \tag{34}$$

and for all $\neg\kappa \in \neg\mathcal{K}^1 \cup (\neg\mathcal{K}^1 \cap \neg\mathcal{K}^2)$ and $\omega \in W$:

$$\mu^4(\neg\kappa)(\omega)$$

$$= \begin{cases} \mu^1(\neg\kappa)(\omega),\ \text{if } \neg\kappa \in \neg\mathcal{K}^1 \setminus (\neg\mathcal{K}^1 \cap \neg\mathcal{K}^2) \\ \mu^3(\neg\kappa)(\omega),\ \text{if } \neg\kappa \in (\neg\mathcal{K}^1 \cap \neg\mathcal{K}^2) \setminus \neg\mathcal{K}^1 = \emptyset \\ \langle \min\{g^1_{\neg\kappa},\ g^3_{\neg\kappa}\},\ \min\{\mu^1_{g^1_{\neg\kappa}},\ \mu^3_{g^3_{\neg\kappa}}\}\rangle,\ \text{if } \neg\kappa \in \neg\mathcal{K}^1 \cap (\neg\mathcal{K}^1 \cap \neg\mathcal{K}^2). \end{cases} \tag{35}$$

$$= \begin{cases} \mu^1(\neg\kappa)(\omega),\ \text{if } \neg\kappa \in \neg\mathcal{K}^1 \setminus (\neg\mathcal{K}^1 \cap \neg\mathcal{K}^2) \\ \langle \min\{g^1_{\neg\kappa},\ \max\{g^1_{\neg\kappa},\ g^2_{\neg\kappa}\}\},\ \min\{\mu^1_{g^1_{\neg\kappa}},\ \max\{\mu^1_{g^1_{\neg\kappa}},\ \mu^2_{g^2_{\neg\kappa}}\}\}\rangle,\ \text{if } \neg\kappa \in \neg\mathcal{K}^1 \cap (\neg\mathcal{K}^1 \cap \neg\mathcal{K}^2). \end{cases} \tag{36}$$

Hence,

$$\psi^4(\kappa)(\omega) = \begin{cases} \psi^1(\kappa)(\omega), & \text{if } \kappa \in \mathcal{K}^1 \setminus (\mathcal{K}^1 \cap \mathcal{K}^2) \\ \psi^1(\kappa)(\omega), & \text{if } \kappa \in \mathcal{K}^1 \cap \mathcal{K}^2 \end{cases} \tag{37}$$

and

$$\mu^4(\neg\kappa)(\omega) = \begin{cases} \mu^1(\neg\kappa)(\omega), & \text{if } \neg\kappa \in \neg\mathcal{K}^1 \setminus (\neg\mathcal{K}^1 \cap \neg\mathcal{K}^2) \\ \mu^1(\neg\kappa)(\omega), & \text{if } \neg\kappa \in \neg\mathcal{K}^1 \cap \neg\mathcal{K}^2. \end{cases} \tag{38}$$

Therefore, $(\psi^1, \mu^1, \mathcal{K}^1, N) \tilde{\cup}_\varepsilon \left((\psi^1, \mu^1, \mathcal{K}^1, N) \tilde{\cap}_r (\psi^2, \mu^2, \mathcal{K}^2, N)\right) = (\psi^1, \mu^1, \mathcal{K}^1, N)$.

The other components can be shown in a similar fashion. □

Proposition 7 Let $(\psi^1, \mu^1, \mathcal{K}^1, N^1)$, $(\psi^2, \mu^2, \mathcal{K}^2, N^2)$, and $(\psi^3, \mu^3, \mathcal{K}^3, N^3)$ be three FN-BSE sets and let $\odot \in \{\tilde{\cup}_\varepsilon, \tilde{\cap}_\varepsilon, \tilde{\cup}_r, \tilde{\cap}_r\}$. Then

1. $(\psi^1, \mu^1, \mathcal{K}^1, N^1) \odot (\psi^2, \mu^2, \mathcal{K}^2, N^2) = (\psi^2, \mu^2, \mathcal{K}^2, N^2) \odot (\psi^1, \mu^1, \mathcal{K}^1, N^1)$.

2. $(\psi^1, \mu^1, \mathcal{K}^1, N^1) \odot \left((\psi^2, \mu^2, \mathcal{K}^2, N^2) \odot (\psi^3, \mu^3, \mathcal{K}^3, N^3)\right) = \left((\psi^1, \mu^1, \mathcal{K}^1, N^1) \odot (\psi^2, \mu^2, \mathcal{K}^2, N^2)\right) \odot (\psi^3, \mu^3, \mathcal{K}^3, N^3)$.

Proof. The proofs are apparent. □

Proposition 8 Let $(\psi^1, \mu^1, \mathcal{K}^1, N^1)$, $(\psi^2, \mu^2, \mathcal{K}^2, N^2)$, and $(\psi^3, \mu^3, \mathcal{K}^3, N^3)$ be three FN-BSE sets. Then

1. $(\psi^1, \mu^1, \mathcal{K}^1, N^1) \tilde{\cup}_\varepsilon \left((\psi^2, \mu^2, \mathcal{K}^2, N^2) \tilde{\cap}_r (\psi^3, \mu^3, \mathcal{K}^3, N^3)\right) = \left((\psi^1, \mu^1, \mathcal{K}^1, N^1) \tilde{\cup}_\varepsilon (\psi^2, \mu^2, \mathcal{K}^2, N^2)\right) \tilde{\cap}_r \left((\psi^1, \mu^1, \mathcal{K}^1, N^1) \tilde{\cup}_\varepsilon (\psi^3, \mu^3, \mathcal{K}^3, N^3)\right)$.

2. $(\psi^1, \mu^1, \mathcal{K}^1, N^1) \tilde{\cap}_r \left((\psi^2, \mu^2, \mathcal{K}^2, N^2) \tilde{\cup}_\varepsilon (\psi^3, \mu^3, \mathcal{K}^3, N^3)\right) = \left((\psi^1, \mu^1, \mathcal{K}^1, N^1) \tilde{\cap}_r (\psi^2, \mu^2, \mathcal{K}^2, N^2)\right) \tilde{\cup}_\varepsilon \left((\psi^1, \mu^1, \mathcal{K}^1, N^1) \tilde{\cap}_r (\psi^3, \mu^3, \mathcal{K}^3, N^3)\right)$.

3. $(\psi^1, \mu^1, \mathcal{K}^1, N^1) \tilde{\cap}_\varepsilon \left((\psi^2, \mu^2, \mathcal{K}^2, N^2) \tilde{\cup}_r (\psi^3, \mu^3, \mathcal{K}^3, N^3)\right) = \left((\psi^1, \mu^1, \mathcal{K}^1, N^1) \tilde{\cap}_\varepsilon (\psi^2, \mu^2, \mathcal{K}^2, N^2)\right) \tilde{\cup}_\mathfrak{R} \left((\psi^1, \mu^1, \mathcal{K}^1, N^1) \tilde{\cap}_\varepsilon (\psi^3, \mu^3, \mathcal{K}^3, N^3)\right)$.

4. $(\psi^1, \mu^1, \mathcal{K}^1, N^1) \tilde{\cup}_r \left((\psi^2, \mu^2, \mathcal{K}^2, N^2) \tilde{\cap}_\varepsilon (\psi^3, \mu^3, \mathcal{K}^3, N^3)\right) = \left((\psi^1, \mu^1, \mathcal{K}^1, N^1) \tilde{\cup}_r (\psi^2, \mu^2, \mathcal{K}^2, N^2)\right) \tilde{\cap}_\varepsilon \left((\psi^1, \mu^1, \mathcal{K}^1, N^1) \tilde{\cup}_r (\psi^3, \mu^3, \mathcal{K}^3, N^3)\right)$.

5. $(\psi^1, \mu^1, \mathcal{K}^1, N^1) \tilde{\cup}_r \left((\psi^2, \mu^2, \mathcal{K}^2, N^2) \tilde{\cap}_r (\psi^3, \mu^3, \mathcal{K}^3, N^3)\right) = \left((\psi^1, \mu^1, \mathcal{K}^1, N^1) \tilde{\cup}_r (\psi^2, \mu^2, \mathcal{K}^2, N^2)\right) \tilde{\cap}_r \left((\psi^1, \mu^1, \mathcal{K}^1, N^1) \tilde{\cup}_r (\psi^3, \mu^3, \mathcal{K}^3, N^3)\right)$.

6. $(\psi^1, \mu^1, \mathcal{K}^1, N^1) \tilde{\cap}_r \left((\psi^2, \mu^2, \mathcal{K}^2, N^2) \tilde{\cup}_r (\psi^3, \mu^3, \mathcal{K}^3, N^3)\right) = \left((\psi^1, \mu^1, \mathcal{K}^1, N^1) \tilde{\cap}_r (\psi^2, \mu^2, \mathcal{K}^2, N^2)\right) \tilde{\cup}_r \left((\psi^1, \mu^1, \mathcal{K}^1, N^1) \tilde{\cap}_r (\psi^3, \mu^3, \mathcal{K}^3, N^3)\right)$.

Proof. (4) Suppose that $\left((\psi^2, \mu^2, \mathcal{K}^2, N^2) \tilde{\cap}_\varepsilon (\psi^3, \mu^3, \mathcal{K}^3, N^3)\right) = (\psi^4, \mu^4, \mathcal{K}^2 \cup \mathcal{K}^3, \max(N^2, N^3))$, then for all $\kappa \in \mathcal{K}^2 \cup \mathcal{K}^3$ and $\omega \in W$:

$$\psi^4(\kappa)(\omega) = \begin{cases} \psi^2(\kappa)(\omega), & \text{if } \kappa \in \mathcal{K}^2 \setminus \mathcal{K}^3 \\ \psi^3(\kappa)(\omega), & \text{if } \kappa \in \mathcal{K}^3 \setminus \mathcal{K}^2 \\ \langle \min\{g^2_\kappa, g^3_\kappa\}, \min\{\psi^2_{g^2_\kappa}, \psi^3_{g^3_\kappa}\} \rangle, & \text{if } \kappa \in \mathcal{K}^2 \cap \mathcal{K}^3 \end{cases} \tag{39}$$

where $\langle g^2_\kappa, \psi^2_{g^2_\kappa} \rangle = \psi^2(\kappa)(\omega)$ and $\langle g^3_\kappa, \psi^3_{g^3_\kappa} \rangle = \psi^3(\kappa)(\omega)$, and for all $\neg\kappa \in \neg\mathcal{K}^2 \cup \neg\mathcal{K}^3$ and $\omega \in W$:

$$\mu^4(\neg\kappa)(\omega) = \begin{cases} \mu^2(\neg\kappa)(\omega), & \text{if } \neg\kappa \in \neg\mathcal{K}^2 \setminus \neg\mathcal{K}^3 \\ \mu^3(\neg\kappa)(\omega), & \text{if } \neg\kappa \in \neg\mathcal{K}^3 \setminus \neg\mathcal{K}^2 \\ \langle \max\{g^2_{\neg\kappa}, g^3_{\neg\kappa}\}, \max\{\mu^2_{g^2_{\neg\kappa}}, \mu^3_{g^3_{\neg\kappa}}\} \rangle, & \text{if } \neg\kappa \in \neg\mathcal{K}^2 \cap \neg\mathcal{K}^3 \end{cases} \tag{40}$$

where $\langle g^2_{\neg\kappa}, \mu^2_{g^2_{\neg\kappa}} \rangle = \mu^2(\neg\kappa)(\omega)$ and $\langle g^3_{\neg\kappa}, \mu^3_{g^3_{\neg\kappa}} \rangle = \mu^3(\neg\kappa)(\omega)$.

Let $(\psi^1, \mu^1, \mathcal{K}^1, N^1) \tilde{\cup}_r (\psi^4, \mu^4, \mathcal{K}^2 \cup \mathcal{K}^3, \max(N^2, N^3)) = (\psi^5, \mu^5, \mathcal{K}^1 \cap (\mathcal{K}^2 \cup \mathcal{K}^3), \max(N^1, \max(N^2, N^3))) = (\psi^5, \mu^5, O \cup P, \max(N^1, N^2, N^3))$ where $O = \mathcal{K}^1 \cap \mathcal{K}^2$ and $P = \mathcal{K}^1 \cap \mathcal{K}^3$, then for all $\kappa \in O \cup P$ and $\omega \in W$:

$$\psi^5(\kappa)(\omega) = \langle \max\{g^1_\kappa, g^4_\kappa\}, \max\{\psi^1_{g^1_\kappa}, \psi^4_{g^4_\kappa}\} \rangle, \tag{41}$$

where $\langle g_\kappa^1,\ \psi_{g_\kappa^1}^1\rangle = \psi^1(\kappa)(\omega)$ and $\langle g_\kappa^4,\ \psi_{g_\kappa^4}^4\rangle = \psi^4(\kappa)(\omega)$, and for all $\neg\kappa \in \neg O \cup \neg P$ and $\omega \in W$:

$$\mu^5(\neg\kappa)(\omega) = \langle \min\{g_{\neg\kappa}^1,\ g_{\neg\kappa}^4\},\ \min\{\mu_{g_{\neg\kappa}^1}^1,\ \mu_{g_{\neg\kappa}^4}^4\}\rangle, \tag{42}$$

where $\langle g_{\neg\kappa}^1,\ \mu_{g_{\neg\kappa}^1}^1\rangle = \mu^1(\neg\kappa)(\omega)$ and $\langle g_{\neg\kappa}^4,\ \mu_{g_{\neg\kappa}^4}^4\rangle = \mu^4(\neg\kappa)(\omega)$.

Hence, for all $\kappa \in O \cup P$ and $\omega \in W$:

$$\psi^5(\kappa)(\omega) = \begin{cases} \langle \max\{g_\kappa^1,\ g_\kappa^2\},\ \max\{\psi_{g_\kappa^1}^1,\ \psi_{g_\kappa^2}^2\}\rangle, & \text{if } \kappa \in O \setminus P \\ \langle \max\{g_\kappa^1,\ g_\kappa^3\},\ \max\{\psi_{g_\kappa^1}^1,\ \psi_{g_\kappa^3}^3\}\rangle, & \text{if } \kappa \in P \setminus O \\ \langle \max\{g_\kappa^1,\ \min\{g_\kappa^2,\ g_\kappa^3\}\},\ \max\{\psi_{g_\kappa^1}^1,\ \min\{\psi_{g_\kappa^2}^2,\ \psi_{g_\kappa^3}^3\}\}\rangle, & \text{if } \kappa \in P \cap O \end{cases} \tag{43}$$

and for all $\neg\kappa \in \neg O \cup \neg P$ and $\omega \in W$:

$$\mu^5(\neg\kappa)(\omega) = \begin{cases} \langle \min\{g_{\neg\kappa}^1,\ g_{\neg\kappa}^2\},\ \min\{\mu_{g_{\neg\kappa}^1}^1,\ \mu_{g_{\neg\kappa}^2}^2\}\rangle, & \text{if } \neg\kappa \in \neg O \setminus \neg P \\ \langle \min\{g_{\neg\kappa}^1,\ g_{\neg\kappa}^3\},\ \min\{\mu_{g_{\neg\kappa}^1}^1,\ \mu_{g_{\neg\kappa}^3}^3\}\rangle, & \text{if } \neg\kappa \in \neg P \setminus \neg O \\ \langle \min\{g_{\neg\kappa}^1,\ \max\{g_{\neg\kappa}^2,\ g_{\neg\kappa}^3\}\},\ \min\{\mu_{g_{\neg\kappa}^1}^1,\ \max\{\mu_{g_{\neg\kappa}^2}^2,\ \mu_{g_{\neg\kappa}^3}^3\}\}\rangle, & \text{if } \neg\kappa \in \neg P \cap \neg O. \end{cases} \tag{44}$$

On the other hand, let $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N^1)\ \tilde{\cup}_r\ (\psi^2,\ \mu^2,\ \mathcal{K}^2,\ N^2) = (\psi^6,\ \mu^6,\ \mathcal{K}^1 \cap \mathcal{K}^2,\ \max(N^1,\ N^2))$, then for all $\kappa \in \mathcal{K}^1 \cap \mathcal{K}^2$ and $\omega \in W$:

$$\psi^6(\kappa)(\omega) = \langle \max\{g_\kappa^1,\ g_\kappa^2\},\ \max\{\psi_{g_\kappa^1}^1,\ \psi_{g_\kappa^2}^2\}\rangle, \tag{45}$$

and for all $\neg\kappa \in \neg O \cup \neg P$ and $\omega \in W$:

$$\mu^6(\neg\kappa)(\omega) = \langle \min\{g_{\neg\kappa}^1,\ g_{\neg\kappa}^2\},\ \min\{\mu_{g_{\neg\kappa}^1}^1,\ \mu_{g_{\neg\kappa}^2}^2\}\rangle. \tag{46}$$

Next, let $(\psi^1,\ \mu^1,\ \mathcal{K}^1,\ N^1)\ \tilde{\cup}_r\ (\psi^3,\ \mu^3,\ \mathcal{K}^3,\ N^3) = (\psi^7,\ \mu^7,\ \mathcal{K}^1 \cap \mathcal{K}^2,\ \max(N^1,\ N^2))$, then for all $\kappa \in \mathcal{K}^1 \cap \mathcal{K}^3$ and $\omega \in W$:

$$\psi^7(\kappa)(\omega) = \langle \max\{g_\kappa^1,\ g_\kappa^3\},\ \max\{\psi_{g_\kappa^1}^1,\ \psi_{g_\kappa^3}^3\}\rangle, \tag{47}$$

and for all $\neg\kappa \in \neg O \cup \neg P$ and $\omega \in W$:

$$\mu^7(\neg\kappa)(\omega) = \langle \min\{g_{\neg\kappa}^1,\ g_{\neg\kappa}^3\},\ \min\{\mu_{g_{\neg\kappa}^1}^1,\ \mu_{g_{\neg\kappa}^3}^3\}\rangle. \tag{48}$$

Now, if $(\psi^6,\ \mu^6,\ \mathcal{K}^1\cap\mathcal{K}^2,\ \max(N^1,\ N^2))\ \tilde{\cap}_\varepsilon\ (\psi^7,\ \mu^7,\ \mathcal{K}^1\cap\mathcal{K}^3,\ \max(N^1,\ N^3)) = (\psi^8,\ \mu^8,\ O\cup P,\ \max(N^1,\ N^2,\ N^3))$, where $O=\mathcal{K}^1\cap\mathcal{K}^2$ and $P=\mathcal{K}^1\cap\mathcal{K}^3$, then for all $\kappa\in O\cup P$ and $\omega\in W$:

$$\psi^8(\kappa)(\omega)=\begin{cases}\psi^6(\kappa)(\omega), & \text{if } \kappa\in O\setminus P,\\ \psi^7(\kappa)(\omega), & \text{if } \kappa\in P\setminus O,\\ \langle\min\{g^6_\kappa,\ g^7_\kappa\},\ \min\{\psi^6_{g^6_\kappa},\ \psi^7_{g^7_\kappa}\}\rangle, & \text{if } \kappa\in P\cap O,\end{cases}\tag{49}$$

where $\langle g^6_\kappa,\ \psi^6_{g^6_\kappa}\rangle=\psi^6(\kappa)(\omega)$ and $\langle g^7_\kappa,\ \psi^7_{g^7_\kappa}\rangle=\psi^7(\kappa)(\omega)$, and hence

$$\psi^8(\kappa)(\omega)=\begin{cases}\langle\max\{g^1_\kappa,\ g^2_\kappa\},\ \max\{\psi^1_{g^1_\kappa},\ \psi^2_{g^2_\kappa}\}\rangle, & \text{if } \kappa\in O\setminus P,\\ \langle\max\{g^1_\kappa,\ g^3_\kappa\},\ \max\{\psi^1_{g^1_\kappa},\ \psi^3_{g^3_\kappa}\}\rangle, & \text{if } \kappa\in P\setminus O,\\ \langle\min\{\max\{g^1_\kappa,\ g^2_\kappa\},\ \max\{g^1_\kappa,\ g^3_\kappa\}\}, & \\ \min\{\max\{\psi^1_{g^1_\kappa},\ \psi^2_{g^2_\kappa}\},\ \max\{\psi^1_{g^1_\kappa},\ \psi^3_{g^3_\kappa}\}\}\rangle & \text{if } \kappa\in P\cap O,\end{cases}\tag{50}$$

and for all $\neg\kappa\in\neg O\cup\neg P$ and $\omega\in W$:

$$\mu^8(\neg\kappa)(\omega)=\begin{cases}\mu^6(\neg\kappa)(\omega),\ \text{if } \neg\kappa\in\neg O\setminus\neg P\\ \mu^7(\neg\kappa)(\omega),\ \text{if } \neg\kappa\in\neg P\setminus\neg O\\ \langle\max\{g^6_{\neg\kappa},\ g^7_{\neg\kappa}\},\ \max\{\mu^6_{g^6_{\neg\kappa}},\ \mu^7_{g^7_{\neg\kappa}}\}\rangle,\ \text{if } \neg\kappa\in\neg P\cap\neg O\end{cases}\tag{51}$$

$$=\begin{cases}\langle\min\{g^1_{\neg\kappa},\ g^2_{\neg\kappa}\},\ \min\{\mu^1_{g^1_{\neg\kappa}},\ \mu^2_{g^2_{\neg\kappa}}\}\rangle, & \text{if } \neg\kappa\in\neg O\setminus\neg P,\\ \langle\min\{g^1_{\neg\kappa},\ g^3_{\neg\kappa}\},\ \min\{\mu^1_{g^1_{\neg\kappa}},\ \mu^3_{g^3_{\neg\kappa}}\}\rangle, & \text{if } \neg\kappa\in\neg P\setminus\neg O,\\ \langle\max\{\min\{g^1_{\neg\kappa},\ g^2_{\neg\kappa}\},\ \min\{g^1_{\neg\kappa},\ g^3_{\neg\kappa}\}\}, & \\ \max\{\min\{\mu^1_{g^1_{\neg\kappa}},\ \mu^2_{g^2_{\neg\kappa}}\},\ \min\{\mu^1_{g^1_{\neg\kappa}},\ \mu^3_{g^3_{\neg\kappa}}\}\}\rangle & \text{if } \neg\kappa\in\neg P\cap\neg O.\end{cases}\tag{52}$$

Since $(\psi^5,\ \mu^5,\ O\cup P,\ \max(N^1,\ N^2,\ N^3))$ and $(\psi^8,\ \mu^8,\ O\cup P,\ \max(N^1,\ N^2,\ N^3))$ are counterpart for all $\kappa\in O\cup P$, $\neg\kappa\in\neg O\cup\neg P$, and $\omega\in W$, we are finished.

The other components can be shown in a similar fashion. □

4. MAGDM application based on fuzzy N-bipolar soft expert sets

MAGDM is a DM process in which multiple participants evaluate a set of alternatives based on various criteria. This approach is particularly useful for complex scenarios where stakeholders may have conflicting perspectives. By incorporating diverse opinions, MAGDM encourages consensus, which is crucial in fields such as business, healthcare, engineering, and public policy, where decisions can have far-reaching impacts.

In this section, we apply the FN-BSE set framework to MAGDM problems to demonstrate its effectiveness in identifying the best alternative based on multi-criteria evaluations. Specifically, we present a case study on selecting the optimal location for a new healthcare facility, considering both the advantages and drawbacks of potential sites.

In MAGDM using FBSE Numbers (FBSENs), it is crucial to quantify both the overall tendency and the decisiveness of expert evaluations. To this end, we introduce the following tools for ranking alternatives:

Definition 31 (Score function). Let $\xi = \langle(\psi_{(\flat,\,\varepsilon,\,1)},\ \psi_{(\flat,\,\varepsilon,\,0)}),\ (\mu_{(\neg\flat,\,\varepsilon,\,1)},\ \mu_{(\neg\flat,\,\varepsilon,\,0)})\rangle$ be an FBSEN. Then the score function $\mathbb{S}(\xi)$ is defined as

$$\mathbb{S}(\xi) = (\psi_{(\flat,\,\varepsilon,\,1)} - \psi_{(\flat,\,\varepsilon,\,0)}) - (\mu_{(\neg\flat,\,\varepsilon,\,1)} - \mu_{(\neg\flat,\,\varepsilon,\,0)}). \tag{53}$$

Theorem 1 (Range of the score function). Let ξ be an FBSEN as above. Then its score function satisfies

$$\mathbb{S}(\xi) \in [-2,\ 2]. \tag{54}$$

Proof. Set

$$x = \psi_{(\flat,\,\varepsilon,\,1)}, \quad y = \psi_{(\flat,\,\varepsilon,\,0)}, \quad u = \mu_{(\neg\flat,\,\varepsilon,\,1)}, \quad v = \mu_{(\neg\flat,\,\varepsilon,\,0)}. \tag{55}$$

From Remark 1 (4), we have $0 \le x + y \le 1$ and $0 \le u + v \le 1$, so $x,\ y,\ u,\ v \in [0,\ 1]$.

The score is

$$\mathbb{S}(\xi) = (x - y) - (u - v). \tag{56}$$

Under $x,\ y \in [0,\ 1]$ with $x + y \le 1$, the difference $x - y$ attains its maximum 1 at $(x,\ y) = (1,\ 0)$ and its minimum -1 at $(x,\ y) = (0,\ 1)$. Hence

$$-1 \le x - y \le 1. \tag{57}$$

Similarly,

$$-1 \le u - v \le 1. \tag{58}$$

Subtracting (58) from (57) yields

$$-2 \leq (x-y)-(u-v) \leq 2, \tag{59}$$

which is exactly $\mathbb{S}(\xi) \in [-2,\ 2]$. In particular, choose $(x,\ y,\ u,\ v) = (1,\ 0,\ 0,\ 1)$ to obtain $\mathbb{S} = 2$, and $(x,\ y,\ u,\ v) = (0,\ 1,\ 1,\ 0)$ to obtain $\mathbb{S} = -2$. Thus the bounds are attained, completing the proof. □

Definition 32 (Natural quasi-ordering of FBSENs). Let $\xi^1 = \langle(\psi^1_{(\flat,\ \varepsilon,\ 1)},\ \psi^1_{(\flat,\ \varepsilon,\ 0)}),\ (\mu^1_{(\neg\flat,\ \varepsilon,\ 1)},\ \mu^1_{(\neg\flat,\ \varepsilon,\ 0)})\rangle$ and $\xi^2 = \langle(\psi^2_{(\flat,\ \varepsilon,\ 1)},\ \psi^2_{(\flat,\ \varepsilon,\ 0)}),\ (\mu^2_{(\neg\flat,\ \varepsilon,\ 1)},\ \mu^2_{(\neg\flat,\ \varepsilon,\ 0)})\rangle$ be two FBSENs. Then a natural quasi-ordering is defined as

$$\xi^1 \leq \xi^2 \iff \begin{cases} \psi^1_{(\flat,\ \varepsilon,\ 1)} \leq \psi^2_{(\flat,\ \varepsilon,\ 1)}, \\ \psi^2_{(\flat,\ \varepsilon,\ 0)} \leq \psi^1_{(\flat,\ \varepsilon,\ 0)}, \\ \mu^2_{(\neg\flat,\ \varepsilon,\ 1)} \leq \mu^1_{(\neg\flat,\ \varepsilon,\ 1)}, \\ \mu^1_{(\neg\flat,\ \varepsilon,\ 0)} \leq \mu^2_{(\neg\flat,\ \varepsilon,\ 0)}. \end{cases} \tag{60}$$

Theorem 1 (Monotonicity of the score function for FBSENs). Let ξ^1 and ξ^2 be two FBSENs, and let $\mathbb{S}(\xi^1)$ and $\mathbb{S}(\xi^2)$ be their score functions. If $\xi^1 \leq \xi^2$, then

$$\mathbb{S}(\xi^1) \leq \mathbb{S}(\xi^2). \tag{61}$$

Proof. Assume $\xi^1 \leq \xi^2$ as in (60). Define

$$\Delta\psi_{(\flat,\ \varepsilon,\ 1)} = \psi^2_{(\flat,\ \varepsilon,\ 1)} - \psi^1_{(\flat,\ \varepsilon,\ 1)} \geq 0, \tag{62}$$

$$-\Delta\psi_{(\flat,\ \varepsilon,\ 0)} = -(\psi^2_{(\flat,\ \varepsilon,\ 0)} - \psi^1_{(\flat,\ \varepsilon,\ 0)}) \geq 0, \tag{63}$$

$$-\Delta\mu_{(\neg\flat,\ \varepsilon,\ 1)} = -(\mu^2_{(\neg\flat,\ \varepsilon,\ 1)} - \mu^1_{(\neg\flat,\ \varepsilon,\ 1)}) \geq 0, \tag{64}$$

and

$$\Delta\mu_{(\neg\flat,\ \varepsilon,\ 0)} = \mu^2_{(\neg\flat,\ \varepsilon,\ 0)} - \mu^1_{(\neg\flat,\ \varepsilon,\ 0)} \geq 0. \tag{65}$$

Then

$$\begin{aligned}\mathbb{S}(\xi^2)-\mathbb{S}(\xi^1) &= (\psi^2_{(\flat,\ \varepsilon,\ 1)}-\psi^1_{(\flat,\ \varepsilon,\ 1)})-(\psi^2_{(\flat,\ \varepsilon,\ 0)}-\psi^1_{(\flat,\ \varepsilon,\ 0)})\\ &\quad -(\mu^2_{(\neg\flat,\ \varepsilon,\ 1)}-\mu^1_{(\neg\flat,\ \varepsilon,\ 1)})+(\mu^2_{(\neg\flat,\ \varepsilon,\ 0)}-\mu^1_{(\neg\flat,\ \varepsilon,\ 0)}) \\ &= \Delta\psi_{(\flat,\ \varepsilon,\ 1)}-\Delta\psi_{(\flat,\ \varepsilon,\ 0)}-\Delta\mu_{(\neg\flat,\ \varepsilon,\ 1)}+\Delta\mu_{(\neg\flat,\ \varepsilon,\ 0)}\geq 0,\end{aligned} \tag{66}$$

which proves (61). □

Remark 2 (Tie-breaking using accuracy function). Sometimes, two FBSENs ξ^1 and ξ^2 may have the same score $\mathbb{S}(\xi^1)=\mathbb{S}(\xi^2)$. To refine the ranking, we introduce an accuracy function $\mathbb{A}(\xi)$, measuring the overall decisiveness of the evaluations.

Definition 33 (Accuracy function). Let $\xi=\langle(\psi_{(\flat,\ \varepsilon,\ 1)},\ \psi_{(\flat,\ \varepsilon,\ 0)}),\ (\mu_{(\neg\flat,\ \varepsilon,\ 1)},\ \mu_{(\neg\flat,\ \varepsilon,\ 0)})\rangle$ be an FBSEN. The accuracy function is defined as

$$\mathbb{A}(\xi)=(\psi_{(\flat,\ \varepsilon,\ 1)}+\psi_{(\flat,\ \varepsilon,\ 0)})+(\mu_{(\neg\flat,\ \varepsilon,\ 1)}+\mu_{(\neg\flat,\ \varepsilon,\ 0)}). \tag{67}$$

Proposition 9 [Range of the accuracy function] For a valid FBSEN, we have

$$\mathbb{A}(\xi)\in[0,\ 2], \tag{68}$$

where 0 occurs when all components are zero, and 2 occurs when the sums of positive and negative components each reach 1.

Example 3 Consider $\xi^1=\langle(\psi^1_{(\flat,\ \varepsilon,\ 1)}=0.6,\ \psi^1_{(\flat,\ \varepsilon,\ 0)}=0.3),\ (\mu^1_{(\neg\flat,\ \varepsilon,\ 1)}=0.2,\ \mu^1_{(\neg\flat,\ \varepsilon,\ 0)}=0.1)\rangle$, and $\xi^2=\langle(\psi^2_{(\flat,\ \varepsilon,\ 1)}=0.5,\ \psi^2_{(\flat,\ \varepsilon,\ 0)}=0.2),\ (\mu^2_{(\neg\flat,\ \varepsilon,\ 1)}=0.1,\ \mu^2_{(\neg\flat,\ \varepsilon,\ 0)}=0.0)\rangle$.

Compute the score functions:

$$\mathbb{S}(\xi^1)=(0.6-0.3)-(0.2-0.1)=0.2, \tag{69}$$

$$\mathbb{S}(\xi^2)=(0.5-0.2)-(0.1-0.0)=0.2. \tag{70}$$

Compute the accuracy functions:

$$\mathbb{A}(\xi^1)=(0.6+0.3)+(0.2+0.1)=1.2, \tag{71}$$

$$\mathbb{A}(\xi^2)=(0.5+0.2)+(0.1+0.0)=0.8. \tag{72}$$

Since $\mathbb{A}(\xi^1)>\mathbb{A}(\xi^2)$, we rank $\xi^1>\xi^2$, resolving the tie.

4.1 *Application algorithm*

Below, we present an algorithm (Algorithm 1) for determining the best alternative in an MAGDM problem using the FN-BSE set framework. The algorithm incorporates expert input, bipolarity, fuzzy data, and multinary evaluations to compute decision scores for each alternative, reflecting both positive and negative opinions. By aggregating assessments from multiple experts and comparing net scores, it ensures transparency, objectivity, and adaptability in addressing complex MAGDM problems.

The flowchart in Figure 1 illustrates the algorithm's procedures, showing how decision scores are calculated and applied to identify the optimal alternative.

Algorithm 1 Selecting the best alternative using FN-BSE sets.

1: **Input:**

• W: Set of alternatives.

• B: Collection of decision parameters.

• E: Group of experts involved in the evaluation.

• $\mathcal{O} = \{0 = \text{disagree},\ 1 = \text{agree}\}$: Binary set of opinions.

• The FN-BSE set $(\psi,\ \mu,\ \mathcal{K},\ N)$, where $\mathcal{K} \subseteq \mathcal{Z}$, with $\mathcal{Z} = B \times E \times \mathcal{O}$.

2: **Procedure:**

1. Identify:

• Positive agree FN-BSE set: $(\psi,\ \mu,\ \mathcal{K},\ N)^{\oplus 1}$.

• Positive disagree FN-BSE set: $(\psi,\ \mu,\ \mathcal{K},\ N)^{\oplus 0}$.

• Negative agree FN-BSE set: $(\psi,\ \mu,\ \mathcal{K},\ N)^{\ominus 1}$.

• Negative disagree FN-BSE set: $(\psi,\ \mu,\ \mathcal{K},\ N)^{\ominus 0}$.

2. For each alternative j, compute the sums:

$$\sigma_j^{\oplus 1} = \sum_i \omega_{ij} \quad \text{from } (\psi,\ \mu,\ \mathcal{K},\ N)^{\oplus 1}, \tag{73}$$

$$\sigma_j^{\oplus 0} = \sum_i \omega_{ij} \quad \text{from } (\psi,\ \mu,\ \mathcal{K},\ N)^{\oplus 0}, \tag{74}$$

$$\sigma_j^{\ominus 1} = \sum_i \omega_{ij} \quad \text{from } (\psi,\ \mu,\ \mathcal{K},\ N)^{\ominus 1}, \tag{75}$$

$$\sigma_j^{\ominus 0} = \sum_i \omega_{ij} \quad \text{from } (\psi,\ \mu,\ \mathcal{K},\ N)^{\ominus 0}. \tag{76}$$

3. Compute the scores for each alternative j:

$$s_j = s_j^{\oplus} - s_j^{\ominus}, \tag{77}$$

where

$$s_j^{\oplus} = \sigma_j^{\oplus 1} - \sigma_j^{\oplus 0} \tag{78}$$

and

$$s_j^{\ominus} = \sigma_j^{\ominus 1} - \sigma_j^{\ominus 0}. \tag{79}$$

3: **Output:** Identify l such that

$$s_l = \max s_j. \tag{80}$$

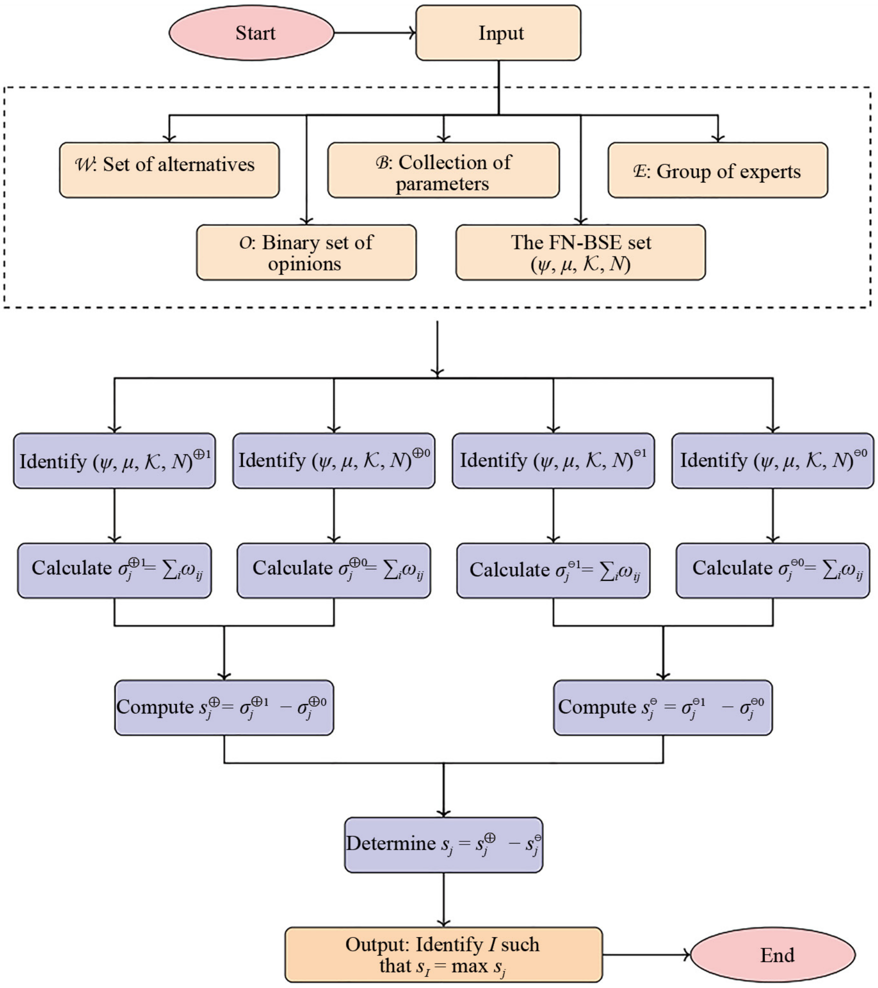

Figure 1. Flowchart of the given Algorithm (Algorithm 1)

4.2 *Optimal healthcare facility allocation*

Healthcare systems worldwide face the challenge of efficiently allocating resources to meet the growing demand for quality care. Selecting the best location for a new healthcare facility is a critical decision involving multiple criteria. A poorly chosen location could lead to inadequate service delivery and high operational costs, while an optimal location can improve access to care and resource utilization.

Suppose a healthcare organization plans to establish one new facility from a list of seven potential locations: $W = \{\omega_1, \ldots, \omega_7\}$. A committee of healthcare planners evaluates the locations based on parameters $B = \{\flat_1 =$ proximity to population, $\flat_2 =$ infrastructure readiness, $\flat_3 =$ cost of development$\}$, along with their positive counterparts: $\neg B = \{\neg\flat_1 =$ distance from population, $\neg\flat_2 =$ poor infrastructure, $\neg\flat_3 =$ high development cost$\}$. The findings are shared with medical experts $E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, who provide their binary opinions $O = \{0 = \text{disagree}, 1 = \text{agree}\}$, summarized in Table 8.

Table 8. Evaluations of healthcare facility location proposals by experts using check-marks

$\mathcal{K} \backslash W$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
$(\flat_1, \varepsilon_1, 1)$	∗∗∗	∗∗	∗∗∗∗	∗∗	∗∗	∗∗∗	∗∗
$(\flat_1, \varepsilon_2, 1)$	∗	∗∗∗	◦	∗∗∗	∗	∗∗∗	∗∗∗
$(\flat_1, \varepsilon_3, 1)$	∗∗	∗	∗∗∗	∗∗∗	∗	∗	∗∗∗
$(\flat_2, \varepsilon_1, 1)$	∗∗∗	∗∗	∗∗	∗∗	∗	∗∗∗	◦
$(\flat_2, \varepsilon_2, 1)$	∗∗∗	◦	∗∗∗	∗∗	∗∗∗	◦	∗∗
$(\flat_2, \varepsilon_3, 1)$	◦	∗∗	∗∗∗∗	∗∗	∗∗	◦	∗∗
$(\flat_3, \varepsilon_1, 1)$	∗∗∗	∗∗∗	∗	∗∗	∗∗∗	◦	∗∗∗
$(\flat_3, \varepsilon_2, 1)$	∗∗	◦	∗∗∗	∗	∗	∗∗∗	∗∗
$(\flat_3, \varepsilon_3, 1)$	∗∗	∗∗∗	∗∗∗	∗∗∗	∗∗	∗∗∗	◦
$(\flat_1, \varepsilon_1, 0)$	∗	∗∗	◦	∗∗	∗	◦	∗∗
$(\flat_1, \varepsilon_2, 0)$	∗∗	∗	∗	∗	∗∗	∗	∗
$(\flat_1, \varepsilon_3, 0)$	∗	∗∗∗	∗	◦	∗∗∗	∗∗	∗
$(\flat_2, \varepsilon_1, 0)$	◦	∗	◦	∗	∗∗∗	∗	◦
$(\flat_2, \varepsilon_2, 0)$	◦	◦	∗	∗	∗	∗∗∗∗	∗∗
$(\flat_2, \varepsilon_3, 0)$	∗∗∗	∗∗	◦	∗∗	∗	∗	∗
$(\flat_3, \varepsilon_1, 0)$	∗	∗	∗∗∗	∗∗	◦	∗∗∗	∗
$(\flat_3, \varepsilon_2, 0)$	∗∗	∗∗	∗	∗	∗∗∗	∗	∗∗
$(\flat_3, \varepsilon_3, 0)$	∗	◦	∗	◦	∗∗	∗	∗∗∗∗
$(\neg\flat_1, \varepsilon_1, 1)$	◦	∗	◦	∗∗	∗∗	∗	∗
$(\neg\flat_1, \varepsilon_2, 1)$	∗∗	∗	∗∗∗	∗	∗∗	◦	◦
$(\neg\flat_1, \varepsilon_3, 1)$	∗∗	∗∗	∗	∗	∗∗	∗∗	∗
$(\neg\flat_2, \varepsilon_1, 1)$	∗	∗	∗∗	∗	∗∗	◦	∗∗∗∗
$(\neg\flat_2, \varepsilon_2, 1)$	◦	∗∗	∗	∗	∗	∗∗∗	∗∗
$(\neg\flat_2, \varepsilon_3, 1)$	∗∗∗∗	∗∗	◦	∗∗	∗	∗∗	∗∗
$(\neg\flat_3, \varepsilon_1, 1)$	∗	∗	∗∗∗	∗∗	◦	∗	◦
$(\neg\flat_3, \varepsilon_2, 1)$	∗	◦	∗	∗∗	∗∗∗	∗	∗∗
$(\neg\flat_3, \varepsilon_3, 1)$	∗	∗	∗	∗	∗∗	∗	∗
$(\neg\flat_1, \varepsilon_1, 0)$	∗∗∗	∗∗	◦	∗∗	∗	∗∗∗	∗∗
$(\neg\flat_1, \varepsilon_2, 0)$	∗	∗∗	∗	∗∗	∗∗	∗	∗
$(\neg\flat_1, \varepsilon_3, 0)$	∗∗	∗	∗∗∗	∗∗∗	◦	∗	∗
$(\neg\flat_2, \varepsilon_1, 0)$	∗∗	∗∗	∗∗	∗∗	◦	∗∗	◦
$(\neg\flat_2, \varepsilon_2, 0)$	∗∗∗∗	∗∗	∗∗	∗∗	∗∗	◦	∗
$(\neg\flat_2, \varepsilon_3, 0)$	◦	∗	∗∗∗	∗∗	∗∗	∗	∗∗
$(\neg\flat_3, \varepsilon_1, 0)$	∗∗	∗∗∗	∗	∗	∗∗	∗	◦
$(\neg\flat_3, \varepsilon_2, 0)$	∗∗	∗	∗∗	∗∗	◦	∗∗	∗∗
$(\neg\flat_3, \varepsilon_3, 0)$	∗∗	∗∗	∗	∗	◦	∗	◦

Checkmarks are translated into numerals (0, 1, 2, 3, 4) for calculations, as shown in Example 1. Consequently, the experts build an F5-BSE set $(\psi, \mu, \mathcal{K}, 5)$, as proposed in Table 9.

Table 9. Evaluations of healthcare facility location proposals by experts using F5-BSE set $(\psi, \mu, \mathcal{K}, 5)$

$(\psi, \mu, \mathcal{K}, 5)$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
$(\flat_1, \varepsilon_1, 1)$	$\langle 3, 0.6\rangle$	$\langle 2, 0.5\rangle$	$\langle 4, 1\rangle$	$\langle 2, 0.5\rangle$	$\langle 2, 0.4\rangle$	$\langle 3, 0.6\rangle$	$\langle 2, 0.4\rangle$
$(\flat_1, \varepsilon_2, 1)$	$\langle 1, 0.2\rangle$	$\langle 3, 0.7\rangle$	$\langle 0, 0\rangle$	$\langle 3, 0.6\rangle$	$\langle 1, 0.2\rangle$	$\langle 3, 0.6\rangle$	$\langle 3, 0.6\rangle$
$(\flat_1, \varepsilon_3, 1)$	$\langle 2, 0.4\rangle$	$\langle 1, 0.3\rangle$	$\langle 3, 0.6\rangle$	$\langle 3, 0.6\rangle$	$\langle 1, 0.3\rangle$	$\langle 1, 0.3\rangle$	$\langle 3, 0.6\rangle$
$(\flat_2, \varepsilon_1, 1)$	$\langle 3, 0.6\rangle$	$\langle 2, 0.5\rangle$	$\langle 2, 0.5\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.2\rangle$	$\langle 3, 0.7\rangle$	$\langle 0, 0\rangle$
$(\flat_2, \varepsilon_2, 1)$	$\langle 3, 0.6\rangle$	$\langle 0, 0.1\rangle$	$\langle 3, 0.6\rangle$	$\langle 2, 0.4\rangle$	$\langle 3, 0.7\rangle$	$\langle 0, 0.1\rangle$	$\langle 2, 0.4\rangle$
$(\flat_2, \varepsilon_3, 1)$	$\langle 0, 0\rangle$	$\langle 2, 0.4\rangle$	$\langle 4, 0.9\rangle$	$\langle 2, 0.4\rangle$	$\langle 2, 0.4\rangle$	$\langle 0, 0\rangle$	$\langle 2, 0.4\rangle$
$(\flat_3, \varepsilon_1, 1)$	$\langle 3, 0.7\rangle$	$\langle 3, 0.6\rangle$	$\langle 1, 0.2\rangle$	$\langle 2, 0.5\rangle$	$\langle 3, 0.7\rangle$	$\langle 0, 0.1\rangle$	$\langle 3, 0.6\rangle$
$(\flat_3, \varepsilon_2, 1)$	$\langle 2, 0.4\rangle$	$\langle 0, 0\rangle$	$\langle 3, 0.7\rangle$	$\langle 1, 0.3\rangle$	$\langle 1, 0.3\rangle$	$\langle 3, 0.6\rangle$	$\langle 2, 0.5\rangle$
$(\flat_3, \varepsilon_3, 1)$	$\langle 2, 0.4\rangle$	$\langle 3, 0.6\rangle$	$\langle 3, 0.6\rangle$	$\langle 3, 0.6\rangle$	$\langle 2, 0.4\rangle$	$\langle 3, 0.7\rangle$	$\langle 0, 0.1\rangle$
$(\flat_1, \varepsilon_1, 0)$	$\langle 1, 0.2\rangle$	$\langle 2, 0.5\rangle$	$\langle 0, 0\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.2\rangle$	$\langle 0, 0.1\rangle$	$\langle 2, 0.4\rangle$
$(\flat_1, \varepsilon_2, 0)$	$\langle 2, 0.4\rangle$	$\langle 1, 0.3\rangle$	$\langle 1, 0.3\rangle$	$\langle 1, 0.2\rangle$	$\langle 2, 0.5\rangle$	$\langle 1, 0.2\rangle$	$\langle 1, 0.3\rangle$
$(\flat_1, \varepsilon_3, 0)$	$\langle 1, 0.2\rangle$	$\langle 3, 0.7\rangle$	$\langle 1, 0.2\rangle$	$\langle 0, 0\rangle$	$\langle 3, 0.6\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.2\rangle$
$(\flat_2, \varepsilon_1, 0)$	$\langle 0, 0\rangle$	$\langle 1, 0.2\rangle$	$\langle 0, 0\rangle$	$\langle 1, 0.2\rangle$	$\langle 3, 0.6\rangle$	$\langle 1, 0.2\rangle$	$\langle 0, 0\rangle$
$(\flat_2, \varepsilon_2, 0)$	$\langle 0, 0.1\rangle$	$\langle 0, 0\rangle$	$\langle 1, 0.2\rangle$	$\langle 1, 0.3\rangle$	$\langle 1, 0.2\rangle$	$\langle 4, 0.8\rangle$	$\langle 2, 0.4\rangle$
$(\flat_2, \varepsilon_3, 0)$	$\langle 3, 0.7\rangle$	$\langle 2, 0.4\rangle$	$\langle 0, 0.1\rangle$	$\langle 2, 0.5\rangle$	$\langle 1, 0.3\rangle$	$\langle 1, 0.2\rangle$	$\langle 1, 0.2\rangle$
$(\flat_3, \varepsilon_1, 0)$	$\langle 1, 0.2\rangle$	$\langle 1, 0.2\rangle$	$\langle 3, 0.6\rangle$	$\langle 2, 0.4\rangle$	$\langle 0, 0\rangle$	$\langle 3, 0.6\rangle$	$\langle 1, 0.2\rangle$
$(\flat_3, \varepsilon_2, 0)$	$\langle 2, 0.4\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.2\rangle$	$\langle 1, 0.2\rangle$	$\langle 3, 0.7\rangle$	$\langle 1, 0.2\rangle$	$\langle 2, 0.5\rangle$
$(\flat_3, \varepsilon_3, 0)$	$\langle 1, 0.2\rangle$	$\langle 0, 0\rangle$	$\langle 1, 0.3\rangle$	$\langle 0, 0\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.2\rangle$	$\langle 4, 0.9\rangle$
$(\neg\flat_1, \varepsilon_1, 1)$	$\langle 0, 0.1\rangle$	$\langle 1, 0.2\rangle$	$\langle 0, 0\rangle$	$\langle 2, 0.4\rangle$	$\langle 2, 0.5\rangle$	$\langle 1, 0.2\rangle$	$\langle 1, 0.2\rangle$
$(\neg\flat_1, \varepsilon_2, 1)$	$\langle 2, 0.4\rangle$	$\langle 1, 0.3\rangle$	$\langle 3, 0.6\rangle$	$\langle 1, 0.2\rangle$	$\langle 2, 0.4\rangle$	$\langle 0, 0\rangle$	$\langle 0, 0.1\rangle$
$(\neg\flat_1, \varepsilon_3, 1)$	$\langle 2, 0.4\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.2\rangle$	$\langle 1, 0.3\rangle$	$\langle 2, 0.4\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.2\rangle$
$(\neg\flat_2, \varepsilon_1, 1)$	$\langle 1, 0.3\rangle$	$\langle 1, 0.2\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.2\rangle$	$\langle 2, 0.5\rangle$	$\langle 0, 0\rangle$	$\langle 4, 0.8\rangle$
$(\neg\flat_2, \varepsilon_2, 1)$	$\langle 0, 0\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.2\rangle$	$\langle 1, 0.2\rangle$	$\langle 1, 0.2\rangle$	$\langle 3, 0.6\rangle$	$\langle 2, 0.4\rangle$
$(\neg\flat_2, \varepsilon_3, 1)$	$\langle 4, 0.9\rangle$	$\langle 2, 0.4\rangle$	$\langle 0, 0\rangle$	$\langle 2, 0.5\rangle$	$\langle 1, 0.3\rangle$	$\langle 2, 0.4\rangle$	$\langle 2, 0.4\rangle$
$(\neg\flat_3, \varepsilon_1, 1)$	$\langle 1, 0.2\rangle$	$\langle 1, 0.2\rangle$	$\langle 3, 0.6\rangle$	$\langle 2, 0.4\rangle$	$\langle 0, 0\rangle$	$\langle 1, 0.2\rangle$	$\langle 0, 0.1\rangle$
$(\neg\flat_3, \varepsilon_2, 1)$	$\langle 1, 0.2\rangle$	$\langle 0, 0\rangle$	$\langle 1, 0.3\rangle$	$\langle 2, 0.4\rangle$	$\langle 3, 0.7\rangle$	$\langle 1, 0.3\rangle$	$\langle 2, 0.5\rangle$
$(\neg\flat_3, \varepsilon_3, 1)$	$\langle 1, 0.2\rangle$	$\langle 1, 0.2\rangle$	$\langle 1, 0.3\rangle$	$\langle 1, 0.3\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.2\rangle$	$\langle 1, 0.2\rangle$
$(\neg\flat_1, \varepsilon_1, 0)$	$\langle 3, 0.6\rangle$	$\langle 2, 0.5\rangle$	$\langle 0, 0\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.2\rangle$	$\langle 3, 0.6\rangle$	$\langle 2, 0.4\rangle$
$(\neg\flat_1, \varepsilon_2, 0)$	$\langle 1, 0.3\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.2\rangle$	$\langle 2, 0.4\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.2\rangle$	$\langle 1, 0.2\rangle$
$(\neg\flat_1, \varepsilon_3, 0)$	$\langle 2, 0.5\rangle$	$\langle 1, 0.3\rangle$	$\langle 3, 0.6\rangle$	$\langle 3, 0.6\rangle$	$\langle 0, 0.1\rangle$	$\langle 1, 0.3\rangle$	$\langle 1, 0.3\rangle$
$(\neg\flat_2, \varepsilon_1, 0)$	$\langle 2, 0.4\rangle$	$\langle 2, 0.4\rangle$	$\langle 2, 0.5\rangle$	$\langle 2, 0.5\rangle$	$\langle 0, 0\rangle$	$\langle 2, 0.5\rangle$	$\langle 0, 0.1\rangle$
$(\neg\flat_2, \varepsilon_2, 0)$	$\langle 4, 0.8\rangle$	$\langle 2, 0.5\rangle$	$\langle 2, 0.4\rangle$	$\langle 2, 0.4\rangle$	$\langle 2, 0.4\rangle$	$\langle 0, 0\rangle$	$\langle 1, 0.2\rangle$
$(\neg\flat_2, \varepsilon_3, 0)$	$\langle 0, 0\rangle$	$\langle 1, 0.2\rangle$	$\langle 3, 0.6\rangle$	$\langle 2, 0.4\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.2\rangle$	$\langle 2, 0.4\rangle$
$(\neg\flat_3, \varepsilon_1, 0)$	$\langle 2, 0.5\rangle$	$\langle 3, 0.7\rangle$	$\langle 1, 0.3\rangle$	$\langle 1, 0.2\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.2\rangle$	$\langle 0, 0\rangle$
$(\neg\flat_3, \varepsilon_2, 0)$	$\langle 2, 0.4\rangle$	$\langle 1, 0.2\rangle$	$\langle 2, 0.4\rangle$	$\langle 2, 0.4\rangle$	$\langle 0, 0\rangle$	$\langle 2, 0.4\rangle$	$\langle 2, 0.5\rangle$
$(\neg\flat_3, \varepsilon_3, 0)$	$\langle 2, 0.4\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.2\rangle$	$\langle 1, 0.2\rangle$	$\langle 0, 0\rangle$	$\langle 1, 0.3\rangle$	$\langle 0, 0.1\rangle$

To begin applying the FN-BSE set algorithm to our healthcare facility allocation case study, we follow the computational procedure outlined in Algorithm 1, as detailed below:

1. **Construct FN-BSE sets:** Based on the inputs-alternatives W, decision parameters B, experts E, binary opinions $\mathcal{O} = \{0 = \text{disagree},\ 1 = \text{agree}\}$, and the F5-BSE set $(\psi, \mu, \mathcal{K}, 5)$-we formed the positive agree, positive disagree, negative agree, and negative disagree FN-BSE sets using Table 9. Grades were omitted in the example, as the FN-BSE model primarily relies on fuzzy membership values in the DM process. The experts organized the data into structured tables (Tables 10-13), where ω_{ij} represents the corresponding membership values used to compute the aggregate scores $\sigma_j^{\oplus 1}$, $\sigma_j^{\oplus 0}$, $\sigma_j^{\ominus 1}$, and $\sigma_j^{\ominus 0}$.

Table 10. Tabulated illustration of a positive agree F5-BSE set $(\psi, \mu, \mathcal{K}, 5)^{\oplus 1}$ and computing $\sigma_j^{\oplus 1}$

$(\psi, \mu, \mathcal{K}, 5)^{\oplus 1}$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
$(\flat_1, \varepsilon_1, 1)$	0.6	0.5	1	0.5	0.4	0.6	0.4
$(\flat_1, \varepsilon_2, 1)$	0.2	0.7	0	0.6	0.2	0.6	0.6
$(\flat_1, \varepsilon_3, 1)$	0.4	0.3	0.6	0.6	0.3	0.3	0.6
$(\flat_2, \varepsilon_1, 1)$	0.6	0.5	0.5	0.4	0.2	0.7	0
$(\flat_2, \varepsilon_2, 1)$	0.6	0.1	0.6	0.4	0.7	0.1	0.4
$(\flat_2, \varepsilon_3, 1)$	0	0.4	0.9	0.4	0.4	0	0.4
$(\flat_3, \varepsilon_1, 1)$	0.7	0.6	0.2	0.5	0.7	0.1	0.6
$(\flat_3, \varepsilon_2, 1)$	0.4	0	0.7	0.3	0.3	0.6	0.5
$(\flat_3, \varepsilon_3, 1)$	0.4	0.6	0.6	0.6	0.4	0.7	0.1
$\sigma_j^{\oplus 1} = \sum_i \omega_{ij}$	$\sigma_1^{\oplus 1} = 3.9$	$\sigma_2^{\oplus 1} = 3.7$	$\sigma_3^{\oplus 1} = 5.1$	$\sigma_4^{\oplus 1} = 4.3$	$\sigma_5^{\oplus 1} = 3.6$	$\sigma_6^{\oplus 1} = 3.7$	$\sigma_7^{\oplus 1} = 3.6$

Table 11. Tabulated illustration of a positive disagree F5-BSE set $(\psi, \mu, \mathcal{K}, 5)^{\oplus 0}$ and computing $\sigma_j^{\oplus 0}$

$(\psi, \mu, \mathcal{K}, 5)^{\oplus 0}$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
$(\flat_1, \varepsilon_1, 0)$	0.2	0.5	0	0.4	0.2	0.1	0.4
$(\flat_1, \varepsilon_2, 0)$	0.4	0.3	0.3	0.2	0.5	0.2	0.3
$(\flat_1, \varepsilon_3, 0)$	0.2	0.7	0.2	0	0.6	0.4	0.2
$(\flat_2, \varepsilon_1, 0)$	0	0.2	0	0.2	0.6	0.2	0
$(\flat_2, \varepsilon_2, 0)$	0.1	0	0.2	0.3	0.2	0.8	0.4
$(\flat_2, \varepsilon_3, 0)$	0.7	0.4	0.1	0.5	0.3	0.2	0.2
$(\flat_3, \varepsilon_1, 0)$	0.2	0.2	0.6	0.4	0	0.6	0.2
$(\flat_3, \varepsilon_2, 0)$	0.4	0.4	0.2	0.2	0.7	0.2	0.5
$(\flat_3, \varepsilon_3, 0)$	0.2	0	0.3	0	0.4	0.2	0.9
$\sigma_j^{\oplus 0} = \sum_i \omega_{ij}$	$\sigma_1^{\oplus 0} = 2.4$	$\sigma_2^{\oplus 0} = 2.7$	$\sigma_3^{\oplus 0} = 1.9$	$\sigma_4^{\oplus 0} = 2.2$	$\sigma_5^{\oplus 0} = 3.5$	$\sigma_6^{\oplus 0} = 2.9$	$\sigma_7^{\oplus 0} = 3.1$

Table 12. Tabulated illustration of a negative agree F5-BSE set $(\psi, \mu, \mathcal{K}, 5)^{\ominus 1}$ and computing $\sigma_j^{\ominus 1}$

$(\psi, \mu, \mathcal{K}, 5)^{\ominus 1}$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
$(\neg\flat_1, \varepsilon_1, 1)$	0.1	0.2	0	0.4	0.5	0.2	0.2
$(\neg\flat_1, \varepsilon_2, 1)$	0.4	0.3	0.6	0.2	0.4	0	0.1
$(\neg\flat_1, \varepsilon_3, 1)$	0.4	0.4	0.2	0.3	0.4	0.4	0.2
$(\neg\flat_2, \varepsilon_1, 1)$	0.3	0.2	0.4	0.2	0.5	0	0.8
$(\neg\flat_2, \varepsilon_2, 1)$	0	0.4	0.2	0.2	0.2	0.6	0.4
$(\neg\flat_2, \varepsilon_3, 1)$	0.9	0.4	0	0.5	0.3	0.4	0.4
$(\neg\flat_3, \varepsilon_1, 1)$	0.2	0.2	0.6	0.4	0	0.2	0.1
$(\neg\flat_3, \varepsilon_2, 1)$	0.2	0	0.3	0.4	0.7	0.3	0.5
$(\neg\flat_3, \varepsilon_3, 1)$	0.2	0.2	0.3	0.3	0.4	0.2	0.2
$\sigma_j^{\ominus 1} = \sum_i \omega_{ij}$	$\sigma_1^{\ominus 1} = 2.7$	$\sigma_2^{\ominus 1} = 2.3$	$\sigma_3^{\ominus 1} = 2.6$	$\sigma_4^{\ominus 1} = 2.9$	$\sigma_5^{\ominus 1} = 3.4$	$\sigma_6^{\ominus 1} = 2.3$	$\sigma_7^{\ominus 1} = 2.9$

Table 13. Tabulated illustration of a negative disagree F5-BSE set $(\psi,\ \mu,\ \mathcal{K},\ 5)^{\ominus 0}$ and computing $\sigma_j^{\ominus 0}$

$(\psi,\ \mu,\ \mathcal{K},\ 5)^{\ominus 0}$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
$(\neg b_1,\ \varepsilon_1,\ 0)$	0.6	0.5	0	0.4	0.2	0.6	0.4
$(\neg b_1,\ \varepsilon_2,\ 0)$	0.3	0.4	0.2	0.4	0.4	0.2	0.2
$(\neg b_1,\ \varepsilon_3,\ 0)$	0.5	0.3	0.6	0.6	0.1	0.3	0.3
$(\neg b_2,\ \varepsilon_1,\ 0)$	0.4	0.4	0.5	0.5	0	0.5	0.1
$(\neg b_2,\ \varepsilon_2,\ 0)$	0.8	0.5	0.4	0.4	0.4	0	0.2
$(\neg b_2,\ \varepsilon_3,\ 0)$	0	0.2	0.6	0.4	0.4	0.2	0.4
$(\neg b_3,\ \varepsilon_1,\ 0)$	0.5	0.7	0.3	0.2	0.4	0.2	0
$(\neg b_3,\ \varepsilon_2,\ 0)$	0.4	0.2	0.4	0.4	0	0.4	0.5
$(\neg b_3,\ \varepsilon_3,\ 0)$	0.4	0.4	0.2	0.2	0	0.3	0.1
$\sigma_j^{\ominus 0} = \sum_i \omega_{ij}$	$\sigma_1^{\ominus 0} = 3.9$	$\sigma_2^{\ominus 0} = 3.6$	$\sigma_3^{\ominus 0} = 3.2$	$\sigma_4^{\ominus 0} = 3.5$	$\sigma_5^{\ominus 0} = 1.9$	$\sigma_6^{\ominus 0} = 2.7$	$\sigma_7^{\ominus 0} = 2.2$

2. **Compute aggregate scores:** For each alternative, the aggregate scores were calculated from the corresponding FN-BSE sets, as summarized in Tables 14 and 15.

Table 14. Positive score table

$\sigma_j^{\oplus 1}$	$\sigma_j^{\oplus 0}$	$s_j^{\oplus} = \sigma_j^{\oplus 1} - \sigma_j^{\oplus 0}$
$\sigma_1^{\oplus 1} = 3.9$	$\sigma_1^{\oplus 0} = 2.4$	$s_1^{\oplus} = 1.5$
$\sigma_2^{\oplus 1} = 3.7$	$\sigma_2^{\oplus 0} = 2.7$	$s_2^{\oplus} = 1$
$\sigma_3^{\oplus 1} = 5.1$	$\sigma_3^{\oplus 0} = 1.9$	$s_3^{\oplus} = 3.2$
$\sigma_4^{\oplus 1} = 4.3$	$\sigma_4^{\oplus 0} = 2.2$	$s_4^{\oplus} = 2.1$
$\sigma_5^{\oplus 1} = 3.6$	$\sigma_5^{\oplus 0} = 3.5$	$s_5^{\oplus} = 0.1$
$\sigma_6^{\oplus 1} = 3.7$	$\sigma_6^{\oplus 0} = 2.9$	$s_6^{\oplus} = 0.8$
$\sigma_7^{\oplus 1} = 3.6$	$\sigma_7^{\oplus 0} = 3.1$	$s_7^{\oplus} = 0.5$

Table 15. Negative score table

$\sigma_j^{\ominus 1}$	$\sigma_j^{\ominus 0}$	$s_j^{\ominus} = \sigma_j^{\ominus 1} - \sigma_j^{\ominus 0}$
$\sigma_1^{\ominus 1} = 2.7$	$\sigma_1^{\ominus 0} = 3.9$	$s_1^{\ominus} = -1.2$
$\sigma_2^{\ominus 1} = 2.3$	$\sigma_2^{\ominus 0} = 3.6$	$s_2^{\ominus} = -1.3$
$\sigma_3^{\ominus 1} = 2.6$	$\sigma_3^{\ominus 0} = 3.2$	$s_3^{\ominus} = -0.6$
$\sigma_4^{\ominus 1} = 2.9$	$\sigma_4^{\ominus 0} = 3.5$	$s_4^{\ominus} = -0.6$
$\sigma_5^{\ominus 1} = 3.4$	$\sigma_5^{\ominus 0} = 1.9$	$s_5^{\ominus} = 1.5$
$\sigma_6^{\ominus 1} = 2.3$	$\sigma_6^{\ominus 0} = 2.7$	$s_6^{\ominus} = -0.4$
$\sigma_7^{\ominus 1} = 2.9$	$\sigma_7^{\ominus 0} = 2.2$	$s_7^{\ominus} = 0.7$

3. **Calculate net scores:** The net scores s_j for each location were computed by combining positive and negative impacts and the results are summarized in Table 16.

Table 16. Final score table

$s_j^{\oplus}$	$s_j^{\ominus}$	$s_j = s_j^{\oplus} - s_j^{\ominus}$
$s_1^{\oplus} = 1.5$	$s_1^{\ominus} = -1.2$	$s_1 = 2.7$
$s_2^{\oplus} = 1$	$s_2^{\ominus} = -1.3$	$s_2 = 2.3$
$s_3^{\oplus} = 3.2$	$s_3^{\ominus} = -0.6$	$s_3 = 3.8$
$s_4^{\oplus} = 2.1$	$s_4^{\ominus} = -0.6$	$s_4 = 2.7$
$s_5^{\oplus} = 0.1$	$s_5^{\ominus} = 1.5$	$s_5 = -1.4$
$s_6^{\oplus} = 0.8$	$s_6^{\ominus} = -0.4$	$s_6 = 1.2$
$s_7^{\oplus} = 0.5$	$s_7^{\ominus} = 0.7$	$s_7 = -0.2$

Output: The alternative with the highest net score, ω_3, is selected as the optimal healthcare facility location.

5. Analysis of fuzzy N-bipolar soft expert set model

In this section, we provide a comprehensive examination of the FN-BSE set model, structured to give a balanced and detailed evaluation. We begin by highlighting the model's key advantages, demonstrating its capacity to incorporate expert input, bipolarity, multinary assessments, and fuzzy data for robust DM. This is followed by a comparative analysis against existing approaches, presented through both qualitative and quantitative perspectives, including tables and figures to illustrate differences and performance in actual DM scenarios. Finally, we discuss the limitations of the FN-BSE framework, identifying potential challenges and areas for future improvement. This structure ensures a thorough assessment of the model's strengths, practical relevance, and constraints in complex decision contexts.

5.1 *Advantages*

The FN-BSE set model offers several notable advantages for DM:

• **Comprehensive Integration:** It seamlessly combines expert input, bipolar evaluations, multinary assessments, and fuzzy data, providing a holistic framework for DM.

• **Flexible Evaluation:** The model is adaptable to a wide range of applications, particularly in environments characterized by uncertainty or imprecision.

• **Balanced Assessment:** By considering both positive and negative aspects, the model ensures well-rounded evaluations, which is particularly valuable in healthcare and risk-sensitive contexts.

• **Suitability for Complex Scenarios:** Its ability to handle various types of evaluations makes it ideal for MAGDM situations.

• **Enhanced Accuracy:** Incorporating expert judgments improves the reliability and domain-specific accuracy of decisions.

These features make the FN-BSE set model a versatile and effective tool for addressing a wide spectrum of DM challenges.

5.2 *Comparative analysis of the proposed model*

In this subsection, we evaluate the proposed FN-BSE set model by comparing it with existing approaches using both qualitative and quantitative criteria. The qualitative comparison highlights the conceptual and methodological distinctions between models, while the quantitative comparison illustrates the impact of these distinctions through actual DM scenarios. Together, these analyses demonstrate the advantages of the FN-BSE set framework in handling complex, bipolar, and multinary decision problems.

5.2.1 *Qualitative comparison*

This part compares the FN-BSE set model with a range of currently available techniques from the literature. The models considered span from simple binary S-set models to advanced frameworks that incorporate Expert Input (EI), Bipolarity Consideration (BC), Multinary Evaluation (ME), and Membership Scale Applicability (MSA). Table 17 summarizes the key characteristics of each approach.

The FN-BSE set model distinguishes itself by integrating all these features into a single, unified framework. While earlier models might focus solely on fuzzy data (FS set), expert input (SE set), or bipolarity (BS set), the FN-BSE set model simultaneously addresses all dimensions, providing a more holistic structure for DM. This comparison underlines the strengths and limitations of existing models and motivates the need for a comprehensive approach like FN-BSE set.

Table 17. Comparison of the proposed model with related available techniques

Approach	EI	BC	ME	MS	Description
S-set [14]	No	No	No	No	Basic binary model lacking advanced features.
FS set [17]	No	No	No	Yes	Extends S-set with fuzzy data.
SE set [42]	Yes	No	No	No	Introduces expert input but remains binary.
FSE set [43]	Yes	No	No	Yes	Extends SE set with fuzzy data.
BS set [28]	No	Yes	No	No	Considers bipolarity but excludes expert input.
FBS set [29]	No	Yes	No	Yes	Extends BS set with fuzzy data.
BSE set [45]	Yes	Yes	No	No	Combines expert input and bipolarity but remains binary.
FBSE set [46]	Yes	Yes	No	Yes	Similar to BSE set, but incorporates fuzzy data.
N-S set [34]	No	No	Yes	No	Basic multinary model without bipolarity or expert input.
FN-S set [35]	No	No	Yes	Yes	Multinary model extended with fuzzy evaluations.
N-SE set [44]	Yes	No	Yes	No	Introduces multinary evaluation with expert input.
FN-SE set [44]	Yes	No	Yes	Yes	Same as N-SE set but with fuzzy data.
N-BS set [50]	No	Yes	Yes	No	Combines bipolarity with multinary evaluation and scale applicability.
FN-BS set [51]	No	Yes	Yes	Yes	Extends N-BS set with fuzzy multinary evaluations, excluding expert input.
N-BSE set [53]	Yes	Yes	Yes	No	Introduces expert involvement into the N-BS set framework.
FN-BSE set (Proposed)	Yes	Yes	Yes	Yes	Comprehensive model integrating all features for detailed DM.

It is important to emphasize not only the structural differences summarized in Table 17, but also the advantages and limitations of these approaches. For example, S-set and FS set models are computationally simple and useful for basic DM problems; however, they cannot capture expert input or bipolar information, which limits their applicability in real-world scenarios. Similarly, SE set and FSE set introduce expert involvement, but their restriction to binary outcomes reduces their effectiveness when criteria require multiple levels of evaluation.

Models such as BS set and FBS set consider bipolarity, which is an advantage for problems that involve both positive and negative assessments, but they overlook expert perspectives and multinary evaluations. On the other hand, N-S set and FN-S set provide multinary evaluations, which improve flexibility, but they lack bipolarity consideration and expert involvement. More advanced models such as N-SE set and FN-SE set integrate expert input with multinary evaluations, yet they still ignore bipolarity, which is often essential in practical applications.

The FN-BSE set model overcomes these limitations by integrating all four essential features: expert input, bipolarity consideration, multinary evaluation, and membership scale applicability. This makes the FN-BSE set a comprehensive and balanced framework, enabling decision-makers to account for complex and uncertain environments where trade-offs between positive and negative aspects must be evaluated systematically. In this way, the FN-BSE set unifies the strengths of earlier models while mitigating their weaknesses.

5.2.2 *Quantitative comparison*

In complex DM scenarios, performance trade-offs are inevitable: an alternative may excel in certain criteria while lagging in others. Evaluating only the positive (desirable) aspects can favor highly specialized options, whereas focusing solely on negative (undesirable) aspects may overly penalize alternatives. Both perspectives are informative but incomplete, and relying on a single dimension may lead to biased or misleading rankings. In such cases, the FN-BSE model generalizes the FN-SE set [44] by incorporating both positive and negative evaluations.

To obtain a comprehensive assessment, we compare three evaluation strategies in the fuzzy environment. For this purpose, we use a synthetic dataset constructed specifically for this study, consisting of seven alternatives and six decision parameters, to illustrate the application of the FN-BSE set model.

1. **Positive evaluation only**, measuring the degree to which each alternative satisfies desirable attributes. In this case, the FN-BSE model reduces to an FN-SE set [44].

2. **Negative evaluation only**, quantifying limiting factors (smaller magnitude is better). Similarly, considering only negative evaluations also corresponds to an FN-SE set [44].

3. **Bipolar evaluation** with the FN-BSE set, integrating both positive and negative perspectives into a single net score to capture the overall balance between strengths and weaknesses (proposed model).

$$\text{Positive scores } s_j^{\oplus}: \ \omega_3 \succ \omega_4 \succ \omega_1 \succ \omega_2 \succ \omega_6 \succ \omega_7 \succ \omega_5, \tag{81}$$

$$\text{Negative scores } s_j^{\ominus} \text{ (smaller is better)}: \ \omega_5 \succ \omega_7 \succ \omega_6 \succ \omega_3 = \omega_4 \succ \omega_1 \succ \omega_2, \tag{82}$$

$$\text{Net scores } s_j = s_j^{\oplus} - s_j^{\ominus}: \ \omega_3 \succ \omega_1 = \omega_4 \succ \omega_2 \succ \omega_6 \succ \omega_7 \succ \omega_5. \tag{83}$$

Table 18 summarizes these results.

Table 18. Comparison of ranking orders based on positive, negative, and net scores for the considered alternatives

Method	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	Ranking order
Positive scores $s_j^{\oplus}$	1.5	1.0	3.2	2.1	0.1	0.8	0.5	$\omega_3 \succ \omega_4 \succ \omega_1 \succ \omega_2 \succ \omega_6 \succ \omega_7 \succ \omega_5$
Negative scores $s_j^{\ominus}$	-1.2	-1.3	-0.6	-0.6	1.5	-0.4	0.7	$\omega_5 \succ \omega_7 \succ \omega_6 \succ \omega_3 = \omega_4 \succ \omega_1 \succ \omega_2$
Net scores s_j	2.7	2.3	3.8	2.7	-1.4	1.2	-0.2	$\omega_3 \succ \omega_1 = \omega_4 \succ \omega_2 \succ \omega_6 \succ \omega_7 \succ \omega_5$

By integrating both positive and negative evaluations through bipolar scoring, the FN-BSE set model provides a more reliable and balanced ranking. Unlike single-perspective evaluations, which may exaggerate strengths or weaknesses, the bipolar approach accounts for both desirable contributions and limiting factors simultaneously. Consequently, alternatives that perform well while maintaining minimal drawbacks rise to the top, offering a nuanced and realistic basis for DM. Figures 2-4 visually illustrate how considering both aspects produces a more robust ranking compared with positive-only or negative-only evaluations.

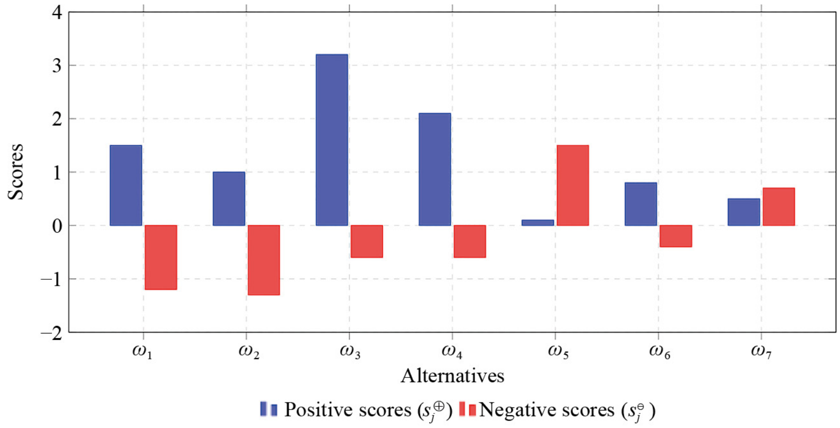

Figure 2. Positive and negative scores from the FN-SE set model [44]

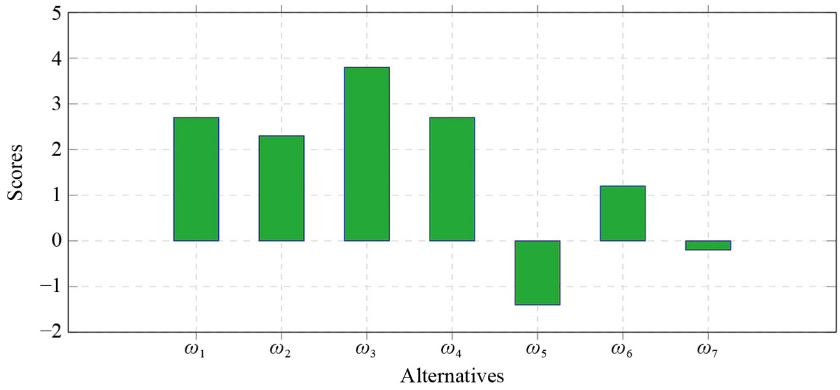

Figure 3. Net scores from the proposed FN-BSE set method

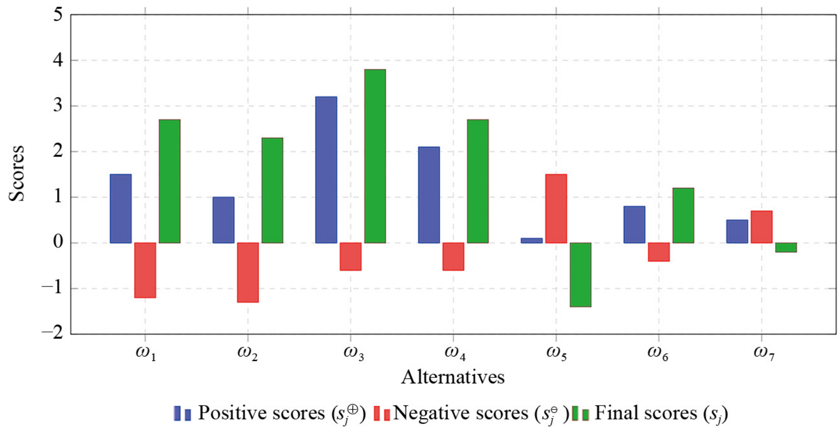

Figure 4. Comparison of positive, negative, and net scores for all alternatives

5.3 *Limitations*

Although the FN-BSE set model provides a comprehensive and unified framework for DM, several limitations should be acknowledged:

• **Dependence on Expert Input:** The framework requires reliable expert assessments. In cases where experts are unavailable, biased, or inconsistent, the quality of results may be affected. Resolving disagreements among multiple experts can also add complexity.

• **Scalability and Computational Complexity:** Aggregating fuzzy, bipolar, and multivalued judgments can be computationally demanding, especially with many criteria, alternatives, or experts.

• **Interpretability:** While mathematically rigorous, the model may be challenging for practitioners unfamiliar with fuzzy or bipolar approaches, potentially limiting adoption in contexts where simplicity is valued.

6. Conclusions and future directions

This study presented the FN-BSE set model as a unified framework for MAGDM under uncertainty, integrating expert opinions, bipolar evaluations, fuzzy numerical data, and multinary assessments. Through the proposed ranking algorithm and operations, the model effectively captured both positive and negative aspects of alternatives.

The performance of the FN-BSE model was demonstrated using a healthcare facility location case study. Quantitative comparisons showed that considering both positive and negative evaluations leads to more balanced and reliable rankings compared with single-perspective approaches. Specifically, the model consistently distinguished among competing alternatives, with the top-ranked options excelling across multiple criteria while maintaining minimal drawbacks.

Future research can address the current limitations in several ways. Hybrid approaches that combine expert knowledge with data-driven methods can reduce dependence on experts and improve the reliability of the model. Scalability and computational efficiency can be enhanced through optimized algorithms, parallel processing, or simplified aggregation techniques. Additionally, extensions such as intuitionistic, picture, interval-valued, or q-rung orthopair FN-BSE sets can enhance interpretability and usability. Together, these directions aim to make the FN-BSE framework more practical, robust, and accessible for complex DM scenarios.

Authorship contribution

Conceptualization, Sagvan Musa; Formal analysis, Sagvan Musa, Zanyar Ameen, Maha Saeed and Baravan Asaad; Funding acquisition, Sagvan Musa, Zanyar Ameen, Maha Saeed and Baravan Asaad; Investigation, Sagvan Musa, Zanyar Ameen, Maha Saeed and Baravan Asaad; Methodology, Sagvan Musa and Baravan Asaad; Validation, Zanyar Ameen, Maha Saeed and Baravan Asaad; Writing-original draft, Sagvan Musa; Writing-review & editing, Sagvan Musa, Zanyar Ameen and Maha Saeed.

Content for publication

During the preparation of this work, the author(s) never used any AI tools.

Conflict of interest

The authors declare no competing financial interest.

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