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Novel Fuzzy Versions of Generalized Fractional Integrals and Related Mathematical Inequalities via Newly Defined Fuzzy Convexity

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Abstract: In this study, the main aim is to establish a set of novel inequalities that enhance the mathematical inequalities discussed. This paper introduces a new fuzzy fractional integral framework along with associated inequalities by defining a novel class of fuzzy-valued convex mappings, termed up and down $(U \cdot D)$ fuzzy-valued generalized strong λ-convex mappings. Some new and classical exceptional cases are also obtained for generalized fuzzy fractional integral operators and $U \cdot D$-fuzzy-valued generalized strong A convex mapping. Some new forms of Hermite-Hadamard inequalities are also derived through fuzzy generalized fractional integrals and extends Pachpatte-type inequalities by applying products of $U \cdot D$-fuzzyvalued generalized strong λ-convex mappings. Additionally, several midpoint inequalities are introduced. Some open problems are also presented for future discussion.

***Keywords*:** up and down generalized strong λ-convex mapping, fuzzy fractional integrals, hermite-hadamard inequalities, pachpatte-type inequalities

MSC: 26A33, 26A51, 26D07, 26D10, 26D15, 26D20

1. Introduction

Fuzzy Theory, also known as Fuzzy Set Theory or Fuzzy Logic, was introduced by mathematician Lotfi Zadeh [1] in 1965 as a way to address uncertainty, imprecision, and vagueness in complex systems. Unlike classical logic and set theory, which require clear-cut, binary distinctions, Fuzzy Theory allows for varying degrees of membership, making it particularly well-suited for handling real-world situations that are not black-and-white. This approach is designed to mirror human reasoning, where things are often "somewhat true" or "almost certain" rather than strictly true or false. Many natural and human-made systems operate in environments filled with ambiguity and uncertainty. Traditional binary logic struggles to represent these complexities, as it assigns absolute values-0 or 1, true or false. For example, defining a "tall person" is not absolute; there is no clear line where "tall" begins. Fuzzy Theory allows elements to have partial

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membership, such as being "60% tall" or "75% warm," offering a mathematical way to represent this imprecision. Fuzzy Theory [2, 3] has transformed the way scientists and engineers approach problems involving uncertainty, bridging the gap between theoretical models and practical, real-world scenarios. By embracing partial truths and degrees of membership, Fuzzy Theory [4] has become an essential tool in technology, machine learning, and complex decision-making [5, 6], pushing the boundaries of classical logic and providing a more human-like approach to problem-solving [7–10].

On the other hand, fuzzy theory has extended the concepts and applications of calculus by providing tools to handle uncertainty, ambiguity, see [11–13], and partial truths in mathematical modeling and analysis. This contribution is especially beneficial in areas where the precision of traditional calculus is limited by real-world variability and imprecision, see [14–17]. Below are some of the key contributions of Fuzzy Theory in calculus: In fuzzy calculus [18, 19], the differentiation of fuzzy-valued functions allows for the analysis of rates of change where values are not sharply defined but have degrees of uncertainty. This is useful in systems where inputs or outputs fluctuate within a range or lack precise boundaries, see [20–23]. Fuzzy integrals [24], including fuzzy Riemann and Lebesgue integrals [25], extend classical integration methods by allowing fuzzy-valued functions over fuzzy sets, see [26–29]. This approach is widely applied in probabilistic reasoning and decision-making models where aggregation over uncertain data is necessary. Fuzzy Theory [30] has significantly expanded the scope of calculus, allowing it to handle complex, real-world systems where uncertainty and imprecision are intrinsic. By introducing concepts like fuzzy differentiation [31], integration [32], and differential equations [33], fuzzy calculus [34] provides a framework to model and solve problems that traditional calculus would find intractable due to a lack of precision, see [35, 36]. The result is a powerful toolkit for applications in diverse fields [37, 38], from engineering and physics to economics and artificial intelligence [39, 40], demonstrating the broad and transformative impact of fuzzy theory on calculus, see [41, 42]. Fractional calculus [41, 42], an extension of classical calculus, has gained significant attention in recent years due to its applicability in modeling real-world phenomena characterized by memory, hereditary properties, and fractal-like structures. When paired with fuzzy set theory, the resulting framework offers a powerful means to analyze systems with inherent uncertainty, vagueness, and imprecision-conditions often present in areas such as engineering, economics, and the natural sciences. Fuzzy fractional integrals [43, 44], therefore, provide a robust mathematical structure for handling these types of problems, where traditional crisp models may fall short.

Within this framework, inequalities play a crucial role, particularly fractional integral inequalities [45, 46]. These inequalities enable researchers to establish bounds, estimates, and constraints on solutions to various fuzzy fractional differential equations, contributing to the reliability and accuracy of models in complex systems. Recent advancements in fuzzy-valued [47] and fuzzy fractional operators [48] have led to generalized forms of classical inequalities [49], including Jensen [50], Hardy [51], and Hermite-Hadamard inequalities [52], adapted to the fuzzy fractional context. These generalized inequalities have broad applications, ranging from stability analysis in control systems to solution estimation in mathematical finance and image processing. Integral inequalities, such as Hermite-Hadamard [53, 54], Hölder [55], and Ostrowski inequalities [56, 57], serve as foundational tools for examining solution behavior in fractional calculus. By establishing these inequalities, researchers can derive error bounds, solution estimates, and stability conditions for fractional differential equations, making them indispensable in the analysis of fractional systems, see [58]. In recent years, there has been considerable interest in generalizing these classical inequalities to the fractional setting, resulting in what are now known as fractional integral inequalities [59]. Fractional integral inequalities [60] are pivotal in the study of fractional calculus, a field that extends traditional calculus to non-integer (fractional) orders, allowing for more flexible modeling of complex, real-world systems. Fractional calculus has proven essential in various scientific and engineering domains, including control theory, physics, biological systems, and finance, due to its ability to capture memory and hereditary properties of processes. Simpson integral inequalities [61] provide valuable tools for bounding and estimating solutions to fractional differential equations, which are often used to describe dynamic systems with these intricate characteristics.

Interval-valued fractional integral inequalities [62, 63] represent a significant advancement in the study of fractional calculus, combining interval analysis and fractional calculus to address uncertainties in modeling complex systems. Fractional calculus, a generalization of classical calculus to non-integer orders, provides robust tools for understanding processes with memory effects and hereditary properties, which are common in diverse fields such as physics, biology, engineering, and economics. The integration of interval-valued functions [64, 65] into this framework allows researchers to model situations where parameters are not fixed but vary within known bounds, adding a layer of flexibility and realism to mathematical models of uncertain or imprecise systems, see [66–70]. For more information, see [71–77] and the references therein.

Inspired by the ongoing research work, in Section 3, this paper introduces new fuzzy fractional integral as well as their related inequalities by newly defined class of fuzzy-valued convex mappings which is known as fuzzy-valued generalized strong λ-convex mapping. In Section 4, Fuzzy generalized fractional integrals are introduced. Additionally, some classical and new exceptional cases are also obtained. Sections 5 and 6 discuss some new version of Hermite-Hadamrd inequalities via fuzzy generalized fractional integrals as well by taking product of fuzzy-valued generalized strong λ-convex mappings, some Pachpatte type inequalities are also obtained. Section 7 presents some of the restrictions and limitations of the proved results. Through this work, we aim to contribute a set of novel inequalities that improve the mathematical inequalities that are discussed in Section 6.

2. Preliminary concepts

Let $\check{R}$ be the set of real numbers. A fuzzy subset A of $\check{R}$ is characterize by a mapping $\tilde{Z}:\check{R}\to[0,\ 1]$ called the membership function, for each fuzzy set and $\gamma\in(0,\ 1]$, then γ-level sets of $\tilde{z}$ is denoted and defined as follows $z_\gamma=\{\mathfrak{z}\in\check{R}\mid\tilde{z}(\mathfrak{z})\geq\gamma\}$. If $\gamma=0$, then $\text{supp}(\tilde{z})=\{\mathfrak{z}\in\tilde{R}\mid\tilde{z}(\mathfrak{z})>0\}$ is called support of $\tilde{z}$. By $[\tilde{z}]^0$ we define the closure of $\text{supp}(\tilde{z})$.

Definition 1 [40] A fuzzy set is defined as a fuzzy number $(F\cdot N)$ possessing the subsequent attributes:

a. $\tilde{z}$ is normal, meaning that $\mathfrak{z}\in\check{R}$ exists and $\tilde{z}(\mathfrak{z})=1$;

b. z is upper semicontinuous, for every $\vartheta\in\check{R}$ that is a part of with $|\mathfrak{z}-\vartheta|<\delta$, there exists $\varepsilon>0$ and $\delta>0$ such that $\tilde{z}(\mathfrak{z})-\tilde{z}(\vartheta)<\varepsilon$;

c. z is fuzzy convex, since $\tilde{z}((1-t\mathfrak{r})\mathfrak{z}+t\vartheta)\geq\min(\tilde{z}(\mathfrak{z}),\ \tilde{z}(\vartheta))\forall\mathfrak{z},\ \vartheta\in\check{R},\ t\in[0,\ 1]$. $[\tilde{z}]^0$ is compact.

All $F\cdot N\mathfrak{S}$ are represented by the set $\mathbb{F}_0$. Distinguishing between the following γ-levels is convenient for $F\cdot Ns$:

$$z_\gamma=\{\mathfrak{z}\in\check{R}\mid\tilde{z}(\mathfrak{z})\geq\gamma\}.$$

From these definitions, we have

$$z_\gamma=[z_*(\gamma),\ z^*(\gamma)],$$

where

$$z_*(\gamma)=\inf\{\mathfrak{z}\in\check{R}\mid\tilde{z}(\mathfrak{z})\geq\gamma\},\ z^*(\gamma)=\sup\{\mathfrak{z}\in\check{R}\mid\tilde{z}(\mathfrak{z})\geq\gamma\}.$$

Since each $e\in\check{R}$ is also a $F\cdot N$, defined as

$$\tilde{e}(\mathfrak{z})=\begin{cases}1\text{ if }\mathfrak{z}=e,\\ \\ 0\text{ if }\mathfrak{z}\neq e.\end{cases}$$

Thus, a $F\cdot N\tilde{z}$ can be identified by a parametrized pair

$$\{(z_*(\gamma),\ z^*(\gamma)) : \gamma \in [0,\ 1]\}.$$

This leads the following characterization of a $F \cdot N$ in terms of the two end point functions $Z_*(\gamma)$ and $Z^*(\gamma)$.

Theorem 1 [40] Suppose that $z_*(.) : [0,\ 1] \to \check{R}$ and $z^*(.) : [0,\ 1] \to \check{R}$ satisfy the following conditions:

(1) $Z_*(\gamma)$ should be non-decreasing function.

(2) $Z^*(\gamma)$ should be non-increasing function.

(3) $z_*(1) \leq z^*(1)$.

(4) $Z_*(\gamma)$ and $Z^*(\gamma)$ are bounded and left continuous on $(0,\ 1]$ and right continuous at $\gamma = 0$.

Furthermore, if $\tilde{Z} : \check{R} \to [0,\ 1]$ is a $F \cdot N$, and its parametrization is $\{(z_*(\gamma),\ z^*(\gamma)) : \gamma \in [0,\ 1]\}$ then the functions $z_*(\gamma)$ and $z^*(\gamma)$ determine the criteria (1) through (4).

Let $\tilde{z},\ \tilde{\mathrm{o}} \in \mathbb{F}_0$ represented parametrically $\{(z_*(\gamma),\ z^*(\gamma)) : \gamma \in [0,\ 1]\}$ and $\{(\mathrm{o}_*(\gamma),\ \mathrm{o}^*(\gamma)) : \gamma \in [0,\ 1]\}$ respectively. We say that $\tilde{z} \leq_{\mathbb{F}} \tilde{\mathrm{o}}$ if for all $\gamma \in (0,\ 1]$, $z^*(\gamma) \leq 0^*(\gamma)$, and $z_*(\gamma) \leq 0_*(\gamma)$. If $\tilde{z} \leq_{\mathbb{F}} \tilde{\mathrm{o}}$, then there exist $\underset{\sim}{\gamma} \in (0,\ 1]$ such that $z^*(\gamma) < \mathrm{o}^*(\gamma)$ or $z_*(\gamma) \leq \mathrm{o}_*(\gamma)$. We say comparable if for any $\tilde{z},\ \tilde{\mathrm{o}} \in \mathbb{F}_0$, we have $\tilde{z} \leq_{\mathbb{F}} \tilde{\mathrm{o}}$ or $\tilde{z} \geq_{\mathbb{F}} \tilde{\mathrm{o}}$ otherwise they are non-comparable.

We can state that $\mathbb{F}_0$ is a partial ordered set under the relation $\leq_{\mathbb{F}}$ if we write $\tilde{z} \leq_{\mathbb{F}} \tilde{0}$ rather than $\tilde{0} \geq_{\mathbb{F}} \tilde{z}$ at times.

In the event that $\tilde{z},\ \tilde{\mathrm{o}} \in \mathbb{F}_0$, then $\tilde{v} \in \mathbb{F}_0$ such that $\tilde{z} = \tilde{\mathrm{o}} \oplus \tilde{v}$. This suggests the presence of a generalized Hukuhara (gH) difference between $\tilde{z}$ and $\tilde{\mathrm{o}}$; this is referred to the $g\mathcal{H}$-difference between $\tilde{z}$ and $\tilde{\mathrm{o}}$, and is represented by $\tilde{z} \ominus_{g\mathcal{H}} \tilde{\mathrm{o}}$, as per [33]. If $g\mathcal{H}$-difference exists, then

$$(\tilde{v})^*(\gamma) = \left(\tilde{z} \ominus_{g\mathcal{H}} \tilde{\mathrm{o}}\right)^*(\gamma) = z^*(\gamma) - o^*(\gamma),\ (\tilde{v})_*(\gamma) = \left(\tilde{z} \ominus_{g\mathcal{H}} \tilde{\mathrm{o}}\right)_*(\gamma) = z_*(\gamma) - o_*(\gamma),$$

and

$$\tilde{z} \ominus_{\mathcal{GH}} \tilde{\mathrm{o}} = \tilde{v} \Leftrightarrow \begin{cases} \tilde{v} = \tilde{z} \ominus_{gH} \tilde{\mathrm{o}}, \\ \\ \tilde{\mathrm{or}}\ \tilde{z} = \tilde{\mathrm{o}} \oplus (-1) \odot \tilde{v}. \end{cases}$$

Now that we've covered a few characteristics of $F \cdot Ns$ under addition and scaler multiplication, we can say that if $0 < \mathrm{e} \in \check{R}$ and $\tilde{z},\ \tilde{\mathrm{o}} \in \mathbb{F}_0$, then $\mathrm{e} \odot \tilde{z}$ and $\tilde{z} \oplus \tilde{\mathrm{o}}$ define as

$$\tilde{z} \oplus \tilde{\mathrm{o}} = \{(z_*(\gamma) + \mathrm{o}_*(\gamma),\ z^*(\gamma) + \mathrm{o}^*(\gamma)) : \gamma \in [0,\ 1]\}, \tag{1}$$

$$\mathrm{e} \odot \tilde{z} = \{(\mathrm{e}z_*(\gamma),\ \mathrm{e}z^*(\gamma)) : \gamma \in [0,\ 1]\}. \tag{2}$$

Remark 1 It is clear that $\mathbb{F}_0$ is closed under scaler addition and the properties that are established above on $\mathbb{F}_0$ correspond to those that arise from the conventional extension idea. In addition, we get for each scalar number $e \in \check{R}$.

$$\tilde{z} \oplus \mathrm{e} = \{(z_*(\gamma) + \mathrm{e},\ z^*(\gamma) + \mathrm{e}) : \gamma \in [0,\ 1]\}. \tag{3}$$

The space $\mathbb{F}_0$ equipped with the supremum metric, represented as $\mathfrak{D}(\tilde{z}, \tilde{o}) = \sup_{0\leq\gamma\leq 1} H([\tilde{z}]^{\gamma}, [\tilde{o}]^{\gamma})$, is commonly known (see, e.g., [71]) to constitute a full metric space with the following properties:

1. $\mathfrak{D}(\tilde{z}\oplus\tilde{v}, \tilde{o}\oplus\tilde{v}) = \mathfrak{D}(\tilde{z}, \tilde{o})$ for all $\tilde{z}, \tilde{o}, \tilde{v} \in \mathbb{F}_0$.
2. $\mathfrak{D}(e\odot\tilde{z}, e\odot\tilde{o}) = |e|\mathfrak{D}(\tilde{z}, \tilde{o})$ for all $\tilde{z}, \tilde{o} \in \mathbb{F}_0$ and $e \in \check{R}$.
3. $\mathfrak{D}(\tilde{z}\oplus\tilde{o}, \tilde{\Theta}\oplus\tilde{v}) = \mathfrak{D}(\tilde{z}, \Theta)\oplus\mathfrak{D}(\tilde{o}, \tilde{v})$ for all $\tilde{z}, \tilde{o}, \Theta, \tilde{v} \in \mathbb{F}_0$.
4. $\mathfrak{D}(\tilde{z}\oplus\tilde{o}, \tilde{0}) \leq \mathfrak{D}(\tilde{z}, \tilde{0})\oplus\mathfrak{D}(\tilde{o}, \tilde{0})$ for all $\tilde{z}, \tilde{o} \in \mathbb{F}_0$, where $\tilde{0}$ is the function $\tilde{0}: \check{R} \to [0, 1]$ as specified by $\tilde{0}(z) = 0$ for all $z \in \check{R}$.
5. $\mathfrak{D}(\tilde{z}, \tilde{o}) = \mathfrak{D}\left(\tilde{z}\ominus_{g\mathcal{H}}\tilde{o}, \tilde{0}\right)$ for all $\tilde{z}, \tilde{o} \in \mathbb{F}_0$.

Definition 2 [71] A $F\cdot\mathcal{V}\cdot M\widetilde{\mathfrak{S}}: [c, r] \to \mathbb{F}_0$ is characterized as $\mathfrak{D}$-continuous, meaning that $z_0 \in \check{R}$, there exist $\varepsilon > 0$ there exist $\delta(\varepsilon, z_0) = \delta > 0$ such that $\mathfrak{D}\left(\widetilde{\mathfrak{S}}(\vartheta), \widetilde{\mathfrak{S}}(z_0)\right) < \varepsilon$ for all $\vartheta \in \check{R}$ with $|\vartheta - z_0| < \delta$.

Definition 3 [71] A mapping $\widetilde{\mathfrak{S}}: [c, r] \to \mathbb{F}_0$ is called $F\cdot\mathcal{V}\cdot M$. For each $\gamma \in [0, 1]$, denote $[\widetilde{\mathfrak{S}}(z)]^{\gamma} = \mathfrak{S}_{\gamma}(z) = [\mathfrak{S}_*(z, \gamma), \mathfrak{S}^*(z, \gamma)]$. Thus, a fuzzy mapping $\widetilde{\mathfrak{S}}$ can be identified by a parametrized triples

$$[\widetilde{\mathfrak{S}}(z)]^{\gamma} = \left\{\left(\widetilde{\mathfrak{S}}_*(z, \gamma), \widetilde{\mathfrak{S}}^*(z, \gamma)\right) : \gamma \in [0, 1]\right\}. \tag{4}$$

Definition 4 [69, 70] Let $L = (m, n)$ and $z \in L$. Then $F\cdot\mathcal{V}\cdot M\widetilde{\mathfrak{S}}: (m, n) \to \mathbb{F}_0$ is characterized as a generalized Hukuhara differentiable (in short, $g\mathcal{H}$-differentiable) at z if there exists an element $\widetilde{\mathfrak{S}}'_{g\mathcal{H}}(z) \in \mathbb{F}_0$ such that for all $0 < v$, sufficiently small, there exist $\widetilde{\mathfrak{S}}(z+v)\ominus_{g\mathcal{H}}\widetilde{\mathfrak{S}}(z)$, $\widetilde{\mathfrak{S}}(z)\ominus_{g\mathcal{H}}\widetilde{\mathfrak{S}}(z-v)$ and the limits (in the metric $\mathfrak{D}$) such that

$$\lim_{v\to 0^+}\frac{\widetilde{\mathfrak{S}}(z+v)\ominus_{g\mathcal{H}}\widetilde{\mathfrak{S}}(z)}{v} = \lim_{v\to 0^+}\frac{\widetilde{\mathfrak{S}}(z)\ominus_{g\mathcal{H}}\widetilde{\mathfrak{S}}(z-v)}{v} = \widetilde{\mathfrak{S}}'_{g\mathcal{H}}(z).$$

Or

$$\lim_{v\to 0^+}\frac{\widetilde{\mathfrak{S}}(z)\ominus_{g\mathcal{H}}\widetilde{\mathfrak{S}}(z+v)}{-v} = \lim_{v\to 0^+}\frac{\widetilde{\mathfrak{S}}(z-v)\ominus_{g\mathcal{H}}\widetilde{\mathfrak{S}}(z)}{-v} = \widetilde{\mathfrak{S}}'_{g\mathcal{H}}(z).$$

Or

$$\lim_{v\to 0^+}\frac{\widetilde{\mathfrak{S}}(z+v)\ominus_{g\mathcal{H}}\widetilde{\mathfrak{S}}(z)}{v} = \lim_{v\to 0^+}\frac{\widetilde{\mathfrak{S}}(z-v)\ominus_{g\mathcal{H}}\widetilde{\mathfrak{S}}(v)}{-v} = \widetilde{\mathfrak{S}}'_{g\mathcal{H}}(z).$$

Or

$$\lim_{v\to 0^+}\frac{\widetilde{\mathfrak{S}}(z)\ominus_{g\mathcal{H}}\widetilde{\mathfrak{S}}(z+v)}{-v} = \lim_{v\to 0^+}\frac{\widetilde{\mathfrak{S}}(z)\ominus_{g\mathcal{H}}\widetilde{\mathfrak{S}}(z-v)}{v} = \widetilde{\mathfrak{S}}'_{GH}(z).$$

Looking at the set $\mathcal{B}\left(\widetilde{\mathfrak{S}}, \mathbb{F}_0\right)$ that contains all bounded fuzzy mappings $\widetilde{\mathfrak{S}}: \nabla \to \mathbb{F}_0$, we can see that $(\mathbb{F}_0, \oplus, \odot)$ is a quasilinear space. As a result, we may create a quasilinear space structure on $\mathcal{B}(\mathfrak{S}, \mathfrak{R}_I)$, where the quasinorm $\|\cdot\|$ is given (see [71]) as follows:

$$\|\widetilde{\mathfrak{S}}\| = \sup_{0\leq\gamma\leq 1}\left\{\left\|\mathfrak{S}_\gamma\right\|\right\} = \sup_{z\in\nabla}\mathfrak{D}(\widetilde{\mathfrak{S}}(z),\ \tilde{0}). \tag{5}$$

Definition 5 [33] Assume that $\widetilde{\mathfrak{S}} : [c,\ r] \subset \check{R} \to \mathbb{F}_0$ is a $F\cdot\mathcal{V}\cdot M$. The fuzzy integral of $\widetilde{\mathfrak{S}}$ over $[c,\ r]$ is then given level-wise by $(FA)\int_c^r \widetilde{\mathfrak{S}}(z)dz$, and is represented by

$$\left[(FA)\int_c^r \widetilde{\mathfrak{S}}(z)dz\right]^\gamma = (IA)\int_c^r \mathfrak{S}_\gamma(z)dz = \left\{\int_c^r \mathfrak{S}(z,\ \gamma)dz : \mathfrak{S}(z,\ \gamma) \in \mathcal{R}_{([c,\ r],\ \gamma)}\right\}, \tag{6}$$

where $\mathcal{R}_{([c,\ r],\ \gamma)}$ indicates the set of Lebesgue-integrable mappings of I-V-Ms for all $\gamma \in (0,\ 1]$. If $(FA)\int_c^r \widetilde{\mathfrak{S}}(z)dz \in \mathbb{F}_0$, then $F\cdot\mathcal{V}\cdot M$ $\widetilde{\mathfrak{S}}$ is FA-integrable over $[c,\ r]$. It should be noted that $\mathfrak{S}$ is fuzzy Aumann-integrable mapping over $[c,\ r]$, if $\mathfrak{S}_*(z,\ \gamma),\ \mathfrak{S}^*(z,\ \gamma)$ are Lebesgue-integrable, see [19].

Theorem 2 [33] Let $\widetilde{\mathfrak{S}} : [c,\ r] \subset \check{R} \to \mathbb{F}_0$ be a $F\cdot\mathcal{V}\cdot M$, it's I-V-Ms are classified according to their γ-levels $\mathfrak{S}_\gamma : [c,\ r] \subset \check{R} \to \mathcal{L}_C$ are given by $\mathfrak{S}_\gamma(z) = [\mathfrak{S}_*(z,\ \gamma),\ \mathfrak{S}^*(z,\ \gamma)]$ for all $z \in [c,\ r]$ and for all $\gamma \in (0,\ 1]$. Then, $\widetilde{\mathfrak{S}}$ is FA-integrable over $[c,\ r]$ if and only if, $\mathfrak{S}_*(z,\ \gamma)$ and $\mathfrak{S}^*(z,\ \gamma)$ are both A-integrable over $[c,\ r]$.

Moreover, if $\widetilde{\mathfrak{S}}$ is FA-integrable over $[c,\ r]$, then

$$\left[(FA)\int_c^r \widetilde{\mathfrak{S}}(z)dz\right]^\gamma = \left[(A)\int_c^r \mathfrak{S}_*(z,\ \gamma)dz,\ (A)\int_c^r \mathfrak{S}^*(z,\ \gamma)dz\right]$$

$$= (IA)\int_c^r \mathfrak{S}_\gamma(z)dz.$$

Definition 6 [57] A mapping $\widetilde{\mathfrak{S}} : I \to \mathbb{R}_J^+$ is considered a $U\cdot D$-fuzzy-valued convex mapping, if

$$\widetilde{\mathfrak{S}}(q\theta + (1-q)y) \supseteq_F \lambda(q) \odot \widetilde{\mathfrak{S}}(\theta) \oplus \lambda(1-q) \odot \widetilde{\mathfrak{S}}(y), \tag{7}$$

for all $q \in (0,\ 1)$ and $\ell,\ c \in I$.

Theorem 3 Let $\widetilde{\mathfrak{S}} : \mathcal{O} = [\ell,\ c] \to \mathbb{F}_0$ be a fuzzy-number mapping, such that from γ-cuts, where we generate the set of IVM_S $\mathfrak{S}_\gamma : [\ell,\ c] \to \mathbb{R}_J$ such that $[\widetilde{\mathfrak{S}}(\theta)]^\gamma = [\underline{\mathfrak{S}}(\theta,\ \gamma),\ \overline{\mathfrak{S}}(\theta,\ \gamma)]$ and $\widetilde{\mathfrak{S}} \in \mathcal{JR}_{([\ell,\ c])}$. If $\widetilde{\mathfrak{S}} : [\ell,\ c] \to \mathbb{R}_J^+$ is a $U\cdot D$-convex fuzzy-valued mapping, then we have

$$\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right) \supseteq_F \frac{\Gamma(\alpha+1)}{2(c-\ell)^\alpha} \odot \left[J_{\ell+}^\alpha \widetilde{\mathfrak{S}}(c) \oplus J_{c-}^\alpha \widetilde{\mathfrak{S}}(\ell)\right] \supseteq_F \frac{\widetilde{\mathfrak{S}}(\ell) \oplus \widetilde{\mathfrak{S}}(c)}{2}. \tag{8}$$

Theorem 4 Let $\widetilde{\mathfrak{S}},\ \widetilde{G} : [\ell,\ c] \to \mathbb{F}_0$ be an fuzzy-number mappings, such that from γ-cuts, where we generate the set of IVM_S $\mathfrak{S}_\gamma,\ G_\gamma : [\ell,\ c] \to \mathbb{R}_J^+$ such that $[\widetilde{\mathfrak{S}}(\theta)]^\gamma = [\underline{\mathfrak{S}}(\theta,\ \gamma),\ \overline{\mathfrak{S}}(\theta,\ \gamma)]$ and $[\widetilde{G}(\theta)]^\gamma = [\underline{G}(\theta,\ \gamma),\ \overline{G}(\theta,\ \gamma)]$. If $\widetilde{\mathfrak{S}},\ \widetilde{G}$ are two $U\cdot D$-fuzzy-valued convex mappings, then we have

$$\frac{\Gamma(\alpha+1)}{2(c-\ell)^{\alpha}}\odot\left[J^{\alpha}_{\ell+}\widetilde{\mathfrak{S}}(c)\otimes\widetilde{G}(c)\oplus J^{\alpha}_{c-}\widetilde{\mathfrak{S}}(\ell)\otimes\widetilde{G}(\ell)\right]$$

$$\supseteq_F\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\odot M(\ell,\ c)\oplus\frac{\alpha}{(\alpha+1)(\alpha+2)}\odot N(\ell,\ c), \tag{9}$$

and

$$2\odot\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\widetilde{G}\left(\frac{\ell+c}{2}\right)$$

$$\supseteq_F\frac{\Gamma(\alpha+1)}{2(c-\ell)^{\alpha}}\odot\left[J^{\alpha}_{\ell+}\widetilde{\mathfrak{S}}(c)\widetilde{G}(c)\oplus J^{\alpha}_{\ell-}\widetilde{\mathfrak{S}}(\ell)\widetilde{G}(\ell)\right] \tag{10}$$

$$\oplus\frac{\alpha}{(\alpha+1)(\alpha+2)}\odot M(\ell,\ c)\oplus\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\odot N(\ell,\ c),$$

where $M(\ell,\ c)=\widetilde{\mathfrak{S}}(\ell)\widetilde{G}(\ell)\oplus\widetilde{\mathfrak{S}}(c)\widetilde{G}(c)$ and $N(\ell,\ c)=\widetilde{\mathfrak{S}}(\ell)\widetilde{G}(c)\oplus\widetilde{\mathfrak{S}}(c)\widetilde{G}(\ell)$.

Refer to [37] for additional fractional inequalities related to convex fuzzy-valued mappings.

3. Fuzzy-valued generalized strong convexities

This section introduces a new category of $U\cdot D$-generalized strong λ-convex fuzzy-valued mappings that rely on a specified mapping. We develop fresh definitions for these generalized strong λ-convex mappings and explore several specific cases to illustrate the concept. In a fuzzy normed quasilinear space $(X,\ \|\cdot\|_I)$, where I denotes a nonempty convex subset of X, we define mappings $H: X\times X\to\mathbb{R}$ and $\lambda:(0,\ 1)\to\mathbb{R}$.

Definition 7 A mapping $\widetilde{\mathfrak{S}}: I\to\mathbb{R}^+_J$ is considered a $U\cdot D$-fuzzy-valued generalized strong λ-convex mapping, if

$$\widetilde{\mathfrak{S}}(q\theta+(1-q)y)\supseteq_F\lambda(q)\odot\widetilde{\mathfrak{S}}(\theta)\oplus\lambda(1-q)\odot\widetilde{\mathfrak{S}}(y)\oplus H(\theta,\ y), \tag{11}$$

for all $q\in(0,\ 1)$ and $\ell,\ c\in I$.

Accordingly, we propose the following new definitions as exceptional cases of Definition 7.

A $U\cdot D$-fuzzy-valued γ-generalized strong λ-convex mapping provides a new form of a generalized strong λ-convex mapping with fuzzy values. This can be defined by setting $H(\theta,\ y)=\varepsilon(\|\theta-y\|)^{\gamma}$ for a given $\varepsilon\in\mathbb{R}$ and $\gamma>1$ within Definition 7.

Definition 8 A mapping $\widetilde{\mathfrak{S}}: I\to\mathbb{R}^+_J$ is considered a $U\cdot D$-fuzzy-valued γ-generalized strong λ-convex mapping, if

$$\widetilde{\mathfrak{S}}(q\theta+(1-q)y)\supseteq_F\lambda(q)\odot\widetilde{\mathfrak{S}}(\theta)\oplus\lambda(1-q)\odot\widetilde{\mathfrak{S}}(y)\oplus\varepsilon(\|\theta-y\|)^{\gamma}, \tag{12}$$

for all $q\in(0,\ 1)$ and $\ell,\ c\in I$.

A $U \cdot D$-fuzzy-valued ε-generalized strong λ-convex mapping provides a new form of a generalized strong λ-convex mapping with fuzzy values. This can be defined by setting $H(\theta,\ y) = \varepsilon(\|\theta - y\|)$ for a given $\varepsilon \in \mathbb{R}$ within Definition 7.

Definition 9 A mapping $\widetilde{\mathfrak{S}}$: $I \to \mathbb{R}_J^+$ is considered a $U \cdot D$-fuzzy-valued ε-generalized strong λ-convex mapping, if

$$\widetilde{\mathfrak{S}}(q\theta + (1-q)y) \supseteq_F \lambda(q) \odot \widetilde{\mathfrak{S}}(\theta) \oplus \lambda(1-q) \odot \widetilde{\mathfrak{S}}(y) \oplus \varepsilon(\|\theta - y\|), \tag{13}$$

for all $q \in (0,\ 1)$ and $\ell,\ c \in I$.

A $U \cdot D$-fuzzy-valued strong λ-convex mapping provides a new form of a generalized strong λ-convex mapping with fuzzy values. This can be defined by setting $H(\theta,\ y) = -\mu q(1-q)\|y - \theta\|^2$ for a given $\mu > 0$ within Definition 7 .

Definition 10 A mapping $\widetilde{\mathfrak{S}} : I \to \mathbb{R}_J^+$ is considered a $U \cdot D$-fuzzy-valued strongly λ-convex mapping, if

$$\widetilde{\mathfrak{S}}(q\theta + (1-q)y) \supseteq_F \lambda(q) \odot \widetilde{\mathfrak{S}}(\theta) \oplus \lambda(1-q) \odot \widetilde{\mathfrak{S}}(y) \ominus \mu q(1-q)\|y - \theta\|^2, \tag{14}$$

for all $q \in (0,\ 1)$ and $\ell,\ c \in I$.

A fuzzy-valued a $U \cdot D$-fuzzy-valued relaxed strong λ-convex mapping provides a new form of a $U \cdot D$-fuzzy-valued generalized strong λ-convex mapping with fuzzy values. This can be defined by setting $H(\theta,\ y) = \mu q(1-q)\|y - \theta\|^2$ for a given $\mu > 0$ within Definition 7.

Definition 11 A mapping $\widetilde{\mathfrak{S}} : I \to \mathbb{R}_J^+$ is considered a $U \cdot D$-fuzzy-valued relaxed λ-convex mapping, if

$$\widetilde{\mathfrak{S}}(q\theta + (1-q)y) \supseteq_F \lambda(q) \odot \widetilde{\mathfrak{S}}(\theta) \oplus \lambda(1-q) \odot \widetilde{\mathfrak{S}}(y) \oplus \mu q(1-q)\|y - \theta\|^2, \tag{15}$$

for all $q \in (0,\ 1)$ and $\ell,\ c \in I$.

Definition 5 is derived by setting $\gamma = 0$ in Definition 4 or $\varepsilon = 0$ in Definition 9 .

4. Fuzzy-valued generalized fractional integral operators

This section explores specific cases of our newly introduced integral operator and establishes a generalized fractional integral operator for mappings with fuzzy values.

Consider defining a mapping $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ that satisfies the necessary requirements:

$$\int_0^1 \frac{\varphi(q)}{q} dq < +\infty \tag{16}$$

$$\frac{1}{A_1} \le \frac{\varphi(s)}{\varphi(r)} \le A_1 \text{ for } \frac{1}{2} \le \frac{s}{r} \le 2 \tag{17}$$

$$\frac{\varphi(r)}{r^2} \le A_2 \frac{\varphi(s)}{q^2} \text{ for } s \le r \tag{18}$$

$$\left| \frac{\varphi(r)}{r^2} - \frac{\varphi(s)}{q^2} \right| \le A_3 |r - s| \frac{\varphi(r)}{r^2} \text{ for } \frac{1}{2} \le \frac{s}{r} \le 2. \tag{19}$$

For $r,\ s>0$, values of $A_1,\ A_2$, and $A_3>0$ remain unaffected. The function φ satisfies conditions (15)-(19) if $\varphi(r)r^{\alpha}$ is non-decreasing for any $\alpha\geq 0$ and $\dfrac{\varphi(r)}{r^{\beta}}$ is non-increasing for some $\beta\geq 0$; refer to [60] for more details. Sarikaya and Ertuğral introduced the following generalized fractional integrals in [59]:

$$
\begin{aligned}
\ell+I_{\varphi}f(\theta)&=\int_{\ell}^{\theta}\frac{\varphi(\theta-q)}{\theta-q}f(q)dq,\ \theta>\ell\\
c-I_{\varphi}f(\theta)&=\int_{\theta}^{c}\frac{\varphi(q-\theta)}{q-\theta}f(q)dq,\ \theta<c.
\end{aligned}
\tag{20}
$$

Accordingly, we propose the following new definitions.

Definition 12 Let $\widetilde{\mathfrak{S}}:\mathcal{O}=[\ell,\ c]\to\mathbb{F}_0$ be a fuzzy-number mapping, such that from γ-cuts, where we generate the set of $IVMs$ $\mathfrak{S}_{\gamma}:[\ell,\ c]\to\mathbb{R}_J$ such that $[\widetilde{\mathfrak{S}}(q)]^{\gamma}=[\underline{\mathfrak{S}}(q,\ \gamma),\ \overline{\mathfrak{S}}(q,\ \gamma)]$ and $\widetilde{\mathfrak{S}}\in\mathcal{JR}_{([\ell,\ c])}$. Then the fuzzy-valued left-sided and right-sided generalized fractional integrals of the mapping $\widetilde{\mathfrak{S}}$, respectively, are given as

$$
\begin{aligned}
\ell+\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(\theta)&=(FA)\int_{\ell}^{\theta}\frac{\varphi(\theta-q)}{\theta-q}\odot\widetilde{\mathfrak{S}}(q)dq,\ \ \theta>\ell\\
c-\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(\theta)&=(FA)\int_{\theta}^{c}\frac{\varphi(q-\theta)}{q-\theta}\odot\widetilde{\mathfrak{S}}(q)dq,\ \ \theta<c.
\end{aligned}
\tag{21}
$$

Corollary 1 Let $\widetilde{\mathfrak{S}}:[\ell,\ c]\to\mathbb{F}_0$ be a fuzzy-number mapping, such that from γ-cuts, where we generate the set of $IVMs$ $\mathfrak{S}_{\gamma}:[\ell,\ c]\to\mathbb{R}_J$ such that $[\widetilde{\mathfrak{S}}(q)]^{\gamma}=[\underline{\mathfrak{S}}(q,\ \gamma),\ \overline{\mathfrak{S}}(q,\ \gamma)]$ and $\widetilde{\mathfrak{S}}\in\mathcal{JR}_{([\ell,\ c])}$. Then, we have

$$
{}_{\ell+}\mathfrak{I}_{\varphi}[\widetilde{\mathfrak{S}}(q)]^{\gamma}=\left[{}_{\ell+}I_{\varphi}\underline{\mathfrak{S}}(\theta,\ \gamma),\ {}_{\ell+}I_{\varphi}\overline{\mathfrak{S}}(\theta,\ \gamma)\right],
$$

and

$$
{}_{c-}\mathfrak{I}_{\varphi}[\widetilde{\mathfrak{S}}(q)]^{\gamma}=\left[{}_{c-}I_{\varphi}\underline{\mathfrak{S}}(\theta,\ \gamma),\ {}_{c-}I_{\varphi}\overline{\mathfrak{S}}(\theta,\ \gamma)\right].
$$

The primary feature of $U\cdot D$-fuzzy-valued generalized fractional integrals is their ability to encompass various types of fractional integrals, including the Hadamard, Katugampola, conformable, RiemannLiouville, and k-Riemann-Liouville fractional integrals. Below, we outline these notable specific cases of integral operators in forms (20) and (21).

i) By setting $\varphi(q)=q$, the fuzzy-valued Riemann integrals are derived by simplifying operators (20) and (21) as follows:

$$
I_{\ell}+\widetilde{\mathfrak{S}}(\theta)=(FA)\int_{\ell}^{\theta}\widetilde{\mathfrak{S}}(q)dq,\ \ \theta>\ell
$$

$$
I_{c}-\widetilde{\mathfrak{S}}(\theta)=(FA)\int_{\theta}^{c}\widetilde{\mathfrak{S}}(q)dq,\ \ \theta<c.
$$

ii) By setting $\varphi(q) = \frac{q^{\alpha}}{\Gamma(\alpha)}$, the fuzzy-valued Riemann fractional integrals are derived by simplifying operators (20) and (21) as follows:

$$J_{\ell^+}^{\alpha}\widetilde{\mathfrak{S}}(\theta) = \frac{1}{\Gamma(\alpha)} \odot (FA)\int_{\ell}^{\theta}(\theta-q)^{\alpha-1} \odot \widetilde{\mathfrak{S}}(q)dq, \ \ \theta > \ell$$

$$J_{c^-}^{\alpha}\widetilde{\mathfrak{S}}(\theta) = \frac{1}{\Gamma(\alpha)} \odot (FA)\int_{\theta}^{c}(q-\theta)^{\alpha-1} \odot \widetilde{\mathfrak{S}}(q)dq, \ \ \theta < c.$$

iii) By setting $\varphi(q) = \frac{q^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, the fuzzy-valued k-Riemann fractional integrals are derived by simplifying operators (20) and (21) as follows:

$$I_{\ell^+,\, k}^{\alpha}\widetilde{\mathfrak{S}}(\theta) = \frac{1}{k\Gamma_k(\alpha)} \odot (FA)\int_{\ell}^{\theta}(\theta-q)^{\frac{\alpha}{k}-1} \odot \widetilde{\mathfrak{S}}(q)dq, \ \ \theta > \ell$$

$$I_{c^-,\, k}^{\alpha}\widetilde{\mathfrak{S}}(\theta) = \frac{1}{k\Gamma_k(\alpha)} \odot (FA)\int_{\theta}^{c}(q-\theta)^{\frac{\alpha}{k}-1} \odot \widetilde{\mathfrak{S}}(q)dq, \ \ \theta < c$$

where

$$\Gamma_k(\alpha) = \int_0^{\infty} q^{\alpha-1}e^{-\frac{q^k}{k}}dq, \ \mathcal{R}(\alpha) > 0,$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right), \ \mathcal{R}(\alpha) > 0; k > 0.$$

iv) By setting $\varphi(q) = q(\theta-q)^{\alpha-1}$, the fuzzy-valued conformable fractional integrals are derived by simplifying operators (20) as follows:

$$I_{\ell}^{\alpha}\widetilde{\mathfrak{S}}(\theta) = (FA)\int_{\ell}^{\theta} q^{\alpha-1} \odot \widetilde{\mathfrak{S}}(q)dq = (FA)\int_{\ell}^{\theta}\widetilde{\mathfrak{S}}(q)d_{\alpha}q, \ \theta > \ell, \ \alpha \in (0,\ 1).$$

5. Fractional integrals inequalities via $U \cdot D$-fuzzy-valued generalized trong λ-convex mapping

This section applies generalized fractional integrals to establish several Hermite-Hadamard type inequalities for $U \cdot D$-fuzzy-valued generalized strong λ-convex mappings. For brevity, the following notations are introduced for use in the forthcoming results:

$$\Lambda(\theta) = \int_0^\theta \frac{\varphi((c-\ell)q)}{q} dq < +\infty,$$

and

$$\psi(\theta) = \int_0^\theta \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q} dq < +\infty.$$

Theorem 5 Let $\widetilde{\mathfrak{S}} : [\ell,\ c] \to \mathbb{F}_0$ be a fuzzy-number mapping, such that from γ-cuts, where we generate the set of IVM_S $\mathfrak{S}_\gamma : [\ell,\ c] \to \mathbb{R}_J^+$. If $\widetilde{\mathfrak{S}}$ is an $U \cdot D$-fuzzy-valued generalized strong $\hbar$-convex mapping such that $[\widetilde{\mathfrak{S}}(q)]^\gamma = [\underline{\mathfrak{S}}(q,\ \gamma),\ \overline{\mathfrak{S}}(q,\ \gamma)]$, then the upcoming inequalities apply to the generalized fractional integrals:

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right) \oplus \frac{1}{2\hbar\left(\frac{1}{2}\right)\Lambda(1)} \int_\ell^c \frac{\varphi(c-\theta)}{c-\theta} H(\theta,\ \ell+c-\theta) d\theta$$

$$\supseteq_F \frac{1}{2\Lambda(1)} \odot \left[\ell + \mathfrak{I}_\varphi \widetilde{\mathfrak{S}}(c) \oplus_{c-} \mathfrak{I}_\varphi \widetilde{\mathfrak{S}}(\ell)\right] \tag{22}$$

$$\supseteq_F \frac{[\widetilde{\mathfrak{S}}(\ell) + \widetilde{\mathfrak{S}}(c)]}{2\Lambda(1)} \odot \int_0^1 [\hbar(q) + \hbar(1-q)] \frac{\varphi((c-\ell)q)}{q} dq + H(\ell,\ c).$$

Proof. Since $\widetilde{\mathfrak{S}}$ is $U \cdot D$-fuzzy-valued generalized strong $\hbar$-convex mapping, we have

$$\frac{1}{\hbar\left(\frac{1}{2}\right)} \mathfrak{S}_\gamma\left(\frac{\theta+y}{2}\right) \supseteq_I \mathfrak{S}_\gamma(\theta) + \mathfrak{S}_\gamma(y) + \frac{1}{\hbar\left(\frac{1}{2}\right)} H(\theta,\ y). \tag{23}$$

By setting $\theta = q\ell + (1-q)c$ and $y = qc + (1-q)\ell$ in (23), we obtain

$$\frac{1}{\hbar\left(\frac{1}{2}\right)} \mathfrak{S}_\gamma\left(\frac{\ell+c}{2}\right) \supseteq_I \mathfrak{S}_\gamma(q\ell + (1-q)c) + \mathfrak{S}_\gamma(qc + (1-q)\ell)$$

$$+ \frac{1}{\hbar\left(\frac{1}{2}\right)} H(q\ell + (1-q)c,\ qc + (1-q)\ell). \tag{24}$$

By multiplying both sides of (24) by $\frac{\varphi((c-\ell)q)}{q}$ and integrating the resulting expression with respect to q over the interval $[0,\ 1]$, we derive:

$$\frac{1}{\lambda\left(\frac{1}{2}\right)}\mathfrak{S}_{\gamma}\left(\frac{\ell+c}{2}\right)\int_{0}^{1}\frac{\varphi((c-\ell)q)}{q}dq$$

$$\supseteq_{I}\left\{(FA)\int_{0}^{1}\frac{\varphi((c-\ell)q)}{q}\mathfrak{S}_{\gamma}(q\ell+(1-q)c)dq+(FA)\int_{0}^{1}\frac{\varphi((c-\ell)q)}{q}\mathfrak{S}_{\gamma}(qc+(1-q)\ell)dq\right. \tag{25}$$

$$\left.+\frac{1}{\lambda\left(\frac{1}{2}\right)}\int_{0}^{1}\frac{\varphi((c-\ell)q)}{q}H(q\ell+(1-q)c,\ qc+(1-q)\ell)dq\right\}.$$

In Eq. (25), we obtain

$$(IA)\int_{0}^{1}\frac{\varphi((c-\ell)q)}{q}\mathfrak{S}_{\gamma}(q\ell+(1-q)c)dq$$

$$=\left[(R)\int_{0}^{1}\frac{\varphi((c-\ell)q)}{q}\underline{\mathfrak{S}}(q\ell+(1-q)c,\ \gamma)dq,\ (R)\int_{0}^{1}\frac{\varphi((c-\ell)q)}{q}\overline{\mathfrak{S}}(q\ell+(1-q)c,\ \gamma)dq\right]$$

$$=\left[(R)\int_{\ell}^{c}\frac{\varphi(c-\theta)}{c-\theta}\underline{\mathfrak{S}}(\theta,\ \gamma)d\theta,\ (R)\int_{\ell}^{c}\frac{\varphi(c-\theta)}{c-\theta}\overline{\mathfrak{S}}(\theta,\ \gamma)d\theta\right]$$

$$=\left[\ell_{+}I_{\varphi}\underline{\mathfrak{S}}(c,\ \gamma),\ \ell_{+}I_{\varphi}\overline{\mathfrak{S}}(c,\ \gamma)\right]$$

$$={}_{\ell+}\mathfrak{I}_{\varphi}\mathfrak{S}_{\gamma}(c).$$

In a similar way, we have

$$(IA)\int_{0}^{1}\frac{\varphi((c-\ell)q)}{q}\mathfrak{S}_{\gamma}(qc+(1-q)\ell)dq$$

$$=\left[(R)\int_{0}^{1}\frac{\varphi((c-\ell)q)}{q}\underline{\mathfrak{S}}(qc+(1-q)\ell,\ \gamma)dq,\ (R)\int_{0}^{1}\frac{\varphi((c-\ell)q)}{q}\overline{\mathfrak{S}}(qc+(1-q)\ell,\ \gamma)dq\right]$$

$$={}_{c-}\mathfrak{I}_{\varphi}\mathfrak{S}_{\gamma}(\ell).$$

Thus, we obtain our initial inequality. Since $\widetilde{\mathfrak{S}}$ is a $U\cdot D$-fuzzy-valued generalized strong λ-convex mapping, the second inequality can be demonstrated by obtaining

$$\mathfrak{S}_\gamma(q\ell+(1-q)c) \supseteq_I \hbar(q)\mathfrak{S}_\gamma(\ell)+\hbar(1-q)\mathfrak{S}_\gamma(c)+H(\ell,\ c), \tag{26}$$

and

$$\mathfrak{S}_\gamma(qc+(1-q)\ell) \supseteq_I \hbar(q)\mathfrak{S}_\gamma(c)+\hbar(1-q)\mathfrak{S}_\gamma(\ell)+H(\ell,\ c). \tag{27}$$

The sum of (26) and (27) gives us

$$\mathfrak{S}_\gamma(q\ell+(1-q)c)+\mathfrak{S}_\gamma(qc+(1-q)\ell) \supseteq_I [\hbar(q)+\hbar(1-q)]\left[\mathfrak{S}_\gamma(\ell)+\mathfrak{S}_\gamma(c)\right]+2H(\ell,\ c) \tag{28}$$

By multiplying both sides of (28) by $\dfrac{\varphi((c-\ell)q)}{q}$ and integrating the resulting expression with respect to q over the interval $[0,\ 1]$, we derive:

$$\begin{aligned}&(FA)\int_0^1 \frac{\varphi((c-\ell)q)}{q}\mathfrak{S}_\gamma(q\ell+(1-q)c)dq+(FA)\int_0^1 \frac{\varphi((c-\ell)q)}{q}\mathfrak{S}_\gamma(qc+(1-q)\ell)dq\\ &\supseteq_I \left[\mathfrak{S}_\gamma(\ell)+\mathfrak{S}_\gamma(c)\right]\int_0^1 \frac{\varphi((c-\ell)q)}{q}[\hbar(q)+\hbar(1-q)]dq+2H(\ell,\ c)\int_0^1 \frac{\varphi((c-\ell)q)}{q}dq.\end{aligned} \tag{29}$$

Thus, the proof is finished.

Corollary 2 By choosing $\varphi(q)=q$ in Theorem 5, the following inequalities are derived.

$$\begin{aligned}&\frac{1}{2\hbar\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\ominus\frac{1}{2\hbar\left(\frac{1}{2}\right)(c-\ell)}\int_\ell^c H(\theta,\ \ell+c-\theta)d\theta\\ &\supseteq_F \frac{1}{c-\ell}\odot(FA)\int_\ell^c\widetilde{\mathfrak{S}}(\theta)d\theta \supseteq_F \frac{\widetilde{\mathfrak{S}}(\ell)\oplus\widetilde{\mathfrak{S}}(c)}{2}\odot\int_0^1[\hbar(q)+\hbar(1-q)]dq+H(\ell,\ c).\end{aligned} \tag{30}$$

Corollary 3 Selecting $\varphi(q)=\dfrac{q^\alpha}{\Gamma(\alpha)}$ in Theorem 5 yields the following inequality for the RiemannLiouville fractional integrals:

$$\begin{aligned}&\frac{1}{2\hbar\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\oplus\frac{\alpha}{2\hbar\left(\frac{1}{2}\right)(c-\ell)^\alpha}\int_\ell^c (c-\theta)^{\alpha-1}H(\theta,\ \ell+c-\theta)d\theta\\ &\supseteq_F \frac{\Gamma(\alpha+1)}{2(c-\ell)^\alpha}\odot\left[J_{\ell+}^\alpha\widetilde{\mathfrak{S}}(c)\oplus J_{c-}^\alpha\widetilde{\mathfrak{S}}(\ell)\right] \supseteq_F \alpha\frac{\widetilde{\mathfrak{S}}(\ell)\oplus\widetilde{\mathfrak{S}}(c)}{2}\odot\int_0^1 q^{\alpha-1}[\hbar(q)+\hbar(1-q)]dq\oplus H(\ell,\ c).\end{aligned} \tag{31}$$

Theorem 6 Let $\widetilde{\mathfrak{S}},\ \widetilde{G}:[\ell,\ c]\to\mathbb{F}_0$ be two fuzzy-number mappings, such that from γ-cuts, where we generate the set of IVM_S $\mathfrak{S}_\gamma,\ G_\gamma:[\ell,\ c]\to\mathbb{R}_J^+$ such that $[\widetilde{\mathfrak{S}}(q)]^\gamma=[\underline{\mathfrak{S}}(q,\ \gamma),\ \overline{\mathfrak{S}}(q,\ \gamma)]$ and $[\widetilde{G}(q)]^\gamma=[\underline{G}(q,\ \gamma),\ \overline{G}(q,\ \gamma)]$. If $\widetilde{\mathfrak{S}},\ \widetilde{G}$ are two $U\cdot D$-fuzzy-valued generalized strong λ-convex mappings, then the upcoming inequality apply to the generalized fractional integrals:

$$\begin{aligned}&\frac{1}{2}\odot\left[{}_{\ell+}\mathfrak{I}_\varphi\widetilde{\mathfrak{S}}(c)\otimes\widetilde{G}(c)\oplus_{c-}\mathfrak{I}_\varphi\widetilde{\mathfrak{S}}(\ell)\otimes\widetilde{G}(\ell)\right]\\&\supseteq_F K_1\odot M(\ell,\ c)\oplus K_2\odot N(\ell,\ c)\oplus K_3\odot P(\ell,\ c)\odot H(\ell,\ c)\oplus\Lambda(1)\odot H^2(\ell,\ c),\end{aligned}\tag{32}$$

where Theorem 4 defines $M(\ell,\ c)$ and $N(\ell,\ c)$, and

$$P(\ell,\ c)=\widetilde{\mathfrak{S}}(\ell)\oplus\widetilde{G}(\ell)\oplus\widetilde{\mathfrak{S}}(c)\oplus\widetilde{G}(c)$$

$$K_1=\frac{1}{2}\int_0^1\frac{\varphi((c-\ell)q)}{q}\left[\lambda^2(q)+\lambda^2(1-q)\right]dq$$

$$K_2=\int_0^1\frac{\varphi((c-\ell)q)}{q}\lambda(q)\lambda(1-q)dq$$

$$K_3=\frac{1}{2}\int_0^1\frac{\varphi((c-\ell)q)}{q}[\lambda(q)+\lambda(1-q)]dq.$$

Proof. Since $\widetilde{\mathfrak{S}}$ and $\widetilde{G}$ are $U\cdot D$-fuzzy-valued generalized strong λ-convex mappings for $q\in[0,\ 1]$, we have

$$\mathfrak{S}_\gamma(q\ell+(1-q)c)\supseteq_I\lambda(q)\mathfrak{S}_\gamma(\ell)+\lambda(1-q)\mathfrak{S}_\gamma(c)+H(\ell,\ c),\tag{33}$$

and

$$G_\gamma(q\ell+(1-q)c)\supseteq_I\lambda(q)G_\gamma(\ell)+\lambda(1-q)G_\gamma(c)+H(\ell,\ c).\tag{34}$$

Multiplying (33) and (34), we get

$$\begin{aligned}&\mathfrak{S}_\gamma(q\ell+(1-q)c)G_\gamma(q\ell+(1-q)c)\\&\supseteq_I\lambda^2(q)\mathfrak{S}_\gamma(\ell)G_\gamma(\ell)+\lambda^2(1-q)\mathfrak{S}_\gamma(c)G_\gamma(c)+\lambda(q)\lambda(1-q)\left[\mathfrak{S}_\gamma(\ell)G_\gamma(c)+\mathfrak{S}_\gamma(c)G_\gamma(\ell)\right]\\&+\lambda(q)H(\ell,\ c)\left[\mathfrak{S}_\gamma(\ell)+G_\gamma(\ell)\right]+\lambda(1-q)H(\ell,\ c)\left[\mathfrak{S}_\gamma(c)+G_\gamma(c)\right]+H^2(\ell,\ c).\end{aligned}\tag{35}$$

Similarly, we obtain

$$\mathfrak{S}_\gamma(qc+(1-q)\ell)G_\gamma(qc+(1-q)\ell)$$

$$\supseteq_I \hat{\lambda}^2(1-q)\mathfrak{S}_\gamma(\ell)G_\gamma(\ell)+\hat{\lambda}^2(q)\mathfrak{S}_\gamma(c)G_\gamma(c)+\hat{\lambda}(q)\hat{\lambda}(1-q)\left[\mathfrak{S}_\gamma(\ell)G_\gamma(c)+\mathfrak{S}_\gamma(c)G_\gamma(\ell)\right] \tag{36}$$

$$+\hat{\lambda}(q)H(\ell,\ c)\left[\mathfrak{S}_\gamma(c)+G_\gamma(c)\right]+\hat{\lambda}(1-q)H(\ell,\ c)\left[\mathfrak{S}_\gamma(\ell)+G_\gamma(\ell)\right]+H^2(\ell,\ c).$$

Adding (35) and (36), we have the following relation:

$$\mathfrak{S}_\gamma(q\ell+(1-q)c)G_\gamma(q\ell+(1-q)c)+\mathfrak{S}_\gamma(qc+(1-q)\ell)G_\gamma(qc+(1-q)\ell)$$

$$\supseteq_I \left[\hat{\lambda}^2(q)+\hat{\lambda}^2(1-q)\right]M(\ell,\ c)+2\hat{\lambda}(q)\hat{\lambda}(1-q)N(\ell,\ c) \tag{37}$$

$$+[\hat{\lambda}(q)+\hat{\lambda}(1-q)]P(\ell,\ c)H(\ell,\ c)+2H^2(\ell,\ c).$$

By multiplying both sides of (37) by $\dfrac{\varphi((c-\ell)q)}{q}$ and integrating the resulting expression with respect to s over the interval $[0,\ 1]$, we derive:

$$(IA)\int_0^1 \frac{\varphi((c-\ell)q)}{q}\mathfrak{S}_\gamma(q\ell+(1-q)c)G_\gamma(q\ell+(1-q)c)dq$$

$$+(IA)\int_0^1 \frac{\varphi((c-\ell)q)}{q}\mathfrak{S}_\gamma(qc+(1-q)\ell)G_\gamma(qc+(1-q)\ell)dq$$

$$\supseteq_I M(\ell,\ c)\int_0^1 \frac{\varphi((c-\ell)q)}{q}\left[\hat{\lambda}^2(q)+\hat{\lambda}^2(1-q)\right]dq \tag{38}$$

$$+2N(\ell,\ c)\int_0^1 \frac{\varphi((c-\ell)q)}{q}\hat{\lambda}(q)\hat{\lambda}(1-q)dq$$

$$+P(\ell,\ c)H(\ell,\ c)\int_0^1 \frac{\varphi((c-\ell)q)}{q}[\hat{\lambda}(q)+\hat{\lambda}(1-q)]dq+2H^2(\ell,\ c)\int_0^1 \frac{\varphi((c-\ell)q)}{q}dq.$$

Now Eq. (38), we have

$$(IA)\int_0^1 \frac{\varphi((c-\ell)q)}{q}\mathfrak{S}_\gamma(q\ell+(1-q)c)G_\gamma(q\ell+(1-q)c)dq = {}_{\ell+}\mathfrak{I}_\varphi \mathfrak{S}_\gamma(c)G_\gamma(c). \tag{39}$$

and

$$(IA)\int_0^1 \frac{\varphi((c-\ell)q)}{q}\mathfrak{S}_\gamma(qc+(1-q)\ell)G_\gamma(qc+(1-q)\ell)dq = {}_{c-}\mathfrak{I}_\varphi \mathfrak{S}_\gamma(\ell)G_\gamma(\ell). \tag{40}$$

We achieve the desired result (32) by substituting equations (39) and (40) into equation (38). The proof is now concluded.

Corollary 4 By assuming that $\varphi(S) = s$ in Theorem 6, we derive the following inequality:

$$\begin{aligned}\frac{1}{c-\ell}\odot(IA)\int_\ell^c \widetilde{\mathfrak{S}}(\theta)\otimes\widetilde{G}(\theta)d\theta \supseteq_F M(\ell,\ c)\odot\int_0^1 \lambda^2(q)dq \oplus N(\ell,\ c)\odot\int_0^1 \lambda(q)\lambda(1-q)dq\\ \oplus P(\ell,\ c)\odot H(\ell,\ c)\odot\int_0^1 \lambda(q)dq\oplus H^2(\ell,\ c).\end{aligned} \tag{41}$$

Corollary 5 The Riemann-Liouville fractional integrals exhibit the following inequality if we apply $\varphi(q) = \dfrac{q^\alpha}{\Gamma(\alpha)}$ in Theorem 6:

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(c-\ell)^\alpha}\odot\left[J^\alpha_{\ell+}\widetilde{\mathfrak{S}}(c)\otimes\widetilde{G}(c)\oplus J^\alpha_{c-}\widetilde{\mathfrak{S}}(\ell)\otimes\widetilde{G}(\ell)\right]\\ &\supseteq_F \frac{\alpha\odot M(\ell,\ c)}{2}\odot\int_0^1 q^{\alpha-1}\left[\lambda^2(q)+\lambda^2(1-q)\right]dq\oplus\alpha\odot N(\ell,\ c)\odot\int_0^1 q^{\alpha-1}\lambda(q)\lambda(1-q)dq\\ &\oplus\frac{\alpha\odot P(\ell,\ c)\odot H(\ell,\ c)}{2}\odot\int_0^1 q^{\alpha-1}[\lambda(q)+\lambda(1-q)]dq + H^2(\ell,\ c).\end{aligned} \tag{42}$$

Theorem 7 Let $\widetilde{\mathfrak{S}},\ \widetilde{G}:[\ell,\ c]\to\mathbb{F}_0$ be an fuzzy-number mappings, such that from γ-cuts, where we generate the set of IVM_S $\mathfrak{S}_\gamma G_\gamma : [\ell,\ c]\to\mathbb{R}^+_J$ such that $[\widetilde{\mathfrak{S}}(q)]^\gamma = [\underline{\mathfrak{S}}(q,\ \gamma),\ \overline{\mathfrak{S}}(q,\ \gamma)]$ and $[\widetilde{G}(q)]^\gamma = [\underline{G}(q,\ \gamma),\ \overline{G}(q,\ \gamma)]$. If $\widetilde{\mathfrak{S}},\ \widetilde{G}$ are two $U\cdot D$-fuzzy-valued generalized strong λ-convex mappings, then the upcoming inequality apply to the generalized fractional integrals:

$$\begin{aligned}&\frac{1}{2\lambda^2\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\otimes\widetilde{G}\left(\frac{\ell+c}{2}\right)\\ &\ominus\frac{1}{2\Lambda(1)\lambda\left(\frac{1}{2}\right)}\odot\left[(FA)\int_\ell^c\frac{\varphi(c-\theta)}{c-\theta}\odot(\widetilde{\mathfrak{S}}(\theta)\oplus\widetilde{G}(\theta))\odot H(\theta,\ \ell+c-\theta)d\theta\right.\\ &\left.\oplus(FA)\int_\ell^c\frac{\varphi(\theta-\ell)}{\theta-\ell}\odot(\widetilde{\mathfrak{S}}(\theta)\oplus\widetilde{G}(\theta))\odot H(\ell+c-\theta,\ \theta)d\theta\right]\end{aligned}$$

$$\supseteq_F \frac{1}{2\Lambda(1)} \odot \left[{}_{\ell-}\mathfrak{I}_\varphi \widetilde{\mathfrak{S}}(c) \otimes \widetilde{G}(c) \oplus_{c-} \mathfrak{I}_\varphi \widetilde{\mathfrak{S}}(\ell) \otimes \widetilde{G}(\ell)\right]$$

$$\oplus \frac{1}{2\Lambda(1)\hbar^2\left(\frac{1}{2}\right)} \odot \int_\ell^c \frac{\varphi(c-\theta)}{c-\theta} H^2(\theta,\ \ell+c-\theta)d\theta$$

$$\oplus \frac{1}{\Lambda(1)} \odot [K_2 \odot M(\ell,\ c) \oplus K_1 \odot N(\ell,\ c) \oplus K_3 \odot P(\ell,\ c) \odot H(\ell,\ c)] \oplus H^2(\ell,\ c), \tag{43}$$

where Theorems 4 and 6 define $M(\ell,\ c)$, $N(\ell,\ c)$ and $K_1,\ K_2,\ K_3,\ P(\ell,\ c)$, respectively.

Proof. For $q \in [0,\ 1]$, we can write

$$\frac{\ell+c}{2} = \frac{(1-q)\ell+qc}{2} + \frac{q\ell+(1-q)c}{2}.$$

Since $\widetilde{\mathfrak{S}}$ and $\widetilde{G}$ are two $U \cdot D$-fuzzy-valued generalized strong $\hbar$-convex mappings, we have

$$\frac{1}{\hbar^2\left(\frac{1}{2}\right)} \mathfrak{S}_\gamma\left(\frac{\ell+c}{2}\right) G_\gamma\left(\frac{\ell+c}{2}\right)$$

$$= \frac{1}{\hbar^2\left(\frac{1}{2}\right)} \mathfrak{S}_\gamma\left(\frac{(1-q)\ell+qc}{2} + \frac{q\ell+(1-q)c}{2}\right) G_\gamma\left(\frac{(1-q)\ell+qc}{2} + \frac{q\ell+(1-q)c}{2}\right)$$

$$\supseteq_I \left[\mathfrak{S}_\gamma((1-q)\ell+qc) + \mathfrak{S}_\gamma(q\ell+(1-q)c) + \frac{1}{\hbar\left(\frac{1}{2}\right)} H((1-q)\ell+qc,\ q\ell+(1-q)c)\right]$$

$$\times \left[G_\gamma((1-q)\ell+qc) + G_\gamma(q\ell+(1-q)c) + \frac{1}{\hbar\left(\frac{1}{2}\right)} H((1-q)\ell+qc,\ q\ell+(1-q)c)\right]$$

$$= \left[\mathfrak{S}_\gamma((1-q)\ell+qc) G_\gamma((1-q)\ell+qc) + \mathfrak{S}_\gamma(q\ell+(1-q)c) G_\gamma(q\ell+(1-q)c)\right]$$

$$+ \left[\mathfrak{S}_\gamma((1-q)\ell+qc) G_\gamma(q\ell+(1-q)c) + \mathfrak{S}_\gamma(q\ell+(1-q)c) G_\gamma((1-q)\ell+qc)\right]$$

$$+ \frac{1}{\hbar\left(\frac{1}{2}\right)} \left[\mathfrak{S}_\gamma(q\ell+(1-q)c) H(q\ell+(1-q)c,\ qc+(1-q)\ell)\right.$$

$$
\begin{aligned}
&+\mathfrak{S}_\gamma(qc+(1-q)\ell)H(q\ell+(1-q)c,\ qc+(1-q)\ell) \\
&+G_\gamma(q\ell+(1-q)c)H(q\ell+(1-q)c,\ qc+(1-q)\ell) \\
&+G_\gamma(qc+(1-q)\ell)H(q\ell+(1-q)c,\ qc+(1-q)\ell)\big] \\
&+\frac{1}{\hbar^2\left(\frac{1}{2}\right)}\left[H^2(q\ell+(1-q)c,\ qc+(1-q)\ell)\right] \\
\supseteq_I &\left[\mathfrak{S}_\gamma((1-q)\ell+qc)G_\gamma((1-q)\ell+qc)+\mathfrak{S}_\gamma(q\ell+(1-q)c)G_\gamma(q\ell+(1-q)c)\right] \\
&+\left[\hbar^2(q)+\hbar^2(1-q)\right]N(\ell,\ c)+2\hbar(q)\hbar(1-q)M(\ell,\ c) \\
&+[\hbar(q)+\hbar(1-q)]P(\ell,\ c)H(\ell,\ c)+2H^2(\ell,\ c) \\
&+\frac{1}{\hbar\left(\frac{1}{2}\right)}\mathfrak{S}_\gamma(q\ell+(1-q)c)H(q\ell+(1-q)c,\ qc+(1-q)\ell) \\
&+\frac{1}{\hbar\left(\frac{1}{2}\right)}\mathfrak{S}_\gamma(qc+(1-q)\ell)H(q\ell+(1-q)c,\ qc+(1-q)\ell) \\
&+\frac{1}{\hbar\left(\frac{1}{2}\right)}G_\gamma(q\ell+(1-q)c)H(q\ell+(1-q)c,\ qc+(1-q)\ell) \\
&+\frac{1}{\hbar\left(\frac{1}{2}\right)}G_\gamma(qc+(1-q)\ell)H(q\ell+(1-q)c,\ qc+(1-q)\ell) \\
&+\frac{1}{\hbar^2\left(\frac{1}{2}\right)}H^2(q\ell+(1-q)c,\ qc+(1-q)\ell).
\end{aligned}
\tag{44}
$$

By multiplying both sides of (44) by $\frac{\varphi((c-\ell)q)}{q}$ and integrating the resulting expression with respect to s over the interval $[0,\ 1]$, we derive:

$$\frac{1}{\lambda^2\left(\frac{1}{2}\right)}(IA)\int_0^1 \frac{\varphi((c-\ell)q)}{q}\mathfrak{S}_\gamma\left(\frac{\ell+c}{2}\right)G_\gamma\left(\frac{\ell+c}{2}\right)dq$$

$$\supseteq_I (IA)\int_0^1 \frac{\varphi((c-\ell)q)}{q}\mathfrak{S}_\gamma((1-q)\ell+qc)G_\gamma((1-q)\ell+qc)dq$$

$$+(IA)\int_0^1 \frac{\varphi((c-\ell)q)}{q}\mathfrak{S}_\gamma(q\ell+(1-q)c)G_\gamma(q\ell+(1-q)c)dq$$

$$+\frac{1}{\lambda^2\left(\frac{1}{2}\right)}\int_0^1 \frac{\varphi((c-\ell)q)}{q}H^2(q\ell+(1-q)c,\ qc+(1-q)\ell)dq$$

$$+\frac{1}{\lambda\left(\frac{1}{2}\right)}(IA)\int_0^1 \frac{\varphi((c-\ell)q)}{q}\mathfrak{S}_\gamma(q\ell+(1-q)c)H(q\ell+(1-q)c,\ qc+(1-q)\ell)dq$$

$$+\frac{1}{\lambda\left(\frac{1}{2}\right)}(IA)\int_0^1 \frac{\varphi((c-\ell)q)}{q}\mathfrak{S}_\gamma(qc+(1-q)\ell)H(q\ell+(1-q)c,\ qc+(1-q)\ell)dq$$

$$+\frac{1}{\lambda\left(\frac{1}{2}\right)}(IA)\int_0^1 \frac{\varphi((c-\ell)q)}{q}G_\gamma(q\ell+(1-q)c)H(q\ell+(1-q)c,\ qc+(1-q)\ell)dq$$

$$+\frac{1}{\lambda\left(\frac{1}{2}\right)}(IA)\int_0^1 \frac{\varphi((c-\ell)q)}{q}G_\gamma(qc+(1-q)\ell)H(q\ell+(1-q)c,\ qc+(1-q)\ell)dq$$

$$+N(\ell,\ c)\int_0^1 \frac{\varphi((c-\ell)q)}{q}\left[\lambda^2(q)+\lambda^2(1-q)\right]dq$$

$$+2M(\ell,\ c)\int_0^1 \frac{\varphi((c-\ell)q)}{q}\lambda(q)\lambda(1-q)dq$$

$$+P(\ell,\ c)H(\ell,\ c)\int_0^1 \frac{\varphi((c-\ell)q)}{q}[\lambda(q)+\lambda(1-q)]dq+2H^2(\ell,\ c)\int_0^1 \frac{\varphi((c-\ell)q)}{q}dq.$$

The desired inequality (43) was achieved by altering the variable of integration.

Corollary 6 By setting $\varphi(q)=q$, in Theorem 7, we derive the following inequality:

$$\frac{1}{2\lambda^2\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right) \otimes \widetilde{G}\left(\frac{\ell+c}{2}\right)$$

$$\oplus \frac{1}{(c-\ell)\lambda\left(\frac{1}{2}\right)} \odot (FA)\int_{\ell}^{c} [\widetilde{\mathfrak{S}}(\theta) \oplus \widetilde{G}(\theta)] \odot H(\theta,\ \ell+c-\theta)d\theta$$

$$\supseteq_F \frac{1}{c-\ell} \odot (FA)\int_{\ell}^{c} \widetilde{\mathfrak{S}}(\theta) \otimes \widetilde{G}(\theta)d\theta \oplus \frac{1}{2(c-\ell)\lambda^2\left(\frac{1}{2}\right)} \int_{\ell}^{c} H^2(\theta,\ \ell+c-\theta)d\theta \tag{45}$$

$$\oplus \left[M(\ell,\ c) \odot \int_0^1 \lambda(q)\lambda(1-q)dq \oplus N(\ell,\ c) \odot \int_0^1 \lambda^2(q)dq\right.$$

$$\left.\oplus P(\ell,\ c) \odot H(\ell,\ c) \odot \int_0^1 \lambda(q)dq\right] \oplus H^2(\ell,\ c).$$

Corollary 7 By setting $\varphi(q) = \frac{q^\alpha}{\Gamma(\alpha)}$, in Theorem 7, we derive the following inequality:

$$\frac{1}{2\lambda^2\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right) \otimes \widetilde{G}\left(\frac{\ell+c}{2}\right)$$

$$\ominus \frac{a}{2(c-\ell)^\alpha\lambda\left(\frac{1}{2}\right)} \odot \left[(FA)\int_{\ell}^{c} (c-\theta)^{\alpha-1} \odot [\widetilde{\mathfrak{S}}(\theta) + \widetilde{G}(\theta)] \odot H(\theta,\ \ell+c-\theta)d\theta\right.$$

$$\left.\oplus (FA)\int_{\ell}^{c} (\theta-\ell)^{\alpha-1} \odot [\widetilde{\mathfrak{S}}(\theta) \oplus \widetilde{G}(\theta)] \odot H(\ell+c-\theta,\ \theta)d\theta\right]$$

$$\supseteq_F \frac{\Gamma(\alpha+1)}{2(c-\ell)^\alpha} \odot \left[J_{\ell+}^{\alpha} \widetilde{\mathfrak{S}}(c) \otimes \widetilde{G}(c) \oplus \widetilde{\mathfrak{S}}(\ell) \otimes \widetilde{G}(\ell)\right]$$

$$\oplus \frac{\alpha}{2(c-\ell)^\alpha\lambda^2\left(\frac{1}{2}\right)} \odot \int_{\ell}^{c} (c-\theta)^{\alpha-1} H^2(\theta,\ \ell+c-\theta)d\theta$$

$$\oplus \alpha \odot \left[M(\ell,\ c) \odot \int_0^1 q^{\alpha-1}\lambda(q)\lambda(1-q)dq \oplus \frac{N(\ell,\ c)}{2} \odot \int_0^1 q^{\alpha-1}\left[\lambda^2(q) + \lambda^2(1-q)\right]dq\right.$$

$$\oplus \frac{P(\ell,\ c) \odot H(\ell,\ c)}{2} \odot \int_0^1 q^{\alpha-1}[\hbar(q)+\hbar(1-q)]dq \Bigg] \oplus H^2(\ell,\ c). \tag{46}$$

Theorem 8 Let $\widetilde{\mathfrak{S}} : [\ell,\ c] \to \mathbb{F}_0$ be a fuzzy-number mapping, such that from γ-cuts, where we generate the set of $IVMs$ $\mathfrak{S}_\gamma : [\ell,\ c] \to \mathbb{R}_J^+$ such that $[\widetilde{\mathfrak{S}}(q)]^\gamma = [\underline{\mathfrak{S}}(q,\ \gamma),\ \overline{\mathfrak{S}}(q,\ \gamma)]$. If $\widetilde{\mathfrak{S}}$ is an $U \cdot D$-fuzzy-valued generalized strong $\hbar$-convex mapping, then the upcoming inequalities apply to the generalized fractional integrals:

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right) \odot \frac{1}{2\psi(1)\hbar\left(\frac{1}{2}\right)} \odot \int_{\frac{\ell+c}{2}}^c \frac{\varphi(c-\theta)}{c-\theta} H(\theta,\ \ell+c-\theta)d\theta$$

$$\supseteq_F \frac{1}{2\psi(1)} \odot \left[_{\left(\frac{\ell+c}{2}\right)} + \mathfrak{I}_\varphi \widetilde{\mathfrak{S}}(c) \oplus \left[_{\left(\frac{\ell+c}{2}\right)} - \mathfrak{I}_\varphi \widetilde{\mathfrak{S}}(\ell)\right]\right. \tag{47}$$

$$\supseteq_F \frac{\widetilde{\mathfrak{S}}(\ell) \oplus \widetilde{\mathfrak{S}}(c)}{2\psi(1)} \odot \int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q} \left[\hbar\left(\frac{2-q}{2}\right) + \hbar\left(\frac{q}{2}\right)\right] dq \oplus H(\ell,\ c).$$

Proof. Since $\widetilde{\mathfrak{S}}$ is $U \cdot D$-fuzzy-valued generalized strong $\hbar$-convex mapping on $[\ell,\ c]$, we have

$$\frac{1}{\hbar\left(\frac{1}{2}\right)} \mathfrak{S}_\gamma\left(\frac{\theta+y}{2}\right) \supseteq_I \mathfrak{S}_\gamma(\theta) + \mathfrak{S}_\gamma(y) + \frac{1}{\hbar\left(\frac{1}{2}\right)} H(\theta,\ y).$$

For $\theta = \frac{q}{2}\ell + \frac{2-q}{2}c$ and $y = \frac{2-q}{2}\ell + \frac{q}{2}c$, we get

$$\frac{1}{\hbar\left(\frac{1}{2}\right)} \mathfrak{S}_\gamma\left(\frac{\ell+c}{2}\right)$$

$$\supseteq_I \mathfrak{S}_\gamma\left(\frac{q}{2}\ell + \frac{2-q}{2}c\right) + \mathfrak{S}_\gamma\left(\frac{2-q}{2}\ell + \frac{q}{2}c\right) + \frac{1}{\hbar\left(\frac{1}{2}\right)} H\left(\frac{q}{2}\ell + \frac{2-q}{2}c,\ \frac{2-q}{2}\ell + \frac{q}{2}c\right). \tag{48}$$

By multiplying both sides of (48) by $\frac{\varphi((c-\ell)q)}{q}$ and integrating the resulting expression with respect to s over the interval $[0,\ 1]$, we derive:

$$\frac{1}{\lambda\left(\frac{1}{2}\right)}\mathfrak{S}_\gamma\left(\frac{\ell+c}{2}\right)\int_0^1\frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}dq$$

$$\supseteq_I (IA)\int_0^1\frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\mathfrak{S}_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)dq+(IA)\int_0^1\frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\mathfrak{S}_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)dq \tag{49}$$

$$+\frac{1}{\lambda\left(\frac{1}{2}\right)}\int_0^1\frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}H\left(\frac{q}{2}\ell+\frac{2-q}{2}c,\ \frac{2-q}{2}\ell+\frac{q}{2}c\right)dq.$$

Equation (49) provides us with

$$(IA)\int_0^1\frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\mathfrak{S}_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)dq$$

$$=\left[(R)\int_0^1\frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\underline{\mathfrak{S}}\left(\frac{q}{2}\ell+\frac{2-q}{2}c,\ \gamma\right)dq,\ (R)\int_0^1\frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\overline{\mathfrak{S}}\left(\frac{q}{2}\ell+\frac{2-q}{2}c,\ \gamma\right)dq\right]$$

$$=\left[(R)\int_{\frac{\ell+c}{2}}^{c}\frac{\varphi(c-u)}{c-u}\underline{\mathfrak{S}}(u,\ \gamma)du,\ (R)\int_{\frac{\ell+c}{2}}^{c}\frac{\varphi(c-u)}{c-u}\overline{\mathfrak{S}}(u,\ \gamma)du\right]$$

$$=\left[{}_{\left(\frac{\ell+c}{2}\right)}+I_\varphi\underline{\mathfrak{S}}(c,\ \gamma),\ {}_{\left(\frac{\ell+c}{2}\right)}+I_\varphi\overline{\mathfrak{S}}(c,\ \gamma)\right]$$

$$={}_{\left(\frac{\ell+c}{2}\right)}+\mathfrak{I}_\varphi\mathfrak{S}_\gamma(c).$$

In the same way, we derive

$$(IA)\int_0^1\frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\mathfrak{S}_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)dq=\left[{}_{\left(\frac{\ell+c}{2}\right)}-I_\varphi\underline{\mathfrak{S}}(\ell,\ \gamma)_{\left(\frac{\ell+c}{2}\right)}-I_\varphi\overline{\mathfrak{S}}(\ell,\ \gamma)\right]={}_{\left(\frac{\ell+c}{2}\right)}-\mathfrak{I}_\varphi\mathfrak{S}_\gamma(\ell).$$

We thereby established the first inequality. As $\widetilde{\mathfrak{S}}$ is a $U\cdot D$-fuzzy-valued generalized strong λ-convex mapping, we first observe that in order to demonstrate the second inequality of (47), we have

$$\mathfrak{S}_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)\supseteq_I \hbar\left(\frac{2-q}{2}\right)\mathfrak{S}_\gamma(\ell)+\hbar\left(\frac{q}{2}\right)\mathfrak{S}_\gamma(c)+H(\ell,\ c), \tag{50}$$

and

$$\mathfrak{S}_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)\supseteq_I \hbar\left(\frac{q}{2}\right)\mathfrak{S}_\gamma(\ell)+\hbar\left(\frac{2-q}{2}\right)\mathfrak{S}_\gamma(c)+H(\ell,\ c). \tag{51}$$

Equations 5.29 and 5.30 added together give us

$$\mathfrak{S}_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)+\mathfrak{S}_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)\supseteq_I \left[\mathfrak{S}_\gamma(\ell)+\mathfrak{S}_\gamma(c)\right]\left[\hbar\left(\frac{2-q}{2}\right)+\hbar\left(\frac{q}{2}\right)\right]+2H(\ell,\ c). \tag{52}$$

By multiplying both sides of (52) by $\frac{\varphi((c-\ell)q)}{q}$ and integrating the resulting expression with respect to s over the interval $[0,\ 1]$, we derive:

$$(IA)\int_0^1 \frac{\varphi\left(\left(\frac{(c-\ell)}{2}q\right)\right)}{q}\mathfrak{S}_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)dq+(IA)\int_0^1 \frac{\varphi\left(\left(\frac{(c-\ell)}{2}q\right)\right)}{q}\mathfrak{S}_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)dq$$

$$\supseteq_I \left[\mathfrak{S}_\gamma(\ell)+\mathfrak{S}_\gamma(c)\right](IA)\int_0^1 \frac{\varphi\left(\left(\frac{(c-\ell)}{2}q\right)\right)}{q}\left[\hbar\left(\frac{2-q}{2}\right)+\hbar\left(\frac{q}{2}\right)\right]dq+2H(\ell,\ c)\int_0^1 \frac{\varphi\left(\left(\frac{(c-\ell)}{2}q\right)\right)}{q}dq.$$

The second inequality, (47), can be obtained by altering the integration variables.

Corollary 8 By selecting $\varphi(q)=q$ in Theorem 8, we derive the following inequalities:

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\ominus\frac{1}{(c-\ell)\hbar\left(\frac{1}{2}\right)}\odot\int_{\frac{\ell+c}{2}}^{c}H(\theta,\ \ell+c-\theta)d\theta$$

$$\supseteq_F \frac{1}{c-\ell}\odot(FA)\int_\ell^c\widetilde{\mathfrak{S}}(\theta)d\theta\supseteq_F\frac{\widetilde{\mathfrak{S}}(\ell)\oplus\widetilde{\mathfrak{S}}(c)}{2}\odot\int_0^1\left[\hbar\left(\frac{2-q}{2}\right)+\hbar\left(\frac{q}{2}\right)\right]dq\oplus H(\ell,\ c). \tag{53}$$

Corollary 9 By selecting $\varphi(q)=\frac{q^\alpha}{\Gamma(\alpha)}$ in Theorem 8, we derive the following inequalities:

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right) \ominus \frac{\alpha 2^{\alpha-1}}{(c-\ell)^{\alpha}\lambda\left(\frac{1}{2}\right)} \odot \int_{\frac{\ell+c}{2}}^{c} (c-\theta)^{\alpha-1} H(\theta,\ \ell+c-\theta) d\theta$$

$$\supseteq_F \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(c-\ell)^{\alpha}} \odot \left[J^{\alpha}_{\left(\frac{\ell+c}{2}\right)+} \widetilde{\mathfrak{S}}(c) \oplus J^{\alpha}_{\left(\frac{\ell+c}{2}\right)-} \widetilde{\mathfrak{S}}(\ell) \right] \tag{54}$$

$$\supseteq_F \frac{\alpha \odot [\widetilde{\mathfrak{S}}(\ell) \oplus \widetilde{\mathfrak{S}}(c)]}{2} \odot \int_0^1 q^{\alpha-1} \left[\lambda\left(\frac{2-q}{2}\right) + \lambda\left(\frac{q}{2}\right) \right] dq \oplus H(\ell,\ c).$$

Theorem 9 Let $\widetilde{\mathfrak{S}},\ \widetilde{G} : [\ell,\ c] \to \mathbb{F}_0$ be two fuzzy-number mappings, such that from γ-cuts, where we generate the set of *IVMs* $\mathfrak{S}_{\gamma},\ G_{\gamma} : [\ell,\ c] \to \mathbb{R}_J^+$ such that $[\widetilde{\mathfrak{S}}(q)]^{\gamma} = [\underline{\mathfrak{S}}(q,\ \gamma),\ \overline{\mathfrak{S}}(q,\ \gamma)]$ and $[\widetilde{G}(q)]^{\gamma} = [\underline{G}(q,\ \gamma),\ \overline{G}(q,\ \gamma)]$. If $\widetilde{\mathfrak{S}},\ \widetilde{G}$ are two $U \cdot D$-fuzzy-valued generalized strong λ-convex mappings, then the upcoming inequality apply to the generalized fractional integrals:

$$\begin{aligned} &\left[{}_{\left(\frac{\ell+c}{2}\right)^+}\mathfrak{I}_{\varphi} \widetilde{\mathfrak{S}}(c) \otimes \widetilde{G}(c) \oplus {}_{\left(\frac{\ell+c}{2}\right)^-}\mathfrak{I}_{\varphi} \widetilde{\mathfrak{S}}(\ell) \otimes \widetilde{G}(\ell) \right] \\ &\supseteq_F K_4 \odot M(\ell,\ c) \oplus K_5 \odot N(\ell,\ c) \oplus H(\ell,\ c) \odot P(\ell,\ c) \odot K_6 \oplus 2\psi(1) \odot H^2(\ell,\ c), \end{aligned} \tag{55}$$

where $M(\ell,\ c)$ and $N(\ell,\ c)$ are specified in Theorem 4, and

$$K_4 = \int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2} q\right)}{q} \left[\lambda^2\left(\frac{q}{2}\right) + \lambda^2\left(\frac{2-q}{2}\right) \right] dq$$

$$K_5 = 2\int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2} q\right)}{q} \lambda\left(\frac{q}{2}\right) \lambda\left(\frac{2-q}{2}\right) dq$$

$$K_6 = \int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2} q\right)}{q} \left[\lambda\left(\frac{q}{2}\right) + \lambda\left(\frac{2-q}{2}\right) \right] dq.$$

Proof. Since $\widetilde{\mathfrak{S}}$ and $\widetilde{G}$ are two $U \cdot D$-fuzzy-valued generalized strong λ-convex mappings,

$$\mathfrak{S}_{\gamma}\left(\frac{2-q}{2}\ell + \frac{q}{2}c\right) \supseteq_I \lambda\left(\frac{2-q}{2}\right) \mathfrak{S}_{\gamma}(\ell) + \lambda\left(\frac{q}{2}\right) \mathfrak{S}_{\gamma}(c) + H(\ell,\ c), \tag{56}$$

and

$$G_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)\supseteq_I \hbar\left(\frac{2-q}{2}\right)G_\gamma(\ell)+\hbar\left(\frac{q}{2}\right)G_\gamma(c)+H(\ell,\, c). \tag{57}$$

By multiplying (56) and (57), we obtain

$$\begin{aligned}
&\mathfrak{S}_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)G_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)\\
&\supseteq_I \hbar^2\left(\frac{2-q}{2}\right)\mathfrak{S}_\gamma(\ell)G_\gamma(\ell)+\hbar^2\left(\frac{q}{2}\right)\mathfrak{S}_\gamma(c)G_\gamma(c)+\hbar\left(\frac{2-q}{2}\right)\hbar\left(\frac{q}{2}\right)\left[\mathfrak{S}_\gamma(\ell)G_\gamma(c)+\mathfrak{S}_\gamma(c)G_\gamma(\ell)\right]\\
&+\hbar\left(\frac{q}{2}\right)H(\ell,\, c)\left[\mathfrak{S}_\gamma(c)+G_\gamma(c)\right]+\hbar\left(\frac{2-q}{2}\right)H(\ell,\, c)\left[\mathfrak{S}_\gamma(\ell)+G_\gamma(\ell)\right]+H^2(\ell,\, c).
\end{aligned} \tag{58}$$

In the similar way, we have

$$\begin{aligned}
&\mathfrak{S}_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)G_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)\\
&\supseteq_I \hbar^2\left(\frac{q}{2}\right)\mathfrak{S}_\gamma(\ell)G_\gamma(\ell)+\hbar^2\left(\frac{2-q}{2}\right)\mathfrak{S}_\gamma(c)G_\gamma(c)+\hbar\left(\frac{q}{2}\right)\hbar\left(\frac{2-q}{2}\right)\left[\mathfrak{S}_\gamma(\ell)G_\gamma(c)+\mathfrak{S}_\gamma(c)G_\gamma(\ell)\right]\\
&+\hbar\left(\frac{2-q}{2}\right)H(\ell,\, c)\left[\mathfrak{S}_\gamma(c)+G_\gamma(c)\right]+\hbar\left(\frac{q}{2}\right)H(\ell,\, c)\left[\mathfrak{S}_\gamma(\ell)+G_\gamma(\ell)\right]+H^2(\ell,\, c).
\end{aligned} \tag{59}$$

By adding (58) and (59), we derive the following relation

$$\begin{aligned}
&\mathfrak{S}_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)G_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)+\mathfrak{S}_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)G_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)\\
&\supseteq_I \hbar^2\left(\frac{2-q}{2}\right)\left[\mathfrak{S}_\gamma(\ell)G_\gamma(\ell)+\mathfrak{S}_\gamma(c)G_\gamma(c)\right]\\
&+\hbar^2\left(\frac{q}{2}\right)\left[\mathfrak{S}_\gamma(\ell)G_\gamma(\ell)+\mathfrak{S}_\gamma(c)G_\gamma(c)\right]+2\hbar\left(\frac{q}{2}\right)\hbar\left(\frac{2-q}{2}\right)\left[\mathfrak{S}_\gamma(\ell)G_\gamma(c)+\mathfrak{S}_\gamma(c)G_\gamma(\ell)\right]\\
&+H(\ell,\, c)\left[\hbar\left(\frac{q}{2}\right)+\hbar\left(\frac{2-q}{2}\right)\right]\left[\mathfrak{S}_\gamma(\ell)+\mathfrak{S}_\gamma(c)+G_\gamma(\ell)+G_\gamma(c)\right]+2H^2(\ell,\, c)\\
&=\left[\hbar^2\left(\frac{2-q}{2}\right)+\hbar^2\left(\frac{q}{2}\right)\right]M(\ell,\, c)+2\hbar\left(\frac{q}{2}\right)\hbar\left(\frac{2-q}{2}\right)N(\ell,\, c)
\end{aligned}$$

$$+H(\ell,\ c)P(\ell,\ c)\left[\lambda\left(\frac{q}{2}\right)+\lambda\left(\frac{2-q}{2}\right)\right]+2H^2(\ell,\ c). \tag{60}$$

By multiplying both sides of (60) by $\dfrac{\varphi((c-\ell)q)}{q}$ and integrating the resulting expression with respect to q over the interval $[0,\ 1]$, we derive:

$$\begin{aligned}
&(IA)\int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\mathfrak{S}_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)G_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)dq\\
&+(IA)\int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\mathfrak{S}_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)G_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)dq\\
&\supseteq_I M(\ell,\ c)\int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\left[\lambda^2\left(\frac{2-q}{2}\right)+\lambda^2\left(\frac{q}{2}\right)\right]dq\\
&+2N(\ell,\ c)\int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\lambda\left(\frac{q}{2}\right)\lambda\left(\frac{2-q}{2}\right)dq\\
&+H(\ell,\ c)P(\ell,\ c)\int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\left[\lambda\left(\frac{q}{2}\right)+\lambda\left(\frac{2-q}{2}\right)\right]dq\\
&+2H^2(\ell,\ c)\int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}dq.
\end{aligned} \tag{61}$$

From Eq. (61), we arrive at the desired inequality.

Corollary 10 By selecting $\varphi(q)=q$ in Theorem 9, we derive the following inequality:

$$\begin{aligned}
&\frac{2}{c-\ell}\odot(FA)\int_\ell^c \widetilde{\mathfrak{S}}(\theta)\otimes\widetilde{G}(\theta)d\theta\\
&\supseteq_F M(\ell,\ c)\odot\int_0^1\left[\lambda^2\left(\frac{q}{2}\right)+\lambda^2\left(\frac{2-q}{2}\right)\right]dq\oplus 2\odot N(\ell,\ c)\odot\int_0^1\lambda\left(\frac{q}{2}\right)\lambda\left(\frac{2-q}{2}\right)dq\\
&\oplus H(\ell,\ c)\odot P(\ell,\ c)\odot\int_0^1\left[\lambda\left(\frac{q}{2}\right)+\lambda\left(\frac{2-q}{2}\right)\right]dq\oplus 2H^2(\ell,\ c).
\end{aligned} \tag{62}$$

Corollary 11 By choosing $\varphi(q)=\dfrac{q^{\alpha}}{\Gamma(\alpha)}$ in Theorem 9, we obtain the following inequality:

$$\begin{aligned}
&\frac{2^{\alpha}\Gamma(\alpha+1)}{(c-\ell)^{\alpha}}\odot\left[J^{\alpha}_{\left(\frac{\ell+c}{2}\right)+}\widetilde{\mathfrak{S}}(c)\otimes\widetilde{G}(c)\oplus J^{\alpha}_{\left(\frac{\ell+c}{2}\right)-}\widetilde{\mathfrak{S}}(\ell)\otimes\widetilde{G}(\ell)\right]\\
&\supseteq_F \alpha\odot M(\ell,\ c)\odot\int_0^1 q^{\alpha-1}\left[\lambda^2\left(\frac{q}{2}\right)+\lambda^2\left(\frac{2-q}{2}\right)\right]dq\\
&\oplus 2\alpha\odot N(\ell,\ c)\odot\int_0^1 q^{\alpha-1}\lambda\left(\frac{q}{2}\right)\lambda\left(\frac{2-q}{2}\right)dq\\
&\oplus \alpha H(\ell,\ c)\odot P(\ell,\ c)\odot\int_0^1 q^{\alpha-1}\left[\lambda\left(\frac{q}{2}\right)+\lambda\left(\frac{2-q}{2}\right)\right]dq\oplus 2H^2(\ell,\ c).
\end{aligned} \tag{63}$$

Theorem 10 Let $\widetilde{\mathfrak{S}},\ \widetilde{G}:[\ell,\ c]\to\mathbb{F}_0$, be two fuzzy-number mapping, such that from γ-cuts, where we generate the set of *IVMs* $\mathfrak{S}_\gamma,\ G_\gamma:[\ell,\ c]\to\mathbb{R}^+_J$ such that $[\widetilde{\mathfrak{S}}(q)]^{\gamma}=[\underline{\mathfrak{S}}(q,\ \gamma),\ \overline{\mathfrak{S}}(q,\ \gamma)]$ and $[\widetilde{G}(q)]^{\gamma}=[\underline{G}(q,\ \gamma),\ \overline{G}(q,\ \gamma)]$. If $\widetilde{\mathfrak{S}},\ \widetilde{G}$ are two $U\cdot D$-fuzzy-valued generalized strong λ-convex mappings, then the upcoming inequality apply to the generalized fractional integrals:

$$\begin{aligned}
&\frac{1}{\lambda^2\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\otimes\widetilde{G}\left(\frac{\ell+c}{2}\right)\\
&\ominus\frac{1}{\lambda\left(\frac{1}{2}\right)\psi(1)}\odot\left[(FA)\int_{\frac{\ell+c}{2}}^{c}\frac{\varphi(c-\theta)}{c-\theta}[\widetilde{\mathfrak{S}}(\theta)\oplus\widetilde{G}(\theta)]H(\theta,\ \ell+c-\theta)d\theta\right.\\
&\left.\oplus(FA)\int_{\ell}^{\frac{\ell+c}{2}}\frac{\varphi(\theta-\ell)}{\theta-\ell}\odot[\widetilde{\mathfrak{S}}(\theta)\oplus\widetilde{G}(\theta)]\odot H(\ell+c-\theta,\ \theta)d\theta\right]\\
&\supseteq_F\frac{1}{\psi(1)}\odot\left[_{\left(\frac{\ell+c}{2}\right)}+\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(c)\otimes\widetilde{G}(c)\oplus_{\left(\frac{\ell+c}{2}\right)}-\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(\ell)\otimes\widetilde{G}(\ell)\right]\\
&\oplus\frac{1}{\psi(1)\lambda^2\left(\frac{1}{2}\right)}\odot\int_{\ell}^{\frac{\ell+c}{2}}\frac{\varphi(\theta-\ell)}{\theta-\ell}H^2(\theta,\ \ell+c-\theta)d\theta\\
&\oplus\frac{1}{\psi(1)}\odot[K_5\odot M(\ell,\ c)\oplus K_4\odot N(\ell,\ c)\oplus K_6\odot P(\ell,\ c)H(\ell,\ c)]\oplus 2H^2(\ell,\ c),
\end{aligned} \tag{64}$$

where Theorems 4 and 9 define $M(\ell,\ c)$, $N(\ell,\ c)$, and $K_4,\ K_5$, and K_6, respectively.

Proof. Since $\widetilde{\mathfrak{S}}$ and $\widetilde{G}$ are two $U\cdot D$-fuzzy-valued generalized strong λ-convex mappings on $[\ell,\ c]$, we have

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\mathfrak{S}_{\gamma}\left(\frac{\theta+y}{2}\right)\supseteq_{I}\mathfrak{S}_{\gamma}(\theta)+\mathfrak{S}_{\gamma}(y)+\frac{1}{\hbar\left(\frac{1}{2}\right)}H(\theta,\ y). \tag{65}$$

For $\theta=\frac{2-q}{2}\ell+\frac{q}{2}c$ and $y=\frac{q}{2}\ell+\frac{2-q}{2}c$, we obtain

$$\begin{aligned}\frac{1}{\hbar\left(\frac{1}{2}\right)}\mathfrak{S}_{\gamma}\left(\frac{\ell+c}{2}\right)\supseteq_{I}&\mathfrak{S}_{\gamma}\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)+\mathfrak{S}_{\gamma}\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)\\&+\frac{1}{\hbar\left(\frac{1}{2}\right)}H\left(\frac{2-q}{2}\ell+\frac{q}{2}c,\ \frac{q}{2}\ell+\frac{2-q}{2}c\right).\end{aligned} \tag{66}$$

In similar way, we acquire

$$\begin{aligned}\frac{1}{\hbar\left(\frac{1}{2}\right)}G_{\gamma}\left(\frac{\ell+c}{2}\right)\supseteq_{I}&G_{\gamma}\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)+G_{\gamma}\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)\\&+\frac{1}{\hbar\left(\frac{1}{2}\right)}H\left(\frac{2-q}{2}\ell+\frac{q}{2}c,\ \frac{q}{2}\ell+\frac{2-q}{2}c\right).\end{aligned} \tag{67}$$

The inequalities (66) and (67), when multiplied, yield

$$\begin{aligned}&\frac{1}{\hbar^{2}\left(\frac{1}{2}\right)}\mathfrak{S}_{\gamma}\left(\frac{\ell+c}{2}\right)G_{\gamma}\left(\frac{\ell+c}{2}\right)\\&\supseteq_{I}\mathfrak{S}_{\gamma}\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)G_{\gamma}\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)\\&\quad+\mathfrak{S}_{\gamma}\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)G_{\gamma}\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)\\&\quad+\mathfrak{S}_{\gamma}\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)G_{\gamma}\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)\\&\quad+\mathfrak{S}_{\gamma}\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)G_{\gamma}\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)\end{aligned}$$

$$+\frac{1}{\lambda^2\left(\frac{1}{2}\right)}H^2\left(\frac{2-q}{2}\ell+\frac{q}{2}c,\ \frac{q}{2}\ell+\frac{2-q}{2}c\right)$$

$$+\frac{1}{\lambda\left(\frac{1}{2}\right)}\mathfrak{S}_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)H\left(\frac{2-q}{2}\ell+\frac{q}{2}c,\ \frac{q}{2}\ell+\frac{2-q}{2}c\right)$$

$$+\frac{1}{\lambda\left(\frac{1}{2}\right)}\mathfrak{S}_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)H\left(\frac{2-q}{2}\ell+\frac{q}{2}c,\ \frac{q}{2}\ell+\frac{2-q}{2}c\right)$$

$$\frac{1}{\lambda\left(\frac{1}{2}\right)}G_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)H\left(\frac{2-q}{2}\ell+\frac{q}{2}c,\ \frac{q}{2}\ell+\frac{2-q}{2}c\right)$$

$$\supseteq_I \mathfrak{S}_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)G_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)+\mathfrak{S}_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)G_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)$$

$$+\left[\lambda\left(\frac{2-q}{2}\right)\mathfrak{S}_\gamma(\ell)+\lambda\left(\frac{q}{2}\right)\mathfrak{S}_\gamma(c)+H(\ell,\ c)\right]$$

$$\times\left[\lambda\left(\frac{q}{2}\right)G_\gamma(\ell)+\lambda\left(\frac{2-q}{2}\right)G_\gamma(c)+H(\ell,\ c)\right]$$

$$+\left[\lambda\left(\frac{q}{2}\right)\mathfrak{S}_\gamma(\ell)+\lambda\left(\frac{2-q}{2}\right)\mathfrak{S}_\gamma(c)+H(\ell,\ c)\right]$$

$$\times\left[\lambda\left(\frac{2-q}{2}\right)G_\gamma(\ell)+\lambda\left(\frac{q}{2}\right)G_\gamma(c)+H(\ell,\ c)\right]$$

$$+\frac{1}{\lambda^2\left(\frac{1}{2}\right)}H^2\left(\frac{2-q}{2}\ell+\frac{q}{2}c,\ \frac{q}{2}\ell+\frac{2-q}{2}c\right)$$

$$+\frac{1}{\lambda\left(\frac{1}{2}\right)}\mathfrak{S}_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)H\left(\frac{2-q}{2}\ell+\frac{q}{2}c,\ \frac{q}{2}\ell+\frac{2-q}{2}c\right)$$

$$+\frac{1}{\lambda\left(\frac{1}{2}\right)}\mathfrak{S}_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)H\left(\frac{2-q}{2}\ell+\frac{q}{2}c,\ \frac{q}{2}\ell+\frac{2-q}{2}c\right)$$

$$
\begin{aligned}
&+\frac{1}{\lambda\left(\frac{1}{2}\right)}G_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)H\left(\frac{2-q}{2}\ell+\frac{q}{2}c,\ \frac{q}{2}\ell+\frac{2-q}{2}c\right)\\
&+\frac{1}{\lambda\left(\frac{1}{2}\right)}G_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)H\left(\frac{2-q}{2}\ell+\frac{q}{2}c,\ \frac{q}{2}\ell+\frac{2-q}{2}c\right)\\
&=\mathfrak{S}_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)G_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)+\mathfrak{S}_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)G_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)\\
&+2M(\ell,\ c)\lambda\left(\frac{2-q}{2}\right)\lambda\left(\frac{q}{2}\right)+\left[\lambda^2\left(\frac{2-q}{2}\right)+\lambda^2\left(\frac{q}{2}\right)\right]N(\ell,\ c)\\
&+\left[\lambda\left(\frac{q}{2}\right)+\lambda\left(\frac{2-q}{2}\right)\right]P(\ell,\ c)H(\ell,\ c)+\frac{1}{\lambda^2\left(\frac{1}{2}\right)}H^2\left(\frac{2-q}{2}\ell+\frac{q}{2}c,\ \frac{q}{2}\ell+\frac{2-q}{2}c\right)\\
&+\frac{1}{\lambda\left(\frac{1}{2}\right)}\mathfrak{S}_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)H\left(\frac{2-q}{2}\ell+\frac{q}{2}c,\ \frac{q}{2}\ell+\frac{2-q}{2}c\right)\\
&+\frac{1}{\lambda\left(\frac{1}{2}\right)}\mathfrak{S}_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)H\left(\frac{2-q}{2}\ell+\frac{q}{2}c,\ \frac{q}{2}\ell+\frac{2-q}{2}c\right)\\
&+\frac{1}{\lambda\left(\frac{1}{2}\right)}G_\gamma\left(\frac{2-q}{2}\ell+\frac{q}{2}c\right)H\left(\frac{2-q}{2}\ell+\frac{q}{2}c,\ \frac{q}{2}\ell+\frac{2-q}{2}c\right)\\
&+\frac{1}{\lambda\left(\frac{1}{2}\right)}G_\gamma\left(\frac{q}{2}\ell+\frac{2-q}{2}c\right)H\left(\frac{2-q}{2}\ell+\frac{q}{2}c,\ \frac{q}{2}\ell+\frac{2-q}{2}c\right)+2H^2(\ell,\ c).
\end{aligned}
\tag{68}
$$

By multiplying both sides of (68) by $\frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}$ and integrating the resulting expression with respect to $\mathfrak{S}$ over the interval $[0,\ 1]$, we derive the inequality (5.43).

Corollary 12 Choosing $\varphi(q)=q$ in Theorem 10, then we acquire the upcoming inequality:

$$
\begin{aligned}
&\frac{1}{\lambda^2\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right) \otimes \widetilde{G}\left(\frac{\ell+c}{2}\right) \\
&\ominus \frac{2}{(c-\ell)\lambda\left(\frac{1}{2}\right)} \odot \left[(FA)\int_{\frac{\ell+c}{2}}^{c} [\widetilde{\mathfrak{S}}(\theta) \oplus \widetilde{G}(\theta)] \odot H(\theta,\ \ell+c-\theta)d\theta \right. \\
&\left. +(FA)\int_{\ell}^{\frac{\ell+c}{2}} [\widetilde{\mathfrak{S}}(\theta) \oplus \widetilde{G}(\theta)] \odot H(\ell+c-\theta,\ \theta)d\theta \right] \\
&\supseteq_F \frac{2}{c-\ell} \odot (FA)\int_{\ell}^{c} \widetilde{\mathfrak{S}}(\theta) \otimes \widetilde{G}(\theta)d\theta \oplus \frac{2}{(c-\ell)\lambda^2\left(\frac{1}{2}\right)} \odot \int_{\frac{\ell+c}{2}}^{c} H^2(\theta,\ \ell+c-\theta)d\theta \\
&\oplus 2 \odot M(\ell,\ c) \odot \int_0^1 \lambda\left(\frac{q}{2}\right)\lambda\left(\frac{2-q}{2}\right)dq \oplus N(\ell,\ c) \odot \int_0^1 \left[\lambda^2\left(\frac{q}{2}\right) + \lambda^2\left(\frac{2-q}{2}\right)\right]dq \\
&\oplus P(\ell,\ c) \odot H(\ell,\ c)\int_0^1 \left[\lambda\left(\frac{q}{2}\right) + \lambda\left(\frac{2-q}{2}\right)\right]dq \oplus 2H^2(\ell,\ c).
\end{aligned} \tag{69}
$$

Corollary 13 We derive the following inequality by applying $\varphi(q) = \frac{q^\alpha}{\Gamma(\alpha)}$ in Theorem 10:

$$
\begin{aligned}
&\frac{1}{\lambda^2\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right) \otimes \widetilde{G}\left(\frac{\ell+c}{2}\right) \\
&\ominus \frac{2^\alpha \alpha}{(c-\ell)^\alpha \lambda\left(\frac{1}{2}\right)} \odot \left[(FA)\int_{\frac{\ell+c}{2}}^{c} (c-\theta)^{\alpha-1} \odot [\widetilde{\mathfrak{S}}(\theta) \oplus \widetilde{G}(\theta)] \odot H(\theta,\ \ell+c-\theta)d\theta \right. \\
&\left. \oplus(FA)\int_{\ell}^{\frac{\ell+c}{2}} (\theta-\ell)^{\alpha-1} \odot [\widetilde{\mathfrak{S}}(\theta) \oplus \widetilde{G}(\theta)] \odot H(\ell+c-\theta,\ \theta)d\theta \right] \\
&\supseteq_F \frac{2^\alpha \Gamma(\alpha+1)}{(c-\ell)^\alpha} \odot \left[J^\alpha_{\left(\frac{\ell+c}{2}\right)+} \widetilde{\mathfrak{S}}(c) \otimes \widetilde{G}(c) \oplus J^\alpha_{\left(\frac{\ell+c}{2}\right)-} \widetilde{\mathfrak{S}}(\ell) \otimes \widetilde{G}(\ell)\right] \\
&\oplus \frac{\alpha 2^\alpha}{(c-\ell)^\alpha \lambda^2\left(\frac{1}{2}\right)} \odot \int_{\frac{\ell+c}{2}}^{c} (c-\theta)^{\alpha-1} H^2(\theta,\ \ell+c-\theta)d\theta
\end{aligned}
$$

$$\oplus \alpha \odot \left\{ 2 \odot M(\ell,\ c) \odot \int_0^1 q^{\alpha-1} \lambda\left(\frac{q}{2}\right) \lambda\left(\frac{2-q}{2}\right) dq \right.$$

$$\oplus N(\ell,\ c) \odot \int_0^1 q^{\alpha-1} \left[\lambda^2\left(\frac{q}{2}\right) + \lambda^2\left(\frac{2-q}{2}\right) \right] dq$$

$$\left. \oplus P(\ell,\ c) \odot H(\ell,\ c) \int_0^1 q^{\alpha-1} \left[\lambda\left(\frac{q}{2}\right) + \lambda\left(\frac{2-q}{2}\right) \right] dq \right\} \oplus 2H^2(\ell,\ c). \tag{70}$$

Theorem 11 Let $\widetilde{\mathfrak{S}} : [\ell,\ c] \to \mathbb{F}_0$ be an fuzzy-number mapping, such that from γ-cuts, where we generate the set of *IVMs* $\mathfrak{S}_\gamma : [\ell,\ c] \to \mathbb{R}_J^+$ such that $[\widetilde{\mathfrak{S}}(q)]^\gamma = [\underline{\mathfrak{S}}(q,\ \gamma),\ \overline{\mathfrak{S}}(q,\ \gamma)]$. If $\widetilde{\mathfrak{S}}$ is $U \cdot D$-fuzzy-valued generalized strong λ-convex mapping, then the upcoming inequalities apply to the generalized fractional integrals:

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right) \ominus \frac{1}{2\psi(1)\lambda\left(\frac{1}{2}\right)} \odot \int_{\frac{\ell+c}{2}}^{c} \frac{\varphi\left(\theta - \frac{\ell+c}{2}\right)}{\theta - \frac{\ell+c}{2}} H(\theta,\ \ell+c-\theta) d\theta$$

$$\supseteq_F \frac{1}{2\psi(1)} \odot \left[{}_{\ell+}\mathfrak{I}_\varphi \widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right) \oplus {}_{\ell-}\mathfrak{I}_\varphi \widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right) \right] \tag{71}$$

$$\supseteq_F \frac{\widetilde{\mathfrak{S}}(\ell) \oplus \widetilde{\mathfrak{S}}(c)}{2\psi(1)} \odot \int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2} q\right)}{q} \left[\lambda\left(\frac{1+q}{2}\right) + \lambda\left(\frac{1-q}{2}\right) \right] dq \oplus H(\ell,\ c).$$

Proof. Since $\widetilde{\mathfrak{S}}$ is an $U \cdot D$-fuzzy-valued generalized strong λ-convex mapping on $[\ell,\ c]$, we have

$$\frac{1}{\lambda\left(\frac{1}{2}\right)} \mathfrak{S}_\gamma\left(\frac{\theta+y}{2}\right) \supseteq_I \mathfrak{S}_\gamma(\theta) + \mathfrak{S}_\gamma(y) + \frac{1}{\lambda\left(\frac{1}{2}\right)} H(\theta,\ y).$$

For $e = \frac{1-q}{2}\ell + \frac{1+q}{2}c$ and $y = \frac{1+q}{2}\ell + \frac{1-q}{2}c$, we get

$$\frac{1}{\lambda\left(\frac{1}{2}\right)} \mathfrak{S}_\gamma\left(\frac{\ell+c}{2}\right) \supseteq_I \mathfrak{S}_\gamma\left(\frac{1-q}{2}\ell + \frac{1+q}{2}c\right) + \mathfrak{S}_\gamma\left(\frac{1+q}{2}\ell + \frac{1-q}{2}c\right)$$

$$+ \frac{1}{\lambda\left(\frac{1}{2}\right)} H\left(\frac{1-q}{2}\ell + \frac{1+q}{2}c,\ \frac{1+q}{2}\ell + \frac{1-q}{2}c\right). \tag{72}$$

By multiplying both sides of (72) by $\dfrac{\varphi\left(\dfrac{(c-\ell)}{2}q\right)}{q}$ and integrating the resulting expression with respect to s over the interval $[0,\ 1]$, we derive

$$\begin{aligned}
&\frac{1}{\lambda\left(\frac{1}{2}\right)}\mathfrak{S}_\gamma\left(\frac{\ell+c}{2}\right)\int_0^1\frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}dq\\
&\supseteq_I (IA)\int_0^1\frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\mathfrak{S}_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)dq\\
&+(IA)\int_0^1\frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\mathfrak{S}_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)dq\\
&+\frac{1}{\lambda\left(\frac{1}{2}\right)}\int_0^1\frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}H\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c,\ \frac{1+q}{2}\ell+\frac{1-q}{2}c\right)dq.
\end{aligned}\tag{73}$$

From Eq. (73), we acquire

$$\begin{aligned}
&(IA)\int_0^1\frac{\varphi\left(\frac{(c-\ell}{2}q\right)}{q}\mathfrak{S}_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)dq\\
&=\left[(R)\int_0^1\frac{\varphi\left(\frac{(c-\ell}{2}q\right)}{q}\underline{\mathfrak{S}}\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c,\ \gamma\right)dq\ \ \text{(R)}\ \int_0^1\frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\overline{\mathfrak{S}}\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c,\ \gamma\right)dq\right]\\
&=\left[(R)\int_{\frac{\ell+c}{2}}^{\ell}\frac{\varphi\left(u\frac{\ell+c}{2}\right)}{u-\frac{\ell+c}{2}}\underline{\mathfrak{S}}(u,\ \gamma)du,\ (R)\int_{\frac{\ell+c}{2}}^{\ell}\frac{\varphi\left(u-\frac{\ell+c}{2}\right)}{u-\frac{\ell+c}{2}}\overline{\mathfrak{S}}(u,\ \gamma)du\right]\\
&=\left[{}_{c-}I_\varphi\underline{\mathfrak{S}}\left(\frac{\ell+c}{2},\ \gamma\right),\ {}_{c-}I_\varphi\overline{\mathfrak{S}}\left(\frac{\ell+c}{2},\ \gamma\right)\right]={}_{\ell-}\mathfrak{I}_\varphi\mathfrak{S}_\gamma\left(\frac{\ell+c}{2}\right).
\end{aligned}$$

In the similar way, we get

$$(IA)\int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\mathfrak{S}_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)dq=\left[{}_{\ell+}I_\varphi\underline{\mathfrak{S}}\left(\frac{\ell+c}{2},\ \gamma\right),\ {}_{\ell+}I_\varphi\overline{\mathfrak{S}}\left(\frac{\ell+c}{2},\ \gamma\right)\right]={}_{\ell+}\mathfrak{I}_\varphi\mathfrak{S}_\gamma\left(\frac{\ell+c}{2},\ \gamma\right).$$

Thus, we have established the first inequality. To demonstrate the second inequality of (71), we begin by observing that, given $\widetilde{\mathfrak{S}}$ is a $U\cdot D$-fuzzy-valued generalized strong λ-convex mapping, we can state that

$$\mathfrak{S}_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)\supseteq_I \lambda\left(\frac{1+q}{2}\right)\mathfrak{S}_\gamma(\ell)+\lambda\left(\frac{1-q}{2}\right)\mathfrak{S}_\gamma(c)+H(\ell,\ c), \tag{74}$$

and

$$\mathfrak{S}_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)\supseteq_I \lambda\left(\frac{1-q}{2}\right)\mathfrak{S}_\gamma(\ell)+\lambda\left(\frac{1+q}{2}\right)\mathfrak{S}_\gamma(c)+H(\ell,\ c). \tag{75}$$

When we add (74) and (75), we get

$$\begin{aligned}&\mathfrak{S}_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)+\mathfrak{S}_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)\\ &\supseteq_I\left[\mathfrak{S}_\gamma(\ell)+\mathfrak{S}_\gamma(c)\right]\left[\lambda\left(\frac{1-q}{2}\right)+\lambda\left(\frac{1+q}{2}\right)\right]+2H(\ell,\ c).\end{aligned} \tag{76}$$

By multiplying both sides of (76) by $\frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}$ and integrating the resulting expression with respect to s over the interval $[0,\ 1]$, we derive

$$\begin{aligned}&(IA)\int_0^1 \frac{\varphi\left(\frac{(c-\ell}{2}q\right)}{q}\mathfrak{S}_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)dq+(IA)\int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\mathfrak{S}_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)dq\\ &\supseteq_I\left[\mathfrak{S}_\gamma(\ell)+\mathfrak{S}_\gamma(c)\right]\int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\left[\lambda\left(\frac{1-q}{2}\right)+\lambda\left(\frac{1+q}{2}\right)\right]dq+2H(\ell,\ c)\int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}dq.\end{aligned}$$

This brings the proof to a close.

Corollary 14 If we select $\varphi(q)=q$ in Theorem 11, the following inequalities result:

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\ominus\frac{1}{(c-\ell)\hbar\left(\frac{1}{2}\right)}\int_{\frac{\ell+c}{2}}^{c}H(\theta,\ \ell+c-\theta)d\theta$$

$$\supseteq_F\frac{1}{c-\ell}\odot(FA)\int_{\ell}^{c}\widetilde{\mathfrak{S}}(\theta)d\theta\supseteq_F\frac{\widetilde{\mathfrak{S}}(\ell)\oplus\widetilde{\mathfrak{S}}(c)}{2}\odot\int_0^1\left[\hbar\left(\frac{1-q}{2}\right)+\hbar\left(\frac{1+q}{2}\right)\right]dq\oplus H(\ell,\ c). \tag{77}$$

Corollary 15 By selecting $\varphi(q)=\dfrac{q^{\alpha}}{\Gamma(\alpha)}$ in Theorem 11, we derive the following inequalities:

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\ominus\frac{\alpha 2^{\alpha-1}}{(c-\ell)^{\alpha}\hbar\left(\frac{1}{2}\right)}\int_{\frac{\ell+c}{2}}^{c}\left(\theta-\frac{\ell+c}{2}\right)^{\alpha-1}H(\theta,\ \ell+c-\theta)d\theta$$

$$\supseteq_F\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(c-\ell)^{\alpha}}\odot\left[J_{\ell+}^{\alpha}\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\oplus J_{c-}^{\alpha}\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\right] \tag{78}$$

$$\supseteq_F\frac{\alpha\odot[\widetilde{\mathfrak{S}}(\ell)+\widetilde{\mathfrak{S}}(c)]}{2}\odot\int_0^1 q^{\alpha-1}\left[\hbar\left(\frac{1-q}{2}\right)+\hbar\left(\frac{1+q}{2}\right)\right]dq\oplus H(\ell,\ c).$$

Theorem 12 Let $\widetilde{\mathfrak{S}},\ \widetilde{G}:[\ell,\ c]\to\mathbb{F}_0$, be an fuzzy-number mapping, such that from γ-cuts, where we generate the set of $IVMs$ $\mathfrak{S}_{\gamma},\ G_{\gamma}:[\ell,\ c]\to\mathbb{R}_J^{+}$ such that $[\widetilde{\mathfrak{S}}(q)]^{\gamma}=[\underline{\mathfrak{S}}(q,\ \gamma),\ \overline{\mathfrak{S}}(q,\ \gamma)]$ and $[\widetilde{G}(q)]^{\gamma}=[\underline{G}(q,\ \gamma),\ \overline{G}(q,\ \gamma)]$. If $\widetilde{\mathfrak{S}},\ \widetilde{G}$ are two $U\cdot D$-fuzzy-valued generalized strong $\hbar$-convex mappings, then the upcoming inequality apply to the generalized fractional integrals:

$$\left[_{\ell+}\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\otimes\widetilde{G}\left(\frac{\ell+c}{2}\right)\oplus_{c^-}\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\otimes\widetilde{G}\left(\frac{\ell+c}{2}\right)\right]$$

$$\supseteq_F K_7\odot M(\ell,\ c)\oplus K_8\odot N(\ell,\ c)\oplus K_9\odot H(\ell,\ c)P(\ell,\ c)\oplus 2\psi(1)H^2(\ell,\ c), \tag{79}$$

where $M(\ell,\ c)$ and $N(\ell,\ c)$ are derived in Theorem 4 and

$$K_7=\int_0^1\frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\left[\hbar^2\left(\frac{1+q}{2}\right)+\hbar^2\left(\frac{1-q}{2}\right)\right]dq$$

$$K_8=2\int_0^1\frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\hbar\left(\frac{1+q}{2}\right)\hbar\left(\frac{1-q}{2}\right)dq$$

$$K_9 = \int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\left[\hbar\left(\frac{1+q}{2}\right)+\hbar\left(\frac{1-q}{2}\right)\right]dq.$$

Proof. Since $\widetilde{\mathfrak{S}}$ and $\widetilde{G}$ are two $U \cdot D$-fuzzy-valued generalized strong $\hbar$-convex mappings,

$$\mathfrak{S}_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right) \supseteq_I \hbar\left(\frac{1-q}{2}\right)\mathfrak{S}_\gamma(\ell)+\hbar\left(\frac{1+q}{2}\right)\mathfrak{S}_\gamma(c)+H(\ell,\ c), \tag{80}$$

and

$$G_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right) \supseteq_I \hbar\left(\frac{1-q}{2}\right)G_\gamma(\ell)+\hbar\left(\frac{1+q}{2}\right)G_\gamma(c)+H(\ell,\ c). \tag{81}$$

By multiplying (80) with (81), we obtain

$$\begin{aligned}
&\mathfrak{S}_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)G_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)\\
&\supseteq_I \hbar^2\left(\frac{1-q}{2}\right)\mathfrak{S}_\gamma(\ell)G_\gamma(\ell)+\hbar^2\left(\frac{1+q}{2}\right)\mathfrak{S}_\gamma(c)G_\gamma(c)\\
&\quad+\hbar\left(\frac{1-q}{2}\right)\hbar\left(\frac{1+q}{2}\right)\left[\mathfrak{S}_\gamma(\ell)G_\gamma(c)+\mathfrak{S}_\gamma(c)G_\gamma(\ell)\right]\\
&\quad+\hbar\left(\frac{1+q}{2}\right)H(\ell,\ c)\left[\mathfrak{S}_\gamma(c)+G_\gamma(c)\right]+\hbar\left(\frac{1-q}{2}\right)H(\ell,\ c)\left[\mathfrak{S}_\gamma(\ell)+G_\gamma(\ell)\right]+H^2(\ell,\ c).
\end{aligned} \tag{82}$$

In the similar way, we acquire

$$\begin{aligned}
&\mathfrak{S}_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)G_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)\\
&\supseteq_I \hbar^2\left(\frac{1+q}{2}\right)\mathfrak{S}_\gamma(\ell)G_\gamma(\ell)+\hbar^2\left(\frac{1-q}{2}\right)\mathfrak{S}_\gamma(c)G_\gamma(c)\\
&\quad+\hbar\left(\frac{1+q}{2}\right)\hbar\left(\frac{1-q}{2}\right)\left[\mathfrak{S}_\gamma(\ell)G_\gamma(c)+\mathfrak{S}_\gamma(c)G_\gamma(\ell)\right]\\
&\quad+\hbar\left(\frac{1-q}{2}\right)H(\ell,\ c)\left[\mathfrak{S}_\gamma(c)+G_\gamma(c)\right]+\hbar\left(\frac{1+q}{2}\right)H(\ell,\ c)\left[\mathfrak{S}_\gamma(\ell)+G_\gamma(\ell)\right]+H^2(\ell,\ c).
\end{aligned} \tag{83}$$

By combining (82) and (83), we arrive at the following relation:

$$
\begin{aligned}
&\mathfrak{S}_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)G_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)\\
&+\mathfrak{S}_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)G_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)\\
&\supseteq_I \hbar^2\left(\frac{1-q}{2}\right)\left[\mathfrak{S}_\gamma(\ell)G_\gamma(\ell)+\mathfrak{S}_\gamma(c)G_\gamma(c)\right]\\
&+\hbar^2\left(\frac{1+q}{2}\right)\left[\mathfrak{S}_\gamma(\ell)G_\gamma(\ell)+\mathfrak{S}_\gamma(c)G_\gamma(c)\right]\\
&+2\hbar\left(\frac{1+q}{2}\right)\hbar\left(\frac{1-q}{2}\right)\left[\mathfrak{S}_\gamma(\ell)G_\gamma(c)+\mathfrak{S}_\gamma(c)G_\gamma(\ell)\right]\\
&+H(\ell,\ c)\left[\hbar\left(\frac{1+q}{2}\right)+\hbar\left(\frac{1-q}{2}\right)\right]\left[\mathfrak{S}_\gamma(\ell)+\mathfrak{S}_\gamma(c)+G_\gamma(\ell)+G_\gamma(c)\right]+2H^2(\ell,\ c)\\
&=\left[\hbar^2\left(\frac{1-q}{2}\right)+\hbar^2\left(\frac{1+q}{2}\right)\right]M(\ell,\ c)+2\hbar\left(\frac{1+q}{2}\right)\hbar\left(\frac{1-q}{2}\right)N(\ell,\ c)\\
&+H(\ell,\ c)P(\ell,\ c)\left[\hbar\left(\frac{1+q}{2}\right)+\hbar\left(\frac{1-q}{2}\right)\right]+2H^2(\ell,\ c).
\end{aligned}
\tag{84}
$$

By multiplying both sides of (84) by $\dfrac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}$ and integrating the resulting expression with respect to s over the interval $[0,\ 1]$, we derive

$$
\begin{aligned}
&(IA)\int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\mathfrak{S}_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)G_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)dq\\
&+(IA)\int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\mathfrak{S}_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)G_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)dq\\
&\supseteq_I M(\ell,\ c)\int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\left[\hbar^2\left(\frac{1-q}{2}\right)+\hbar^2\left(\frac{1+q}{2}\right)\right]dq
\end{aligned}
$$

$$+2N(\ell,\, c)\int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q} \lambda\left(\frac{1+q}{2}\right)\lambda\left(\frac{1-q}{2}\right)dq$$

$$+H(\ell,\, c)P(\ell,\, c)\int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\left[\lambda\left(\frac{1+q}{2}\right)+\lambda\left(\frac{1-q}{2}\right)\right]dq$$

$$+2H^2(\ell,\, c)\int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}dq. \tag{85}$$

From Equation (85), we derive the desired inequality.

Corollary 16 If we select $\varphi(q) = q$ in Theorem 12, we obtain the following inequality:

$$\begin{aligned}
&\frac{2}{c-\ell}\odot(IA)\int_\ell^c \widetilde{\mathfrak{S}}(\theta)\otimes\widetilde{G}(\theta)d\theta \\
&\supseteq_F M(\ell,\, c)\odot\int_0^1\left[\lambda^2\left(\frac{1+q}{2}\right)+\lambda^2\left(\frac{1-q}{2}\right)\right]dq\oplus 2N(\ell,\, c)\odot\int_0^1 \lambda\left(\frac{1+q}{2}\right)\lambda\left(\frac{1+q}{2}\right)dq \\
&\oplus H(\ell,\, c)\odot P(\ell,\, c)\odot\int_0^1\left[\lambda\left(\frac{1+q}{2}\right)+\lambda\left(\frac{1-q}{2}\right)\right]dq\oplus 2H^2(\ell,\, c).
\end{aligned} \tag{86}$$

Corollary 17 By choosing $\varphi(q) = \dfrac{q^\alpha}{\Gamma(\alpha)}$ in Theorem 12, we derive the following inequality:

$$\begin{aligned}
&\frac{2^\alpha\Gamma(\alpha+1)}{(c-\ell)^\alpha}\odot\left[J^\alpha_{\left(\frac{\ell+c}{2}\right)+}\widetilde{\mathfrak{S}}(c)\otimes\widetilde{G}(c)\oplus J^\alpha_{\left(\frac{\ell+c}{2}\right)-}\widetilde{\mathfrak{S}}(\ell)\otimes\widetilde{G}(\ell)\right] \\
&\supseteq_F \alpha\odot\left\{M(\ell,\, c)\odot\int_0^1 q^{\alpha-1}\left[\lambda^2\left(\frac{1+q}{2}\right)+\lambda^2\left(\frac{1-q}{2}\right)\right]dq\right. \\
&\oplus 2\odot N(\ell,\, c)\odot\int_0^1 q^{\alpha-1}\lambda\left(\frac{1+q}{2}\right)\lambda\left(\frac{1-q}{2}\right)dq \\
&\left.\oplus H(\ell,\, c)\odot P(\ell,\, c)\odot\int_0^1 q^{\alpha-1}\left[\lambda\left(\frac{1+q}{2}\right)+\lambda\left(\frac{1-q}{2}\right)\right]dq\right\}\oplus 2H^2(\ell,\, c).
\end{aligned} \tag{87}$$

Theorem 13 Let $\widetilde{\mathfrak{S}},\ \widetilde{G}:[\ell,\ c]\to\mathbb{F}_0$ be two fuzzy-number mappings, such that from γ-cuts, where we generate the set of $IVMs$ $\mathfrak{S}_\gamma,\ G_\gamma:[\ell,\ c]\to\mathbb{R}_J^+$ such that $[\widetilde{\mathfrak{S}}(q)]^\gamma = [\underline{\mathfrak{S}}(q,\ \gamma),\ \overline{\mathfrak{S}}(q,\ \gamma)]$ and $[\widetilde{G}(q)]^\gamma = [\underline{G}(q,\ \gamma),\ \overline{G}(q)]$. If $\widetilde{\mathfrak{S}},\ \widetilde{G}$ are two $U\cdot D$-fuzzy-valued generalized strong λ-convex mappings, then the upcoming inequality apply to the generalized fractional integrals:

$$
\begin{aligned}
&\frac{1}{\lambda^2\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right) \otimes \widetilde{G}\left(\frac{\ell+c}{2}\right) \\
&\ominus \frac{1}{\lambda\left(\frac{1}{2}\right)\psi(1)} \odot \left[(FA)\int_{\frac{\ell+c}{2}}^{\ell} \frac{\varphi\left(\theta-\frac{\ell+c}{2}\right)}{\theta-\frac{\ell+c}{2}} [\widetilde{\mathfrak{S}}(\theta) \oplus \widetilde{G}(\theta)] H(\theta,\ \ell+c-\theta)d\theta \right. \\
&\left. \oplus (FA)\int_{\ell}^{\frac{\ell+c}{2}} \frac{\varphi\left(\frac{\ell+c}{2}-\theta\right)}{\frac{\ell+c}{2}-\theta} \odot [\widetilde{\mathfrak{S}}(\theta) \oplus \widetilde{G}(\theta)] \odot H(\ell+c-\theta,\ \theta)d\theta \right] \\
&\supseteq_F \frac{1}{\psi(1)} \odot \left[{}_{\ell+}\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right) \otimes \widetilde{G}\left(\frac{\ell+c}{2}\right) \oplus {}_{\ell-}\ \mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right) \otimes \widetilde{G}\left(\frac{\ell+c}{2}\right) \right] \\
&\oplus \frac{1}{\psi(1)\lambda^2\left(\frac{1}{2}\right)} \odot \int_{\frac{\ell+c}{2}}^{\ell} \frac{\varphi\left(\theta-\frac{\ell+c}{2}\right)}{\theta-\frac{\ell+c}{2}} H^2(\theta,\ \ell+c-\theta)d\theta \\
&\oplus \frac{1}{\psi(1)} \odot [K_8 \odot M(\ell,\ c) + K_7 \odot N(\ell,\ c) + K_9 \odot P(\ell,\ c) \odot H(\ell,\ c)] + 2H^2(\ell,\ c),
\end{aligned} \tag{88}
$$

where $M(\ell,\ c)$, $N(\ell,\ c)$ and $K_7,\ K_8,\ K_9$ are defined in Theorem 4 and Theorem 12, respectively.

Proof. Since $\widetilde{\mathfrak{S}}$ and $\widetilde{G}$ are two $U \cdot D$-fuzzy-valued generalized strong λ-convex mappings on $[\ell,\ c]$, we have

$$
\frac{1}{\lambda\left(\frac{1}{2}\right)}\mathfrak{S}_{\gamma}\left(\frac{\theta+y}{2}\right) \supseteq_I \mathfrak{S}_{\gamma}(\theta) + \mathfrak{S}_{\gamma}(y) + \frac{1}{\lambda\left(\frac{1}{2}\right)}H(\theta,\ y). \tag{89}
$$

For $\theta = \frac{1-q}{2}\ell + \frac{1+q}{2}c$ and $y = \frac{1+q}{2}\ell + \frac{1-q}{2}c$, we acquire

$$
\begin{aligned}
\frac{1}{\lambda\left(\frac{1}{2}\right)}\mathfrak{S}_{\gamma}\left(\frac{\ell+c}{2}\right) \supseteq_I &\mathfrak{S}_{\gamma}\left(\frac{1-q}{2}\ell + \frac{1+q}{2}c\right) + \mathfrak{S}_{\gamma}\left(\frac{1+q}{2}\ell + \frac{1-q}{2}c\right) \\
&+ \frac{1}{\lambda\left(\frac{1}{2}\right)}H\left(\frac{1-q}{2}\ell + \frac{1+q}{2}c,\ \frac{1+q}{2}\ell + \frac{1-q}{2}c\right).
\end{aligned} \tag{90}
$$

Using same steps, we achieve

$$\frac{1}{\hbar\left(\frac{1}{2}\right)} G_\gamma\left(\frac{\ell+c}{2}\right) \supseteq_I G_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)+G_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)$$

$$+\frac{1}{\hbar\left(\frac{1}{2}\right)} H\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c,\ \frac{1+q}{2}\ell+\frac{1-q}{2}c\right). \tag{91}$$

By multiplying inequalities (90) and (91), we arrive at

$$\frac{1}{\hbar^2\left(\frac{1}{2}\right)} \mathfrak{S}_\gamma\left(\frac{\ell+c}{2}\right) G_\gamma\left(\frac{\ell+c}{2}\right)$$

$$\supseteq_I \mathfrak{S}_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right) G_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)+\mathfrak{S}_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right) G_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)$$

$$+\mathfrak{S}_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right) G_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)+\mathfrak{S}_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right) G_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)$$

$$+\frac{1}{\hbar^2\left(\frac{1}{2}\right)} H^2\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c,\ \frac{1+q}{2}\ell+\frac{1-q}{2}c\right)$$

$$+\frac{1}{\hbar\left(\frac{1}{2}\right)} \mathfrak{S}_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right) H\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c,\ \frac{1+q}{2}\ell+\frac{1-q}{2}c\right)$$

$$+\frac{1}{\hbar\left(\frac{1}{2}\right)} \mathfrak{S}_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right) H\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c,\ \frac{1+q}{2}\ell+\frac{1-q}{2}c\right)$$

$$+\frac{1}{\hbar\left(\frac{1}{2}\right)} G_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right) H\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c,\ \frac{1+q}{2}\ell+\frac{1-q}{2}c\right)$$

$$+\frac{1}{\hbar\left(\frac{1}{2}\right)} G_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right) H\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c,\ \frac{1+q}{2}\ell+\frac{1-q}{2}c\right)$$

$$
\begin{aligned}
&\supseteq_I \mathfrak{S}_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)G_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)+\mathfrak{S}_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)G_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)\\
&\quad+\left[\hbar\left(\frac{1-q}{2}\right)\mathfrak{S}_\gamma(\ell)+\hbar\left(\frac{1+q}{2}\right)\mathfrak{S}_\gamma(c)+H(\ell,\,c)\right]\times\left[\hbar\left(\frac{1+q}{2}\right)G_\gamma(\ell)+\hbar\left(\frac{1-q}{2}\right)G_\gamma(c)+H(\ell,\,c)\right]\\
&\quad+\left[\hbar\left(\frac{1+q}{2}\right)\mathfrak{S}_\gamma(\ell)+\hbar\left(\frac{1-q}{2}\right)\mathfrak{S}_\gamma(c)+H(\ell,\,c)\right]\times\left[\hbar\left(\frac{1-q}{2}\right)G_\gamma(\ell)+\hbar\left(\frac{1+q}{2}\right)G_\gamma(c)+H(\ell,\,c)\right]\\
&\quad+\frac{1}{\hbar^2\left(\frac{1}{2}\right)}H^2\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c,\ \frac{1+q}{2}\ell+\frac{1-q}{2}c\right)\\
&\quad+\frac{1}{\hbar\left(\frac{1}{2}\right)}\mathfrak{S}_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)H\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c,\ \frac{1+q}{2}\ell+\frac{1-q}{2}c\right)\\
&\quad+\frac{1}{\hbar\left(\frac{1}{2}\right)}\mathfrak{S}_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)H\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c,\ \frac{1+q}{2}\ell+\frac{1-q}{2}c\right)\\
&\quad+\frac{1}{\hbar\left(\frac{1}{2}\right)}G_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)H\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c,\ \frac{1+q}{2}\ell+\frac{1-q}{2}c\right)\\
&\quad+\frac{1}{\hbar\left(\frac{1}{2}\right)}G_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)H\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c,\ \frac{1+q}{2}\ell+\frac{1-q}{2}c\right)\\
&=\mathfrak{S}_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)G_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)\\
&\quad+\mathfrak{S}_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)G_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)\\
&\quad+2M(\ell,\,c)\hbar\left(\frac{1-q}{2}\right)\hbar\left(\frac{1+q}{2}\right)+\left[\frac{\hbar^2}{2}\left(\frac{1-q}{2}\right)+\hbar^2\left(\frac{1+q}{2}\right)\right]N(\ell,\,c)\\
&\quad+\left[\frac{1+q}{2}\right)+\hbar\left(\frac{1-q}{2}\right)\right]P(\ell,\,c)H(\ell,\,c)
\end{aligned}
$$

$$+\frac{1}{\hat{\lambda}^2\left(\frac{1}{2}\right)}H^2\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c,\ \frac{1+q}{2}\ell+\frac{1-q}{2}c\right)$$

$$+\frac{1}{\hat{\lambda}\left(\frac{1}{2}\right)}\mathfrak{S}_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)H\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c,\ \frac{1+q}{2}\ell+\frac{1-q}{2}c\right)$$

$$+\frac{1}{\hat{\lambda}\left(\frac{1}{2}\right)}\mathfrak{S}_\gamma\left(\frac{1+q}{2}\ell+\frac{1-q}{2}c\right)H\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c,\ \frac{1+q}{2}\ell+\frac{1-q}{2}c\right)$$

$$+\frac{1}{\hat{\lambda}\left(\frac{1}{2}\right)}G_\gamma\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c\right)H\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c,\ \frac{1+q}{2}\ell+\frac{1-q}{2}c\right)$$

$$\ell+\frac{1-q}{2}c\Bigg)H\left(\frac{1-q}{2}\ell+\frac{1+q}{2}c,\ \frac{1+q}{2}\ell+\frac{1-q}{2}c\right)+2H^2(\ell,\ c). \tag{92}$$

By multiplying both sides of (92) by $\dfrac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}$ and integrating the resulting expression with respect to s over the interval $[0,\ 1]$, we derive the inequality (88).

Corollary 18 By setting $\varphi(q)=q$ in Theorem 13, we derive the following inequality:

$$\frac{1}{\hat{\lambda}^2\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\widetilde{G}\left(\frac{\ell+c}{2}\right)$$

$$\oplus\frac{2}{(c-\ell)\hat{\lambda}\left(\frac{1}{2}\right)}\odot\left[(FA)\int_{\frac{\ell+c}{2}}^{c}[\widetilde{\mathfrak{S}}(\theta)\oplus\widetilde{G}(\theta)]\odot H(\theta,\ \ell+c-\theta)d\theta\right.$$

$$\left.\oplus(FA)\int_{\ell}^{\frac{\ell+c}{2}}[\widetilde{\mathfrak{S}}(\theta)\oplus\widetilde{G}(\theta)]\odot H(\ell+c-\theta,\ \theta)d\theta\right]$$

$$\supseteq_F\frac{2}{c-\ell}\odot(FA)\int_{\ell}^{c}\widetilde{\mathfrak{S}}(\theta)\widetilde{G}(\theta)d\theta\oplus\frac{2}{(c-\ell)\hat{\lambda}^2\left(\frac{1}{2}\right)}\int_{\frac{\ell+c}{2}}^{c}H^2(\varphi,\ \ell+c-\theta)d\theta$$

$$\oplus 2\odot M(\ell,\ c)\odot\int_0^1\hat{\lambda}\left(\frac{1+q}{2}\right)\hat{\lambda}\left(\frac{1-q}{2}\right)dq\oplus N(\ell,\ c)\odot\int_0^1\left[\hat{\lambda}^2\left(\frac{1+q}{2}\right)+\hat{\lambda}^2\left(\frac{1-q}{2}\right)\right]dq$$

$$\oplus P(\ell,\, c) \odot H(\ell,\, c) \odot \int_0^1 \left[\text{ꝉ} \left(\frac{1+q}{2} \right) + \text{ꝉ} \left(\frac{1-q}{2} \right) \right] dq \oplus 2H^2(\ell,\, c). \tag{93}$$

Corollary 19 By choosing $\varphi(q) = \dfrac{q^\alpha}{\Gamma(\alpha)}$ in Theorem 14, we derive the following inequality:

$$\frac{1}{\text{ꝉ}^2 \left(\frac{1}{2} \right)} \odot \widetilde{\mathfrak{S}} \left(\frac{\ell+c}{2} \right) \widetilde{G} \left(\frac{\ell+c}{2} \right)$$

$$\ominus \frac{2^\alpha \alpha}{(c-\ell)^\alpha \text{ꝉ} \left(\frac{1}{2} \right)} \odot \left[(FA) \int_{\frac{\ell+c}{2}}^{c} \left(\theta - \frac{\ell+c}{2} \right)^{\alpha-1} \odot [\widetilde{\mathfrak{S}}(\theta) + \widetilde{G}(\theta)] \odot H(\theta,\, \ell+c-\theta) d\theta \right.$$

$$\left. \oplus (FA) \int_{\ell}^{\frac{\ell+c}{2}} \left(\frac{\ell+c}{2} - \theta \right)^{\alpha-1} \odot [\widetilde{\mathfrak{S}}(\theta) \oplus \widetilde{G}(\theta)] \odot H(\ell+c-\theta,\, y) d\theta \right]$$

$$\supseteq_F \frac{2^\alpha \Gamma(\alpha+1)}{(c-\ell)^\alpha} \odot \left[J^\alpha_{\left(\frac{\ell+c}{2}\right)+} \widetilde{\mathfrak{S}}(c) \widetilde{G}(c) \oplus J^\alpha_{\left(\frac{\ell+c}{2}\right)-} \widetilde{\mathfrak{S}}(\ell) \widetilde{G}(\ell) \right] \tag{94}$$

$$\oplus \frac{\alpha 2^\alpha}{(c-\ell)^\alpha \text{ꝉ}^2 \left(\frac{1}{2} \right)} \odot \int_{\frac{\ell+c}{2}}^{c} \left(\theta - \frac{\ell+c}{2} \right)^{\alpha-1} H^2(\theta-\ell+c-\theta) d\theta$$

$$\oplus \alpha \odot \left\{ 2 \odot M(\ell,\, c) \odot \int_0^1 q^{\alpha-1} \text{ꝉ} \left(\frac{1+q}{2} \right) \text{ꝉ} \left(\frac{1-q}{2} \right) dq \right.$$

$$\oplus N(\ell,\, c) \odot \int_0^1 q^{\alpha-1} \left[\text{ꝉ}^2 \left(\frac{1+q}{2} \right) + \text{ꝉ}^2 \left(\frac{1-q}{2} \right) \right] dq$$

$$\left. \oplus P(\ell,\, c) \odot H(\ell,\, c) \odot \int_0^1 q^{\alpha-1} \left[\text{ꝉ} \left(\frac{1+q}{2} \right) + \text{ꝉ} \left(\frac{1-q}{2} \right) \right] dq \right\} + 2H^2(\ell,\, c).$$

6. Some exceptional cases

In this section, we examine specific instances derived from our main findings. Based on Theorem 5, we present the following result.

Corollary 20 If $\widetilde{\mathfrak{S}} : [\ell,\, c] \to \mathbb{F}_0$ is $U \cdot D$-fuzzy-valued γ-generalized strong ꝉ-convex mapping, then

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right) \ominus \frac{\varepsilon}{2\lambda\left(\frac{1}{2}\right)\Lambda(1)} \int_{\ell}^{c} \frac{\varphi(c-\theta)}{c-\theta} (\|2\varphi-(\ell+c)\|)^{\gamma} d\theta$$

$$\supseteq_F \frac{1}{2\Lambda(1)} \odot \left[{}_{\ell+}\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(c) \oplus_{c-} \mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(\ell)\right] \tag{95}$$

$$\supseteq_F \frac{\widetilde{\mathfrak{S}}(\ell)\oplus\widetilde{\mathfrak{S}}(c)}{2\Lambda(1)} \odot \int_0^1 [\lambda(q)+\lambda(1-q)] \frac{\varphi((c-\ell)q)}{q} dq \oplus \varepsilon(\|c-\ell\|)^{\gamma}.$$

Corollary 21 If $\widetilde{\mathfrak{S}},\ \widetilde{G}:[\ell,\ c]\to\mathbb{F}_0$ are two $U\cdot D$-fuzzy-valued γ-generalized strong λ-convex mappings, then

$$\frac{1}{2} \odot \left[{}_{\ell+}\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(c)\widetilde{G}(c) \oplus_{c-} \mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(\ell)\widetilde{G}(\ell)\right]$$

$$\supseteq_F K_1 \odot M(\ell,\ c) \oplus K_2 \odot N(\ell,\ c) \oplus K_3 \odot P(\ell,\ c)\varepsilon(\|c-\ell\|)^{\gamma} \oplus \lambda(1)\varepsilon^2(\|c-\ell\|)^{2\gamma}. \tag{96}$$

Corollary 22 If $\widetilde{\mathfrak{S}},\ \widetilde{G}:[\ell,\ c]\to\mathbb{F}_0$ are two $U\cdot D$-fuzzy-valued ε-generalized strong λ-convex mappings, then

$$\frac{1}{2} \odot \left[{}_{\ell+}\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(c)\widetilde{G}(c) \oplus_{c-} \mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(\ell)\widetilde{G}(\ell)\right]$$

$$\supseteq_F K_1 \odot M(\ell,\ c) \oplus K_2 \odot N(\ell,\ c) \oplus K_3 \odot P(\ell,\ c) \odot \varepsilon(\|c-\ell\|) \oplus \Lambda(1)\varepsilon^2(\|c-\ell\|)^2. \tag{97}$$

We get the following outcome from Theorem 7.
Corollary 23 If $\widetilde{\mathfrak{S}},\ \widetilde{G}:[\ell,\ c]\to\mathbb{F}_0$ are two $U\cdot D$-fuzzy-valued γ-generalized strong λ-convex mappings, then

$$\frac{1}{2\lambda^2\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\widetilde{G}\left(\frac{\ell+c}{2}\right)$$

$$\ominus \frac{\varepsilon}{2\Lambda(1)\lambda\left(\frac{1}{2}\right)} \odot \left[(FA)\int_{\ell}^{c} \frac{\varphi(c-\theta)}{c-\theta} \odot (\widetilde{\mathfrak{S}}(\theta)\oplus\widetilde{G}(\theta)) \odot (\|2\theta-(\ell+c)\|)^{\gamma} d\theta\right.$$

$$\left.\oplus (FA)\int_{\ell}^{c} \frac{\varphi(\theta-\ell)}{\theta-\ell} \odot (\widetilde{\mathfrak{S}}(\theta)\oplus\widetilde{G}(\theta)) \odot (\|\ell+c-2\theta\|)^{\gamma} d\theta\right]$$

$$\supseteq_F \frac{1}{2\Lambda(1)} \odot \left[{}_{\ell+}\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(c)\widetilde{G}(c) \oplus_{c-} \mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(\ell)\widetilde{G}(\ell)\right]$$

$$\oplus \frac{\varepsilon^2}{2\Lambda(1)\lambda^2\left(\frac{1}{2}\right)} \odot \int_\ell^c \frac{\varphi(c-\theta)}{c-\theta}(\|2\theta-(\ell+c)\|)^{2\gamma}d\theta$$

$$\oplus \frac{1}{\Lambda(1)} \odot [K_2 \odot M(\ell,\ c) \oplus K_1 \odot N(\ell,\ c) \oplus K_3 \odot P(\ell,\ c) \odot \varepsilon(\|c-\ell\|)^\gamma] \oplus \varepsilon^2(\|c-\ell\|)^{2\gamma}. \tag{98}$$

Corollary 24 If $\widetilde{\mathfrak{S}},\ \widetilde{G} : [\ell,\ c] \to \mathbb{F}_0$ are two $U \cdot D$-fuzzy-valued ε-generalized strong λ-convex mappings, then

$$\frac{1}{2\lambda^2\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\widetilde{G}\left(\frac{\ell+c}{2}\right)$$

$$\ominus \frac{\varepsilon}{2\Lambda(1)\lambda\left(\frac{1}{2}\right)} \odot \left[(FA)\int_\ell^c \frac{\varphi(c-\theta)}{c-\theta} \odot (\widetilde{\mathfrak{S}}(\theta) \oplus \widetilde{G}(\theta)) \odot (\|2\varphi-(\ell+c)\|)d\theta\right.$$

$$\left.\oplus (FA)\int_\ell^c \frac{\varphi(\theta-\ell)}{\theta-\ell} \odot (\widetilde{\mathfrak{S}}(\theta) \oplus \widetilde{G}(\theta))(\|\ell+c-2\theta\|)d\theta\right] \tag{99}$$

$$\supseteq_F \frac{1}{2\Lambda(1)} \odot \left[{}_{\ell+}\mathfrak{I}_\varphi\widetilde{\mathfrak{S}}(c)\widetilde{G}(c) \oplus_{c-} \mathfrak{I}_\varphi\widetilde{\mathfrak{S}}(\ell)\widetilde{G}(\ell)\right]$$

$$\oplus \frac{\varepsilon^2}{2\Lambda(1)\lambda^2\left(\frac{1}{2}\right)} \odot \int_\ell^c \frac{\varphi(c-\theta)}{c-\theta}(\|2\theta-(\ell+c)\|)^2 d\theta$$

$$\oplus \frac{1}{\Lambda(1)} \odot [K_2 \odot M(\ell,\ c) + K_1 \odot N(\ell,\ c) + K_3 \odot P(\ell,\ c) \odot \varepsilon(\|c-\ell\|)] \oplus \varepsilon^2(\|c-\ell\|)^2.$$

We get the following outcome from Theorem 8.

Corollary 25 If $\widetilde{\mathfrak{S}} : [\ell,\ c] \to \mathbb{F}_0$ is $U \cdot D$-fuzzy-valued γ-generalized strong λ-convex mapping, then

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right) \ominus \frac{\varepsilon}{2\psi(1)\lambda\left(\frac{1}{2}\right)} \int_{\frac{\ell+c}{2}}^c \frac{\varphi(c-\theta)}{c-\theta}(\|2\varphi-(\ell+c)\|)^\gamma d\theta$$

$$\supseteq_F \frac{1}{2\psi(1)} \odot \left[{}_{\left(\frac{\ell+c}{2}\right)}+\mathfrak{I}_\varphi\widetilde{\mathfrak{S}}(c) \oplus_{\left(\frac{\ell+c}{2}\right)} -\mathfrak{I}_\varphi\widetilde{\mathfrak{S}}(\ell)\right]$$

$$\supseteq_F \frac{[\widetilde{\mathfrak{S}}(\ell)\oplus\widetilde{\mathfrak{S}}(c)]}{2\psi(1)}\odot\int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\left[\lambda\left(\frac{2-q}{2}\right)+\lambda\left(\frac{q}{2}\right)\right]dq\oplus\varepsilon(\|c-\ell\|)^{\gamma}. \tag{100}$$

Corollary 26 If $\widetilde{\mathfrak{S}}:[\ell,\ c]\to\mathbb{F}_0$ is $U\cdot D$-fuzzy-valued ε-generalized strong λ-convex mapping, then

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\odot\frac{\varepsilon}{2\psi(1)\lambda\left(\frac{1}{2}\right)}\int_{\frac{\ell+c}{2}}^{c}\frac{\theta(c-\theta)}{c-\theta}(\|2\theta-(\ell+c)\|)d\theta$$

$$\supseteq_F \frac{1}{2\psi(1)}\odot\left[{}_{\left(\frac{\ell+c}{2}\right)^+}\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(c)\oplus{}_{\left(\frac{\ell+c}{2}\right)^-}\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(\ell)\right] \tag{101}$$

$$\supseteq_F \frac{[\widetilde{\mathfrak{S}}(\ell)\oplus\widetilde{\mathfrak{S}}(c)]}{2\psi(1)}\odot\int_0^1 \frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\left[\lambda\left(\frac{2-q}{2}\right)+\lambda\left(\frac{q}{2}\right)\right]dq\oplus\varepsilon(\|c-\ell\|).$$

We get the following outcome from Theorem 9.

Corollary 27 If $\widetilde{\mathfrak{S}},\ \widetilde{G}:[\ell,\ c]\to\mathbb{F}_0$ are two $U\cdot D$-fuzzy-valued γ-generalized strong λ-convex mappings, then

$$\begin{aligned}&\left[{}_{\left(\frac{\ell+c}{2}\right)}+\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(c)\widetilde{G}(c)\oplus{}_{\left(\frac{\ell+c}{2}\right)}-\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(\ell)\widetilde{G}(\ell)\right]\\&\supseteq_F K_4\odot M(\ell,\ c)\oplus K_5\odot N(\ell,\ c)\oplus\varepsilon^2(\|c-\ell\|)^{\gamma}\odot P(\ell,\ c)\odot K_6\oplus 2\psi(1)\varepsilon^2(\|c-\ell\|)^{2\gamma}.\end{aligned} \tag{102}$$

Corollary 28 If $\widetilde{\mathfrak{S}},\ \widetilde{G}:[\ell,\ c]\to\mathbb{F}_0$ are two $U\cdot D$-fuzzy-valued ε-generalized strong λ-convex mappings, then

$$\begin{aligned}&\left[{}_{\left(\frac{\ell+c}{2}\right)}+\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(c)\widetilde{G}(c)\oplus{}_{\left(\frac{\ell+c}{2}\right)}-\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(\ell)\widetilde{G}(\ell)\right]\\&\supseteq_F K_4\odot M(\ell,\ c)\oplus K_5\odot N(\ell,\ c)\oplus\varepsilon^2(\|c-\ell\|)\odot P(\ell,\ c)\odot K_6\oplus 2\psi(1)\varepsilon^2(\|c-\ell\|)^{2}.\end{aligned} \tag{103}$$

We get the following outcome from Theorem 10.

Corollary 29 If $\widetilde{\mathfrak{S}},\ \widetilde{G}:[\ell,\ c]\to\mathbb{F}_0$ are two $U\cdot D$-fuzzy-valued γ-generalized strong λ-convex mappings, then

$$\frac{1}{\lambda^2\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\widetilde{G}\left(\frac{\ell+c}{2}\right)$$

$$\ominus\frac{\varepsilon}{\lambda\left(\frac{1}{2}\right)\psi(1)}\odot\left[(FA)\int_{\frac{\ell+c}{2}}^{\ell}\frac{\varphi(c-\theta)}{c-\theta}\odot[\widetilde{\mathfrak{S}}(\theta)\oplus\widetilde{G}(\theta)]\odot(\|2\theta-(\ell+c)\|)^{\gamma}d\theta\right.$$

$$\oplus (FA)\int_{\ell}^{\frac{\ell+c}{2}} \frac{\varphi(\theta-\ell)}{\theta-\ell} \odot [\widetilde{\mathfrak{S}}(\theta) \oplus \widetilde{G}(\theta)] \odot (\|\ell+c-2\theta\|)^{\gamma} d\theta\Bigg]$$

$$\supseteq_F \frac{1}{\psi(1)} \odot \left[_{\left(\frac{\ell+c}{2}\right)} + \mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(c)\widetilde{G}(c) \oplus_{\left(\frac{\ell+c}{2}\right)} - \mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(\ell)\widetilde{G}(\ell)\right]$$

$$\oplus \frac{1}{\psi(1)\lambda^2\left(\frac{1}{2}\right)} \odot \int_{\frac{\ell+c}{2}}^{c} \frac{\varphi(c-\theta)}{c-\theta} \varepsilon^2 (\|2\theta-(\ell+c)\|)^{2\gamma} d\theta$$

$$\oplus \frac{1}{\psi(1)} \odot [K_5 \odot M(\ell,\ c) \oplus K_4 \odot N(\ell,\ c) \oplus \varepsilon(\|c-\ell\|)^{\gamma} \odot K_6 \odot P(\ell,\ c)] \oplus 2\varepsilon^2(\|c-\ell\|)^{2\gamma}. \tag{104}$$

Corollary 30 If $\widetilde{\mathfrak{S}},\ \widetilde{G} : [\ell,\ c] \to \mathbb{F}_0$ are two $U \cdot D$-fuzzy-valued ε-generalized strong λ-convex mappings, then

$$\frac{1}{\lambda^2\left(\frac{1}{2}\right)} \odot \widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\widetilde{G}\left(\frac{\ell+c}{2}\right)$$

$$\ominus \frac{\varepsilon}{\lambda\left(\frac{1}{2}\right)\psi(1)} \odot \Bigg[(FA)\int_{\frac{\ell+c}{2}}^{c} \frac{\varphi(c-\theta)}{c-\theta} \odot [\widetilde{\mathfrak{S}}(\theta) \oplus \widetilde{G}(\theta)] \odot (\|2\theta-(\ell+c)\|) d\theta$$

$$\oplus (FA)\int_{\ell}^{\frac{\ell+c}{2}} \frac{\varphi(\theta-\ell)}{\theta-\ell} \odot [\widetilde{\mathfrak{S}}(\theta) \oplus \widetilde{G}(\theta)] \odot (\|\ell+c-2\theta\|) d\theta\Bigg] \tag{105}$$

$$\supseteq_F \frac{1}{\psi(1)} \odot \left[_{\left(\frac{\ell+c}{2}\right)} + \mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(c)\widetilde{G}(c) \oplus_{\left(\frac{\ell+c}{2}\right)} - \mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}(\ell)\widetilde{G}(\ell)\right]$$

$$\oplus \frac{1}{\psi(1)\lambda^2\left(\frac{1}{2}\right)} \int_{\ell}^{c} \frac{\varphi(c-\theta)}{c-\theta} \varepsilon^2 (\|2\theta-(\ell+c)\|)^{2} d\theta$$

$$\oplus \frac{1}{\psi(1)} \odot [K_5 \odot M(\ell,\ c) + K_4 \odot N(\ell,\ c) \oplus \varepsilon(\|c-\ell\|) K_6 \odot P(\ell,\ c)] \oplus 2\varepsilon^2(\|c-\ell\|)^{2}.$$

We get the following outcome from Theorem 11.
Corollary 31 If $\widetilde{\mathfrak{S}} : [\ell,\ c] \to \mathbb{F}_0$ is $U \cdot D$-fuzzy-valued γ-generalized strong λ-convex mapping, then

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\ominus\frac{\varepsilon}{2\psi(1)\lambda\left(\frac{1}{2}\right)}\int_{\frac{\ell+c}{2}}^{c}\frac{\varphi\left(\theta-\frac{\ell+c}{2}\right)}{\theta-\frac{\ell+c}{2}}(\|2\theta-(\ell+c)\|)^{\gamma}d\theta$$

$$\supseteq_F\frac{1}{2\psi(1)}\odot\left[_{\ell+}\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\oplus_{c-}\mathcal{J}_{\varphi}\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\right] \tag{106}$$

$$\supseteq_F\frac{[\widetilde{\mathfrak{S}}(\ell)\oplus\widetilde{\mathfrak{S}}(c)]}{2\psi(1)}\odot\int_0^1\frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\left[\lambda\left(\frac{1+q}{2}\right)+\lambda\left(\frac{1-q}{2}\right)\right]dq\oplus\varepsilon(\|c-\ell\|)^{\nu}.$$

Corollary 32 If $\widetilde{\mathfrak{S}}:[\ell,\ c]\to\mathbb{F}_0$ is $U\cdot D$-fuzzy-valued ε-generalized strong λ-convex mapping, then

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\ominus\frac{\varepsilon}{2\psi(1)\lambda\left(\frac{1}{2}\right)}\int_{\frac{\ell+c}{2}}^{c}\frac{\varphi\left(\theta-\frac{\ell+c}{2}\right)}{\theta-\frac{\ell+c}{2}}(\|2\theta-(\ell+c)\|)d\theta$$

$$\supseteq_F\frac{1}{2\psi(1)}\odot\left[_{\ell+}J_{\varphi}\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\oplus_{c-}J_{\varphi}\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\right] \tag{107}$$

$$\supseteq_F\frac{[\widetilde{\mathfrak{S}}(\ell)\oplus\widetilde{\mathfrak{S}}(c)]}{2\psi(1)}\odot\int_0^1\frac{\varphi\left(\frac{(c-\ell)}{2}q\right)}{q}\left[\lambda\left(\frac{1+q}{2}\right)+\lambda\left(\frac{1-q}{2}\right)\right]dq\oplus\varepsilon(\|c-\ell\|).$$

We get the following outcome from Theorem 12.

Corollary 33 If $\widetilde{\mathfrak{S}},\ \widetilde{G}:[\ell,\ c]\to\mathbb{F}_0$ are two $U\cdot D$-fuzzy-valued γ-generalized strong λ-convex mappings, then

$$\left[_{\ell}+\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\widetilde{G}\left(\frac{\ell+c}{2}\right)\oplus_{c}-\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\widetilde{G}\left(\frac{\ell+c}{2}\right)\right]$$

$$\supseteq_F K_7\odot M(\ell,\ c)\oplus K_8\odot N(\ell,\ c)\oplus\varepsilon(\|c-\ell\|)^{\gamma}K_9\odot P(\ell,\ c)\oplus 2\psi(1)\varepsilon^2(\|c-\ell\|)^{2\gamma}. \tag{108}$$

Corollary 34 If $\widetilde{\mathfrak{S}},\ \widetilde{G}:[\ell,\ c]\to\mathbb{F}_0$ are two $U\cdot D$-fuzzy-valued ε-generalized strong λ-convex mappings, then

$$\left[_{\ell+}\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\widetilde{G}\left(\frac{\ell+c}{2}\right)\oplus_{c-}\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\widetilde{G}\left(\frac{\ell+c}{2}\right)\right]$$

$$\supseteq_F K_7\odot M(\ell,\ c)\oplus K_8\odot N(\ell,\ c)\oplus\varepsilon(\|c-\ell\|)K_9\odot P(\ell,\ c)\oplus 2\psi(1)\varepsilon^2(\|c-\ell\|)^2. \tag{109}$$

We get the following outcome from Theorem 13.

Corollary 35 If $\widetilde{\mathfrak{S}},\ \widetilde{G}:[\ell,\ c]\to\mathbb{F}_0$ are two $U\cdot D$-fuzzy-valued γ-generalized strong λ-convex mappings, then

$$\frac{1}{\lambda^2\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\widetilde{G}\left(\frac{\ell+c}{2}\right)$$

$$\ominus\frac{\varepsilon}{\lambda\left(\frac{1}{2}\right)\psi(1)}\odot\left[(FA)\int_{\frac{\ell+c}{2}}^{\ell}\frac{\varphi\left(\theta-\frac{\ell+c}{2}\right)}{\theta-\frac{\ell+c}{2}}\odot[\widetilde{\mathfrak{S}}(\theta)+\widetilde{G}(\theta)]\odot(\|2\theta-(\ell+c)\|)^{\gamma}d\theta\right.$$

$$\left.\oplus(FA)\int_{\ell}^{\frac{\ell+c}{2}}\frac{\varphi\left(\frac{\ell+c}{2}-\theta\right)}{\frac{\ell+c}{2}-\theta}\odot[\widetilde{\mathfrak{S}}(\theta)\oplus\widetilde{G}(\theta)]\odot(\|\ell+c-2\theta\|)^{\gamma}d\theta\right] \tag{110}$$

$$\supseteq_F\frac{1}{\psi(1)}\odot\left[{}_{\ell+}\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\widetilde{G}\left(\frac{\ell+c}{2}\right)\oplus_{c-}\mathfrak{I}_{\varphi}\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\widetilde{G}\left(\frac{\ell+c}{2}\right)\right]$$

$$\oplus\frac{\varepsilon^2}{\psi(1)\lambda^2\left(\frac{1}{2}\right)}\int_{\frac{\ell+c}{2}}^{\ell}\frac{\varphi\left(\theta-\frac{\ell+c}{2}\right)}{\theta-\frac{\ell+c}{2}}(\|2\theta-(\ell+c)\|)^{2\gamma}d\theta$$

$$\oplus\frac{1}{\psi(1)}\odot[K_8\odot M(\ell,\ c)\oplus K_7\odot N(\ell,\ c)\oplus\varepsilon(\|c-\ell\|)^{\gamma}K_9\odot P(\ell,\ c)]\oplus 2\varepsilon^2(\|c-\ell\|)^{2\gamma}.$$

Corollary 36 If $\widetilde{\mathfrak{S}},\ \widetilde{G}:[\ell,\ c]\to\mathbb{F}_0$ are two $U\cdot D$-fuzzy-valued ε-generalized strong λ-convex mappings, then

$$\frac{1}{\lambda^2\left(\frac{1}{2}\right)}\odot\widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\widetilde{G}\left(\frac{\ell+c}{2}\right)$$

$$\ominus\frac{\varepsilon}{\lambda\left(\frac{1}{2}\right)\psi(1)}\odot\left[(FA)\int_{\frac{\ell+c}{2}}^{c}\frac{\varphi\left(\theta-\frac{\ell+c}{2}\right)}{\theta-\frac{\ell+c}{2}}\odot[\widetilde{\mathfrak{S}}(\theta)\oplus\widetilde{G}(\theta)]\odot(\|2\theta-(\ell+c)\|)d\theta\right.$$

$$\left.\oplus(FA)\int_{\ell}^{\frac{\ell+c}{2}}\frac{\varphi\left(\frac{\ell+c}{2}-\theta\right)}{\frac{\ell+c}{2}-\theta}\odot[\widetilde{\mathrm{G}}(\theta)\oplus\widetilde{G}(\theta)]\odot(\|\ell+c-2\theta\|)d\theta\right]$$

$$\supseteq_F \frac{1}{\psi(1)} \odot \left[\ell + \mathfrak{I}_\varphi \widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\widetilde{G}\left(\frac{\ell+c}{2}\right) \oplus_{c-} \mathfrak{I}_\varphi \widetilde{\mathfrak{S}}\left(\frac{\ell+c}{2}\right)\widetilde{G}\left(\frac{\ell+c}{2}\right)\right]$$

$$\oplus \frac{\varepsilon^2}{\psi(1)\lambda^2\left(\frac{1}{2}\right)} \int_{\frac{\ell+c}{2}}^{c} \frac{\varphi\left(\theta - \frac{\ell+c}{2}\right)}{\theta - \frac{\ell+c}{2}} (\|2\theta - (\ell+c)\|)^2 d\theta$$

$$\oplus \frac{1}{\psi(1)} \odot [K_8 \odot M(\ell,\ c) \oplus K_7 \odot N(\ell,\ c) \oplus \varepsilon(\|c-\ell\|) \odot K_9 \odot P(\ell,\ c)] \oplus 2\varepsilon^2(\|c-\ell\|)^2. \tag{111}$$

Remark 2 By utilizing Theorems 5-13, where $(\theta,\ y) = -\mu q(1-q)\|y-\theta\|^2$ and $H(\theta,\ y) = \mu q(1-q)\left\|\frac{1}{y} - \frac{1}{\theta}\right\|^2$ for some $\mu > 0$, we can derive new inequalities for $U \cdot D$-fuzzy-valued strongly and relaxed λ convex mappings through $U \cdot D$-fuzzy-valued generalized fractional integrals.

7. Limitations of strong convex fuzzy mappings

The "strong convexity" condition is often too restrictive for real-world fuzzy problems. Many practical fuzzy mappings may only satisfy weak convexity or non-convexity, thus excluding a large class of useful models. By requiring strong convexity, the mapping becomes mathematically elegant but less general. This reduces its applicability to decision-making or optimization problems where criteria behave non-convexly. On the other hand, ensuring or verifying strong convexity in fuzzy mappings is computationally demanding, especially in high-dimensional fuzzy environments. For large-scale fuzzy systems, this can be impractical. Strong convex fuzzy mappings assume structured uncertainty. In reality, fuzzy data may exhibit irregular, non-symmetric, or even discontinuous behavior, which strong convexity cannot capture. In Multi-Criteria Decision-Making (MCDM), economics, and engineering optimization, imposing strong convexity may oversimplify problems and lead to less realistic solutions. Strong convexity is mainly advantageous for proving existence, uniqueness, and stability results in fuzzy fixed-point theory and optimization. However, its direct practical interpretability in applied decision-making remains limited.

8. Conclusion

A new class of mappings, termed $U \cdot D$-fuzzy-valued generalized strong λ-convex, is introduced and applied here to derive multiple results relevant to convex analysis and optimization theory. Additionally, new fractional integral operators are also introduced. Moreover, Hermite-Hadamard type, Hermite-Hadamard Fejér type, and Pachpatte type inequalities are also acquired. Some new and classical exceptional cases are also obtained. The authors anticipate that these findings will inspire further research across various domains in both pure and applied sciences like we can derive additional inequalities for $U \cdot D$-fuzzy-valued generalized strong λ-convex mappings by utilizing Theorems 5-13 with suitable alternatives for the mapping $\varphi(q) = \frac{q^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$; $\varphi(q) = q(\theta - q)^{\alpha-1}$ for $\alpha \in (0,\ 1)$; $\varphi(q) = \frac{q}{\alpha}\exp(-Aq)$, where $A = \frac{1-\alpha}{\alpha}$ for $\alpha \in (0,\ 1)$, where $A = \frac{1-\alpha}{\alpha}$ for $\alpha \in (0,\ 1)$. Furthermore, generalized fractional integrals can be employed to derive novel and interesting inequalities for appropriate selections of the mapping $\lambda(q) = 1$; $\lambda(q) = q$; $\lambda(q) = q^s$; $\lambda(q) =$

q^{-s}; $\lambda(q) = q(1-q)$; $\lambda(q) = \dfrac{\sqrt{q}}{2\sqrt{1-q}}$, and so forth. Interested readers are invited to explore these details as we do not provide the proofs here.

Conflict of interest

The authors declare no competing financial interest.

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