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Feng-Liu's Type Fixed Point Theorem in Modular Function Spaces

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Abstract: We establish a Feng-Liu's type fixed point theorem within the framework of modular function spaces. Using this result, we investigate the existence of solutions to a Volterra-type integral inclusion in modular function spaces. The generalized setting enables us to handle multivalued mappings and broader classes of integral inclusions, thus extending classical fixed point theorems to a more flexible analytical environment.

***Keywords*:** Feng-Liu's fixed point theorem, multivalued mapping, modular function spaces

MSC: 47H10, 54H25

1. Introduction

Many real-world phenomena, such as optimal control systems, economics, and population dynamics, are governed by differential inclusions. In ordinary differential equations of the form $y'(s) = g(s, y(s))$, the derivative $y'(s)$ naturally inherits the regularity properties of the function g and the function $y(s)$. However, unlike standard differential equations, differential inclusions of the form $y'(s) \in G(s, y(s))$ introduce additional difficulties due to their set-valued nature, requiring advanced mathematical tools for analysis. This topic, first studied by Zaremba [1] in 1934 and Marchaud [2] in 1938, gained renewed interest in the 1960s with applications in control theory and optimization. A detailed discussion on this can be found in [3].

A fundamental approach to proving the existence of solutions to differential inclusions involves fixed point principles. Several researchers have investigated integral equations in the space of all ρ-continuous functions from $[0, 1]$ into L_ρ, employing methods such as the Banach contraction principle and Brouwer's fixed point theorem (see references [4–7]). Taleb-Hanebaly [5] utilized the Banach contraction principle to address integral equations in modular function spaces, considering the Δ_2-type condition. Hajji-Hanebaly [6, 7] and Bachar [8] used the argument in [5].

Since integral equations in modular function spaces often arise in the study of dynamic systems with uncertainty, it is natural to extend these problems to integral inclusions, where the right-hand side is set-valued. This generalization allows for more flexibility in modeling real-world phenomena, such as differential inclusions and optimization problems with constraints. Specifically, while a classical integral equation takes the form:

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$$u(t) = v(t) + \int_0^t f(t, s, h(s))ds,$$

where f is a single-valued function, many applications require considering multi-valued mappings $F(t, s, h(s))$, leading to integral inclusion:

$$u(t) \in v(t) + \int_0^t F(t, s, h(s))ds, \tag{1}$$

where $u, v \in \mathscr{C}_\rho([0, A], L_\rho)$, and $F : [0, A] \times [0, A] \times L_\rho \twoheadrightarrow R$ is lower semi-continuous.

Our work extends previous results on integral equations.

The modern study of fixed point theorems for multivalued mappings was initiated by Nadler, whose result in metric spaces inspired a large number of extensions and applications. In this line, Feng–Liu [9] introduced an important fixed point theorem for multivalued contractive mappings and multivalued Caristi-type mappings, which has since generated an extensive body of research. Several authors have worked on generalizing the Feng–Liu framework. For instance, Nicolae [10] established fixed point theorems for multivalued mappings of Feng–Liu type, while Mohammad et al. [11] investigated multivalued contractive and Caristi-type mappings in more applied contexts. More recently, further extensions of Feng–Liu-type results have appeared, including contributions in Nashine et al. [12] and Türkoğlu et al. [13], which emphasize the continued interest in relaxing contractive conditions and expanding applicability. These studies collectively highlight the central role of Feng–Liu-type theorems in advancing the theory of multivalued fixed points. The present work distinguishes itself by establishing a Feng–Liu-type fixed point theorem in the setting of modular function spaces. First introduced systematically by Kozlowski, modular function spaces generalize Banach spaces by replacing the norm structure with a modular, while still retaining important analytical tools such as the Δ_2-condition and the Fatou property. This flexibility allows one to handle broader classes of mappings and integral inclusions than are possible in Banach or even w-distance spaces. In particular, modular function spaces provide a natural and powerful framework for problems where weaker topological assumptions are more appropriate. In summary, our contributions are as follows:

• We establish a Feng–Liu-type fixed point theorem in modular function spaces under a new modular contractive condition.

• We apply the result to prove the existence of solutions to a Volterra-type integral inclusion.

The rest of this paper is organized as follows. In Section 2, we present the necessary background on modular function spaces and their properties. Section 3 develops our main results concerning the existence of solutions to Volterra-type integral inclusion.

2. Preliminaries

Before presenting our main results, we introduce key concepts from modular theory, following the work of Kozlowski [14].

Definition 1 [14, Preliminaries, p.88]

A functional $\rho : \mathscr{V} \to [0, \infty]$ defined on a vector space called a **pseudomodular** if there holds for arbitrary $u, v \in \mathscr{V}$:

(A) $\rho(0) = 0$;

(B) $\rho(cu) = \rho(u)$ for every $c \in K$ ($K = C$ or $K = R$) such that $|c| = 1$;

(C) $\rho(cu + dv) \leq \rho(u) + \rho(v)$ for every $c, d \geq 0$, $c + d = 1$.

If in place of (C) there holds

(C') $\rho(cu+dv) \leq c^s\rho(u) + d^s\rho(v)$ for every $c, d \geq 0, c^s + d^s = 1, s \in (0, 1]$, then ρ is called **s-convex**; when $s = 1$, it is **convex**.

Replacing condition (A) with

(A') $\rho(0) = 0$ and $\rho(\lambda u) = 0$ for every $\lambda > 0$ implies $u = 0$, then ρ is called a **semimodular**.

Additionally, if

(A'') $\rho(0) = 0$ and $\rho(u) = 0$ implies $u = 0$, then ρ is called a **modular**.

Definition 2 [15, Definition 1.4]

If ρ is a pseudomodular in $\mathscr{V}$, then the **modular space** is defined as

$$\mathscr{V}_\rho = \{u \in \mathscr{V}\colon \lim_{\lambda \to 0} \rho(\lambda u) = 0\}.$$

Remark 1 The modular ρ lacks subadditivity and therefore does not exhibit the characteristics of a norm or a distance. We will require the following notions.

Definition 3 [15, p. 2]

$\|\cdot\| : \mathscr{V} \to [0, \infty]$ is an F-pseudonorm if

(i) $\|0\| = 0$.

(ii) For $c \in K$ with $|c| = 1$, $\|cu\| = \|u\|$.

(iii) $\|u+v\| \leq \|u\| + \|v\|$.

(iv) If $c_k \to c$ and $\|u_k - u\| \to 0$, then $\|c_k u_k - cu\| \to 0$.

If, moreover (i') $\|u\| = 0$ implies $u = 0$, then $\|\cdot\|$ is called an F-norm.

If $\|\cdot\|$ satisfies the above conditions (i) – (iii) and the condition (iv') $\|\alpha u\| = |\alpha|^s \|u\|$, for $0 < s \leq 1$, then $\|\cdot\|$ is called an s-pseudonorm and adding (i') we obtain an s-norm in $\mathscr{V}$ which is denoted $\|\cdot\|^s$. If $s = 1$, then $\|\cdot\|$ is called a norm.

Definition 4 [14, Preliminaries, p. 88]

If ρ is modular (pseudomodular) in $\mathscr{V}$, then

$$|u|_\rho = \inf\{\alpha > 0\colon \rho(u/\alpha) \leq \alpha\}$$

is an F-norm (F-pseudonorm) in $\mathscr{V}_\rho$.

If ρ is s-convex modular (s-convex pseudomodular) in $\mathscr{V}$, then the functional

$$\|u\|_\rho^s = \inf\{\alpha > 0\colon \rho(u/\alpha^{1/s}) \leq 1\}$$

is an s-norm (s-pseudonorm) in $\mathscr{V}_\rho$ (a norm (pseudonorm) for $s = 1$ for which $\|u\|_\rho^1 = \|u\|_\rho$).

Assume $\Omega \neq \emptyset$, a nontrivial σ-algebra Σ and a nontrivial σ-ring P of subsets of Ω. Let $A \cap B \in P$ for any $B \in \Sigma$ and $A \in P$. Assume that there exists an increasing sequence of sets $H_n \in P$ such that $\Omega = \bigcup H_n$. Define E as the linear space of all simple functions with supports from P. Let M denote the space of all measurable functions. The characteristic function of A is denoted by 1_A.

Definition 5 [16, Definition 3]

If a functional $\rho : E \times \Sigma \to [0, \infty]$ satisfies:

(P_1) for any $A \in \Sigma$, $\rho(0, A) = 0$,

(P_2) $\rho(p, A) \leq \rho(q, A)$ whenever $|p(w)| \leq |q(w)|$ for each $w \in \Omega$, $A \in \Sigma$ and $p,\ q \in E$,

(P_3) for any $p \in E$, $\rho(p, .) : \Sigma \to [0, \infty]$ is a σ-subadditive measure,

(P_4) $\rho(\gamma, B) \to 0$ as $\gamma \downarrow 0$ for every $B \in P$, where $\rho(\gamma, B) = \rho(\gamma 1_B, B)$,

(P_5) if there exits $\gamma > 0$ such that $\rho(\gamma, B) = 0$, then $\rho(\eta, B) = 0$ for each $\eta > 0$,

(P_6) for any $\gamma > 0$, $\rho(\gamma, .)$ is order continuous on P, that is $\rho(\gamma, B_n) \to 0$ for every sequence $\{B_n\} \subset P$ such that $B_n \downarrow \emptyset$,

Then ρ is called **function modular**. To extend the function modular ρ to the space of measurable functions M, we approximate each p in M using simple functions. Since every measurable function can be expressed as the pointwise limit of an increasing sequence of simple functions [17, p. 62], the definition of ρ is then extended to measurable p in M as

$$\rho(p, A) = \sup\{\rho(q, A)\colon q \in E,\ |q(w)| \leq |p(w)| \forall w \in \Omega\}.$$

This ensures that the modular preserves the fundamental properties established for simple functions in E and extends naturally to the larger space M. This allows us to define $\rho(\gamma, A)$ for sets A that are not in P. For convenience, we denote it simply as $\rho(p)$ instead of $\rho(p, \Omega)$.

Definition 6 [18, Definition 2.1.3]

If for all $\gamma > 0$, $\rho(\gamma, A) = 0$ then the set A is called ρ-null. A property $\mathfrak{b}(w)$ holds ρ-almost everywhere if the exceptional set of elements in Ω such that $\mathfrak{b}(w)$ does not hold is ρ-null.

Theorem 1 [18, Theorem 2.1.4]

The functional ρ from M to $[0, \infty]$ is a modular. The modular function space induced by function modular ρ is given by

$$L_\rho = \{p \in M\colon \rho(\alpha p) \to 0 \,\text{as}\, \alpha \to 0\}.$$

Definition 7 [19, Definition 2.3]

A function modular ρ satisfies the Δ_2-condition if $\{p_n\} \subset M$, $E_i \in \Sigma$, $E_i \downarrow \emptyset$ and $\sup_{n\geq 1} \rho(p_n, E_i) \to 0$ as $i \to \infty$, then

$$\sup_{n\geq 1} \rho(2p_n, E_i) \to 0.$$

Definition 8 [19, Definition 2.4]

A function modular ρ satisfies the Δ_2-type condition if there is an $M > 0$ such that

$$\rho(2p) \leq M\rho(p) \quad \text{for each} \quad p \in L_\rho.$$

The Δ_2-type condition guarantees the Δ_2-condition. However, the reverse implication does not necessarily hold.

Definition 9 [20, Definition 3.4]

1. $\{p_n\} \subset L_\rho$ is ρ-convergent to $p \in L_\rho$ if $\rho(p_n - p) \to 0$ and written as $p_n \to p\ (\rho)$.

2. A sequence $\{p_n\} \subset L_\rho$ is called ρ-Cauchy if $\rho(p_n - p_m) \to 0$ as $n, m \to \infty$. 3. A set $B \subset L_\rho$ is called ρ-closed if for any sequence $\{p_n\}$ in B, the convergence $p_n \to p\ (\rho)$ implies that $p \in B$. 4. A set $B \subset L_\rho$ is called ρ-bounded if

$$\delta_\rho(B) = \sup\{\rho(p - q)\colon p \in B, q \in B\} < \infty.$$

5. ρ has the Fatou property if $\rho(p - q) \leq \rho(p_n - q_n)$ whenever $p_n \to p\ (\rho)$ and $q_n \to q\ (\rho)$.

Theorem 2 [18, Theorem 2.3.7]

$(L_\rho, \|\cdot\|_\rho)$ is a complete metric space.

Proposition 1 [18, Proposition 3.1.6]

If ρ satisfies the Δ_2-condition, then convergence in norm and convergence in modular are equivalent. This equivalence extends to cases where the Δ_2-type condition is satisfied. In the following discussion, we assume that ρ is a convex function modular with the Δ_2-type condition.

Definition 10 [20, Definition 3.7]

Define a growth function ψ as follows

$$\psi(z) = \sup\left\{\frac{\rho(zp)}{\rho(p)} \colon p \in L_\rho, 0 < \rho(p) < \infty\right\}, \quad \text{for all} \quad 0 \le z < \infty.$$

Lemma 1 [20, Lemma 3.1]

The following are the properties of the growth function ψ:

1. for all $z \in [0, \infty)$, $\psi(z) < \infty$.
2. ψ from $[0, \infty)$ to $[0, \infty)$ is both convex and strictly increasing. Therefore, ψ is continuous.
3. For all $\gamma, \eta \in [0, \infty)$, $\psi(\gamma\eta) \le \psi(\gamma)\psi(\eta)$.
4. For all $\gamma, \eta \in [0, \infty)$, $\psi^{-1}(\gamma)\psi^{-1}(\eta) \le \psi^{-1}(\gamma\eta)$, where the function ψ^{-1} is the inverse of ψ.

The subsequent lemma demonstrates that ψ can provide an upper bound.

Lemma 2 [20, Lemma 3.1]

Let ρ be as above. Then

$$\|p\|_\rho \le \frac{1}{\psi^{-1}\left(\frac{1}{\rho(p)}\right)} \text{ whenever } p \in L_\rho \setminus \{0\}.$$

Let $C \subseteq L_\rho$. $\mathscr{C}_\rho(C)$ represents the collection of all nonempty ρ-closed subsets of C. The map $H : \mathscr{C}_\rho(L_\rho) \times \mathscr{C}_\rho(L_\rho) \to R^+$ defined as

$$H_\rho(A, B) = \max\{\sup_{p\in A} dist_\rho(p, B), \sup_{q\in B} dist_\rho(q, A)\}, \quad A, B \in \mathscr{C}_\rho(L_\rho)$$

is the generalized Hausdorff distance over $\mathscr{C}_\rho(L_\rho)$, where $dist_\rho(p, B) = \inf\{\rho(p-q) \colon q \in B\}$. A fixed point of a multivalued mapping $T : C \to \mathscr{C}_\rho(C)$ is a point satisfying $p \in Tp$. A mapping $\lambda : C \to R$ is ρ-lower semi-continuous [20, remark 3.4] if, for every sequence $\{p_n\}$ in C such that $p_n \to p$ (ρ) implies

$$\lambda(p) \le \lim_{n\to\infty} \inf \lambda(p_n).$$

Dhompongsa [19] generalized contraction theorem in modular function spaces in the following manner (also see [21]).

Theorem 3 [19, Theorem 3.1]

Let ρ be as above and C be a non-empty ρ-bounded ρ-closed subset of L_ρ. Assume that $T : C \to \mathscr{C}_\rho(C)$ be a ρ-contraction mapping, i.e., there is $k \in [0, 1)$ such that

$$H_\rho(Tp, Tq) \leq k\rho(p-q), \quad p, q \in C. \tag{2}$$

Then T has a fixed point.

3. Main results

This section focuses on the solution of a Volterra-type integral inclusion in $\mathscr{C}_\rho([0, A], L_\rho)$. We will solve the integral inclusion

$$u(t) \in v(t) + \int_0^t F(t, s, h(s))ds, \tag{3}$$

where $u, v \in \mathscr{C}_\rho([0, A], L_\rho)$, and $F: [0, A] \times [0, A] \times L_\rho \twoheadrightarrow P_{cv}(R)$ is lower semi-continuous, $P_{cv}(R)$ refers to the class of nonempty closed and convex subset of R. We use some notions from [7, 8].

Definition 11 [8, Definition 3.1]

Consider ρ to be a convex function modular and $u: [0, A] \to L_\rho$. Then u is ρ-continuous at $t_0 \in [0, A]$, if $\rho(u(t) - u(t_0))$ as $t \to t_0$.If ρ satisfies the Δ_2-condition, then any ρ-continuous function in L_ρ is also $\|\cdot\|_\rho$-continuous function in L_ρ. Let $\mathscr{C}_\rho([0, A], L_\rho)$ be the space of all ρ-continuous mappings from $[0, A]$ into L_ρ. Now define $\rho_{\mathscr{C}_\rho}: \mathscr{C}_\rho([0, A], L_\rho) \to [0, \infty]$ by

$$\rho_{\mathscr{C}_\rho}(u) = \sup_{t \in [0, A]} \rho(u(t)), \tag{4}$$

for any $u \in \mathscr{C}_\rho([0, A], L_\rho)$. For any nonempty subset $B \subset L_\rho$, the set $\mathscr{C}_\rho([0, A], B)$ represents the collection of functions $u \in \mathscr{C}_\rho([0, A], L_\rho)$ such that $u([0, A]) \subset B$.

Now consider the following lemma.

Lemma 3 [5, Proposition II.1]

Let $B \subset L_\rho$ be a nonempty ρ-closed bounded subset of L_ρ. Then:

1. $\rho_{\mathscr{C}_\rho}$ is a convex function modular satisfying Δ_2-type condition and the Fatou property.
2. $(\mathscr{C}_\rho([0, A], L_\rho)$ is $\rho_{\mathscr{C}_\rho}$-complete.
3. $(\mathscr{C}_\rho([0, A], B)$ is $\rho_{\mathscr{C}_\rho}$-closed, bounded subset of $(\mathscr{C}_\rho([0, A], L_\rho)$.

Before proving the principal result, we first establish the preliminary theorem, which will play a crucial role in its proof. First, let us define the set I_b^p.

Assume that $T: L_\rho \to N(L_\rho)$ be a multivalued mapping. Define a function $f: L_\rho \to R^+$ as

$$f(p) = dist_\rho(p, Tp).$$

For a constant $b \in (0, 1)$, define the set $I_b^p \subset L_\rho$ as

$$I_b^p = \{q \in T(p) \,|\, b\rho(p-q) \leq dist_\rho(p, Tp)\}.$$

Let us analyze the set I_b^p:

Case I

(a) Let $p \in Tp$ and $p = q$, then

$$b\rho(p-q) = b\rho(p-p) \le dist_\rho(p, Tp) = 0.$$

(b) Let $p, q \in Tp$ and $p \ne q$, then

$$dist_\rho(p, Tp) = 0$$

but $\rho(p-q) \ne 0$ which means that the inequality $b\rho(p-q) \le dist_\rho(p, Tp)$ cannot be satisfied for any $b \in (0, 1)$, that is the set $I_b^p = \{p\}$ if $p \in Tp$.

Case II

Assume that $p \notin Tp$, then there exists an element $q \in Tp$ such that $q \ne p$. By definition of infimum there exists $\varepsilon > 0$ such that

$$\rho(p-q) \le dist_\rho(p, Tp) + \varepsilon.$$

Choose $\varepsilon = (1-b)\rho(p-q)$, then

$$b\rho(p-q) \le dist_\rho(p, Tp),$$

this implies that $q \in I_b^p$. Thus $I_b^p \ne \emptyset$.

Theorem 4 Let ρ be a convex function modular satisfying Δ_2-type condition and having the Fatou property. Let C be a non-empty ρ-bounded, ρ-closed subset of L_ρ and $T: C \to \mathscr{C}_\rho(C)$. If there is a constant $c \in (0, 1)$ such that for any $p \in C$ there exists $q \in I_b^p$ satisfying

$$dist_\rho(q, Tq) \le c\rho(p-q), \tag{5}$$

then T has a fixed point in C provided $c < b$ and f is lower semi-continuous.

Proof. Since $Tp \in \mathscr{C}_\rho(C)$ for any $p \in C$, $I_b^p \ne \emptyset$ for any $b \in (0, 1)$. For $p_0 \in C$, there exists $p_1 \in I_b^{p_0}$ then from (5)

$$dist_\rho(p_1, Tp_1) \le c\rho(p_0 - p_1).$$

Similarly, for $p_1 \in C$, there is $p_2 \in I_b^{p_1}$ satisfying

$$dist_\rho(p_2, Tp_2) \le c\rho(p_1 - p_2).$$

Proceeding in this manner, we obtain a sequence $\{p_n\}_{n=0}^{\infty}$, where $p_{n+1} \in I_b^{p_n}$ and

$$dist_{\rho}(p_{n+1}, Tp_{n+1}) \leq c\rho(p_n - p_{n+1}), \quad n = 0, 1, 2, \ldots. \tag{6}$$

Now, we show that $\{p_n\}_{n=0}^{\infty}$ is a Cauchy sequence. On the other hand, $p_{n+1} \in I_b^{p_n}$ implies

$$b\rho(p_n - p_{n+1}) \leq dist_{\rho}(p_n, Tp_n), \quad n = 0, 1, 2, \ldots. \tag{7}$$

From (6) and (7) we have

$$dist_{\rho}(p_n, Tp_n) \leq \frac{c}{b} dist_{\rho}(p_{n-1}, Tp_{n-1}) \leq \ldots \leq \frac{c^n}{b^n} dist_{\rho}(p_0, Tp_0), \quad n = 0, 1, 2, \ldots. \tag{8}$$

$$\rho(p_n - p_{n+1}) \leq \frac{c}{b}\rho(p_{n-1} - p_n) \leq \ldots \leq \frac{c^n}{b^n}\rho(p_0 - p_1) \quad n = 0, 1, 2, \ldots. \tag{9}$$

Without loss of generality, we may assume that $\rho(p_n - p_{n+1}) \neq 0$. Hence

$$\frac{1}{\left(\frac{c}{b}\right)^n \rho(p_0 - p_1)} \leq \frac{1}{\rho(p_n - p_{n+1})}.$$

By Lemma 1, we get

$$\psi^{-1}\left(\frac{1}{\frac{c}{b}}\right)^n \psi^{-1}\left(\frac{1}{\rho(p_0 - p_1)}\right) \leq \psi^{-1}\left(\frac{1}{\rho(p_n - p_{n+1})}\right).$$

Therefore,

$$\frac{1}{\psi^{-1}\left(\frac{1}{\rho(p_n - p_{n+1})}\right)} \leq \frac{1}{\psi^{-1}\left(\frac{1}{\frac{c}{b}}\right)^n \psi^{-1}\left(\frac{1}{\rho(p_0 - p_1)}\right)}.$$

By Lemma 2, we have

$$\begin{aligned} \|p_n - p_{n+1}\|_{\rho} &\leq \frac{1}{\psi^{-1}\left(\frac{1}{\rho(p_n - p_{n+1})}\right)} \\ &\leq \left(\frac{1}{\psi^{-1}\left(\frac{1}{\frac{c}{b}}\right)}\right)^n \left(\frac{1}{\psi^{-1}\left(\frac{1}{\rho(p_0 - p_1)}\right)}\right). \end{aligned}$$

Since ψ^{-1} is strictly increasing and $c < b$, we have $\frac{c}{b} < 1$, which implies

$$frac1\psi^{-1}\left(\frac{1}{\frac{c}{b}}\right) < 1.$$

This implies that $\{p_n\}$ is a Cauchy sequence in $(L_\rho, \|.\|_\rho)$. By Theorem 2 , there exists $\zeta \in L_\rho$ such that $\{p_n\}$ is $\|.\|_\rho$-convergent to ζ. Because the Δ_2-type condition holds then $\{p_n\} \to \zeta\ (\rho)$ and $\zeta \in C$ since C is ρ-closed. Now, we must demonstrate that ζ is a fixed point of mapping T. From (8)

$$dist_\rho(p_n, Tp_n) \leq \frac{c}{b} dist_\rho(p_{n-1}, Tp_{n-1}) \tag{10}$$

$$< dist_\rho(p_{n-1}, Tp_{n-1}). \tag{11}$$

Hence, $\{dist_\rho(p_n, Tp_n)\}$ is a decreasing sequence, so it converges to some non-negative real number, say b. By taking limit in (10) we get $b = 0$.

Since f is lower semi-continuous, then

$$dist_\rho(p, Tp) = f(p) \leq \liminf_n f(p_n) = \liminf_n dist_\rho(p_n, Tp_n) = 0.$$

Hence, $p \in Tp$, since Tp is closed, it follows that p is fixed point of T. □

Note: The strict requirement $c < b$ is analogous to the condition in the classical Feng–Liu theorem and is essential to preserve the contraction effect; the weaker form $c \leq b$ would in general fail to guarantee convergence.

Remark 2 Since $dist_\rho(q, Tq) \leq H_\rho(Tp, Tq)$ for $q \in Tp$ and $I_b^p \subset Tp$, we have subsequent result.

Corollary 1 Let ρ be a convex function modular satisfying the Δ_2-type condition and let C be a non-empty ρ-bounded ρ-closed subset of L_ρ. Suppose $T : C \to \mathscr{C}_\rho(C)$ be multivalued mapping. If there is $c \in (0, 1)$ such that

$$dist_\rho(q, Tq) \leq c\rho(p - q) \quad \text{for each } p \in C \text{ and } q \in Tp,$$

then T has a fixed point in C provided f is lower semi-continuous.

Remark 3 Corollary 1 extends the results of Theorem 3.

Now let us provide an example to validate Theorem 4.

Example 1 The real number system R is a space modulared by $\rho(p) = |p|$. Let $C = \{\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\} \cup \{0, 1\}$. Clearly, C is a non-empty ρ-closed and ρ-bounded subset of R. Define $T : C \to \mathscr{C}_\rho(C)$ as

$$Tp = \begin{cases} \{\frac{1}{2^{n+1}}, 1\} & \text{if} \quad p = \frac{1}{2^n},\ n = 0, 1, 2, \ldots, \\ \\ \{0, \frac{1}{2}\} & \text{if} \quad p = 0. \end{cases}$$

The function f is given by

$$f(p) = dist_\rho(p, Tp) = \begin{cases} \frac{1}{2^{n+1}} & \text{if} \quad p = \frac{1}{2^n},\ n = 1, 2, \ldots, \\ \\ 0 & \text{if} \quad p = 0, 1, \end{cases}$$

f is continuous. Moreover, For instance, if $p = \frac{1}{2}$, then we may choose $q = \frac{1}{4} \in Tp$. In this case

$$dist_\rho(q, Tq) = \frac{1}{8} = \frac{1}{2}\rho(p-q).$$

Therefore, there exists $q \in I_{0.7}^p$ for any $p \in X$ such that

$$dist_\rho(q, Tq) = \frac{1}{2}\rho(p-q).$$

Note that all the requirements of Theorem 4 are satisfied. Therefore, T has a fixed point. Observe that the mapping T does not satisfy the assumptions of Theorem 3. However, observe

$$H_\rho\left(T\left(\frac{1}{2^n}\right), T(0)\right) = \frac{1}{2} \geq \frac{1}{2^n} = \left|\frac{1}{2^n} - 0\right| = dist_\rho\left(\frac{1}{2^n}, 0\right), \quad n = 1, 2, \ldots.$$

Now we present our main result.

We shall consider (3) under the circumstances as follows:

(1) $s \mapsto F(t, s, h(s))$ is Lebesgue measurable over $[0, A]$ and $t \mapsto \int_0^t F(t, s, h(s)) \in \mathscr{C}_\rho([0, A], L_\rho)$,

(2) there is $\beta > 1$ such that

$$H_\rho(F(t, t_i, h(t_i)) - F(t, t_i, h'(t_i))) \leq \frac{1}{\beta}(\rho(h(t_i) - h'(t_i))),$$

(3) $\sum_{i=0}^{n-1}(t_{i+1} - t_i) = 1$.

Consider the multivalued mapping $\mathscr{T} : \mathscr{C}_\rho([0, A], L_\rho) \twoheadrightarrow \mathscr{C}_\rho([0, A], L_\rho)$ defined by

$$\mathscr{T}(h)(t) = \{u \in \mathscr{C}_\rho([0, A], L_\rho) \colon u(t) \in v(t) + \int_0^t F(t, s, h(s))ds,\ t \in [0, A]\} \tag{12}$$

The solution of (3) is equivalent to determining a fixed point of $\mathscr{T}$.

Let $h, g \in \mathscr{C}_\rho([0, A], L_\rho)$ and let $m \in \mathscr{T}(h)$, then we have

$$m(t) \in v(t) + \gamma\int_0^t F(t, s, h(s))ds.$$

By Michael's selection theorem [22], there is a continuous function $f_h : [0, \text{A}] \times [0, A] \to R$ so that $f_h(t, s) \in F_h(t, s)$ so

$$m(t) = v(t) + \gamma \int_0^t f_h(t, s)ds.$$

Let $\mathfrak{P} = \{0 = t_0, t_1, \ldots, t_n = t\}$ be a subdivision of $[0, A]$. Then,

$$\int_0^t f_h(t, s)ds = \sum_{i=0}^{n-1} (t_{i+1} - t_i) f(t, t_i, h(t_i))$$

$\sum_{i=0}^{n-1} (t_{i+1} - t_i) f_h(t, ti)$ is $\|.\|_\rho$ convergent hence ρ-convergent to $\int_0^t f_h(t, s)ds$ when $|\mathfrak{P}| = \sup_{i=0, 1, \cdots, n-1} |t_{i+1} - t_i|$ $\to 0$ as $n \to \infty$. By the Fatou property

$$\rho(m(t) - n(t)) \leq \liminf \rho\left(\sum_{i=0}^{n-1} (t_{i+1} - t_i)\left(f_h(t, t_i) - f_{h'}(t, t_i)\right)\right).$$

By convexity of ρ, we obtain

$$\leq \liminf \sum_{i=0}^{n-1} (t_{i+1} - t_i)\left(\rho(f_h(t, t_i) - f_{h'}(t, t_i)\right)$$

$$\leq \frac{1}{\beta}\rho(h(t_i) - h'(t_i)) \liminf \sum_{i=0}^{n-1} (t_{i+1} - t_i)$$

$$\leq \frac{1}{\beta} \sup_{t_i \in [0, A]} \rho(h(t_i) - h'(t_i))$$

$$= \frac{1}{\beta}\rho_{\mathscr{C}_\rho}(h - h').$$

Now, by changing the rules of h and h', we have

$$dist_\rho(h', \mathscr{T}h') \leq \frac{1}{\beta}\rho_{\mathscr{C}_\rho}(h - h'), \quad \forall\, h' \in \mathscr{T}h.$$

Thus, all the requirements of Theorem 4 are satisfied. Therefore, $\mathscr{T}$ has a fixed point which is the solution of (3).

4. Conclusion

In this work, we generalized Feng–Liu's fixed point theorem within modular function spaces. By constructing appropriate contraction-type conditions and using properties of the modular ρ, we established sufficient criteria for the existence of fixed points for multivalued mappings. We then applied this result to the Volterra-type integral inclusion and proved the existence of solutions under mild assumptions.

This study highlights the effectiveness of modular function spaces for handling multivalued mappings and integral inclusions. Future research may explore generalizations to fractional and stochastic integral equations, iterative schemes such as the variational iteration method, and nonlinear frequency formulations [23], further demonstrating the analytical strength and broad applicability of our results.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Zaremba S. On an extension of the concept of differential equation. *Proceedings of the Academy of Sciences (Paris)*. 1934; 199: 1278-1280.

[2] Marchaud A. On the derivatives and differences of functions of real variables. *Journal of Pure and Applied Mathematics*. 1938; 9(17): 337-425.

[3] Aubin JP, Frankowska H. *Set-Valued Analysis*. Birkhäuser; 2009.

[4] Bardaro C, Musielak J, Vinti G. *Nonlinear Integral Operators and Applications*. Vol. 9. Walter de Gruyter; 2003.

[5] Taleb A, Hanebaly E. A fixed point theorem and its application to integral equations in modular function spaces. *Proceedings of the American Mathematical Society*. 1999; 127(8): 2335-2342.

[6] Hajji A, Hanebaly E. Perturbed integral equations in modular function spaces. *Electronic Journal of Qualitative Theory of Differential Equations*. 2003; 2003: 1-7.

[7] Hajji A, Hanebaly E. Fixed point theorem and its application to perturbed integral equations in modular function spaces. *Electronic Journal of Differential Equations*. 2005; 2005: 1-11.

[8] Bachar M. Nonlinear Fredholm equations in modular function spaces. *Electronic Journal of Differential Equations*. 2019; 2019: 1-9.

[9] Feng Y, Liu S. Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings. *Journal of Mathematical Analysis and Applications*. 2006; 317(1): 103-112.

[10] Nicolae A. Fixed point theorems for multi-valued mappings of Feng-Liu type. *Fixed Point Theory*. 2011; 12(1): 145-154.

[11] Mohammad M, Bhardwaj R, Kabir QA, Barve SK, Dassani M. Fixed point theorems for multi-valued contractive mappings and Caristi-type mappings. *Materials Today: Proceedings*. 2020; 29: 625-632.

[12] Nashine HK, Pant R. Feng-Liu type fixed point theorems for w-distance spaces and applications. *Filomat*. 2022; 36(11): 3899-3917.

[13] Türkoğlu D, Manav N. Feng-Liu type fixed point results for multivalued mappings in GMMS. *Mathematics*. 2019; 7(11): 1031.

[14] Kozlowski WM. Notes on modular function spaces. I. *Mathematical Commentaries*. 1988; 28(1): 91-104.

[15] Musielak J. *Orlicz and Modular Spaces*. Vol. 1034. Springer; 1983.

[16] Khamsi MA. Fixed point theory in modular function spaces. In: *Recent Advances on Metric Fixed Point Theory (Seville, 1995)*. Universidad de Sevilla, Sevilla; 1996. p.31-57.

[17] Royden HL, Fitzpatrick PM. *Real Analysis*. 4th ed. Prentice-Hall Inc.; 2010.

[18] Kozlowski WM. *Modular Function Spaces*. Vol. 122. Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker Inc.; 1988.

[19] Dhompongsa S, Domínguez Benavides T, Kaewcharoen A, Panyanak B. Fixed point theorems for multivalued mappings in modular function spaces. *Japanese Journal of Mathematical Sciences*. 2006; 63(2): 161-170.

[20] Khamsi MA, Kozlowski WM. *Fixed Point Theory in Modular Function Spaces*. Birkhäuser; 2015.

[21] Kutbi MA, Latif A. Fixed points of multivalued maps in modular function spaces. *Fixed Point Theory and Applications*. 2009; 2009: 1-12.

[22] Michael E. Continuous selections. In: *Encyclopedia of General Topology*. Elsevier; 2003. p.107-109.

[23] Petković M, Petronijević M, Mijalković N. On frequency formulation of nonlinear vibration systems. *Facta Universitatis, Series: Mechanical Engineering*. 2023; 21(3): 137-150.